Finance Theory

Homework 2, due 04/17/2008

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Class meet: Room 204, Management Building 1

1. Do exercise 1 in Lecture 2.

2. Consider the following economy with two investors U and V. There are two equally likely states at date 1, denoted by $\omega_1$ and $\omega_2$. U and V have the following date-1 random endowments: U is endowed with 1 unit of consumption in state $\omega_1$ and nothing in state $\omega_2$; and V is endowed with 1 unit of consumption in both date-1 states. U and V only care about date-1 consumption, and each of them seeks to maximize the expected utility from date-1 consumption. Let $u(\cdot)$ and $v(\cdot)$ be respectively U’s and V’s VNM utility functions. It is assumed that at date 0, the Arrow-Debreu securities for state $\omega_1$ and state $\omega_2$ are both traded, with their equilibrium prices being denoted by 1 and $\phi$ respectively.
(i) Write down respectively U’s and V’s maximization problems. Form the Lagrangian for each of the two maximization programs, and denote by $\mu_u$ and $\mu_v$ the Lagrange multipliers for respectively U’s and V’s budget constraints. From now on, assume that $u(\cdot) = v(\cdot) = \log(\cdot)$, and show that $\mu_u = 2$ and $\mu_v = \frac{2}{1+\phi}$.
(ii) Solve the Walrasian equilibrium for this economy. That is, find $(c^*_u(\omega_1), c^*_u(\omega_2), c^*_v(\omega_1), c^*_v(\omega_2), \phi)$, such that (A) given $\phi$, the date-1 consumption plan $(c^*_u(\omega_1), c^*_u(\omega_2))$ is budget feasible for U and it maximizes U’s expected utility from date-1 random consumption; (B) given $\phi$, the date-1 consumption plan $(c^*_v(\omega_1), c^*_v(\omega_2))$ is budget feasible for V and it maximizes V’s expected utility from date-1 random consumption; and (C) (the markets-clearing
condition) $c_u^*(\omega_1) + c_v^*(\omega_1) = 1 + 1 = 2$ and $c_u^*(\omega_2) + c_v^*(\omega_2) = 0 + 1 = 1$.

(iii) Show that in equilibrium, $\mu_v = \frac{2}{3}$, and

$$c_u^*(\omega_i) = \frac{1}{4}e(\omega_i) \equiv s_u(e(\omega_i)), \quad c_v^*(\omega_i) = \frac{3}{4}e(\omega_i) \equiv s_v(e(\omega_i)), \quad i = 1, 2,$$

where $e(\omega_i)$ is the aggregate date-1 wealth in state $\omega_i$. That is, the competitive equilibrium allocation can be re-produced by implementing the sharing rule $\{s_u(\cdot), s_v(\cdot)\}$.

(iv) Now, imagine an artificial economy with one single investor whose date-1 endowment is $e(\omega_i)$ in state $\omega_i$. This investor’s VNM utility function for date-1 consumption is defined as

$$\frac{1}{\mu_u}u(s_u(\cdot)) + \frac{1}{\mu_v}v(s_v(\cdot))$$

Again, assume that both Arrow-Debreu securities are traded at date 0, but denote the equilibrium prices by 1 and $\psi$ instead. Show that $\phi = \psi$.\footnote{This is the content of the Representative Agent Theorem. Note that in a competitive equilibrium only relative prices matter, and hence we can always pick one Arrow-Debreu security as the numeraire. Here we have picked the Arrow-Debreu security that corresponds to state $\omega_1$ as the numeraire.}

3. Suppose that you are an investment consultant and you want to help a risk averse client choose an optimal portfolio at date 0. Your client can invest 1 dollar in $n$ risky assets at date 0, and the assets generate cash flows at date 1. Let $\tilde{r}_j$ be the rate of return on asset $j$. Suppose that the joint density function of $(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n)$ is symmetric about the vector $\mu \mathbf{1}_{n \times 1}$, where $\mu \geq 0$, and $\mathbf{1}$ is the vector with all its elements equal to 1.

(i) Show that if your client spends $w_i$ dollars on asset $i$, then the rate of return on this portfolio strategy is $\sum_{i=1}^n w_i \tilde{r}_i$. (Hint: Use the definition of rate of return. See section 8 of Lecture 1.)

(ii) Suppose that your only information about your client is that he has no time preference (i.e., he feels indifferent about one dollar at date 0 and one dollar at date 1), and he is risk averse (i.e., his VNM utility function is concave). If you want to recommend a portfolio of the $n$ traded assets that maximizes his expected utility, what should be your recommendation?
4. A firm needs a random amount \( \tilde{x} \) of some input, where \( \tilde{x} \in [0, +\infty) \) has density \( f(x, \rho) \) and distribution function \( F(x, \rho) \), with \( \rho \) indicating a mean-preserving spread: \( \rho \geq \rho' \Rightarrow F(x, \rho) \geq_{SSD} F(x, \rho') \). The firm can purchase some amount \( x_1 \) of the input now at the price of 1 before learning the realization of \( \tilde{x} \). After learning the realization \( x \) of \( \tilde{x} \), the firm must order an additional amount \( x - x_1 \) in case \( x > x_1 \), which will cost \( (x - x_1)^2 \); but if instead \( x \leq x_1 \), then the firm will do nothing and leave idle the un-needed amount \( (x_1 - x) \). (We are assuming that this input cannot be resold.) The firm wants to determine the optimal amount \( x_1 \) to be purchased right now, in order to minimize the expected total input costs. Show that the optimal \( x_1^* \) increases with \( \rho \).

5. A manufacturer offers a two-part tariff \((w, f)\) to a retailer that is interested in selling the product that the manufacturer produces, where \( w \) is the unit (wholesale) price for the product and \( f \) is a fixed franchise fee. Without bargaining power, the retailer can either reject the manufacturer’s offer (so that the game ends) or he can accept it. In case the manufacturer’s offer is accepted, the retailer then learns his unit cost \( c \), which is drawn from the distribution function \( F(c; r) \), where \( r \) is a parameter such that \( F(c; r) \) second-order stochastically dominates \( F(c; r') \) if \( r < r' \). That is, a higher \( r \) implies a higher uncertainty in the retailer’s unit marketing cost \( c \). Given \((w, f, c)\), and faced with the demand curve \( q(p) \), where \( q'(\cdot) < 0 \), the retailer then choose the retail price \( p \) to maximize his own profit. When making the offer \((w, f)\), the manufacturer has rational expectations about the retailer’s subsequent behavior, and the manufacturer knows that the retailer is willing to cooperate as long as cooperation generates non-negative profits. Show that an increase in \( r \) benefits the manufacturer if the manufacturer and the retailer are both risk neutral (i.e. caring only about expected profit).

6. Consider two risky assets with rates of return \( \tilde{r}_1 \) and \( \tilde{r}_2 \), and denote the expected value and standard deviation of \( \tilde{r}_j \) by \( \mu_j \) and \( \sigma_j \). Suppose that

\[
\mu_1 > \mu_2, \quad \sigma_1 > \sigma_2.
\]

A portfolio of these two assets can be conveniently denoted by \((w, 1 - \)
where $w$ is the portfolio weight assigned to (the percentage of the initial wealth spent on) asset 1.

(i) Find the portfolio for the two assets with the smallest variance of rate of return. From now on, we refer to this portfolio the *minimum variance portfolio* of assets 1 and 2, or simply the *mvp*. Can the *mvp* turn out to be asset 1 alone? If it can, when does this happen? Can it be asset 2 alone? If it can, when does this happen?

(ii) Now, suppose that the two risky assets are the only two traded assets at date 0. Suppose also that every investor is endowed with a mean-variance utility function (as defined in Problem 2 of Homework 1). That is, every investor’s welfare is increasing in the expected value and decreasing in the variance of the rate of return on the portfolio that he chooses to hold at date 0. Suppose furthermore that

$$\mu_1 < \mu_2, \sigma_1 > \sigma_2.$$ 

Can there be an investor with a mean-variance utility function that is willing to take a long position in asset 1 at date 0? Does your answer depend on whether the two assets are in positive supply?

7. Assume that the Sharpe-Lintner CAPM holds in the date-0 financial markets. We shall derive a pricing formula for an asset with date-1 random cash flow $x$. Let its date-0 price be $P_x$. Recall from Lecture 1 that

$$r_x \equiv \frac{x}{P_x} - 1.$$ 

Now since the CAPM holds, we have

$$E(r_x) = r_f + \frac{\text{cov}(r_x, r_m)}{\text{var}(r_m)}[E(r_m) - r_f].$$

(i) From the above two equations, deduce the following *certainty equivalent* pricing formula for asset $x$:

$$P_x = \frac{E(x) - \lambda \text{cov}(x, r_m)}{1 + r_f},$$

Here you must detail the first-order and second-order conditions.
where the constant
\[ \lambda = \frac{E(r_m) - r_f}{\text{var}(r_m)}. \]

(ii) Now we apply the certainty equivalent formula to an investment problem. Mr. X is the CEO of a large company, considering taking one of the following two mutually exclusive investment projects, A and B. The features of these two projects can be summarized as follows.

- Both incur a date-0 cash outflow of $1,000;
- Both generate a sure date-1 cash revenue equal to $1,500;
- The two projects differ in their date-1 cash expenses (denoted by $C_A$ and $C_B$ respectively). There are three equally likely date-1 states, referred to as boom, average, and recession. The following table summarizes the date-1 cash expenses of the two projects and the realized rate of return on the market portfolio in each of the three date-1 states.

<table>
<thead>
<tr>
<th>states</th>
<th>prob.</th>
<th>$C_A$</th>
<th>$C_B$</th>
<th>$r_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>boom</td>
<td>1/3</td>
<td>500</td>
<td>600</td>
<td>20%</td>
</tr>
<tr>
<td>average</td>
<td>1/3</td>
<td>400</td>
<td>400</td>
<td>10%</td>
</tr>
<tr>
<td>recession</td>
<td>1/3</td>
<td>300</td>
<td>200</td>
<td>0%</td>
</tr>
</tbody>
</table>

Which project between A and B has a higher variance of date-1 cash flow? Which project has a higher NPV (net present value) at date 0?

8. Suppose that in a two-period economy there are 3 traded assets, labeled 1, 2 and 3. Suppose that the following data are valid and short sale is completely prohibited (that is, the portfolio weights are all required to be non-negative).

\[
\mathbf{e} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.4 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0.01 & -0.02 & -0.03 \\ -0.02 & 0.04 & 0.06 \\ -0.03 & 0.06 & 0.16 \end{bmatrix}.
\]

Find the portfolio frontier (i.e. the frontier portfolio $\mathbf{w}^*(\mu)$ for each target expected rate of return $\mu \in \mathbb{R}$). (Hint: Set up a minimization program with inequality and equality constraints, and apply Kuhn-Tucker theorem.)