Finance Theory

Homework 1, due 04/03/08.

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1. Consider an investor endowed with the VNM utility function $u(W) = \frac{W^{1-\rho}}{1-\rho}$, $0 < \rho < 1$, where $W$ is the investor’s date-1 wealth. The investor is faced with two traded assets at date 0, one with risky rate of return $\bar{r}$, and the other with sure rate of return $r_f$ (which is a lending and borrowing opportunity). Returns are generated at date 1. Let $W_0$ and $\alpha^*$ be the investor’s initial wealth and the amount of money that she chooses to spend on the risky asset.

(i) Show that if $\rho = \frac{1}{2}$, $r_f = 0$, and $\bar{r} = \left\{\begin{array}{l} 2, \quad \text{with probability } \frac{1}{2}, \\ -\frac{1}{2}, \quad \text{with probability } \frac{1}{2}, \end{array}\right.$

then the following portfolio is optimal for the investor:

$$
\begin{bmatrix}
\frac{\alpha^*}{W_0} \\
1 - \frac{\alpha^*}{W_0}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}.
$$

(ii) Continue with part (i). If $W_0$ is equal to 100,000, how much money does the investor have to lend or borrow at date 0 in order to implement the above optimal portfolio strategy?

2. Recall the sets $\mathbf{Z}$ and $\mathbf{P}$ defined in section 2 of Lecture 1. We shall assume that $\mathbf{Z}$ is a finite subset of $\mathbb{R}$, so that we can arrange its $n$ elements into a column vector $\mathbf{z}$. Define the column vector $\mathbf{z}^2$ as such that its $i$-th element is equal to the square of the $i$-th element of $\mathbf{z}$. Recall that $\mathbf{P}$ is the set of all possible lotteries with outcomes in the set $\mathbf{Z}$. With $\mathbf{Z}$ having $n$ possible outcomes, a lottery $\mathbf{p} \in \mathbf{P}$ is an $n$-vector with its (non-negative) elements summing up to one (that is, $\mathbf{p}$ is a prob. distribution on $\mathbf{Z}$). Given $\mathbf{p}$, the expected prize and the variance of the prize are respectively $\mathbf{p}'\mathbf{z}$ and $\mathbf{p}'\mathbf{z}^2 - (\mathbf{p}'\mathbf{z})^2$. We say that
an investor has a *mean-variance utility function* \( V(\cdot) \) defined on on \( P \), if for all lotteries \( p \in P \), for a given parameter \( \rho > 0 \),

\[
V(p) = p'z - \rho[p'z^2 - (p'z)^2].
\]

Define a binary relation \( \succ \) as such that

\[
\forall p, q \in P, \; p \succ q :\iff V(p) > V(q).
\]

Determine if \( \succ \) satisfies respectively Axiom 1, Axiom 2, and Axiom 3 listed in section 2 of Lecture 1.

3. Suppose that the price system \((X, p)\) in a two-period frictionless economy is such that

\[
X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \; p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},
\]

where \( p_2 > p_1 > 0 \) are two given constants. Note that there are three possible date-1 states and 2 traded assets at date 0.

(i) Show that if the price system admits of no arbitrage opportunities, then an infinite number of state price vectors exist, but each state price vector \( f_{3 \times 1} \) is a convex combination\(^1\) of the following two weak state price vectors:

\[
f_1 = \begin{bmatrix} p_1 \\ 0 \\ p_2 \end{bmatrix}, \; f_2 = \begin{bmatrix} 0 \\ p_1 \\ p_2 - p_1 \end{bmatrix}.
\]

(ii) Suppose that the three date-1 states are equally likely. Mr. A seeks to maximize

\[
E[\log(\hat{c})] = \sum_{k=1}^{3} \frac{1}{3} \log(c_k),
\]

where

\[
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = e_{3 \times 1} + \sum_{j=1}^{2} q_jx_j,
\]

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\(^1\)Let \( x, y \) be two elements of a real vector space \( V \). We say that \( ax + by \) is a *linear combination* of \( x \) and \( y \), if \( a, b \in \mathbb{R} \). We say that \( ax + by \) is a *convex combination* of \( x \) and \( y \), if \( a, b \in [0, 1] \) and \( a + b = 1 \).
with
\[ x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \]
and
\[ e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

Here recall that $e$ is Mr. A’s date-1 endowed wealth. Find the optimal portfolio strategy $q_{2 \times 1}$ and the corresponding optimal consumption plan $c_{3 \times 1}$ for Mr. A.

4. There are three agents (investors) in a two-period frictionless economy. There are three traded assets at date 0, each with a net supply of one unit. Agent $i$ is endowed with the entire unit of asset $i$, $i = 1, 2, 3$. Let asset 1 be the numeraire (so that its equilibrium price is one), and let the equilibrium price for asset 2 and asset 3 be denoted by $p$ and $q$. Asset $i$ pays a random cash flow $z_i > 0$ at date 1. The three agents have identical CRRA utility function $u(W) = W^{1-\rho}/(1-\rho)$, where $0 < \rho < 1$. Derive explicit formulae for the equilibrium prices $p$ and $q$.

5. Suppose that there are two investors in the date-0 frictionless financial market, both having von Neumann-Morgenstern utility function $u(\cdot) = \log(\cdot)$.

\[ u(\cdot) = \log(\cdot). \]

You can first find the optimal consumption plan for Mr. A by solving the following maximization program:
\[ \max_{c_1, c_2, c_3} \sum_{k=1}^{3} \frac{1}{3} \log(c_k), \]
subject to
\[ c_1' f_1 \leq e' f_1, \]
\[ c_2' f_2 \leq e' f_2. \]

Then, upon obtaining $c^*$, you can find $q^*$ by solving the system of equations
\[ c^* = e + Xq^*. \]

This is the so-called Cox and Huang approach.
There are one risky asset and one riskless asset available for trading at date 0. The riskless rate of return is $r_f$. The risky asset is a common stock, which has 3 shares outstanding. Each share of the stock generates a random cash flow $\tilde{x} > 0$ at date . For $i = 1, 2$, investor $i$ is endowed with $i$ shares of the stock. Nobody is endowed with the riskless asset. (Think of the riskless asset as a lending and borrowing opportunity, so that the net supply of the riskless asset is zero.) Let $D_i(P)$ be investor $i$’s date-0 demand function for the stock (risky asset), where $P$ denotes the date-0 stock price.

(i) Show that given the stock price $P$, $D_i(P)$ must satisfy the following first-order condition for the investor’s maximization problem:

$$E[\frac{\tilde{x}}{P} - 1 - r_f \frac{1}{(i - D_i)P(1 + r_f) + D_i \tilde{x}}] = 0.$$  

(Again, we are assuming that such an interior solution exists. 3)

(ii) A competitive equilibrium for the date-0 financial markets is a tuple $(P^*, D_1(\cdot), D_2(\cdot))$ such that $D_1(P)$ and $D_2(P)$ satisfy the first-order condition in part (i) given any stock price $P$, and $D_1(P^*) + D_2(P^*) = 3$ (the market-clearing condition). Show that in a competitive equilibrium the two investors must hold the same portfolio, and moreover, the equilibrium stock price is

$$P^* = \frac{1}{E[\frac{1+r_f}{\tilde{x}}]}.$$  

6. Suppose that there are two investors in the two-period frictionless economy, where for $i = 1, 2$, investor $i$ has von Neumann-Morgenstern utility

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3Recall that in the two-asset portfolio problem, a necessary condition for the maximization problem

$$\max_{a \in \mathbb{R}} f(a) \equiv E[u(W_0(1 + r_f) + a(\bar{r} - r_f))]$$

to have an interior solution $a^*$ (that satisfies $f'(a^*) = 0$) is that the two events $\{\bar{r} > r_f\}$ and $\{\bar{r} < r_f\}$ both can occur with a strictly positive probability. A sufficient condition for the existence of an interior solution $a^*$ is, by applying the Intermediate Value Theorem to the continuous function $f'()$, the following Inada condition:

$$f'(0) > 0 > \lim_{y \to \pm\infty} f'(y).$$

So, we assume that the Inada condition holds for our current problem.
function \( u(z) = -e^{-\rho z}, \quad \forall z \in \mathbb{R}, \)

where \( \rho_2 > \rho_1 > 0. \)

There are one risky asset and one riskless asset available for trading at date 0. The riskless rate of return is \( r_f. \) The risky asset is a common stock, which has 2 shares outstanding, with the two investors each holding one share before trading starts at date 0. Nobody is endowed with the riskless asset, so that the riskless asset is in zero net supply.

Let \( x \sim N(\mu, \sigma^2) \) be the date-1 cash flow generated by one share of the common stock, where \( \mu > r_f \geq 0. \)

(i) Let \( P \) be the date-0 stock price. Let \( D_i(P) \) be investor \( i \)'s demand for the common stock at date 0, given that the stock price is \( P. \) Find \( D_1(\cdot) \) and \( D_2(\cdot). \)

(ii) Write down the market clearing condition, and obtain the equilibrium stock price \( P^*. \)

(iii) Plug \( P^* \) into \( D_1(\cdot) \) and \( D_2(\cdot) \) and determine which investor is buying the stock, and which investor is selling the stock in equilibrium at date 0.

(iv) Which one between the two investors is borrowing in equilibrium at date 0? Which one is lending? Why?

7. There are 3 agents in a two-period frictionless economy with a single consumption good. Agent 0 owns a firm, which generates 1 unit of consumption at date 1, and he wants to consume at date 0 only. Agents 1 and 2 seek to maximize

\[ u(c_0) + E[u(c_1)], \]

where \( u(c) = \log(c). \) There are two equally likely states at date 1. In state \( i, \) agent \( i \) is endowed with \( X > 0 \) units of consumption, but agent \( j \) is endowed with nothing, where \( i, j = 1, 2. \) Both agents are also endowed with one unit of consumption at date 0.

(i) First suppose that the entrepreneur issues one share of common stock to sell the firm. Show that the equilibrium firm value is \( \frac{2 + 2X}{2 + 3X}, \)
and both agents 1 and 2 hold $\frac{1}{2}$ shares of the common stock.

(ii) Next, suppose instead that the entrepreneur issues the two Arrow-Debreu securities to the market, with prices $p_1$ and $p_2$. Let $a_{ij}$ be the number of shares of the $i$-th Arrow-Debreu security held by agent $j$. Show that in equilibrium

$$a_{11} = a_{22} = \frac{1 - X}{2}, \quad a_{12} = a_{21} = \frac{1 + X}{2}, \quad p_1 = p_2 = \frac{1}{2 + X}$$

so that the firm value in this case is

$$\frac{2}{2 + X},$$

and verify that

$$\frac{2}{2 + X} < \frac{2 + 2X}{2 + 3X}.$$ 

(iii) Conclude that a firm’s changing its capital structure may change its value in equilibrium and that a firm’s value may be higher when markets are incomplete than complete. Interpret.

(iv) Now suppose instead that $X < 5$ and $u(c) = 100c - \frac{1}{3}c^3$, for $c \in [0, 10]$. Determine if the firm value is higher when the firm issues the two Arrow-Debreu securities than when it issues common stock only.

8. An investor is endowed with a VNM utility function $u : \mathbb{R} \to \mathbb{R}$, with $u' > 0 > u''$. Moreover, it is known that for some constant $x > 0$,

$$-\frac{u''(z)}{u'(z)} > -\frac{u''(x + z)}{u'(x + z)}, \quad \forall z \in \mathbb{R}.$$ 

Show that there exists $f' > 0 > f''$ such that

$$u(z) = f(u(x + z)), \quad \forall z \in \mathbb{R}.$$