Finance Theory

Lecture 1: Expected Utility Theory, Walrasian Equilibrium, and Radner Equilibrium

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1. A decision maker facing a set $A$ of feasible alternatives is said to be rational if she is endowed with a weak preference relation (a binary relation that is transitive and complete) $\succeq$ on $A$. From the weak preference $\succeq$, we can derive the strict preference $\succ$ as follows. For all $a, b \in A$, $a \succ b$ if it is not true that $b \succeq a$. We say that a utility function $u : A \to \mathbb{R}$ represents $\succeq$ if and only if for all $a, b \in A$,

$$u(a) > u(b) \iff a \succ b.$$ 

2. Let $P$ be the set of all feasible lotteries that take values in a finite consumption set $Z$. A lottery $p$ is simply a probability distribution over the set $Z$ which generates consumption level $z \in Z$ with probability $p(z)$. We shall assume that $Z$ has $N$ elements. An investor facing $P$ is rational if she is endowed with a weak preference $\succeq$ on $P$. Consider the following axioms.

**Axiom 1** $\succeq$ is a preference relation on $P$.

**Axiom 2** For all $p, q, r \in P$ and $a \in (0, 1)$, $p \succ q \Rightarrow ap + (1 - a)r > aq + (1 - a)r$.\(^1\)

**Axiom 3** For all $p, q, r \in P$, $p \succ q \succ r \Rightarrow ap + (1 - a)r \succ q \succ bp + (1 - b)r$ for some $a, b \in (0, 1)$.

3. The following lemmas follow from the above three axioms.

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\(^1\)Here $ap + (1 - a)r$ represents the lottery which yields the consumption level $z \in Z$ with probability $ap(z) + (1 - a)r(z)$. Throughout this note, we shall assume that the following axiom of reduction holds: the investor feels indifferent about the simple lottery $ap + (1 - a)r$ and the compound lottery that with probability $a$ he gets to take the lottery $p$ and with probability $1 - a$ he gets to take the lottery $r$. Note that a (two-stage) compound lottery is simply a probability distribution on $P$. 

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Lemma 1  \( p \succ q \) and \( 0 \leq a < b \leq 1 \) imply that \( bp + (1-b)q \succ ap + (1-a)q \).

Proof  Let \( w = \frac{b-a}{1-a} \in (0,1) \), axiom 2 implies that \( r \equiv wp + (1-w)q \succ q \). Applying axiom 2 again, we have

\[
bp + (1-b)q = ap + (1-a)r \succ ap + (1-a)q.
\]

Lemma 2  \( p \succeq q \succeq r \) and \( p \succ r \) imply the unique existence of \( a \in [0,1] \) such that \( q \sim ap + (1-a)r \).

Proof  If \( p \sim q \) then \( a = 1 \) will do, and by lemma 1, for all \( b \in [0,1) \), \( q \succ bp + (1-b)r \), showing the uniqueness. The case of \( p \succ q \sim r \) is similar: \( a = 0 \) will do, and by lemma 1, for all \( b \in (0,1] \), we have \( bp + (1-p)r \succ 0 \cdot p + (1-0) \cdot r = r \sim q \), showing the uniqueness. It remains to consider the case of \( p \succ q \succ r \). Define

\[
U = \{ a \in [0,1] : ap + (1-a)r \succ q \},
\]

\[
I = \{ a \in [0,1] : ap + (1-a)r \sim q \},
\]

\[
L = \{ a \in [0,1] : q \succ ap + (1-a)r \}. \]

By axiom 1, \( U, I, \) and \( L \) form a partition of \([0,1]\). Apparently \( 1 \in U \) and \( 0 \in L \). By axiom 3, \( U \) and \( L \) respectively have elements different from 0 and 1. Moreover, \( U \) and \( L \) both have the continuum property: If \( a_1 < a_2 \) and \( a_1, a_2 \in U \), then by lemma 1 \( a \in U \) for all \( a \in (a_1, a_2) \). The same is true for \( L \). It follows that \( U \) and \( L \) are both intervals. Observe also that by lemma 1 \( I \) is either empty or a singleton. Suppose that \( I \) were empty and we would have a contradiction. Since \([0,1]\) is connected and \( U \) and \( L \) form a partition for \([0,1]\), one between \( U \) and \( L \) must be closed: either for some \( a, L = [0,a] \) and \( U = (a,1] \) or \( L = [0,a] \) and \( U = [a,1] \). Suppose that the former were true. Then \( u \in U \) if and only if \( u > a \), and we have

\[
uu p + (1-u)r \succ q \succ ap + (1-a)r;
\]

by the fact that \( u \in U \) and \( a \in L \). Axiom 3 then implies that there exists a \( b \in (0,1) \) such that

\[
q \succ b[uv + (1-u)r] + (1-b)[ap + (1-a)r] \in U,
\]
a contradiction! The case where $L = [0, a)$ and $U = [a, 1]$ can be ruled out analogously. It follows that $I$ is nonempty, and hence a singleton.

**Lemma 3** $p > q$ and $r > s$ and $a \in [0, 1]$ imply that $ap + (1 - a)r > aq + (1 - a)s$.

**Proof** Repeatedly applying axiom 2, we have

$$ap + (1 - a)r > aq + (1 - a)r > aq + (1 - a)s.$$  

**Lemma 4** $p \sim q$ and $a \in [0, 1]$ imply that $p \sim ap + (1 - a)q$.

**Proof** Suppose instead that $p > ap + (1 - a)q$. Then $q > ap + (1 - a)q$ also. By lemma 3, we have

$$ap + (1 - a)q > a[ap + (1 - a)q] + (1 - a)[ap + (1 - a)q] = ap + (1 - a)q,$$

a contradiction. The case where $ap + (1 - a)q > p$ can be ruled out analogously.

**Lemma 5** $p \sim q$ and $a \in [0, 1]$ imply that $ap + (1 - a)r \sim aq + (1 - a)r$ for all $r \in P$.

**Proof** Fix $p \sim q$. For $r \in P$ such that $p \sim q \sim r$, the assertion follows easily from lemma 4. Thus we consider the case $p \sim q > r$ (the remaining case is similar). Suppose that $ap + (1 - a)r > aq + (1 - a)r$, and we shall demonstrate a contradiction. By lemma 1, we have

$$ap + (1 - a)r > aq + (1 - a)r > r.$$  

By lemma 2, there must exist $b \in (0, 1)$ such that

$$b[ap + (1 - a)r] + (1 - b)[r] > aq + (1 - a)r;$$  

or equivalently,

$$abp + (1 - ab)r > aq + (1 - a)r.$$
However, by lemma 1, we also have

\[ q \sim p \succ bp + (1 - b)r, \]

so that by axiom 2, we have

\[ aq + (1 - a)r \succ a[bp + (1 - b)r] + (1 - a)r = abp + (1 - ab)r, \]

a contradiction.

**Lemma 6** There exist \( z^*, z_\ast \in \mathbf{Z} \) such that \( P_{z^*} \succeq p \succeq P_{z_\ast} \) for all \( p \in \mathbf{P} \), where \( P_z \) denotes the lottery that generates consumption \( z \) with probability one.

**Proof** Since \( \mathbf{Z} \) is finite, we must have by axiom 1

\[ P_{z^*} \succeq P_z \succeq P_{z_\ast}, \quad \forall z \in \mathbf{Z}. \]

Next consider \( p \in \mathbf{P} \) with support consisting of 2 elements in \( \mathbf{Z} \). The assertion is true by lemma 1 and lemma 4. We can finish the proof by induction: Each \( p \in \mathbf{P} \) with support composed of \( n \) elements, \( n \geq 3 \), can be represented as a compound lottery which is composed of two lotteries each of which has a support consisting of no more than \( n - 1 \) elements of \( \mathbf{Z} \).

4. Now we are ready to state our first main result.

**Theorem 1** In case \( \mathbf{Z} \) is finite, then \( \succeq \) has an expected utility function representation if and only if axioms 1-3 hold; that is, there exists a utility function \( u : \mathbf{Z} \to \mathbb{R} \) such that

\[ \forall p, q \in \mathbf{P}, \quad H(p) \equiv \sum_{z \in \mathbf{Z}} p(z)u(z) \equiv E_p[u] > E_q[u] \equiv \sum_{z \in \mathbf{Z}} q(z)u(z) \equiv H(q) \]

\[ \iff p \succ q. \]

Moreover, the utility function \( u \) is unique up to a positive affine transformation in the sense that if \( v : \mathbf{Z} \to \mathbb{R} \) is such that

\[ \forall p, q \in \mathbf{P}, \quad \sum_{z \in \mathbf{Z}} p(z)v(z) > \sum_{z \in \mathbf{Z}} q(z)v(z) \iff p \succ q, \]
then there exist \( a \in \mathbb{R} \) and \( b \in \mathbb{R}_{++} \) such that for all \( z \in \mathbb{Z} \),

\[
  u(z) = a + b v(z).
\]

**Proof.** In case \( P_{z^*} \sim P_{z_*} \), then all lotteries are indifferent to the agent. We can define \( u(z) = k \in \mathbb{R} \), and \( E_p[u(z)] = \sum_{z \in \mathbb{Z}} p(z) u(z) = k \) for all \( p \in \mathbf{P} \) serves the purpose. In case \( P_{z^*} \succ P_{z_*} \), then define \( H(p) \) as such that

\[
  H(p) P_{z^*} + (1 - H(p)) P_{z_*} \sim p,
\]

where by lemma 2, \( H(p) \) is uniquely defined for all \( p \in \mathbf{P} \). In fact, lemma 1 and lemma 4 show that \( H(p) \) is a utility function representing \( \succeq \): \( H(p) \succ H(q) \) if and only if \( p \succ q \) for all \( p, q \in \mathbf{P} \). By lemma 5 and the definition of \( H(p) \), we have

\[
  H(p) = H\left( \sum_{z \in \mathbb{Z}} p(z) P_z \right) = \sum_{z \in \mathbb{Z}} p(z) H(P_z) = \sum_{z \in \mathbb{Z}} p(z) u(z).
\]

Thus \( \succeq \) does have an expected utility function representation.

Now we show that that \( u(\cdot) \) is determined up to a positive affine transform. Suppose that \( G(p) = \sum_{z \in \mathbb{Z}} p(z) v(z) \) is another expected utility function representing \( \succeq \). By the fact that \( p \sim H(p) P_{z^*} + [1 - H(p)] P_{z_*} \), we have

\[
  G(p) = G(H(p) P_{z^*} + [1 - H(p)] P_{z_*}) = H(p) G(P_{z^*}) + [1 - H(p)] G(P_{z_*})
\]

\[
  = v(z_*) + [v(z^*) - v(z_*)] H(p),
\]

\footnote{Here note that \( H(\cdot) \) is a ordinal utility function in the sense that any monotonic transform of \( H(\cdot) \) also represents \( \succeq \).}
and hence $G(p)$ must be an affine transform of $H(p)$. To see that this transform is positive, note that $P_{x^*} \succ P_{z_*}$ and hence $v(z^*) - v(z_*) > 0$. The necessity in Theorem 1 is much easier. Suppose that $\succeq$ has an expected utility function representation. We must show that the three axioms are satisfied by $\succeq$. Axiom 1 follows from the fact that “$\succeq$” is a preference ordering on $\mathbb{R}$. The fact that $H(p)$ is linear implies that axiom 2 holds also. Axiom 3 follows from the Archimedes’ principle on $\mathbb{R}$. ||

5. The above $u(\cdot)$ is referred to as a von Neumann-Morgenstern (VNM) utility function (cf. von Neumann and Morgenstern (1953)), which is defined on $\mathbb{Z}$. Note that $u$ is a cardinal utility function. We usually take $\mathbb{Z}$ to be $\mathbb{R}_+$, representing the set of (non-negative) consumption or wealth levels. In a dynamic setting where an investor faces a stream of lotteries, we usually assume that the investor maximizes a discounted sum of temporal VNM utility functions. (We say that the investor’s utility function is time-additive or time-separable.) For example, if $T = [0, T]$ denotes the time span, the investor’s objective function is represented as $E_0[\int_{0 \in T} e^{-\rho t} u(\bar{c}_t)dt]$, where $E_0[\cdot]$ is the expectation operator conditional on the time-0 information, and $\rho$ and $e^{-\rho t}$ are referred to as respectively the discount rate and the discount factor. For another example, if $T = \{0, 1, 2, \ldots, t, t + 1, \ldots, T\}$, then we usually assume that the investor seeks to maximize $E_0[\sum_{t=0}^T \delta^t u(\bar{c}_t)]$, where $\delta \in (0, 1)$ is the discount factor.

6. In the remainder of this note, investors are assumed to maximize expected utility when making investment decisions. An investor is represented by a pair $(W_0, u)$, where $W_0$ is the investor’s (non-random)
initial wealth, and $u(\cdot)$ is the investor’s VNM utility function for terminal wealth.\(^5\) Consider the decision-making problem facing an investor endowed with VNM utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$. A fair gamble is a lottery that implies zero expected profit. An investor with VNM utility function $u$ is risk neutral (respectively, risk averse and risk seeking) if given any initial wealth $W_0$ and any fair gamble $\tilde{z}$,

$$E[u(W_0 + \tilde{z})] = (\text{respectively, } \leq \text{ and } \geq) u(W_0).$$

**Theorem 2** The investor with VNM utility function $u(\cdot)$ is risk averse if and only if $u(\cdot)$ is concave.

**Proof.** Consider necessity. For any $x, y \in \mathbb{R}$ and any $\lambda \in [0, 1]$, we want to show

$$u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y).$$

Define $W_0 = \lambda x + (1 - \lambda)y$ and consider a fair gamble

$$\tilde{z} = \begin{cases} 
(1 - \lambda)(x - y), & \text{with probability } \lambda; \\
\lambda(y - x), & \text{with probability } (1 - \lambda).
\end{cases}$$

Since the investor is risk averse, we have

$$u(\lambda x + (1 - \lambda)y) = u(W_0) \leq E[u(W_0 + \tilde{z})] = \lambda u(x) + (1 - \lambda)u(y),$$

and since $x, y$, and $\lambda$ are chosen arbitrarily, this proves that $u(\cdot)$ is concave.

On the other hand, suppose that $u(\cdot)$ is concave, and hence its right-hand and left-hand derivatives both exist everywhere in the domain of definition.\(^6\) It can be shown (see for example Tiel (1984)) that for all $x, y \in \mathbb{R}$,

$$u(x) \leq u(y) + u'_+(y)(x - y).$$

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\(^5\)We say that the investor is faced with background risks if $W_0$ is random. We shall provide some results regarding background risks in Lecture 2.

\(^6\)The following concave function is not differentiable at $z = 0$:

$$u(z) = \begin{cases} 
z, & z \geq 0; \\
2z, & z < 0.
\end{cases}$$
For any initial wealth $W_0$ and fair gamble $\tilde{z}$, putting $x = W_0 + \tilde{z}$ and $y = W_0 + E[\tilde{z}]$, we have

$$u(W_0 + \tilde{z}) \leq u(W_0 + E[\tilde{z}]) + u'(W_0 + E[\tilde{z}])(\tilde{z} - E[\tilde{z}]),$$

which, upon taking expectations on both sides and recognizing that $E[\tilde{z}] = 0$ for a fair gamble, implies that

$$E[u(W_0 + \tilde{z})] \leq u(W_0),$$

and hence the investor is risk averse.

7. Given a VNM utility function $u : \mathbb{R} \to \mathbb{R}$, define two new functions $R^u_A$ and $R^u_R$ with the same domain of definition and range as $u$ as follows.

$$R^u_A(x) = -\frac{u''(x)}{u'(x)}, \quad R^u_R(x) = -\frac{xu''(x)}{u'(x)}, \quad \forall x \in \mathbb{R}_+.$$

These two are referred to as the *Arrow-Pratt measures for absolute and for relative risk aversion* respectively; see Arrow (1970) and Pratt (1964). Observe that if $R^u_A(\cdot) = \rho > 0$ is a constant function, then $u(x) = -e^{-\rho x}$, and in this case, we refer to $u(\cdot)$ as a CARA (constant absolute risk aversion) utility function with $\rho$ being the associated *coefficient of absolute risk aversion*. (If $\rho = 0$, then apparently $u(z) = z$, so that the investor is risk neutral.) Similarly, $u(\cdot)$ is a CRRA (constant relative risk aversion) utility function if $R^u_R(\cdot)$ is a constant function. In this case either $u(x) = \log(x)$ or $u(x) = x^p$ for some $p \in (0, 1)$ (up to a positive affine transform). If $R^u_A(\cdot)$ is an increasing (decreasing) function, then $u(\cdot)$ is referred to as an IARA (DARA) utility function. Some evidence has suggested that most investors have DARA preferences, and in this case if $u'''$ exists, one can show that $u'' \geq 0$.

8. In a static setting, investors trade assets at date 0 and receive cash inflows at date 1. Let $p$ and $\tilde{x}$ be the date-0 price of an asset and the date-1 (random) cash flow generated by the asset. We refer to $\frac{\tilde{x}}{p}$, $\frac{\tilde{x}}{p} - 1$ and $E[\frac{\tilde{x}}{p} - 1]$ as respectively the *return*, the *rate of return*, and the *expected rate of return* on the asset. The asset is referred to as *riskless* if $\tilde{x}/p$ is non-random, and in this case we denote $E[\frac{\tilde{x}}{p} - 1]$ by $r_f$. We refer to $\frac{\tilde{x}}{p} - (1 + r_f)$ and $E[\frac{\tilde{x}}{p}] - (1 + r_f)$ as respectively the *excess rate of
return and the risk premium on the asset (assuming the presence of a riskless asset).

9. Suppose that an investor with initial wealth $W_0$ can trade a riskless asset and a risky asset at date 0, and let $\hat{r}$ be the rate of return on the risky asset. The investor is assumed to be a price-taker. Let $a$ be the amount of her initial wealth to be invested in the risky asset. Define $f(a) = E[u((W_0 - a)(1 + r_f) + a(1 + \hat{r}))]$. Assume $u' > 0 > u''$. Under some mild conditions, we have

$$f'(a) = E\left[\frac{\partial}{\partial a} u((W_0 - a)(1 + r_f) + a(1 + \hat{r}))\right]$$

and

$$f''(a) = E\left[\frac{\partial^2}{(\partial a)^2} u((W_0 - a)(1 + r_f) + a(1 + \hat{r}))\right]$$

where note that $u''(W_0(1 + r_f) + a(\hat{r} - r_f))$ is a negative random variable and $(\hat{r} - r_f)^2$ is a positive random variable, so that $f''(a) < 0$. This shows that $f(a)$ is concave in $a$. The investor seeks to

$$\max_{a \in \mathbb{R}} f(a);$$

that is, the investor would like to find the optimal amount of money that should be spent on the risky asset, where optimality means that the investor’s expected utility is maximized. Since $f(\cdot)$ is concave, our mathematical review shows that the optimal amount $a$ to be invested in the risky asset can be found by solving the first-order condition.

**Theorem 3** Other things being equal, an increase in $W_0$ leads to an increase (respectively, a decrease) in $a$ if $R_A$ is a decrease (respectively, increasing) function. Other things being equal, an increase in $W_0$ leads to an increase (respectively, a decrease) in $\frac{1}{W_0}$ if $R_R$ is a decreasing (respectively, increasing) function.

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7In general, what is needed is a uniform integrability condition; see Theorem 2 of Lecture 5 of my notes on Stochastic Processes.

8For more discussions in this respect, see Huang and Litzenberger (1988).
Proof. Recall that
\[ f(a, W_0) = E[u((W_0 - a)(1 + r_f) + a(1 + \tilde{r})]. \]
The optimal amount \(a^*\) to be invested in the risky asset is such that
\[ L(a^*, W_0) \equiv \frac{\partial f}{\partial a}(a^*, W_0) = 0. \]
The implicit function theorem says that
\[ \frac{da^*}{dW_0} = -\frac{\frac{\partial L}{\partial W_0}}{\frac{\partial L}{\partial a}}, \]
as long as \(L\) is continuously differentiable in \(a^*\) and \(W_0\) and the denominator on the right-hand side is non-zero. Note that except for degenerate cases when \(u'' < 0\),
\[ \frac{\partial^2 f}{(\partial a)^2}(a, W_0) = E[u''(\tilde{W})(\tilde{r} - r_f)^2] < 0, \]
where \(\tilde{W} = W_0(1 + r_f) + a(\tilde{r} - r_f)\), and that the aforementioned denominator is exactly \(\frac{\partial^2 f}{(\partial a)^2}(a^*, W_0)\). Thus the sign of \(\frac{da^*}{dW_0}\) is the same as the sign of
\[ \frac{\partial L}{\partial W_0} = E[u''(\tilde{W})(\tilde{r} - r_f)(1 + r_f)], \]
or equivalently given that \((1 + r_f) > 0\), the sign of
\[ E[u''(\tilde{W})(\tilde{r} - r_f)]. \]
Suppose that \(R_A^u\) is a decreasing function so that when \(\tilde{W} \geq W_0(1 + r_f)\), we have
\[ R_A^u(\tilde{W}) \leq R_A^u(W_0(1 + r_f)), \]
and in the event that \(\tilde{W} < W_0(1 + r_f)\),
\[ R_A^u(\tilde{W}) \geq R_A^u(W_0(1 + r_f)). \]
Now multiplying both sides of the above last two inequalities by \(-u'(\tilde{W})(\tilde{r} - r_f)\), we have
\[ u''(\tilde{W})(\tilde{r} - r_f) \geq -R_A^u(W_0(1 + r_f))u'(\tilde{W})(\tilde{r} - r_f). \]
Taking expectations on both sides of the last inequality, and using the first-order condition, we conclude that

\[ E[u^n(\hat{W})(\hat{r} - r_f)] \geq 0. \]

The case where \( R_{\lambda}^u(\cdot) \) is an increasing function is similar. This proves the first assertion. The second assertion is left as an exercise.

10. For other properties of concave VNM utility functions, see my note “Investors’ Behavior under Uncertainty, Part I.”

11. The preceding expected utility theory was derived under the assumption that for each lottery in \( P \), there exists an exogenous probability distribution on the set \( Z \). Now we briefly go over Savage’s (1954, The Foundations of Statistics, Wiley) subjective expected utility theory and its extension by Gul (1992, Journal of Economic Theory), where there may not exist objective probability measures; instead, the probability distribution associated with a lottery will be derived from the decision maker’s preference relation. Again, Kreps (1988) and Fishburn (1970) are good references to look up the details. We first set the stage by defining mixture space and qualitative probability.

12. A set \( P \) together with a family of functions \( h_a : P \times P \to P, \ a \in [0, 1] \) is called a mixture space if for all \( p, q \in P \) and all \( a, b \in [0, 1] \)

(i) \( h_1(p, q) = p; \)

(ii) \( h_a(p, q) = h_{1-a}(q, p); \) and

(iii) \( h_a(h_b(p, q), q) = h_{ab}(p, q). \)

We have the following theorem:

**Mixture Space Theorem.** (Herstein and Milnor, 1953, Econometrica.) Suppose \( P \) is a mixture space and \( \succeq \) a binary relation on \( P \). Then (a),(b),(c) are equivalent to (d),(d1),(d2):

(a) \( \succeq \) is a preference relation;

(b) (Independence Axiom) \( p, q \in P \) and \( p \succ q \) implies that for all \( a \in (0, 1] \) and all \( r \in P, h_a(p, r) \succeq h_a(q, r); \)

(c) (Archimedean Axiom) \( p, q, r \in P \) and \( p \succ q \succ r \) implies the existence of \( a, b \in (0, 1] \) such that \( h_a(p, r) \succeq q \succ h_b(p, r); \)

(d) There exists a real-valued function \( F : P \to \mathbb{R} \) such that

(d1) \( p \succ q \) iff \( F(p) > F(q); \)
(d2) \( F(h_a(p, q)) = aF(p) + (1 - a)F(q) \). Moreover, \( F \) is unique up to a strictly positive affine transformation.

13. The proof of equivalent conditions for an objective expected utility function representation follows from the above mixture space theorem.

14. Let \( S \) be the state space and \( A \) a \( \sigma \)-algebra. A binary relation \( \succ \) on \( A \) is a qualitative probability if
   (i) \( \succ \) is asymmetric and negative transitive;
   (ii) \( a \succeq \phi \) for all \( a \in A \);
   (iii) \( S \succeq \phi \); and
   (iv) \( a \cap c = b \cap c = \phi \) implies that \( a \succ b \) iff \( a \cup c \succ b \cup c \).

15. Fact: A probability measure on \( A \) is a qualitative probability but the converse may not be true.

16. \( 2^n \)-Partition Property: The state space is said to have this property if it is able to be partitioned into \( 2^n \) equally likely events \( \{a_1^n, a_2^n, \ldots, a_{2^n}^n\} \) for all positive integer \( n \). When this is true, for any event \( b \), for each \( n \), there is a smallest \( k(n) \) such that

   \[
   \bigcup_{j=1}^{k(n)} a_j^n \succ b.
   \]

   We define \( p(b) = \lim_{n \to +\infty} \frac{k(n)}{2^n} \) whenever the limit exists.

17. Proposition 8.6 of Kreps (1988) If \( \succ \) is a qualitative probability and if \( S \) has the \( 2^n \)-partition property, then the above limit exists for all \( b \in A \), and \( p(\cdot) \) is indeed a probability measure that represents \( \succ \) in a weak sense, i.e. \( a \succ b \) implies \( p(a) \geq p(b) \). (What we want is \( a \succ b \) iff \( p(a) > p(b) \).)

18. Caveat: It can happen that \( a \succ b \) and yet \( p(a) = p(b) \).

   Example. Let \( S = S_1 \cup S_2 \), where \( S_1 = [0, 1] \) and \( S_2 = [2, 3] \). Let \( A_1 \) be the set containing all finite unions of subintervals of \( S_1 \). Let \( A \) be a set containing such subsets \( a \) of \( S \) that \( a \cap S_1 \in A_1 \). Let \( a_i = a \cap S_1 \) and \( a_i = (a_1 \cup a_2 \cup b_1 \cup b_2) \), we define \( a \succ b \) if \( \lambda(a_1) > \lambda(b_1) \) or if \( \lambda(a_1) = \lambda(a_2) \) and \( \lambda(a_2) > \lambda(b_2) \), where \( \lambda(a_i) \) is the ‘length’ of \( a_i \).
(i) Verify that $\succ$ is a qualitative probability.
(ii) Consider the equi-partitioned sets

$$a^n_j = ([j - 1/2^n, j/2^n] \cup [2 + j/2^n, 2 + j/2^n])$$

except for $j = 2^n$. Verify that, in this case,

$$p(a_1 \cup a_2) = \lambda(a_1).$$

So, consider $a = [0, 1/2] \cup [2, 2.5]$ and $b = [0, 1/2] \cup [2, 2.4]$. We have $a \succ b$, but $p(a) = p(b) = 1/2$.

19. **Fine and Tight Qualitative probability** For all $a, b \in A$ such that $a \succ b$ there exists a finite partition $C(a, b) \equiv \{c_1, c_2, \cdots, c_n\}$ for $S$ such that for all $k = 1, 2, \cdots, n$, $a \succ b \cup c_k$. This axiom will be referred to as (FT) later on.

20. **Fact:** Under (FT), the probability $p(\cdot)$ represents $\succ$. One can show that $\succ$ defined in the above example is not fine and tight.

21. Now we introduce Savage’s theory. There is a set of prizes or consequences, denoted by $Z$. There is an infinite state space $S$ whose power set is $A$. An individual has preference $\succ$ over the set of acts or actions $F = \{f : S \rightarrow Z\}$. First we record Savage’s axioms.

**(S1)** $\succ$ is a strict preference relation on $F$;

**(S2)** There exist $x, y \in Z$ such that $x \succ y$;

**(S3) (Independence Axiom)** Suppose $f, f', g, g' \in F$ and $a \in A$ are such that

1. $f(s) = f'(s), \ g(s) = g'(s)$ for all $s \in a$ and
2. $f(s) = g(s), \ f'(s) = g'(s)$ for all $s \in a^c$.

Then, $f \succ g$ iff $f' \succ g'$.

Here we pause to give 2 definitions.

**Conditional Preference.** Define $f \succ_a g$ if $f' \succ g'$ for all $f', g' \in F$ such that $f' = f, g' = g$ on $a$ and $f = g, f' = g'$ on $a^c$. One can show

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9This is a set of measurable functions; see Lecture 4 of my notes on Stochastic Processes. We do not need a probability measure to define measurable functions.

10This is motivated by (S3), which says that $f$ is better than $g$ on event $a$ if $f \succ g$ once $f$ and $g$ are both replaced by any other lottery $h$ on $a^c$. 

13
that $\succ_a$ is indeed a strict preference relation, from which we can define $\succeq_a$ and $\sim_a$ accordingly.

**Null Event:** An event $a \in A$ is null if for all $f, g \in F$, $f \sim_a g$.

(S4) **(Utility is not state-dependent)** If $a \in A$ is not null, and if there exist $x, y \in Z$ such that $f(s) = x$, $g(s) = y$ for all $s \in a$, then $f \succ_a g$ iff $x \succ y$.

(S5) Suppose

1. $x, y, x', y' \in Z$ are such that $x \succ y$ and $x' \succ y'$;
2. $f, f' \in F$ and $a \in A$ are such that $f(s) = x$, $f'(s) = x'$ on $a$ and $f(s) = y$, $f'(s) = y'$ on $a^c$;
3. $g, g' \in F$ and $b \in A$ are such that $g(s) = x$, $g'(s) = x'$ on $b$ and $g(s) = y$, $g'(s) = y'$ on $b^c$;

Then, $f \succ g$ iff $f' \succ g'$.

(S6) **(Sure Thing Principle)** For all $a \in A$,

1. $f \succ_a g(s)$ for all $s \in a$ implies $f \succeq_a g$;
2. $g(s) \succ_a f$ for all $s \in a$ implies $g \succeq_a f$.

(S7) For all $f, g \in F$, $f \succ g$, and all $z \in Z$, there is a finite partition of $S$ such that for each $a$ in the partition,

1. $f'(s) = z$ for $s \in a$, $f'(s) = f(s)$ for $s \in a^c$ implies $f' \succ g$; and
2. $g'(s) = z$ for $s \in a$, $g'(s) = g(s)$ for $s \in a^c$ implies $f \succ g'$.

22. To understand Savage axioms, consider the following pairs of choices, where $f, g, f', g', f'', g''$ are acts, $a, b, a^c, b^c$ are events, and $w, x, y, z \in Z$ are prizes. We assume that $w \succ z$ and $x \succ y$.

<table>
<thead>
<tr>
<th>acts/prizes/events</th>
<th>acts/events</th>
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<tbody>
<tr>
<td>$f$</td>
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<td>$a^c \cap b^c$</td>
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<tr>
<td>$g$</td>
<td>$y$</td>
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<td>$x$</td>
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</tbody>
</table>

11Note that if $x \succ y$, then $x \succ_a y$ on any event $a$!

12Since $x \succ y$ $f$ is a win if $a$ happens and $g$ is a win if $b$ happens. Presumably, $f \succ g$ because $a$ is considered more likely than $b$, but then $f' \succ g'$ for similar reasons. Note that implicit in this axiom is that acts do not affect probabilities (no moral hazard problems; J. Dreze did some work in this respect.).

13Note carefully that $g(s) \in Z$ while $g \in F$. 

---

14
Note that (S3) says that

\[ f \succeq g \Rightarrow f' \succeq g', \]

and (S5) says that

\[ f \succeq g \Rightarrow f'' \succeq g''. \]

23. Now we explain how Savage’s theory develops. First, from preference ordering on \( F \), he derives preference orderings on \( F \) conditional on different events (using (S3) and the defined conditional preferences), from which he in turn deduces ordering on consequences, i.e. \( Z \), using mainly (S4). From ordering on \( Z \), he then deduces ordering on events, i.e. \( A \), using mainly (S5). Then, the above qualitative probability theorem is used to show that the lattermost ordering is a fine and tight qualitative probability, and hence a probability measure \( p \) on \( A \) exists. Finally, given \( p \), the mixture space theorem is used to deduce a state-independent utility function.

24. At first, for \( a, b \in A \), define \( a \succ b \) (read ‘\( a \) is more likely to \( b \)’) if for all \( x, y \in Z \) with \( x \succ y \), we have \( f \succ g \), where \( f(s) = x1_{[s \in a]}(s) + y1_{[s \in a^c]}(s) \) and \( g(s) = x1_{[s \in b]}(s) + y1_{[s \in b^c]}(s) \), where recall that the indicator function \( 1_B(\cdot) \) is such that \( 1_B(s) = 1 \) if \( s \in B \) and \( 1_B(s) = 0 \) if otherwise. Note that by (S5), choice of such \( x \) and \( y \) can be arbitrary.\(^{14}\) One can show that \( \succ \) is a qualitative probability, using (S1),(S2),(S3), (S4) and (S5). In fact, by (S7), one can show that (FT) holds, and hence there exists a unique probability measure \( p \) on the measurable space \((S, A)\) such that \( a \succ b \) iff \( p(a) > p(b) \). Moreover, for all \( a \in A \), \( r \in [0,1] \), there exists another event \( b \) such that \( p(b) = rp(a) \).

\(^{14}\)This is certainly a requirement: \( \succ \) should not depend on the chosen pair \((x, y)\).
25. Having obtained the probability, now we turn to the VNM utility function. This part is very difficult and lengthy, and so let me only provide a general idea. We can use \( \succeq \) to define a binary relation \( \succeq^s \) on \( Q_s \), which is the set of all simple lotteries on \( Z \) (those with finite supports). More precisely, if \( q, q' \in Q_s \) we can find acts \( f, f' \in F \) such that \( q = p(f) \) and \( q' = p(f') \).\(^{15}\) Then, we define

\[
q \succeq^s q' \text{ if } f \succ f'.
\]

Now, one can show (with no ease) that \( \succeq^s \) satisfies the mixture space axioms, and hence the mixture space theorem applies. This completes the proof for simple act-simple probability case. To complete the proof for the general case, there are more to be done. Interested students should refer to Fishburn (1970).

26. Gul (1992, JET) extends Savage’s theory to a finite state space. His contribution rests on a continuity axiom. We state Gul’s 4 axioms as follows:

(G1) \( \succeq \) is a preference relation on \( F \);

(G2) (Independence Axiom) \( f'(s) \sim f(s)1_{s \in a}(s) + h(s)1_{s \not\in a}(s) \) and \( g'(s) \sim g(s)1_{s \in a}(s) + h(s)1_{s \not\in a}(s) \) and \( a \) is not null implies that \( f \succ g \) iff \( f' \succ g' \).

(G3) (Strict Monotonicity and Equally Likely Partitions) \( x > y \) implies that \( x \succ y \). For all \( x, y \in Z \), there exists an event \( a(x, y) \) such that

\[
x1_{[s \in a(x, y)\cap\{s\}]}(s) + y1_{[s \in a(x, y)\setminus\{s\}]}(s) \sim y1_{[s \in a(x, y)\cap\{s\}]}(s) + x1_{[s \in a(x, y)\setminus\{s\}]}(s).
\]

(G4) (Continuity) Since we are dealing with finite state space, let \( |S| = N \). Endow \( F \) with the standard metric of \( R^N \) space. Now this axiom states that \( \succeq \) is continuous with respect to this metric.

27. Gul’s results are quite strong. Because of the assumed strict monotonicity and continuity, the utility function is continuous and strictly increasing. Once again, the utility function is state-independent and is unique up to a strictly positive affine transformation.

\(^{15}\)These are the push-forward measures or distribution functions of \( f \) and \( f' \); see Lecture 4 of my notes on Stochastic Processes.

29. Machina shows that assuming that $H(p)$ is smooth in a proper sense (see below) is able to obtain a local expected utility representation for preference even without independence axiom. Moreover, most implications of the expected utility theory about investment behavior remain valid under the local expected utility theory. Since many experiment results have suggested that people’s choice behavior violate the independence axiom, Machina’s theory is important in reassuring that the expected utility theory is a useful theory in generating correct implications about investment behavior.

30. Let $E$ be a normed linear space and $E'$ its topological dual. Let $x_0 \in E$ and $f : E \to \mathbb{R} \cup \{+\infty, -\infty\}$ with $f(x_0) < +\infty$. $f$ is said to be Fréchet-differentiable at $x_0$ if there exists $x' \in E'$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \langle x - x_0, x' \rangle}{\|x - x_0\|} = 0.$$ 

$f$ is said to be Gâteaux-differentiable at $x_0$ if there exists $x' \in E'$ such that for all $x_0 \in E$

$$\lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon) - f(x_0)}{\epsilon} = \langle x_0, x' \rangle.$$ 

Fréchet-differentiability implies Gâteaux-differentiability but not vice versa.

31. Suppose the prize or consequence space is $Z = [0, M]$. Assume an objective probability exists. Let $D([0, M])$ be the set of acts; each act is a probability distribution over $[0, M]$. To talk about continuity of the preference ordering $\succeq$, define on $D$ the $L^1$-norm:

$$\|F\| = \int_0^M |F(x)| dx,$$
where $F$ (a c.d.f.) is a generic element of $D$. From this norm, define the metric for any two elements of $D$, say $F$ and $G$, by

$$
\|F - G\| = \int_0^M |F(x) - G(x)|dx.
$$

We shall assume that $\succeq$ is continuous with respect to this metric. By Debreu’s theorem (1954, ‘Representation of a preference ordering by a numerical function,’ in Decision Processes, edited by Thrall, Coombs, and Davis, and published by Wiley), $\succeq$ admits a continuous utility function representation,

$$
V : D([0, M]) \to \mathcal{R}.
$$

We shall assume that $V$ is not only continuous but also Fréchet-differentiable at every $F \in D([0, M])$ with respect to $\| \cdot \|$. Note that Riesz representation theorem states that any continuous linear functional $h$ on $D$ is essentially bounded, i.e. $h \in L^\infty$. Indeed, the functional value of $h$ at $F$ is

$$
<F, h> = \int_0^M F(x)h(x)dx.
$$

Hence, the assumed Fréchet-differentiability of $V$ implies the existence of an essentially bounded function $h(x, F)$ such that for $F^*$ in a small neighborhood of $F$,

$$
V(F^*) - V(F) = \int_0^M [F^*(x) - F(x)]h(x, F)dx + o(\|F^* - F\|).
$$

Now define

$$
u(x, F) = -\int_0^x h(s, F)ds,
$$
then given $F$, $u$ is absolutely continuous in $x$ (following essential boundedness of $h$). Hence $u$ is differentiable almost everywhere. Since $[0, M]$ is the common support of $F^*$ and $F$, integrating by parts yields

$$
\int_0^M [F^*(x) - F(x)]h(x, F)dx = \int_0^M u(x, F)[dF^*(x) - dF(x)].
$$

Hence, we have a local expected utility representation for $\succeq$. 18
32. In calculus, several important aspects of a differentiable function are described by its local linear approximation. For example, a differentiable function is increasing if and only if its linear approximation is increasing. Based on this observation, Machina shows that

**Theorem M1** Let \( F^*, F \in D([0, M]) \). Consider smooth \( V \)'s such that \( u(x, F) \) is increasing in \( x \) for all \( F \in D \). Then, \( V(F^*) \geq V(F) \) iff \( F^*(x) \leq F(x) \ \forall x \in [0, M]. \)

**Theorem M2** Let \( F^*, F \in D([0, M]) \). Consider smooth \( V \)'s such that \( u(x, F) \) is concave in \( x \) for all \( F \in D \). Then, \( V(F^*) \geq V(F) \) iff \( \int_0^y [F^*(x) - F(x)] dx \leq 0 \ \forall y \in [0, M] \).

**Theorem M3** Let \( F^*, F \in D([0, M]) \) with \( F^* \) has a higher mean and risk (in the sense of a mean preserving spread) than \( F \). Suppose that \( V^*(F) = V^*(F^*) \). Then \( V(F) \geq V(F^*) \) iff for all \( F \), \( u(x, F) \) is more concave than \( u^*(x, F) \), if

\[
\forall F, \ \forall x, \quad -\frac{u_{11}(x, F)}{u_1(x, F)} \geq -\frac{u^*_{11}(x, F)}{u^*_1(x, F)}.
\]

16 Let us give an example. Suppose instead that \( Z = \{z_1, z_2, z_3\} \), and in this case each lottery is represented by a point \( p \) in the following Machina's triangle

\[
P = \{p \in \mathbb{R}_+^3 : p'1 = 1\}.
\]

Now, suppose that \( H(p) = H(p_1, p_2, p_3) \) is a smooth function, we can define, at each \( q \in P \),

\[
u(z_i; q) \equiv \frac{\partial H}{\partial q_i}(q), \ \forall i = 1, 2, 3.
\]

Then for \( p \) close enough to \( q \) in terms of the standard metric on \( \mathbb{R}^3 \), we have approximately

\[
H(p) - H(q) = \sum_{i=1}^3 u(z_i; q)(p_i - q_i),
\]

so that a local expected utility function does represent the ranking of lotteries very similar to \( q \). Moreover, suppose that \( z_1 < z_2 < z_3 \), and consider the collection \( \mathcal{H} \) of functions \( H \) that satisfy

\[
\frac{\partial H}{\partial q_1}(q) < \frac{\partial H}{\partial q_2}(q) < \frac{\partial H}{\partial q_3}(q), \ \forall q \in P.
\]

Then if all investors with utility function \( H \in \mathcal{H} \) agree that \( p \) is better than \( q \), then the distribution function of \( p \) must lie below the distribution function of \( q \). This is what Theorem M1 means, which has a counterpart in von Neumann-Morgenstern expected utility theory; see Lecture 2.
33. Machina’s Resolution to Friedman-Savage Paradox. He shows the following two hypotheses together resolve the paradox:

(H1) $\forall F \in D([0, M])$, $R_A(x, F)$ decreases with $x$;
(H2) $\forall x, F, F^*$ such that $F^* \geq_{FSD} F$ implies that $R_A(x, F^*) \geq R_A(x, F)$.

34. Having obtained some ideas about expected utility theory, now we turn our attention to asset prices that arise in the equilibrium of financial markets. Consider the following two-period (pure exchange) economy where $m$ assets are traded in perfect financial markets at $t = 0$ which pay dividends denominated in a single consumption good at $t = 1$. Assume that these traded assets are in zero net supply. There are $n$ possible states at $t = 1$, and state $k$ can occur with a strictly positive probability $\pi_k$, with $\sum_{k=1}^{n} \pi_k = 1$. Let $X_{n \times m}$ be such that its $(i, j)$-element represents the dividend per share of asset $j$ distributed in state $i$ at $t = 1$. For simplicity, assume that $\rho(X) = m \leq n$; that is, the column vectors of $X$ are linearly independent. There are $I$ agents (consumers, investors) in this economy. Agent $i$ is represented by a pair $(U_i, e_i)$, where $U_i : R^n_+ \to \mathcal{R}$ is a strictly increasing utility function for date-1 consumption and $e_i \in R^n_+$ the agent’s date-1 endowment in the single consumption good (we are assuming for simplicity that the agent never consumes at date 0 and the agent has no date-0 endowments, but this is inessential). Thus we have assumed for simplicity that no agents are endowed with non-zero initial positions in the $m$ traded assets.

\footnote{This paradox is about the seemingly contradictory phenomenon that most people buy insurance on one hand and take gambles on the other hand.}

\footnote{For our purpose, you can assume that}

\[ U_i(c_{i1}, c_{i2}, \ldots, c_{ik}, \ldots, c_{in}) = \sum_{k=1}^{n} \pi_k u_i(c_{ik}), \]

where $u_i(\cdot)$ is investor $i$’s VNM utility function. Note that if asset markets are closed, then investors are forced to consume their date-1 endowments, so that investor $i$’s gets a utility $U_i(c_{i1}, c_{i2}, \ldots, c_{ik}, \ldots, c_{in})$. Here, we have $m$ traded assets in zero net supply. (Recall that futures and options have zero net supply.) By taking a portfolio $q = (q_1, q_2, \ldots, q_m)'$, an investor’s terminal wealth at date 1 becomes $e_{ik} + \sum_{j=1}^{m} q_j x_{kj}$ if state $k$ occurs at date 1. This implies that the investor’s date-0 utility from taking the portfolio $q$, is

\[ U_i(e_{i1} + \sum_{j=1}^{m} q_j x_{1j}, e_{i2} + \sum_{j=1}^{m} q_j x_{2j}, \ldots, e_{in} + \sum_{j=1}^{m} q_j x_{nj}) \equiv U_i(e_i + Xq). \]
The economy is compactly written as $E = (X, (U_i, e_i); i = 1, 2, \cdots, I)$. Let $p_{m \times 1}$ be the vector of date-0 prices of the $m$ assets. Given $p$ and $e_i$, the budget set facing agent $i$ at date 0 is the set of feasible date-1 consumption plans that the agent can attain by taking positions in the $m$ assets at date 0, which is defined as

$$B(p, e_i) = \{e_i + Xq \in \mathbb{R}^n_+ : q \in \mathbb{R}^m, p'q \leq 0\}.$$ 

Being rational, agent $i$ seeks to solve problem $(G_i)$ at date 0:

$$\sup_{c \in B(p, e_i)} U_i(c_i).$$

**Definition 1** A feasible consumption allocation is an $I$-tuple of $n$-vectors $d = (d_i; i = 1, 2, \cdots, I)$ in $\mathbb{R}^n_+$, such that $\sum_{i=1}^I d_i \leq \sum_{i=1}^I e_i$. A feasible consumption allocation $d = (d_i; i = 1, 2, \cdots, I)$ is Pareto optimal (Pareto efficient) if there exist no other feasible consumption allocations $c = (c_i; i = 1, 2, \cdots, I)$ such that for all $i$, $U_i(d_i) \leq U_i(c_i)$ and for some $i$, $U_i(d_i) < U_i(c_i)$.

**Definition 2** A Walrasian equilibrium for $E$ is a price $p$ and a feasible allocation $Q = (q_1, q_2, \cdots, q_I)$ of the $m$ traded assets (meaning that $\sum_{i=1}^I q_i = 0$) such that given $p$, for all $i = 1, 2, \cdots, I$, $c_i = e_i + Xq_i$ solves $(G_i)$.

The pair $(X, p)$ will be referred to as a price system, where note that $X$ and $p$ are respectively exogenous and endogenous variables.\(^{19}\)

To prove our next theorem, we shall need to use the following *Theorem of Alternatives*, which is a version of the well-known Farka’s Lemma:\(^{20}\) For each real-valued matrix $A_{m \times n}$, either (I) or (II) holds but they

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\(^{19}\)From now on, for all $n \times 1$-vector $a$, we write $a \geq 0$, if each element of $a$ is non-negative; we write $a > 0$ if $a \geq 0$ and at least one element of $a$ is non-zero; and we write $a >> 0$ if every element of $a$ is strictly positive.

\(^{20}\)Farka’s Lemma says that given $b_{n \times 1}$ and $A_{m \times n}$, if for all $x_{n \times 1}$ such that $Ax \geq 0_{m \times 1}$, it must be that $bx \geq 0$, then there must exist some $t_{m \times 1} \geq 0_{m \times 1}$ such that $b = A't$. Let us now prove Farka’s Lemma.
(Step 1: von Neumann’s Theorem) Let \( n \) be a positive integer and let \( X \subseteq \mathbb{R}^n \) be non-empty, closed and convex. Let \( y \in \mathbb{R}^n \setminus X \). Then there exists \( p \in \mathbb{R}^n \setminus \{0\} \) and \( \alpha \in \mathbb{R} \) such that \( p^T x \geq \alpha > p^T y \) for all \( x \in X \).

**Proof.** The minimization problem has a solution \( a \) because \( X \cap B(y, \|y - z\|) \) is compact, and \( \|x - y\| \) is continuous in \( x \). It is also clear that

\[
\|x - y\| \geq \|a - y\|
\]

for all \( x \in X \cap B(y, \|y - z\|)^c \). Hence we have

\[
\|x - y\| \geq \|a - y\|
\]

for all \( x \in X \).

Now define \( p = a - y \) and \( \alpha = (a - y)^T a \). For each \( x \in X \), we have for each \( t \in [0, 1] \), since \( tx + (1 - t)a \in X \),

\[
\frac{\|tx + (1 - t)a - y\|^2 - \|a - y\|^2}{t} \geq 0,
\]

so that by passing \( t \downarrow 0 \), we have from the above inequality that

\[
-2(a - y)^T (a - y) + 2(a - y)^T (x - y) \geq 0, \quad \forall x \in X.
\]

This implies that for all \( x \in X \),

\[
\alpha \equiv p^T a = (a - y)^T a \leq (a - y)^T x = p^T x.
\]

On the other hand, we have

\[
p^T y = \alpha - \|p\|^2 < \alpha.
\]

This finishes the proof.

(Step 2: Minkowski’s Theorem) Let \( n \) be a positive integer and let \( A, B \subseteq \mathbb{R}^n \) be disjoint, non-empty, and convex. Then there exists \( p \in \mathbb{R}^n \setminus \{0\} \) such that \( p^T a \geq p^T b \) for all \( a \in A \) and \( b \in B \). Moreover, if \( A \) is closed and \( B \) is compact, or the other way around, then \( p^T a > p^T b \) for all \( a \in A \) and \( b \in B \).

**Proof.** It is easy to verify that \( X \) is convex and non-empty. If \( X \) is closed, then applying Step 1 we can conclude that there exist a non-zero vector \( p \) and some scalar \( \alpha \) such that \( p^T x \geq \alpha > p^T y \) for all \( x \in X \), and hence the assertion is proved. In case \( X \) is not closed, and yet \( y \) is not contained in the closure of \( X \), then Step 1 is still applicable, with \( X \) replaced by the closure of \( X \), and the same conclusion follows. Finally, if \( X \) is not closed, and \( y \) is contained in the closure of \( X \) but not \( X \) itself, then \( y \) must be a boundary point of \( X \), and hence there must exist a sequence \( \{y_n; n \in \mathbb{Z}_+\} \) in the closure of \( X^c \), such that the sequence converges to \( y \), and Step 1 can be applied to each \( y_n \) so that there exist correspondingly a sequence of non-zero vectors \( \{p_n; n \in \mathbb{Z}_+\} \) with unit length such that for all \( x \in X \), we have for all \( n \),

\[
p_n^T x \geq p_n^T y_n.
\]
never hold at the same time:

(I) There exists \( x_{n \times 1} \in \mathcal{R}_{++}^m \) such that \( Ax = 0_{m \times 1} \).

(II) There exists \( y \in \mathcal{R}^m \) such that \( A'y > 0_{n \times 1} \).

**Definition 3** A portfolio is an \( m \)-vector \( q \), of which the (date-0) mar-

Since the sequence \( \{p_n; n \in \mathbb{Z}^+\} \) is sequentially compact (because \( p_n, p_0 = 1 \) for all \( n \) by assumption), it has a convergent subsequence \( \{p_{n_k}; k \in \mathbb{Z}^+\} \) with limit \( p \) where \( p'p = 1 \) also. Now by passing \( k \) and \( n_k \) to infinity, we have

\[ p'x \geq p'y, \quad \forall x \in X. \]

Hence the same conclusion holds.

(Step 3: Farka's Lemma) Let \( n \) be a positive integer and \( a_1, a_2, \ldots, a_m, b \in \mathbb{R}^n \) where \( b \neq 0_{n \times 1} \). Let

\[ K \equiv \{ \sum_{j=1}^{m} t_j a_j : t_j \in \mathbb{R}^+, \forall j = 1, 2, \ldots, m \}, \]

and

\[ X \equiv \{ x \in \mathbb{R}^n : x'a_j \geq 0, \forall j = 1, 2, \ldots, m \}. \]

Suppose that \( x'b \geq 0 \) for all \( x \in X \). Then \( b \in K \).

**Proof.** It is easy to verify that \( K \) is non-empty, closed and convex, and if \( b \) is not contained in \( K \), then Step 1 implies the existence of a non-zero vector \( x \) and some scalar \( \alpha \) such that \( x'k \geq \alpha > x'b \) for all \( k \in K \). It follows that \( x'k \geq 0 \) for all \( k \in K \); if instead for some \( k \), we have \( x'k < 0 \), then given any \( \alpha, \) for \( t \in \mathbb{R}^+ \) sufficiently large, \( x'[tk] < \alpha \) whereas \( tk \in K \), a contradiction. Hence \( \alpha \leq 0 \) and \( b'x < 0 \). This completes the proof.

\[ 21 \] Here is a lengthy but elementary proof. Let \( a_i, i = 1, 2, \ldots, n \) be the \( i \)-th column vector of \( A \). Let \( co(A) \) be the convex hull generated by the \( a_i \)'s. The convex hull \( co(A) \) is generated by a finite number of points in \( \mathcal{R}^m \), and is referred to as a convex polytope, which is closed and bounded (compact) in \( \mathcal{R}^m \). There are 2 possibilities. Either \( co(A) \) contains or does not contain \( 0_{m \times 1} \). In the latter case, fact 1 above implies that there is a hyperplane separating strictly \( A \) and \( 0 \), and hence (II) has to hold. In case (I) holds also, we shall have a contradiction. Suppose there exists \( x > 0 \) such that \( Ax = 0 \). Define \( z = \frac{x}{x'1} \). We have \( Az = 0, z > 0 \), \( z'1 = 1 \), so that \( 0 \in co(A) \), which is a contradiction to the assumption that \( co(A) \) does not contain \( 0 \).

In the former case, in case (I) holds then (II) cannot hold: If they both hold, then we have

\[ 0 = y'0 = y'Ax = [y'A]x > 0, \]

an obvious contradiction. It remains to show that when (I) fails then (II) must hold, given that \( 0 \in co(A) \). In this case, suppose without loss of generality that for some \( l < n \), for some \( (x_1, x_2, \ldots, x_l)' \gg 0_{l \times 1} \),

\[ 0 = \sum_{i=1}^{l} x_ia_i, \]

\[ 23 \]
ket value is $p'q$ and the (date-1) payoff is $Xq$. An arbitrage (opportunity) is a portfolio $q$ such that either $p'q \leq 0$ and $Xq > 0$ or $p'q < 0$ and $Xq \geq 0$.

**Definition 4** An Arrow-Debreu security is an asset that pays 1 unit of consumption at $t = 1$ if exactly one prespecified state occurs and nothing otherwise. Given that there are $n$ possible states at $t = 1$, there are correspondingly $n$ Arrow-Debreu securities. The vector of the prices of Arrow-Debreu securities will be referred to as the state price vector, which is an $n$-vector $f \in \mathbb{R}^n_{++}$ such that $p' = Xf$. The price system $(X, p)$ is arbitrage free if it admits no arbitrage opportunities, and it is weakly arbitrage free, if for all portfolios $q$, $Xq \geq 0$ implies $p'q \geq 0$.\(^{23}\)

The linear span of $\{a_i; i = 1, 2, \ldots, l\}$ is the set

$$L = \{ \sum_{i=1}^{l} x_i a_i : x_i \in \mathbb{R}, \forall i = 1, 2, \ldots, l \}.$$

Let $C$ be the convex hull generated by those $a_j$’s not contained in $L$. We claim that $C$ is non-empty. Suppose not. Then for all $j = l + 1, l + 2, \ldots, n$, we can represent

$$a_j = \sum_{i=1}^{l} \lambda_{ji} a_i.$$

But then, we have

$$0 = Az,$$

where $z_{n \times 1}$ is such that

$$z_i = x_i - \epsilon \sum_{j=l+1}^{n} \lambda_{ji}, \ i = 1, 2, \ldots, l,$$

and

$$z_i = \epsilon, \ \forall i = l + 1, l + 2, \ldots, n.$$  

By choosing $\epsilon > 0$ small enough, we have $z \in \mathbb{R}^n_{++}$, a contradiction to the assumption that (I) fails. It follows that $C$ and $L$ are both non-empty, convex, closed. Note that $C$ is compact and $L$ and $C$ are disjoint. Minkowski’s Theorem (Step 2 in the preceding footnote) then implies the presence of a linear functional $F : \mathbb{R}^m \to \mathbb{R}$ such that for all $j = 1, 2, \ldots, n$, $F(a_j) = y' a_j > 0$ if $a_j \in C$ (hence for all $j = l + 1, l + 2, \ldots, n$) and $F(a_j) = 0$ for all $j = 1, 2, \ldots, l$. Thus (II) holds.

\(^{23}\)It is a direct consequence of Farkas’ lemma that the price system is weakly arbitrage free if and only if there exists a weak state price vector $f \in \mathbb{R}^n_{++}$ such that $p = Xf$.  

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Theorem 4 The price system \((X, p)\) is arbitrage free if and only if there exists a state price vector \(f \in \mathcal{R}_+^n\) such that \(p = X'f\).

Proof

We shall apply the Theorem of Alternatives. Define

\[
A_{m \times (n+1)} \equiv [-p, X].
\]

If for some \(q_{m \times 1}\) we have

\[
A'q > 0,
\]

then \(q\) is an arbitrage opportunity. Thus the absence of arbitrage opportunities from the price system \((X, p)\) implies the existence of some \(f \in \mathcal{R}_+^n\) such that

\[
A \begin{bmatrix} 1 \\ f \end{bmatrix} = 0_{m \times 1},
\]

or equivalently

\[
p = X'f.
\]

Hence a state price vector does exist. ||

When do we have a unique state price vector? If \(X\) is of rank \(n\), then uniqueness obtains. We shall refer to this special case the complete markets case. Conversely, when the rank of \(X\), \(\rho(X) < n\), then there are infinitely many state price vectors. To see this, consider the system of equations \(p = X'f\), where \(f\) represents the unknowns. When markets are incomplete \((\rho(X) = m < n)\), there are more unknowns than equations. It can be shown that the dimension of the set of \(f\) compatible with the equation \(p = X'f\) is the same as the dimension of the orthogonal complement of the linear span of \(X\), which is \(n - \rho(X)\). In fact, the set of \(f\) is an \((n - \rho(X))\)-simplex with \(n - \rho(X) + 1\) vertices.

For example, suppose that there are 3 states and the only traded asset is the pure discount bond with face value of 1 dollar. Suppose that the bond price today is also 1 dollar. Then \(f \in \mathcal{R}_+^3\) are such that

\[
f_1 + f_2 + f_3 = 1,
\]

\[
f_1 + f_2 + f_3 = 1,
\]
which is two-dimensional; recall that if a subset of $\mathbb{R}^n$ can be written as $a + L = \{a + x : x \in L\}$, where $L$ is a linear subspace of $\mathbb{R}^n$ with dimension $k < n$, then the set $a + L$ is said to be an affine set, and its dimension is defined as the dimension of $L$, and in general, the dimension of a subset of $\mathbb{R}^n$ is defined as the dimension of the smallest affine set containing that set. The closure of the state price vectors is thus a two-dimensional simplex, and we know that a $k$-dimensional simplex has $k + 1$ vertices.\(^{24}\)

35. **Theorem 5** If the economy $E$ has a Walrasian equilibrium at $t = 0$, and if there exists a portfolio $q^*$ such that $Xq^* > 0$ (which holds if markets are complete), then there exist no arbitrage opportunities. The equilibrium consumption allocation is Pareto optimal if markets are complete; i.e. if the rank of the matrix $X$, denoted by $\rho(X)$, is equal to $n$.\(^{25}\)

**Proof** By definition, for all $i = 1, 2, \cdots, I$, $(G_i)$ has a solution in any equilibrium $(p, Q)$. This rules out immediately any feasible portfolio $q$ with $p'q \leq 0$ and $Xq > 0$. Given the existence of portfolio $q^*$, there

\(^{24}\)Continue with this example, and assume that there are two linearly independent traded assets. Then there again will be infinitely many state price vectors which are consistent with the absence of arbitrage, but all those vectors are convex combinations of two fixed state price vectors! For example, suppose that

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}. $$

Then any state price vector $f_{3 \times 1}$ is a convex combination of the following two weak state price vectors (their elements are non-negative):

$$f_1 = \begin{bmatrix} p_1 \\ 0 \\ p_2 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ p_1 \\ p_2 - p_1 \end{bmatrix}. $$

\(^{25}\)In the current economy every conceivable asset can be represented by an $n \times 1$-vector, which specifies the cash flow generated by one share of the asset in the $n$ possible states at $t = 1$. Hence $\mathbb{R}^n$ consists of all possible assets that investors may want to trade at $t = 0$. If there are $n$ linearly independent assets traded at $t = 0$, or equivalently if $\rho(X) = n$, then every conceivable asset can be replicated by a portfolio of those $n$ assets. That is why we say that markets are complete when $\rho(X) = n$. 

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can be no portfolio \( q \) such that \( p'q < 0 \) and \( Xq \geq 0 \) either: if instead such \( q \) exists, then one can buy a tiny share of \( q^* \) and a tiny share of \( q \) at the same time, which will be budget feasible, which improves on any supposed equilibrium portfolio for agent \( i \).

Next, suppose that markets are complete and the economy has a Walrasian equilibrium \((p, Q)\), but there also exists a non-equilibrium allocation \( d = (d_i; i = 1, 2, \ldots, I) \in \mathbb{R}_{+}^{I \times n} \) such that
\[
\sum_{i=1}^{I} d_i = \sum_{i=1}^{I} c_i = \sum_{i=1}^{I} e_i,
\]
where for all \( i = 1, 2, \ldots, I \), \( c_i = e_i + Xq_i \), and
\[
U_i(d_i) \geq U_i(c_i),
\]
with the inequality being strict for some agent \( j \). We shall demonstrate a contradiction.

Since markets are complete, there exists a unique state price vector \( f = [X]^{-1}p \in \mathbb{R}_{+}^{I} \) (recall that we have assumed that all column vectors of \( X \) are linearly independent), and moreover, the pure discount bond \( 1_{n \times 1} \) is marketed, and to rule out arbitrage opportunities, the bond has a strictly positive price, \( f(1) = f'1 > 0 \). Apparently, we must have
\[
U_j(d_j) > U_j(c_j) \Rightarrow f'd_j > f'c_j.
\]
We claim that for all \( i \neq j \),
\[
U_i(d_i) \geq U_i(c_i) \Rightarrow f'd_i \geq f'c_i.
\]
To see this, suppose instead that for some \( i \neq j \), \( f'd_i < f'c_i \leq f'e_i \). For \( \epsilon \in \mathbb{R}_{+} \) small enough, we have
\[
U_i(d_i + \epsilon 1) > U_i(d_i) \geq U_i(c_i),
\]
and yet
\[
f(d_i + \epsilon 1) = \epsilon f(1) + f'd_i < f'c_i \leq f'e_i,
\]
so that \( c_i \), being dominated by \( d_i + \epsilon 1 \), does not solve \((G_i)\), a contradiction. Now it follows from \( f'd_j > f'c_j \) and \( f'd_i \geq f'c_i \) for all \( i \neq j \) that
\[
f' \sum_{i=1}^{I} d_i > f' \sum_{i=1}^{I} c_i = f' \sum_{i=1}^{I} e_i,
\]
a contradiction to the assumption that \( d \) is a feasible allocation.
36. **Lemma 7** Suppose that \(a, b \in \mathbb{R}^n\) are such that for all \(x \in \mathbb{R}^n\),

\[
a'x = 0 \iff b'x = 0.
\]

Then, there exists \(\lambda \in \mathbb{R}\) such that \(\lambda a = b\).

**Proof** The assertion is obvious if either \(a\) or \(b\) is the zero vector. Assume that neither is zero then. The null space for the linear functional \(f : \mathbb{R}^n \to \mathbb{R}\) defined by \(f(x) = a'x\) is the set \(\mathcal{N} \equiv \{z \in \mathbb{R}^n : f(z) = 0\}\), which is a linear subspace of \(\mathbb{R}^n\) with dimension \(n-1\) (the fundamental theorem of linear algebra). Suppose that \(\{z_j; j = 1, 2, \ldots, n-1\}\) is a basis for \(\mathcal{N}\). Then \(\{a, z_j; j = 1, 2, \ldots, n-1\}\) is a basis for \(\mathbb{R}^n\), and hence for some \(\{\lambda, \alpha_j \in \mathbb{R}; j = 1, 2, \ldots, n-1\}\) we have

\[
b = \lambda a + \sum_{j=1}^{n-1} \alpha_j z_j.
\]

Since \(b'z_j = 0\) for all \(j = 1, 2, \ldots, n-1\), we conclude that

\[
b = \lambda a.
\]

37. **Theorem 6** Suppose that there exists a Walrasian equilibrium for \(\mathcal{E}\). Suppose that for some \(q^* \in \mathbb{R}^m\), \(Xq^* > 0\). Suppose that for some \(i\), \(c_i \in \mathbb{R}_{++}^n\) and the gradient \(\partial U_i(c_i)\) exists and \(\partial U_i(c_i) >> 0\). Then there exists some \(\lambda > 0\) such that \(\lambda \partial U_i(c_i)\) is a state price vector.

**Proof** The fact that \((G_i)\) has a solution implies that the price system admits no arbitrage opportunities (verify!), and hence there exists a state price vector \(\mathbf{f}\).

Moreover, since the optimal consumption \(c_i\) for agent \(i\) is an interior solution to program \((G_i)\), for any \(q\) with \(p'q = 0\) there exists \(k \in \mathbb{R}_{++}\) such that the following problem

\[
\max_{a \in [-k, k]} g(a; q) \equiv U_i(c_i + aXq)
\]

has a solution \(a = 0\). The first-order condition is

\[
0 = g'(0; q) = [\partial U_i(c_i)]'Xq.
\]
Thus for all $q$ with $p'q = 0$, we have $[\partial U_i(c_i)]'Xq = 0$.

Next, suppose that $q$ is such that $[\partial U_i(c_i)]'Xq = 0$. We claim that $p'q = 0$. To see this, suppose instead that $p'q < 0$. By assumption, there exists $q^*$ such that $Xq^* > 0$. To rule out arbitrage opportunities, we must have $p'q^* > 0$, and hence we can buy $-\frac{p'q}{p'q^*}$ units of portfolio $q^*$, so that

$$p'(q + \frac{-p'q}{p'q^*}q^*) = p'q = 0.$$ 

Note however that in this case $g'(0; q) = [\partial U_i(c_i)]'Xq > 0$, a contradiction to the assumption that $c_i$ is optimal. The case where $p'q > 0$ can be analogously ruled out.

We conclude that for all portfolios $q$, $p'q = 0$ if and only if $[\partial U_i(c_i)]'Xq = 0$. It follows from Lemma 7 that there exists some scalar $\lambda$ such that

$$\lambda X'\partial U_i(c_i) \equiv X'f^* = p.$$ 

We claim that $\lambda > 0$. Suppose instead that $\lambda \leq 0$ so that

$$f^* = \lambda \partial U_i(c_i) \leq 0,$$

and consider the date-0 market value of portfolio $q^*$:

$$0 \geq [f^*]'Xq^* = p'q^* = f'Xq^* > 0,$$

which is a contradiction. Thus we conclude that $\lambda > 0$, and hence

$$f^* = \lambda \partial U_i(c_i) >> 0$$

is also a state price vector.

38. For $w \in \mathbb{R}_+ - \{0\}$, define

$$U_w(x) = \sup_{(d_i; i=1,2,..,I)} \sum_{i=1}^I w_i U_i(d_i)$$

subject to

$$\sum_{i=1}^I d_i \leq x.$$
Theorem 7 Suppose that for all $i$, $U_i$ is strictly increasing and concave. Then (i) a feasible consumption allocation $(d_i; i = 1, 2, \ldots, I)$ is Pareto optimal if and only if there exists $w \in \mathbb{R}_+^I - \{0\}$ such that $(d_i; i = 1, 2, \ldots, I)$ solves (1) at $x = \sum_{i=1}^I e_i$. Consequently, (ii) if markets are complete in $\mathcal{E}$ and $\mathcal{E}$ has a Walrasian equilibrium, then there must exist a single-agent economy where the single agent has a utility function defined by (1) for some $w \in \mathbb{R}_+^I - \{0\}$ and endowments equal to the aggregate endowments of the original economy, such that this single-agent economy has a Walrasian equilibrium with the same equilibrium asset prices as in the equilibrium of the original economy, and in equilibrium the single agent always consumes his endowments. Finally, given the $w$, the equilibrium consumption allocation of the original economy solves (1) at $x = \sum_{i=1}^I e_i$.26

26A simple example can illustrate the ideas of this theorem. Suppose that there are two investors A and B, and two possible time-1 states. The two investors have random endowments

$$e_A = \begin{bmatrix} e_{A1} \\ e_{A2} \end{bmatrix}, \quad e_B = \begin{bmatrix} e_{B1} \\ e_{B2} \end{bmatrix},$$

where for $i \in \{A, B\}$ and $j = 1, 2$, $e_{ij}$ is investor $i$'s endowed time-1 wealth in state $j$. Let

$$e_j = e_{Aj} + e_{Bj}, \quad j = 1, 2;$$

that is, $e_j$ is the aggregate time-1 consumption in state $j$. First, we look for the Pareto optimal allocations. Consider

$$\max_{c_{A1}, c_{A2}, c_{B1}, c_{B2}} U_A(c_{A1}, c_{A2})$$

subject to

$$c_{A1} + c_{B1} = e_j, \quad \forall j = 1, 2,$$

$$U_B(c_{B1}, c_{B2}) \geq \overline{\pi},$$

where $\overline{\pi}$ is the reservation utility for investor B. It can be shown that each Pareto optimal allocation can be obtained by solving the above maximization program with some given $\overline{\pi}$. The standard way to solve such a problem is to form the Lagrangian:

$$L = U_A(c_{A1}, c_{A2}) + \lambda[U_B(c_{B1}, c_{B2}) - \overline{\pi}],$$

and maximize $L$ over $c_{A1}, c_{A2}, c_{B1}, c_{B2}$ and $\lambda$, subject to the remaining constraints

$$c_{A1} + c_{B1} = e_j, \quad \forall j = 1, 2.$$
Using these constraints to replace $c_{ Bj}$ by $c_{ Aj}$ in $L$, and writing down the first-order conditions with respect to $c_{ A1}, c_{ A2}$ and $\lambda$, we have

$$U_{ B}(c_{ B1}, c_{ B2}) = \overline{u},$$

$$\frac{\partial U_{ A}}{\partial c_{ A1}} = \lambda = \frac{\partial U_{ B}}{\partial c_{ B1}}.$$

That is the rate of marginal substitution between the consumption in state 1 and in state 2 for investor A must equal that for investor B. Let the Pareto optimal allocation corresponding to $\overline{u}$ be denoted by

$$\{c^*_{ A1}(e_1; \overline{u}), c^*_{ A2}(e_2; \overline{u}), c^*_{ B1}(e_1; \overline{u}), c^*_{ B2}(e_2; \overline{u})\}.$$  

Let the solution for $\lambda$ be correspondingly denoted by $\lambda^*(e_1, e_2, \overline{u})$.

Now, the Pareto optimal allocation so obtained can be obtained in another way: Let $w_{ A} = 1$ and $w_{ B} = \lambda^*(e_1, e_2, \overline{u})$, and solve the following maximization program.

$$\max_{c_{ A1}, c_{ A2}, c_{ B1}, c_{ B2}} \ w_{ A} U_{ A}(c_{ A1}, c_{ A2}) + w_{ B} U_{ B}(c_{ B1}, c_{ B2}),$$

subject to

$\ c_{ Aj} + c_{ Bj} = e_j, \ \forall j = 1, 2.$

This is the first assertion of the theorem.

Next, suppose that both Arrow-Debreu securities are traded in frictionless time-0 financial markets. Then the equilibrium state price vector $f$ must be such that

$$\frac{\partial U_{ A}}{\partial c_{ A1}} = f_1 = \frac{\partial U_{ B}}{\partial c_{ B1}},$$

$$c_{ Aj} + c_{ Bj} = e_j, \ \forall j = 1, 2,$$

$$f_1 c_{ A1} + f_2 c_{ A2} = f_1 e_{ A1} + f_2 e_{ A2},$$

$$f_1 c_{ B1} + f_2 c_{ B2} = f_1 e_{ B1} + f_2 e_{ B2}.$$  

Let the equilibrium allocation be

$$\{c'_{ A1}(e_{ A1}, e_{ A2}, e_{ B1}, e_{ B2}), c'_{ A2}(e_{ A1}, e_{ A2}, e_{ B1}, e_{ B2}), c'_{ B1}(e_{ A1}, e_{ A2}, e_{ B1}, e_{ B2}), c'_{ B2}(e_{ A1}, e_{ A2}, e_{ B1}, e_{ B2})\}.$$  

Now imagine that there is only one investor in an artificial economy, where the single investor’s utility function is defined by

$$U(c_1, c_2) = w_{ A} U_{ A}(c'_{ A1}(e_{ A1; \overline{u}}), c'_{ A2}(e_{ A2; \overline{u}})) + w_{ B} U_{ B}(c'_{ B1}(e_{ B1; \overline{u}}), c'_{ B2}(e_{ B2; \overline{u}})),$$

where

$$\overline{u} = U_{ B}(c'_{ B1}(e_{ A1}, e_{ A2}, e_{ B1}, e_{ B2}), c'_{ B2}(e_{ A1}, e_{ A2}, e_{ B1}, e_{ B2})).$$
Proof First we prove the necessity of statement (i). Suppose that \((d_i; i = 1, 2, \cdots, I)\) is Pareto optimal and define

\[
U \equiv \{y \in \mathbb{R}^I : y_i \leq U_i(c_i) - U_i(d_i), \ c \in \mathcal{A}\},
\]

where \(y_i\) is the \(i\)-th element of \(y\) and \(\mathcal{A}\) is the set of all feasible consumption allocations. We claim that \(U\) is a convex subset of \(\mathbb{R}^I\). To see this, suppose that \(y_1, y_2 \in U\) so that there exist \(c^1, c^2 \in \mathcal{A}\) such that

\[
y_i = \begin{cases} U_i(c^1) & \text{if } i = 1,2, \cdots, I \end{cases}
\]

Moreover, for \(j = 1, 2\), the single investor is assumed to be endowed with \(e_j\) units of time-1 consumption in state \(j\). Again assume that both Arrow-Debreu securities are traded in this economy and consider the competitive equilibrium. It is easy to see that in equilibrium the single investor must consume her endowments. Thus the equilibrium state price vector \(f^*\) must be such that

\[
\frac{\partial U}{\partial c_1}(e_1, e_2) = f_1^*, \quad \frac{\partial U}{\partial c_2}(e_1, e_2) = f_2^*.
\]

The second assertion of the theorem says that

\[
\frac{f_1^*}{f_2^*} = \frac{f_1}{f_2}.
\]

Let us verify if this is true. The left-hand side is equal to

\[
\frac{f_1^*}{f_2^*} = \frac{\partial U}{\partial c_1}(e_1, e_2) = \frac{w_A \frac{\partial U_A}{\partial c_{A1}} \frac{\partial c_{A1}}{\partial c_1} + w_B \frac{\partial U_B}{\partial c_{B1}} \frac{\partial c_{B1}}{\partial c_1}}{w_A \frac{\partial U_A}{\partial c_{A2}} \frac{\partial c_{A2}}{\partial c_1} + w_B \frac{\partial U_B}{\partial c_{B2}} \frac{\partial c_{B2}}{\partial c_1}}
\]

\[
= \frac{\left[w_B \frac{\partial U_B}{\partial c_{B1}} \frac{\partial c_{B1}}{\partial c_1} \right] + \left[w_B \frac{\partial U_B}{\partial c_{B2}} \frac{\partial c_{B2}}{\partial c_1} \right] - \left[w_B \frac{\partial U_B}{\partial c_{B1}} \frac{\partial c_{B1}}{\partial c_1} \right] + \left[w_B \frac{\partial U_B}{\partial c_{B2}} \frac{\partial c_{B2}}{\partial c_1} \right]}{w_A \frac{\partial U_A}{\partial c_{A1}} \frac{\partial c_{A1}}{\partial c_1} + w_B \frac{\partial U_B}{\partial c_{B1}} \frac{\partial c_{B1}}{\partial c_1}} = \frac{f_1}{f_2},
\]

where the 3rd equality follows from the fact that

\[
w_A \frac{\partial U_A}{\partial c_{A1}} = w_B \frac{\partial U_B}{\partial c_{B1}}, \quad \forall j = 1, 2;
\]

and the 4th equality follows from the fact that

\[
\frac{\partial c^*_{A1}}{\partial c_j} + \frac{\partial c^*_{B1}}{\partial c_j} = 1, \quad \forall j = 1, 2.
\]

Hence the equilibrium asset prices in the artificial economy are the same as those prevailing in the equilibrium of the original economy. We refer to this theorem the representative agent theorem, and this theorem has been widely used in the literature of empirical finance.
that, for all $i = 1, 2, \cdots, I$,

$$y_{ji} \leq U_i(c^j_i) - U_i(d_i), \ j = 1, 2,$$

where $y_{ji}$ is the $i$-th element of $y_j$. Observe that for all $\lambda \in (0, 1)$,

$$\lambda y_{1i} + (1 - \lambda)y_{2i} \leq \lambda U_i(c^1_i) + (1 - \lambda)U_i(c^2_i) - U_i(d_i)$$

$$\leq U_i(\lambda c^1_i + (1 - \lambda)c^2_i) - U_i(d_i), \ \forall i = 1, 2, \cdots, I,$$

where $\lambda c^1_i + (1 - \lambda)c^2_i \in \mathbb{R}^n_+$ and

$$\sum_{i=1}^{I}[\lambda c^1_i + (1 - \lambda)c^2_i] = \lambda \sum_{i=1}^{I} c^1_i + (1 - \lambda) \sum_{i=1}^{I} c^2_i \leq \sum_{i=1}^{I} e_i,$$

and hence $(\lambda c^1_i + (1 - \lambda)c^2_i; i = 1, 2, \cdots, I)$ is a feasible allocation, proving that $\lambda y_{1i} + (1 - \lambda)y_{2i} \in \mathcal{U}$. Thus $\mathcal{U}$ is convex in $\mathbb{R}^I$.

Define

$$\mathcal{J} \equiv \{ y \in \mathbb{R}^I : y \neq 0 \}.$$ 

Since $(d_i; i = 1, 2, \cdots, I)$ is Pareto optimal, $\mathcal{U} \cap \mathcal{J} = \emptyset$. To see this, suppose instead that $y \in \mathcal{U} \cap \mathcal{J}$. This implies the existence of some feasible consumption allocation $c$ with for all $i$

$$U_i(c_i) - U_i(d_i) \geq y_i \geq 0,$$

where the last inequality is strict for at least one $i$, and hence $d$ cannot be a Pareto optimal allocation. Apparently, $\mathcal{J}$ is non-empty and convex. Moreover, $0 \in \mathcal{U}$ so that $\mathcal{U}$ is non-empty either. Thus it follows from the Minkowski’s Theorem that there exists $w \in \mathbb{R}^I - \{0\}$ such that

$$w'y \leq w'z, \ \forall y \in \mathcal{U}, \ \forall z \in J.$$

Since $0 \in \mathcal{U}$ and since for all $i = 1, 2, \cdots, I$, the standard basis vectors of $\mathbb{R}^n$, $u_i \in \mathcal{J}$, we have

$$0 = w'0 \leq w'u_i = w_i, \ \forall i = 1, 2, \cdots, I,$$

so that with $w \neq 0$, we have $w > 0$. 

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Next we prove the sufficiency of statement (i). Suppose that there exists \( w \in \mathbb{R}^I_+ \setminus \{0\} \) such that \((d_i; i = 1, 2, \cdots, I)\) solves (1) at \( x = \sum_{i=1}^I e_i \).

It must be that \( \sum_{i=1}^I (e_i - d_i) = 0 \) and \( w_i = 0 \Rightarrow d_i = 0, \ \forall i = 1, 2, \cdots, I. \)

Define \( H \equiv \{ i \in \{1, 2, \cdots, I\} : w_i = 0 \} \) and \( H^c \equiv \{1, 2, \cdots, I\} - H. \)

Suppose that \((c_i; i = 1, 2, \cdots, I)\) is such that \( \sum_{i=1}^I (c_i - e_i) \leq 0 \) and

\[
U_i(d_i) \leq U_i(c_i)
\]

with the inequality being strict for some agent \( j. \) Since \( w \in \mathbb{R}^I_+ \), we have

\[
\sum_{i=1}^I w_i U_i(d_i) \leq \sum_{i=1}^I w_i U_i(c_i).
\]

On the other hand, since \((d_i; i = 1, 2, \cdots, I)\) solves (1) at \( x = \sum_{i=1}^I e_i \), we have

\[
\sum_{i=1}^I w_i U_i(d_i) = \max_{(b_i \in \mathbb{R}^n_+; i = 1, 2, \cdots, I), \sum_{i=1}^I b_i \leq \sum_{i=1}^I e_i} \sum_{i=1}^I w_i U_i(b_i)
\geq \max_{(b_i \in \mathbb{R}^n_+; i = 1, 2, \cdots, I), \sum_{i=1}^I b_i \leq \sum_{i=1}^I e_i} \sum_{i=1}^I w_i U_i(c_i),
\]

where the first inequality follows from the strict monotonicity of \( U_i(\cdot) \), for all \( i, \) and \( \sum_{i=1}^I c_i \leq \sum_{i=1}^I e_i \). Thus we conclude that

\[
\sum_{i=1}^I w_i U_i(d_i) = \sum_{i=1}^I w_i U_i(c_i).
\]

We shall demonstrate a contradiction to this equality. Observe that

\[
U_j(d_j) < U_j(c_j) \Rightarrow j \in H.
\]
(If agent \( j \in H^c \), then we have \( \sum_{i=1}^{I} w_i U_i(c_i) > \sum_{i=1}^{I} w_i U_i(d_i) \), a contradiction.) Now \( \mathbf{0} = \sum_{i \in H} d_i \) and hence the existence of the aforementioned agent \( j \) implies

\[
\sum_{i \in H} (c_i - d_i) \equiv g > 0.
\]

Now observe that

\[
\sum_{i=1}^{I} w_i U_i(c_i) = \sum_{i \in H^c} w_i U_i(c_i)
\]

\[
\leq \max_{(k_i; i = 1, 2, \ldots, I), \sum_{i \in H^c} k_i \leq \sum_{i=1}^{I} e_i} \left[ \sum_{i \in H^c} w_i U_i(k_i) \right]
\]

\[
< \max_{(k_i; i = 1, 2, \ldots, I), \sum_{i \in H^c} k_i \leq \sum_{i=1}^{I} e_i} \left[ \sum_{i \in H^c} w_i U_i(k_i) \right]
\]

\[
= \sum_{i \in H^c} w_i U_i(d_i)
\]

\[
= \sum_{i=1}^{I} w_i U_i(d_i),
\]

where the strict inequality follows from the fact that \( U_i(\cdot) \) is strictly increasing for all \( i = 1, 2, \ldots, I \). Thus we have a contradiction.

Finally, consider statement (ii). Since \( \mathcal{E} \) has a Walrasian equilibrium, the price system \((X, p)\) admits no arbitrage opportunities, and since markets are complete in \( \mathcal{E} \), there exists a unique state price vector \( f \). In the original economy \( \mathcal{E} \), agent \( i \)'s problem \((G_i)\) can now be expressed as

\[
\sup_{c_i \in \mathbb{R}^n_+} U_i(c_i)
\]

subject to

\[
f'c_i \leq f'e_i.
\]

Since \( U_i \) is concave and \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by \( g_i(c_i) = f'(c_i - e_i) \) is affine (therefore convex), (2) presents a concave program. Here we can

\footnote{Once we solve for the optimal \( c_i \), we can recover the optimal portfolio \( q_i = X^{-1}[e_i - c_i] \); this is the Cox-Huang martingale approach used to solve for a consumer's optimal consumption and investment policies.}

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assume without loss of generality that for all $i = 1, 2, \cdots, I$, $e_i > 0$, for agents without any endowments must consume the zero consumption plan, and therefore irrelevant to our analysis. With this assumption, since $c_i = 0$ is feasible and $f > 0$, we have

$$f'0 < f'e_i,$$
which is the Slater condition required in the saddle-point theorem.\textsuperscript{28}

\textsuperscript{28}Recall that a concave program is a triple \((U, X, g)\) of the form
\[
\sup_{x \in X} U(x) \text{ subject to } g(x) \leq 0,
\]
where \(X\) is a convex subset of some real linear space, \(U : X \to \mathbb{R}\) is concave, and \(g : X \to \mathbb{R}^m\) is convex. The Lagrangian for the concave program is \(L : X \times \mathbb{R}^m_+ \to \mathbb{R}\) defined by
\[
L(x, \lambda) = U(x) - \lambda \cdot g(x),
\]
where “\(\cdot\)" denotes inner product. A pair \((x_0, \lambda_0)\) is a saddle point of \(L\) if for all \((x, \lambda) \in X \times \mathbb{R}^m_+\),
\[
L(x, \lambda) \leq L(x_0, \lambda_0) \leq L(x_0, \lambda).
\]
If \((x_0, \lambda_0)\) is a saddle point, we call \(\lambda_0\) a Lagrange multiplier for the concave program.

**The Saddle-point Theorem** Suppose that there exists \(x'_0\) such that \(g(x'_0) << 0\) (Slater condition).
1. (Necessity) If \(x_0\) is a solution to the concave program, then there exists \(\lambda_0 \in \mathbb{R}^m_+\) such that \((x_0, \lambda_0)\) is a saddle point of \(L\) and \(\lambda_0 \cdot g(x_0) = 0\) (complementary slackness).
2. (Sufficiency) If \((x_0, \lambda_0)\) is a saddle point of \(L\) then \(x_0\) is a solution to the concave program.

**Proof.** First consider necessity. Define \(L = \mathbb{R} \times \mathbb{R}^m\), with subsets
\[
A = \{(r, z) : r \leq U(x), \ z \geq g(x), \text{ for some } x \in X\},
\]
\[
B = \{(r, z) : r > U(x_0), \ z << 0\}.
\]
Both \(A\) and \(B\) are convex. If \((r, z) \in A \cap B\), then there is some \(x \in X\) with \(g(x) \leq z << 0\) but \(U(x) \geq r > U(x_0)\), which is contradiction. Thus \(A \cap B = \emptyset\). Since \(B\) is open, we conclude that there exists a linear functional \(F(r, z) = \alpha r + \lambda \cdot z\) such that
\[
F(v) < F(w), \forall v \in A, \ w \in B.
\]
Moreover, for all \(v \in A\) and all \(w \in B\), we have
\[
F(v) \leq F(w).
\]
Note that \((U(x_0), 0) \in A \cap \overline{B}\), \((U(x'_0), g(x'_0)) \in A\), and \((U(x_0) + \epsilon, g(x')) \in B\) given \(\epsilon > 0\).
Thus we have
\[
aU(x_0) + \alpha \epsilon + \lambda \cdot g(x') > aU(x'_0) + \lambda \cdot g(x').
\]
Thus we have
\[
a[U(x_0) - U(x'_0) + \epsilon] > 0,
\]
implying that \(\alpha > 0\). Since \((U(x_0), 0) \in A\) and \((U(x_0) + \epsilon, -h) \in B\) if \(\epsilon > 0, h >> 0\), we have
\[
aU(x_0) + \alpha \epsilon - \lambda \cdot h > aU(x_0).
\]
By letting $\epsilon, h_j \downarrow 0$ for all $j \neq i$, and $h_i = 1$, we have

$$-\lambda_i \geq 0,$$

showing that

$$\lambda_i \leq 0, \forall i = 1, 2, \ldots, m.$$ 

Since $(U(x_0), 0) \in \overline{B}$ and $(U(x_0), g(x_0)) \in A$, we have

$$\alpha U(x_0) \geq \alpha U(x_0) + \lambda \cdot g(x_0),$$

showing that

$$\lambda \cdot g(x_0) \leq 0.$$ 

Since $\lambda, g(x_0) \leq 0$, we conclude that

$$\lambda \cdot g(x_0) = 0.$$ 

As $\alpha > 0$, we can define $\lambda_0 = \frac{1}{\alpha}\lambda$. Now, for all $x \in X$, $(U(x), g(x)) \in A$, and since $(U(x_0), 0) \in \overline{B}$, we have

$$U(x_0) - \lambda_0 \cdot g(x_0) = U(x_0) \geq U(x) - \lambda_0 \cdot g(x).$$

Moreover, since $g(x_0) \leq 0$, for any $\lambda \in \mathbb{R}^m_+$, we have $-\lambda \cdot g(x_0) \geq 0 = -\lambda_0 \cdot g(x_0)$, and hence

$$U(x_0) - \lambda \cdot g(x_0) \geq U(x_0) - \lambda_0 \cdot g(x_0).$$

This finishes the proof for the necessity.

Now consider sufficiency. Suppose that $(x_0, \lambda_0)$ is a saddle point for the Lagrangian. First I claim that $x_0$ is feasible to the concave program. Suppose instead that for some $j$, $g_j(x_0) > 0$. Then choosing $\lambda$ as such that $\lambda_j > \max(0, \frac{\lambda_0 \cdot g_j(x_0)}{g_j(x_0)})$ and $\lambda_k = 0$ for all $k \neq j$, we have

$$U(x_0) - \lambda \cdot g(x_0) = U(x_0) - \lambda_j g_j(x_0) < U(x_0) - \lambda_0 \cdot g(x_0),$$

violating the assumption that $(x_0, \lambda_0)$ is a saddle point for the Lagrangian. It follows that $\lambda_0 \cdot g(x_0) \leq 0$. Next we claim that $\lambda_0 \cdot g(x_0) = 0$. Suppose instead that $\lambda_0 \cdot g(x_0) < 0$. Then choosing $\lambda = \frac{1}{2}\lambda_0$, and we have

$$U(x_0) - \lambda \cdot g(x_0) = U(x_0) - \frac{1}{2}\lambda_0 \cdot g(x_0) < U(x_0) - \lambda_0 \cdot g(x_0),$$

which again violates the assumption that $(x_0, \lambda_0)$ is a saddle point for the Lagrangian. Now, for all $x \in X$ such that $g(x) \leq 0$, we have

$$U(x_0) = U(x_0) - \lambda_0 \cdot g(x_0) \geq U(x) - \lambda_0 \cdot g(x) \geq U(x),$$

and hence $x_0$ solves the concave program $(U, X, g)$. 

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Now by the saddle-point theorem, if \( c_i \) solves (2), there must exist some \( \lambda_i \in \mathbb{R}_+ \) such that given \( \lambda_i \), \( c_i \) solves
\[
\sup_{d_i \in \mathbb{R}_+^n} L_i(d_i, \lambda_i) = U_i(d_i) - \lambda_i g_i(d_i). \tag{3}
\]

We claim that \( \lambda_i > 0 \). To see this, suppose instead that \( \lambda_i = 0 \). Since \( U_i \) is strictly increasing, we have
\[
L_i(c_i + 1, \lambda_i) > L_i(c_i, \lambda_i),
\]
so that \((c_i, \lambda_i)\) is not a saddle point of \( L_i(\cdot, \cdot) \), a contradiction. Thus we can define \( \lambda_i = \frac{1}{\lambda_i} \), for all \( i = 1, 2, \cdots, I \), and define accordingly \( w \in \mathbb{R}_+^I \) as such that its \( i \)-th element is \( w_i \). We claim that with this \( w \), the equilibrium consumption allocation \( c \equiv (c_i; i = 1, 2, \cdots, I) \) solves (1) at \( x = \sum_{i=1}^I e_i \). To see this, note that if \( d \equiv (d_i; i = 1, 2, \cdots, I) \) is any other feasible consumption allocation given the aggregate endowment \( \sum_{i=1}^I e_i \), then we have
\[
\sum_{i=1}^I w_i U_i(d_i) \\
\leq \sum_{i=1}^I w_i U_i(d_i) - f'|\sum_{i=1}^I (d_i - e_i)] \\
= \sum_{i=1}^I w_i U_i(d_i) - f'|\sum_{i=1}^I \lambda_i w_i(d_i - e_i)] \\
= \sum_{i=1}^I w_i U_i(d_i) - \lambda_i f'[d_i - e_i] \\
\leq \sum_{i=1}^I w_i U_i(c_i) - \lambda_i f'[c_i - e_i] \\
= \sum_{i=1}^I w_i U_i(c_i),
\]
where the last equality follows from the complementary slackness property of the saddle points \((c_i, \lambda_i)\), for all \( i = 1, 2, \cdots, I \).

It remains to show that when a single agent is endowed with the utility function \( U_w(\cdot) \) and endowments \( \sum_{i=1}^I e_i \), \( f \) remains an equilibrium state.
price vector with the agent’s optimal consumption being equal exactly to $\sum_{i=1}^{I} e_i$. Suppose instead that this is not true. This means that given the price vectors $f$, there exists $\sum_{i=1}^{I} x_i \in \mathbb{R}_+^n$ such that

$$\sum_{i=1}^{I} w_i U_i(x_i) = U_w(\sum_{i=1}^{I} x_i) > U_w(\sum_{i=1}^{I} e_i) = \sum_{i=1}^{I} w_i U_i(c_i),$$

and

$$f'[\sum_{i=1}^{I} x_i] \leq f'[\sum_{i=1}^{I} e_i] = f'[\sum_{i=1}^{I} c_i],$$

where the last equality follows from strict monotonicity of $U_i$ for all $i = 1, 2, \cdots, I$. We shall demonstrate a contradiction. Note that

$$\sum_{i=1}^{I} w_i U_i(x_i) - f'[\sum_{i=1}^{I} (x_i - e_i)] > \sum_{i=1}^{I} w_i U_i(e_i) - f'[\sum_{i=1}^{I} (c_i - e_i)]$$

$$\Rightarrow \sum_{i=1}^{I} w_i \{U_i(x_i) - \lambda f'[x_i - e_i]\} > \sum_{i=1}^{I} w_i \{U_i(e_i) - \lambda f'[c_i - e_i]\},$$

which, by the fact that $w_i > 0$ for all $i = 1, 2, \cdots, I$, contradicts the fact that $(c_i, \lambda_i)$ is a saddle point for $L_i(\cdot, \cdot)$, for all $i = 1, 2, \cdots, I$.

39. Now we give the first introduction to the martingale pricing theorem. Suppose in a two-period (dates 0 and 1) economy there is one single consumption good and one single agent with date-t endowment $e_t$, $t = 0, 1$. The agent has time-additive state-independent von Neumann-Morgenstern utility function

$$u_0(c_0) + u_1(c_1).$$

The set of date-1 possible states of nature is denoted by $\Omega$ (a finite set), of which a typical element is denoted $\omega$. Suppose markets are complete

Note that we do not require that the $j$-th element $\sum_{i=1}^{I} x_{ij}$ of $\sum_{i=1}^{I} x_i$ be less than or equal to the $j$-th element of $\sum_{i=1}^{I} e_i$; that is, we do not require the single agent to choose only among feasible allocations. This follows from the price-taking individual agent’s behavior: the agent first finds the market value of his date-1 endowment, and he is free to choose any consumption plan with a market value not exceeding the market value of his endowment.
with the Arrow-Debreu security for state $\omega$ having a date-0 price $\phi_{\omega}$. The agent’s consumption problem is at date 0
\[
\max_{c_0,c_1(\omega)} u_0(c_0) + E[u_1(c_1(\omega))],
\]
subject to
\[
c_0 + \sum_{\omega \in \Omega} \phi_{\omega} c_1(\omega) = e_0 + \sum_{\omega \in \Omega} \phi_{\omega} e_1(\omega).
\]
It can be shown that
\[
\frac{\pi_{\omega} u'_1(c_1(\omega))}{u'_0(e_0)} = \phi_{\omega}, \quad \forall \omega \in \Omega,
\]
where $\pi_{\omega}$ is the prob. for the state $\omega$. Next, markets clearing condition requires that $c_0 = e_0$ and for all $\omega \in \Omega$, $c_1(\omega) = e_1(\omega)$. (As there are no others in this economy, the equilibrium prices of assets must be such that the single agent consumes her endowments at each date in each state.) Now consider an asset that pays $x(\omega)$ in state $\omega$. Its date-0 price must be
\[
P_x = \sum_{\omega \in \Omega} \phi_{\omega} x(\omega)
\]
\[
= E\left[\frac{u'_1(\tilde{e}_1)}{u'_0(e_0)} \tilde{x}\right].
\]
In particular, the price of a pure discount bond with one dollar face value is
\[
\frac{1}{1 + r_f} = E\left[\frac{u'_1(\tilde{e}_1)}{u'_0(e_0)} \tilde{x}\right].
\]
Thus, for all assets $x$,
\[
P_x = \frac{E\left[\frac{u'_1(\tilde{e}_1)}{u'_0(e_0)} \tilde{x}\right]}{1 + r_f}.
\]
Define
\[
\tilde{\xi} = \frac{E\left[\frac{u'_1(\tilde{e}_1)}{u'_0(e_0)} \tilde{x}\right]}{E\left[\frac{u'_1(\tilde{e}_1)}{u'_0(e_0)}\right]},
\]

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We can define the martingale probabilities as
\[ \forall \omega \in \Omega, \quad \pi^*_\omega = \pi_\omega \xi(\omega). \]

Note that these are indeed well-defined probabilities. We have obtained the following martingale pricing formula for any asset \( x \),
\[ P_x = \frac{E[\xi \tilde{x}]}{1 + r_f} = \frac{E^*[\tilde{x}]}{1 + r_f}. \]

That is, the original prob. measure is related to the martingale prob. measure by multiplying a Radon-Nikodym derivative \( \xi \). Note that \( \xi \) is defined by the agent’s equilibrium (random) intertemporal marginal rate of substitution (which depends on the agent’s preferences and endowments) and the agent’s original probabilities. The purpose of changing the probability measures is to get rid of risk aversion: if the agent were risk neutral, then \( \xi = 1 \) and the original prob. measure would coincide with the equivalent martingale measure.

40. **Theorem 8** Suppose that all agents are strictly risk averse and prefer more wealth to less; i.e. for all \( j \), \( u''_j < 0 < u'_j \). Then a Pareto efficient allocation \( \{c^1, c^2, \ldots, c^I\} \) has the property that for all \( j \), \( c^i_j < c^i_j' \) whenever \( C_i < C'_i \), where \( C_i \) is the aggregate consumption in state \( i \). In particular, in any two states \( i \) and \( i' \) where \( C_i = C'_i \), it must be that \( c^i_j = c^i_j' \) for all \( j \). For this reason, corresponding to each Pareto efficient allocation \( \{c^1, c^2, \ldots, c^I\} \), there exist \( n \) strictly increasing functions \( \{s_1(\cdot), s_2(\cdot), \ldots, s_I(\cdot)\} \) such that \( c^i_j = s_j(C_i) \) for all \( i \) and all \( j \). That is, \( c^i_j \) depends only on \( C_i \) but not on \( i \) directly. These \( n \) functions \( \{s_j(\cdot)\} \) are referred to as a Pareto optimal (risk) sharing rule.

The above theorem says that for Pareto efficiency to prevail, everyone must be allowed to consume more in states where there are more to consume. To see the idea, suppose that \( C_i > C'_i \). The feasibility constraint then implies that for at least one \( j \), \( c^i_j > c^i_j' \). Since the marginal rate of substitution over \( c_i \) and \( c'_i \) must be identical across agents, and \( u''_j(\cdot) \) is strictly decreasing for each \( j \), it must be that \( c^i_j > c^i_j' \) holds for all \( j \).

41. Suppose that in the two-date economy, the \( I \) agents have common beliefs \( \{\pi_i\}_{i=1}^n \) for the \( n \) states (there is a single physical consumption good
at date 1). The equivalence condition for Pareto efficient allocations is, as we recall,
\[
\frac{\partial U_j}{\partial c_j} = \frac{\partial U_{j'}}{\partial c_{j'}},
\]
for any two agents \(j\) and \(j'\) and for any two states of nature \(i\) and \(i'\). This is referred to as the Borch rule for Pareto efficient risk sharing. To take a closer look at this result, assume \(I = 2\) with agent 1 being risk neutral and agent 2 strictly risk averse with VNM utility function \(u(\cdot)\). We conclude from the Borch rule that, for any two distinct states \(i\) and \(i'\),
\[
\frac{u'(c_i^2)}{u'(c_{i'}^2)} = 1.
\]
Since \(u'(\cdot)\) is a strictly decreasing function, this implies that \(c_i^2\) equals some constant \(\tau\) for all \(i\). That is, agent 2 should have a sure amount of consumption at date 1 regardless of which state will occur. The risk neutral agent thus bears all the risk.

42. Let us give an example for Pareto optimal sharing rules. Suppose that the economy has one single consumption good, \(I = 2\), and the aggregate consumption \(\tilde{C}\) has a compact support \([0, 1]\). Agent \(j\) has von Neumann Morgenstern utility function
\[
u_j(c) = e^{-a_j c}, \quad j = 1, 2,
\]
where \(a_j > 0\) are constants. We are looking for two increasing functions \(s_1(C)\) and \(s_2(C) = C - s_1(C)\) such that for some positive constant \(\pi \in [0, 1]\), \(s_1(\cdot)\) is the solution to
\[
\max_{s(\cdot)} J(s(\cdot)) = \pi E[-e^{-a_1 s(C)}] + (1 - \pi) E[-e^{-a_2(\tilde{C} - s(C))}].
\]

Note that for an arbitrary function \(t(\cdot)\),
\[
\max_{a \in \mathbb{R}} H(a) = J(s_1(\cdot) + at(\cdot))
\]
assumes maximum at \(a = 0\), as \(s_1(\cdot)\) is Pareto efficient. Thus, in case \(H(\cdot)\) is smooth enough,
\[
H'(0) = 0.
\]
Expanding, we see that $s_1(C) = \alpha + \beta C$ for some constants $\alpha$ and $\beta > 0$. That is, for CARA people, the optimal sharing rules are linear. Note that to implement a linear sharing rule by competitive markets system, we only need to open two markets, one for a risky asset that pays dividends proportional to $\bar{C}$, and the other a riskless asset. This says that two fund separation holds for CARA people.\(^{30}\)

43. Now we review the concept of Radner equilibrium in a 2-period economy. There are two dates, 0 and 1, and consumption takes place only at date 1. We call date 0 ex-ante and date 1 ex-post. There are $S$ possible states at date 1 and there are $L$ consumption goods. There are $I$ agents maximizing expected utilities, which are derived from their date-1 consumptions. Agents trade assets at date 0 and may be able to trade in the date-1 spot markets for consumption goods. First suppose that markets are Debreu complete: there are $SL$ contingent assets, where the $(s,l)$-th asset delivers 1 unit of good $l$ if state $s$ occurs at date 1 and nothing otherwise. Assume that agent $i$’s von Neumann-Morgenstern utility function $v^i : \mathbb{R}^L_+ \to \mathcal{R}$ is continuously differentiable, increasing, and strictly concave, her endowments are $(w_{sl})$, and she holds the beliefs that state $s$ may occur with prob. $\pi_s^i$.

**Definition 5** An Arrow-Debreu contingent market equilibrium\(^{31}\) is a pair $\{ (p_{sl}), (x_{sl})^i \equiv x^i \}$ such that

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\(^{30}\)Optimal sharing rules are linear if and only if the $I$ agents have linear risk tolerances and identical risk cautiousness; i.e. their von Neumann-Morgenstern utility functions are in the HARA class. See for example Huang and Litzenberger, chapter 5. Interested readers should refer to Cass and Stiglitz (1970).

\(^{31}\)If there is a complete set of Arrow-Debreu commodities traded at date 0, we say markets are Debreu-complete and the pure exchange economy is referred to as the Arrow-Debreu contingent markets economy. If at date 0 we have a complete set of traded contingent commodities, and we have rational expectations (agreeing upon all relevant measurable functions in the sense of Roy Radner (1972)), then no trade would occur even if markets reopen. Things are different if we do not have a complete set of Arrow-Debreu commodities traded at date 0. In the latter case, trades may occur when markets reopen after date 0, and in this case markets may or may not be dynamically complete. It is clear that price expectations are important elements in the latter market structure; see Radner (1972, Econometrica).
(i) $\forall i = 1, 2, \ldots, I$, $x^i = \arg\max u^i(x^i) \equiv \sum_{s=1}^{S} \pi^i_s v^i(x^i_s)$ subject to
\[ \sum_s \sum_l p_{sl} (x^i_{sl} - w^i_{sl}) \leq 0; \]

(ii) $\sum_i (x^i - w^i) = 0$.

As an earlier theorem shows, the contingent market equilibrium allocation is Pareto efficient. This is referred to as the *ex-ante efficiency*. It implies the following *ex-post efficiency*: for all $s$, the equilibrium in the date-1 spot market for consumption goods is such that $(x^1_s, x^2_s, \ldots, x^I_s)$ is Pareto efficient relative to the set of utility functions $(v^1, v^2, \ldots, v^I)$. It can be shown that ex-ante efficiency implies ex-post efficiency. The converse is not true in general.

44. Now assume that only $K < SL$ markets are open ($K$ assets in zero supply are traded) at date $0$, but spot markets (for goods) are all open at date $1$. Thus the market structure at date $0$ is incomplete. This is where the concept of Radner equilibrium arises.

First we define notation. An asset is denoted by a matrix

\[ A = [a_{sl}] = [a_1, a_2, \ldots, a_S], \]

where $a_s$ is an $L$-vector specifying the amount of good $l$ to be delivered at date 1 in case state $s$ occurs.\(^{32}\) Let $q^k$, $k = 1, 2, \ldots, K$, be the date-0 price of asset $k$. Let $z^{ik}$ be agent $i$’s position in asset $k$. We call $\mathbf{z}^i$ the portfolio of agent $i$. The timing is as follows. At date 0, agents first trade assets and form portfolios. At date 1, the assets first pay (liquidation) dividends (in goods; see the above definition). These dividends are added to endowments and then agents trade in the spot markets. Define

\[ \overline{w}_s^i = w_s^i + \sum_{k=1}^{K} z^{ik} a_s^k, \]

which will be agent $i$’s pre-trade endowments in state $s$ at date 1.

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\(^{32}\)For example, for the $(s, l)$ Arrow-Debreu contingent good (Arrow-Debreu security), $a_{sl} = e_l \cdot 1_{[l'=s]}$ where $e_l$ is the $l$-th standard basis vector in $\mathbb{R}^L$. A bond in good $l'$ is such that $a_{sl} = 1_{[l=l']}$. 
Definition 6  A Radner equilibrium is a tuple
\[
\{ q^* \in \mathcal{R}^K_+; p^*_s \in \mathcal{R}^L_+ (\forall s); (z^{s*1}, z^{s*2}, \ldots, z^{s*I}); (x^{s*1}, x^{s*2}, \ldots, x^{s*I}) \}
\]
such that
(i) \( \forall i, (z^{s*i}, x^{s*i}) \) maximizes \( u^i(x^i) \) subject to
(ii-1) \( q^* \cdot z^{s*i} \leq 0 \);
(ii-2) \( p^*_s \cdot x^{s*i} \leq \bar{p}^{s*i} = p^*_s \cdot (w^{s*i}_s + \sum_k z^{s*k} a^{s*k}_s), \forall s \);
(iii) \( \sum_i z^{s*i} = 0 \);
(iv) \( \forall s, \sum_i x^{s*i} = \sum_i w^{s*i}_s \).

Note that agents have perfect foresight in a Radner equilibrium, in the sense that at date 0 when they make portfolio decisions they correctly predict the measurable functions \( p^i_s = p^i_l(s) \) even if they may hold different prob. measures. (Agent \( i \)'s information structure is represented by the prob. space \( (S = \{1, 2, \ldots, S\}, 2^S, \pi^i) \).)\(^{33}\)

45. With incomplete markets, Radner equilibrium allocations are usually Pareto inefficient. For example, let \( L = 1, S = 3, K = 2 \). Let \( a^1 = (1 + r, 1 + r, 1 + r) \) and \( a^2 = (0, 0, 1) \). Comparing the two equilibrium definitions, we see that inefficiency arises in the Radner equilibrium because with incomplete markets an agent is faced with \( S + 1 \) budget constraints rather than one single constraint as in the case of complete markets.

46. Unlike in the Arrow-Debreu economy where there will be no trade even if markets reopen after date 0, trades will generally take place at both dates 0 and 1 in a Radner equilibrium. At date 0, agents have perfect foresight in the sense that they correctly expect the terms of trade at date 1 and based on these price expectations, they carry out part of the trading process at date 0 and postpone the rest till date 1. It follows that for a Radner equilibrium to implement the Pareto efficient allocation of a contingent market equilibrium, we only need to ensure that each agent gets the right amount of pre-trade wealth in state \( s \): since the ex-ante efficient contingent market equilibrium allocation implies that the allocation is ex-post state-\( s \) efficient for all \( s \), and each

\(^{33}\)Note that the the contingent market equilibrium is a special case of Radner equilibrium: simply let \( q^* = p^*, z^{s*i} = x^{s*i} - w^i \).
ex-post efficient allocation is sustained by a competitive equilibrium (actually at the contingent market equilibrium prices), we only need to make sure that the date-0 asset structure is such that in every state \( s \) the optimal portfolio for each agent generates an additional wealth which is exactly the difference between the value of the agent’s initial state-\( s \) endowments and the value of her contingent market equilibrium consumption bundle. This motivates the idea of \textit{Arrow completeness}.

47. Arrow proposes that Pareto efficiency is ensured if at date 0 there are properly chosen \( S \) asset markets and all \( L \) spot markets are available at date 1. When markets meet Arrow’s requirement, we say that they are \textit{Arrow complete}; see Arrow’s papers listed in the references.\textsuperscript{34} The idea is as follows. Take good 1 as numeraire and rewrite the definition for Radner equilibrium. Let \((p^*, (x^s))\) be an Arrow-Debreu contingent market equilibrium were we to have a complete set of traded contingent goods. Now, create \( S \) assets such that asset \( s \) (the contingent money in state \( s \)) pays 1 unit of good 1 in state \( s \) and nothing otherwise. Consider the price expectations \( p_s = \frac{p_s}{p_s^1} \) for date-1 spot markets and the date-0 asset prices \( q_s = p_s^1 \). Given these prices, agent \( i \)'s portfolio decision is such that \( z^i_s = p_s (x^i_s - w^i_s) \). It can now be verified that together with \( x^s \) this forms a Radner equilibrium.\textsuperscript{35}

48. Does a Radner equilibrium exist always?

\textbf{Theorem 9} Suppose that agent \( i \)'s consumption plan must be contained in \( X^i \subset \mathbb{R}^{S\mathbb{R}} \), which is closed, convex, and bounded from below (implying that the set \( \{ \sum_i x^i \leq \sum_i w^i \} \) is compact). Suppose that \( u^i \) is continuous and concave. Suppose that for all \( i \) and all \( x^i \), there exists \([x^i]'\) differing from \( x^i \) only in one state such that \( u'(x^i) > u'(x^i) \). Suppose that for all \( i \), there exists at least one \( x^i_0 \in X^i \) such that \( x^i_0 \ll w^i \). Finally, suppose there exists \( D > 0 \) such that for all \( i \) and all \( k \), \( z^i_k \geq -D \) (so that \( \{ z| \sum_i z^i \} \) is compact). Then there exists a Radner equilibrium.

\textsuperscript{34}Arrow’s idea is further extended by Harrison and Kreps (1979) into a notion commonly referred to as \textit{dynamic completeness}.

\textsuperscript{35}Note that in each state at date 1, each agent will use up her pre-trade wealth (because \( v^i \) is strictly increasing). It follows that essentially each agent is facing one single budget constraint just like in the Arrow-Debreu economy.
The lower bound $D$ represents one type of short sale constraint. Without this assumption, Radner equilibrium need not exist. For example, suppose $S = K = 2 \leq L$. Hart (1975; Journal of Economic Theory) gives an example where for all $(p_1, p_2) \neq (p_1', p_2')$ (subscripts for states), $(p_1 \cdot a_1, p_2 \cdot a_2)$ is linearly independent of $(p_1 \cdot b_1, p_2 \cdot b_2)$ (where $a, b$ stand for the two assets), but the two vectors are linearly dependent at $(p_1', p_2')$. In this case, except at the $(p_1', p_2')$, the attainable consumption plans span $\mathbb{R}^2$ (without the bound $D$!) so that markets are complete, but markets are incomplete at $(p_1', p_2')$. It follows that the budget correspondence is not upper hemi-continuous, and a fixed point fails to exist. If $L = 1$, however, this is no problem, because the bounds on $X^i$ translate into bounds on $z^i$, and hence an equilibrium can exist; this is Peter Diamond’s stock market economy (1967). Or else, if assets, instead of being real, express their returns in units of account, or they only deliver some numeraire good, then the non-existence problem disappears. Still, Duffie and Shafer (1986) show that Hart’s example is non-generic.\(^{36}\)

49. Let us give two examples for Radner equilibrium.

**Example 1: Commodity Futures** A farmer plants wheat at date 0 and harvests at date 1. The farmer has the production function

$$q = \sqrt{x},$$

where $x \geq 0$ is the amount of input (taken as numeraire) and $q$ the amount of the output. The date-1 wheat price is a random variable $\tilde{p} \sim N(\mu_p, \sigma^2)$. The farmer can sell the output in the date-0 forward market at the forward price $f$ (to be endogenously determined below). Let $f$ be the units of the forward contract sold by the farmer. In the date-0 forward market, there is another trader called the urbanite. The farmer and the urbanite are both price-takers and expected utility maximizers with their von Neumann-Morgenstern utility functions

\(^{36}\)Although the non-existence problem is non-generic, a Radner equilibrium, when it exists, is generically constrained inefficient. This means that conditional on the fact of missing markets, the allocation in equilibrium is still less than Pareto efficient. It is shown that monetary policy may be used to improve efficiency in this world. It all depends on whether assets pay real dividends (or in units of account). If assets are all real, there is no role for money. On the other hand, monetary policy has real effects if assets are nominal.
being respectively

\[ U_{\phi}(w) = -e^{-\phi w}, \quad U_{u}(w) = -e^{-uw}, \quad \phi, u > 0. \]

Being price-takers with rational expectations, the farmer and the urbanite both know \( p_f \) and \( \tilde{p} \sim N(\mu, \sigma^2) \), and they take these as given. In particular, in choosing \( x \) and \( f \), the farmer ignores the fact that his choice of \( x \) could affect the distribution of \( \tilde{p} \).

(i) Given the farmer’s beliefs about \( \tilde{p} \) and \( p_f \), find his optimal date-0 choices for \( x \) and \( f \). Show that, in the presence of the forward market, the farmer in choosing \( x \) behaves as if all his output were hedged in the forward market (i.e. as if \( f = q(x) \)). Consequently, \( f \) can be decomposed into a fully hedged component plus a bet.

(ii) Find the urbanite’s demand for the forward contract. Find the date-0 forward market equilibrium \((p_f, f)\) using the market clearing condition. Suppose that as a true relationship, \( \mu = 1 - q(x) \), or equivalently,

\[ \tilde{p} = 1 - q(x) - \epsilon, \quad \epsilon \sim N(0, \sigma^2). \]

Now impose rational expectations to remove the dependence of equilibrium variables on \( \mu \). What happens to the forward price \( p_f \) if \( \sigma^2 \) (or \( \phi \) or \( u \)) tends to zero?

(iii) What is the role of the urbanite? Argue that there exists a level of \( \sigma^2 \) most preferred by the urbanite.

(iv) Does the existence of a forward market stabilize the date-1 spot price for wheat?

Solution. Consider part (i). The first order condition of the farmer’s problem gives

\[ E[U_{\phi}' \cdot (\tilde{p}q' - 1)] = 0 = E[U_{\phi}' \cdot (-\tilde{p} + p_f)], \]

implying that

\[ \frac{1}{q'(x)} = p_f, \]

which means that \( x \) depends only on \( p_f \) but not on \( \tilde{p} \). This result can be alternatively obtained by solving

\[ \max_x p_f q(x) - x; \]
that is, $x$ is chosen as if the farmer will sell all the output in the current forward market!

It is straightforward to show that

$$f = \frac{p_f}{2} + \frac{p_f - \mu_p}{\phi \sigma^2}$$

$$= q(x) + \frac{p_f - \mu_p}{\phi \sigma^2},$$

so that the position the farmer takes in the forward market is the output plus a bet (the above second term). In particular, the bet is a short (resp. a long) if and only if $p_f \geq (\text{resp. } \leq) \mu_p$.

In part (ii), it is easy to show that the urbanite’s position is $\frac{\mu_p - p_f}{u \sigma^2}$. Markets clearing and rational expectations then imply that

$$p_f = \frac{2(\phi + u)}{3(\phi + u) + \phi u \sigma^2}, \quad x = \frac{\phi + u}{3(\phi + u) + \phi u \sigma^2}.$$

If one among $\phi$, $u$, and $\sigma$ tends to zero, then $p_f$ is asymptotically independent of all these three variables.

In part (iii), the urbanite’s role is to provide insurance to the farmer, and being risk averse herself, the urbanite needs to be compensated (so that $p_f < \mu_p$, a phenomenon known as normal backwardation). As to the urbanite’s preferences over $\sigma^2$, risk aversion implies that a too high $\sigma^2$ hurts the urbanite, but on the other hand as $\sigma^2$ tends to zero the urbanite ends up with no profits. The answer to part (iv) is obviously negative given the linear specification in (ii), where $x$ affects only the drift term of $\bar{p}$.

50. (Example 2: Pareto inferior trade) There are $n$ identical farmers and $m$ identical consumers, all price-takers. Farmers can plant either a safe crop (s) or a risky crop (r). His problem is, at date 0, to determine a fraction $x$ of his land to be planted with r, and $1 - x$ to be planted with s. Assume one unit of land for each farmer. Each unit of land generates 1 unit of safe crop, or alternatively, $z$ units of risky crop, where $z \sim N(1, \sigma^2)$. Each consumer has an indirect utility function

$$V(I, p, q) = \ln(I) - a \ln(p) - b \ln(q).$$

50
where $a, b$ are strictly positive constants and $p, q$ are respectively the spot prices for $r$ and $s$ after harvest at date 1.

(i) Using Roy’s identity to show that the demands for $r$ and $s$ are respectively $\frac{d}{p}$ and $\frac{d}{q}$.

(ii) Show that, given the equilibrium $x$, the date-1 spot prices for $r$ and $s$ are respectively

$$\bar{p} = \frac{ay}{x \bar{z}}, \quad q = \frac{by}{1-x},$$

where $y = \frac{ml}{n}$.

(iii) Show that farmer’s problem at date 0 is

$$\max_x E[U(x \bar{p} \bar{z} + q(1-x))]$$

with $U' > 0 > U''$ and $q$ and the distribution of $\bar{p}$ given. (Simply because farmers are price-takers.)

(iv) Find the first-order condition for the farmer at date 0. Now impose rational expectations: the optimal $x$ solving the first-order condition must be such that it gives rise to the future prices $q$ and $\bar{p}$.

(v) Conclude that in equilibrium, a farmer’s welfare is $U((a+b)y)$ and a consumer’s is $\ln(I) - (a+b)\ln((a+b)y) + aE[\ln(\bar{z})]$. 

**Solution.** Consider part (i). Using Roy’s identity, we have

$$d_r = -\frac{\partial V}{\partial p} = \frac{aI}{p}.$$ 

Similarly, $d_s = \frac{bI}{q}$. From this, we have the aggregate demands

$$Q_r = m d_r, \quad Q_s = m d_s.$$ 

The aggregate supplies are price-inelastic (simply because the production decisions are made before farmers see the equilibrium spot prices):

$$S_r = nx \bar{z}, \quad S_s = n(1-x).$$

It is straightforward to get the equilibrium prices $\bar{p}$ and $q$. This is part (ii). Note that the price and quantity of the risky crop are inversely related! More important, the revenue from the risky crop is sure to be $ay = \frac{ml}{n}$ for each farmer! The revenue from the safe crop for
each farmer is also nonrandom, which is \( by = \frac{\ln l}{n} \). This gives clues regarding why opening new markets may reduce farmers’ welfare: if opening new markets prevents farmers from fully insuring themselves, then as \( U'' < 0 \), farmers will suffer from excessive risk bearing.

Now, in part (iii), a price-taking farmer is to choose \( x \) given the constant \( q \) and the random variables \( \hat{p} \) and \( \hat{z} \). The first-order condition is necessary and sufficient for the farmer’s maximization problem: \( x^* \) solves

\[
E[U'(x\hat{p}\hat{z} + q(1 - x))(\hat{p}\hat{z} - q)] = 0.
\]

The rational price expectations condition can now be imposed. We require that \( x^*, q, \hat{p} \) be mutually consistent in the sense that (i) given \( \hat{p} \) and \( q \), \( x^* \) satisfies the above first-order condition; and (ii) given \( x^*, \hat{p} \) and \( q \) respectively satisfy

\[
\hat{p} = \frac{ay}{a^*z}, \quad q = \frac{by}{1 - x^*}.
\]

This implies that

\[
\hat{p}\hat{z} = q.
\]

Hence we have

\[
x^* = \frac{a}{a + b}, \quad q = (a + b)y.
\]

These are parts (iii) and (iv). Part (v) is obvious.

Now imagine that there is another economy \( E' \) identical to the above economy \( E \) except that the random variable \( z \) in \( E \) is replaced by the random variable \( z' \) in \( E' \), and \( z, z' \) are such that \( z + z' = 2 \) for sure. Now if trade is allowed, there will be trade (why?). We look for a symmetric free trade equilibrium where \( x \) is the fraction of land planted with the risky crop. Now we have \( S_r = 2nx \), \( S_s = 2n(1 - x) \) for sure. One can verify that now a farmer in economy \( E \) (respectively, \( E' \)) has profit \( (az + b)y \) (respectively, \( (az' + b)y \)), which is a mean-preserving spread of the farmer’s profit without trade. Assume now that each consumer has a CRRA utility function with \( R \) the measure of relative risk aversion. It can be shown that, if the farmers do not change their production plans, then

\[
\frac{\Delta U}{u'} = \frac{u([a + b]y) - E[u([a\theta + b]y)]}{u'} = \frac{Rya^2\sigma^2}{2(a + b)}
\]
However, farmers do have different production plans in the presence of both $E$ and $E'$, and although a consumer faces no risk in the new scenario, she is hurt by the change in $x$ (which is made by the farmers because of an increase in the risk of planting the risky crop). For $R > \sigma$, one can show that consumers are also worse off. Hence the opening of a new market hurts all farmers and consumers in this example.

References


