1. Consider an investor endowed with the VNM utility function $u(W) = \frac{W^{1-\rho}}{1-\rho}$, $0 < \rho < 1$, where $W$ is the investor’s date-1 wealth. The investor is faced with two traded assets at date 0, one with risky rate of return $\tilde{r}$, and the other with sure rate of return $r_f$ (which is a lending and borrowing opportunity). Returns are generated at date 1. Let $W_0 > 0$ and $a^*$ be the investor’s initial wealth and the amount of money that she chooses to spend on the risky asset.

(i) Show that if $\rho = \frac{1}{2}$, $r_f = 0$, and $\tilde{r} = \begin{cases} 2, & \text{with probability } \frac{1}{2}, \\ -\frac{1}{2}, & \text{with probability } \frac{1}{2}, \end{cases}$ then the following portfolio is optimal for the investor:

$$\left[ \frac{a^*}{W_0}, \frac{W_0}{1-a^*} \right] = \left[ \frac{\frac{a^*}{2}}{2}, \frac{\frac{1}{2}}{2} \right].$$

(ii) Continue with part (i). If $W_0$ is equal to 100,000, how much money does the investor have to lend or borrow at date 0 in order to implement the above optimal portfolio strategy?

Solution. Note that for all $W > 0$,

$$u'(W) = W^{-\rho} > 0 > -\rho W^{-\rho-1} = u''(W),$$

implying that

$$f(a) \equiv E[u(W_0(1+r_f) + a(\tilde{r} - r_f))]$$

is concave in $a$, as long as $W_0(1+r_f) + a(\tilde{r} - r_f) > 0$ with probability one. (This is true for $a \geq 0$.) Now, for part (i), it suffices to verify that $a^* = \frac{3W_0}{2}$ satisfies the first-order condition. Note that

$$f'(a^*) = W_0^{-\rho} \left[ \frac{1}{2} \times (1 + 2 \frac{a^*}{W_0})^{-\rho} \times 2 + \frac{1}{2} \times (1 - \frac{a^*}{W_0})^{-\rho} \times (-\frac{1}{2}) \right]$$
\[ W_0 a^* \times \frac{1}{2} \left[ \frac{2}{\sqrt{1 + 2 \frac{a^*}{W_0}}} - \frac{1}{2 \sqrt{1 - \frac{a^*}{W_0}}} \right] \]

\[ = W_0 a^* \times \frac{1}{2} \times 0 = 0. \]

Since the terminal wealth is strictly positive with \( a^* = \frac{3W_0}{2}, a^* = \frac{3W_0}{2} \)
is indeed the optimal solution.

Now, for part (ii), it is easy to see that the investor has to borrow
\[ \frac{1}{2} \times 100,000 = 50,000 \]
in order to implement the optimal portfolio.

2. Recall the sets \( Z \) and \( P \) defined in section 2 of Lecture 1. We shall assume that \( Z \) is a finite subset of \( \mathbb{R} \), so that we can arrange its \( n \) elements into a column vector \( z \). Define the column vector \( z^2 \) as such that its \( i \)-th element is equal to the square of the \( i \)-th element of \( z \). Recall that \( P \) is the set of all possible lotteries with outcomes in the set \( Z \). With \( Z \) having \( n \) possible outcomes, a lottery \( p \in P \) is an \( n \)-vector with its (non-negative) elements summing up to one (that is, \( p \) is a prob. distribution on \( Z \)). Given \( p \), the expected prize and the variance of the prize are respectively \( p'z \) and \( p'z^2 - [p'z]^2 \). We say that an investor has a mean-variance utility function \( V(\cdot) \) defined on on \( P \), if for all lotteries \( p \in P \), for a given parameter \( \rho > 0 \),

\[ V(p) = p'z - \rho[p'z^2 - [p'z]^2]. \]

Define a binary relation \( \succ \) as such that

\[ \forall p, q \in P, \ p \succ q \iff V(p) > V(q). \]

Determine if \( \succeq \) satisfies respectively Axiom 1, Axiom 2, and Axiom 3 listed in section 2 of Lecture 1.

\textbf{Solution}. Because \( \succeq \) is complete and transitive on \( \mathbb{R} \), it is immediate that \( \succeq \) is complete and transitive on \( P \). Moreover, because \( V(\cdot) \) is actually a polynomial defined on \( \mathbb{R}^n \), \( V(\cdot) \) is a continuous function, and this implies immediately that Axiom 3 (the continuity axiom) is satisfied. We claim that \( \succeq \) fails the independence axiom (Axiom 2).

It suffices to give a counter-example. Without loss of generality, let \( \rho = 1 \). Let \( Z = \{-1, 1\} \). Consider there lotteries \( p, q, r \) defined as follows.
Lotteried/Elements of $Z$

<table>
<thead>
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<tbody>
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<td>$p$</td>
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<tr>
<td>$q$</td>
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<td>$\frac{1}{2}$</td>
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<td>$r$</td>
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Let $a = \frac{1}{2}$. Then it is straightforward to verify that

$$V(p) > V(q),$$

and yet

$$V(aq + (1 - a)r) > V(ap + (1 - a)r).$$

3. Suppose that the price system $(X, p)$ in a two-period frictionless economy is such that

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

where $p_2 > p_1 > 0$ are two given constants. Note that there are three possible date-1 states and 2 traded assets at date 0.

(i) Show that if the price system admits of no arbitrage opportunities, then an infinite number of state price vectors exist, but each state price vector $f_{3\times1}$ is a convex combination\(^1\) of the following two weak state price vectors:

$$f_1 = \begin{bmatrix} p_1 \\ 0 \\ p_2 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ p_1 \\ p_2 - p_1 \end{bmatrix}.$$  

(ii) Suppose that the three date-1 states are equally likely. Mr. A seeks to maximize

$$E[\log(\bar{c})] = \sum_{k=1}^{3} \frac{1}{3} \log(c_k),$$

where

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = e_{3\times1} + \sum_{j=1}^{2} q_j x_j,$$

---

\(^1\)Let $x, y$ be two elements of a real vector space $V$. We say that $ax + by$ is a **linear combination** of $x$ and $y$, if $a, b \in \mathbb{R}$. We say that $ax + by$ is a **convex combination** of $x$ and $y$, if $a, b \in [0, 1]$ and $a + b = 1$.  

3
with

\[ x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \]

and

\[ e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

Here recall that \( e \) is Mr. A’s date-1 endowed wealth. Find the optimal portfolio strategy \( q^*_2 \) and the corresponding optimal consumption plan \( c^*_3 \) for Mr. A.

**Solution.** By definition, \( f \) is a state price vector if and only if all its elements are strictly positive, and it satisfies

\[ p = Xf, \]

or equivalently, \( f = (f_1, f_2, f_3)' \) must satisfy

\[
\begin{aligned}
p_1 &= f_1 + f_2 \\
p_2 &= f_2 + f_3 \\
f_1, f_2, f_3 &> 0.
\end{aligned}
\]

Let \( \Phi \) be the set of weak state price vectors; that is,

\[ \Phi = \{(f_1, f_2, f_3)' \in \mathbb{R}^3_+ : p_1 = f_1 + f_2, \ p_2 = f_2 + f_3\}. \]

Since \( p_2 > p_1 > 0 \), if \( (f_1, f_2, f_3)' \) is an element of \( \Phi \), we must have \( f_3 > 0 \). Thus the set \( \Phi \) has two extreme points, corresponding to respectively \( f_1 = 0 \) and \( f_2 = 0 \). Solving, we obtain \( f_1 \) and \( f_2 \). This concludes part (i).

For part (ii), recall that \( c \) is budget feasible if and only if there is \( q \in \mathbb{R}^2 \) such that

\[ c = e + Xq, \quad p'q \leq 0. \]

---

\(^2\)Or else, \( f_2 = p_2 \), implying that \( f_1 < 0 \), a contradiction.

\(^3\)Note that \( f_1 \) and \( f_2 \) are *not* state price vectors, for each of them has an element equal to zero. This is why we refer to them *weak state price vectors*. It can be shown that the set of state price vectors is the interior of \( \Phi \), which is the largest open set contained in \( \Phi \).
Equivalently, this requires that for each state price vector \( f \) (i.e., with \( f_i \gg 0 \) and \( p = X'f \)),
\[
f'(c - e) = f'Xq = p'q \leq 0.
\]
It can be shown that this equivalence condition is also equivalent to\(^4\)
\[
f_1'(c - e) \leq 0, \quad f_2'(c - e) \leq 0.
\]
Now we demonstrate two ways to solve part (ii). The first one is Merton’s approach, which solves the optimal investment policy \( q^* \) and optimal consumption plan \( c^* \) at the same time.\(^5\) Mr. A’s problem is to
\[
\text{max} \quad \sum_{k=1}^{3} \frac{1}{3} \log(c_k),
\]
or equivalently, to
\[
\text{max} \quad \log(c_1) + \log(c_2) + \log(c_3),
\]
subject to
\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  1
\end{bmatrix} + \begin{bmatrix}
  1 & 0 \\
  1 & 1 \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix},
\]
and
\[
p_1q_1 + p_2q_2 \leq 0.
\]
It is easy to see that at optimum, the last inequality (which is the date-0 budget constraint) must hold as an equality (and we say that the budget constraint is \emph{binding} at optimum). Define
\[
x \equiv \frac{p_1}{p_2} \in (0, 1).
\]
\(^4\)The sufficiency follows from the fact that each state price vector \( f \) is a convex combination of \( f_1 \) and \( f_2 \). The necessity can be proved by taking a sequence of state price vectors \( \{f_n^i; \ n \in \mathbb{Z}_+\} \) converging to \( f_i \), \( i = 1, 2 \), using the continuity of \( G(f) \equiv f'(c - e) \).
The above problem can be re-stated as
\[
\max_{q_1} \log(q_1) + \log((1 - x)q_1) + \log(1 - xq_1)
\]
subject to
\[
q_2 = -xq_1, \quad c_1 = q_1, \quad c_2 = (1 - x)q_1, \quad c_3 = 1 - xq_1.
\]
Solving, we have
\[
q_1^* = \frac{2}{3x}, \quad q_2^* = \frac{2}{3}, \quad c_1^* = \frac{2}{3x}, \quad c_2^* = \frac{2(1 - x)}{3x}, \quad c_3^* = \frac{1}{3}.
\]

The next approach is due to John Cox and Chi-fu Huang, known as the Cox-Huang approach.\(^6\) It first requires a description of the set of feasible consumption plans using the set \(\Phi\) of state price vectors, and then asks the investor to first solve for his optimal consumption plan. Then, given the optimal consumption plan, it asks the investor to find one portfolio that finances that consumption plan.

For the current problem, the set of feasible consumption plans is
\[
C = \{c \in \mathbb{R}_+^3 : f_1'(c - \mathbf{e}) \leq 0, \ f_2'(c - \mathbf{e}) \leq 0\}.
\]

Now, Mr. A’s problem is to
\[
\max_{c_1,c_2,c_3} \sum_{k=1}^{3} \frac{1}{3} \log(c_k),
\]
subject to
\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{bmatrix} \in C.
\]

Using the definitions of \(f_1, f_2,\) and \(x,\) the above problem can be re-stated as
\[
\max_{c_1,c_2,c_3} \log(c_1) + \log(c_2) + \log(c_3),
\]
subject to 
\[ xc_1 + c_3 \leq 1, \quad c_3 + \frac{x}{1-x}c_2 \leq 1. \]

It is easy to see that at optimum, both constraints must be binding. Hence we can equivalently solve the following maximization problem:

\[ \max_{c_3} \log\left(\frac{1}{x}(1-c_3)\right) + \log((1-c_3)(\frac{1}{x} - 1)) + \log(c_3). \]

The necessary and sufficient first-order condition yields

\[ c_3^* = \frac{1}{3}, \]

so that

\[ c_1^* = \frac{1}{x}(1-c_3^*) = \frac{2}{3x}, \quad c_2^* = (1-c_3^*)(\frac{1}{x} - 1) = \frac{2(1-x)}{3x}. \]

Now, the last step is to find one feasible portfolio that produces this optimal consumption plan. We must solve for the \(q^*\) that satisfies

\[
\begin{bmatrix}
  c_1^* \\
  c_2^* \\
  c_3^*
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  1
\end{bmatrix}
+ \begin{bmatrix}
  1 & 0 \\
  1 & 1 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  q_1^* \\
  q_2^*
\end{bmatrix},
\]

and the solution is

\[ q_1^* = \frac{2}{3x}, \quad q_2^* = -\frac{2}{3}. \]

The two approaches may seem equally good in the static setting, but the second approach has proven to be computational more efficient in a dynamic continuous-time setting.\textsuperscript{7}

4. There are three agents (investors) in a two-period frictionless economy. There are three traded assets at date 0, each with a net supply of one unit. Agent \(i\) is endowed with the entire unit of asset \(i, i = 1, 2, 3\). Let asset 1 be the numeraire (so that its equilibrium price is one), and let the equilibrium price for asset 2 and asset 3 be denoted by \(p\) and \(q\).

Asset \( i \) pays a random cash flow \( \tilde{z}_i > 0 \) at date 1. The three agents have identical CRRA utility function \( u(W) = \frac{W^{1-\rho}}{1-\rho} \), where \( 0 < \rho < 1 \). Derive explicit formulae for the equilibrium prices \( p \) and \( q \).

**Solution.** The three agents must hold the same portfolio in equilibrium (why?). Let \( \mathbf{a} \) be agent 1’s equilibrium holdings of the three traded assets, then there exist positive constants \( \beta \) and \( \gamma \) such that agents 2’s and 3’s equilibrium holdings can be denoted by \( \beta \mathbf{a} \) and \( \gamma \mathbf{a} \). The markets-clearing condition requires that

\[
(1 + \beta + \gamma) \mathbf{a} = \mathbf{1}.
\]

It follows that

\[
\mathbf{a} = \frac{1}{1 + \beta + \gamma} \mathbf{1},
\]

where note that the denominator is non-zero. Note that the three agents’ budget constraints must be binding in the Walrasian equilibrium (why?), and so we have

\[
\begin{bmatrix}
\beta \mathbf{a}' \\
\gamma \mathbf{a}'
\end{bmatrix}
\begin{bmatrix}
1 \\
p
q
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
p
q
\end{bmatrix},
\]

which together with \( \mathbf{a} = \frac{1}{1 + \beta + \gamma} \mathbf{1} \) implies that

\[
\beta = p, \quad \gamma = q.
\]

Now, replacing \( \mathbf{a} \) into agent 1’s first-order conditions for expected utility maximization, we have

\[
p = E[(\tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3)^{-\rho} \tilde{z}_2] / E[(\tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3)^{-\rho} \tilde{z}_1], \quad q = E[(\tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3)^{-\rho} \tilde{z}_3] / E[(\tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3)^{-\rho} \tilde{z}_1].
\]

5. Suppose that there are two investors in the date-0 frictionless financial market, both having von Neumann-Morgenstern utility function

\[
u(\cdot) = \log(\cdot).
\]

There are one risky asset and one riskless asset available for trading at date 0. The riskless rate of return is \( r_f \). The risky asset is a
common stock, which has 3 shares outstanding. Each share of the stock generates a random cash flow \( \hat{x} > 0 \) at date 0. For \( i = 1,2 \), investor \( i \) is endowed with \( i \) shares of the stock. Nobody is endowed with the riskless asset. (Think of the riskless asset as a lending and borrowing opportunity, so that the net supply of the riskless asset is zero.) Let \( D_i(P) \) be investor \( i \)'s date-0 demand function for the stock (risky asset), where \( P \) denotes the date-0 stock price.

(i) Show that given the stock price \( P \), \( D_i(P) \) must satisfy the following first-order condition for the investor’s maximization problem:

\[
E\left[ \frac{\hat{x}}{P} - 1 - r_f \right] = 0.
\]

(Again, we are assuming that such an interior solution exists.\(^8\))

(ii) A competitive equilibrium for the date-0 financial markets is a tuple \((P^*, D_1(\cdot), D_2(\cdot))\) such that \( D_1(P) \) and \( D_2(P) \) satisfy the first-order condition in part (i) given any stock price \( P \), and \( D_1(P^*) + D_2(P^*) = 3 \) (the markets-clearing condition). Show that in a competitive equilibrium the two investors must hold the same portfolio, and moreover, the equilibrium stock price is

\[ P^* = \frac{1}{E[\frac{1+r_f}{\hat{x}}]}. \]

**Solution.** Consider part (i). Investor \( i \)'s problem is

\[
\max_{D_i \in \mathbb{R}} E[\log((i - D_i)P(1 + r_f) + D_i\hat{x})],
\]

\(^8\)Recall that in the two-asset portfolio problem, a necessary condition for the maximization problem

\[
\max_{a \in \mathbb{R}} f(a) \equiv E[u(W_0(1 + r_f) + a(\bar{r} - r_f))]
\]

to have an interior solution \( a^* \) (that satisfies \( f'(a^*) = 0 \)) is that the two events \{\bar{r} > r_f\} and \{\bar{r} < r_f\} both can occur with a strictly positive probability. A sufficient condition for the existence of an interior solution \( a^* \) is, by applying the Intermediate Value Theorem to the continuous function \( f'(\cdot) \), the following Inada condition:

\[
f'(0) > 0 > \lim_{y \to +\infty} f'(y).
\]

So, we assume that the Inada condition holds for our current problem.
and since log(\cdot) is a concave function on \( \mathbb{R}_+ \), if the optimal \( D_i \) exists, it must satisfy the following first-order condition:

\[
E\left[ \frac{\hat{x} - P(1 + r_f)}{(i - D_i)P(1 + r_f) + D_i \hat{x}} \right] = 0.
\]

Now, since \( \hat{x} > 0 \), and since in equilibrium there can be no arbitrage opportunity (Theorem 5 of Lecture 1), we must have \( P > 0 \) in equilibrium. Dividing the above first-order condition by \( P \), we get the desired result for part (i).

Now, consider part (ii). Conjecture that \( i - D_i \neq 0 \), and define

\[
b_i \equiv \frac{D_i}{i - D_i}, \quad i = 1, 2.
\]

Then investor \( i \)'s first-order condition can be re-written as

\[
\frac{1}{i - D_i} \left( \frac{\hat{x} - P(1 + r_f)}{P(1 + r_f) + b_i \hat{x}} \right) = 0 \Leftrightarrow \frac{\hat{x} - P(1 + r_f)}{P(1 + r_f) + b_i \hat{x}} = 0.
\]

It follows that \( b_1 = b_2 \).

Now we can solve for the equilibrium \( (P^*, D_1(\cdot), D_2(\cdot)) \) by first solving \( D_1(P^*) \) and \( D_2(P^*) \). Note that

\[
\frac{D_1(P^*)}{1 - D_1(P^*)} = \frac{D_2(P^*)}{2 - D_2(P^*)},
\]

and the markets-clearing condition requires that

\[
D_1(P^*) + D_2(P^*) = 3.
\]

Hence we have

\[
D_i(P^*) = i, \quad i = 1, 2.
\]

Plugging this result in the first-order condition, we have

\[
\frac{1}{D_i(P^*)} \left( \frac{\hat{x} - P(1 + r_f)}{\hat{x}} \right) = 0 \Leftrightarrow P = \frac{1}{E[\frac{\hat{x}}{\hat{x}}]}.
\]
6. Suppose that there are two investors in the two-period frictionless econ-
omy, where for \( i = 1, 2 \), investor \( i \) has von Neumann-Morgenstern utility
function
\[
u(z) = -e^{-\rho_i z}, \; \forall z \in \mathbb{R},
\]
where
\[
\rho_2 > \rho_1 > 0.
\]
There are one risky asset and one riskless asset available for trading at
date 0. The riskless rate of return is \( r_f \). The risky asset is a common
stock, which has 2 shares outstanding, with the two investors each
holding one share before trading starts at date 0. Nobody is endowed
with the riskless asset, so that the riskless asset is in zero net supply.
Let \( \tilde{x} \sim \mathcal{N}(\mu, \sigma^2) \) be the date-1 cash flow generated by one share of the
common stock, where
\[
\mu > r_f \geq 0.
\]
(i) Let \( P \) be the date-0 stock price. Let \( D_i(P) \) be investor \( i \)'s demand
for the common stock at date 0, given that the stock price is \( P \). Find
\( D_1(\cdot) \) and \( D_2(\cdot) \).
(ii) Write down the markets-clearing condition, and obtain the equilib-
rium stock price \( P^* \).
(iii) Plug \( P^* \) into \( D_1(\cdot) \) and \( D_2(\cdot) \) and determine which investor is buy-
ing the stock, and which investor is selling the stock in equilibrium at
date 0.
(iv) Which one between the two investors is borrowing in equilibrium
at date 0? Which one is lending? Why?
Solution. Consider part (i). Investor \( i \)'s problem is
\[
\max_{D_i \in \mathbb{R}} E[-e^{-\rho_i [D_i \tilde{x} + (1-D_i)P(1+r_f)]}] = e^{-\rho_i [(1-D_i)P(1+r_f)]} E[-e^{-\rho_i D_i \tilde{x}}].
\]
Note that the random variable
\[
-\rho_i D_i \tilde{x} \sim \mathcal{N}(-\rho_i D_i \mu, \rho_i^2 D_i^2 \sigma^2),
\]
so that, by the formula of moment generating function for a Gaussian
random variable,\(^9\) we have
\[
E[-e^{-\rho_i D_i \tilde{x}}] = e^{-\rho_i D_i \mu + \frac{1}{2} \rho_i^2 D_i^2 \sigma^2}.
\]
\(^9\)It says that for \( \tilde{z} \sim \mathcal{N}(\mu, V) \),
\[
M_{\tilde{z}}(t) \equiv E[e^{i\tilde{z}t}] = e^{te + \frac{t^2 V}{2}}.
\]
Thus investor \( i \) seeks to

\[
\max_{D_i \in \mathbb{R}} -e^{-\rho_i[D_i \mu + (1-D_i)P(1+r_f) - \frac{1}{2}\rho_i D_i^2 \sigma^2]}
\]

\[
= u(D_i \mu + (1 - D_i)P(1 + r_f) - \frac{1}{2}\rho_i D_i^2 \sigma^2),
\]

where recall that \( u(\cdot) \) is strictly increasing. Thus in searching for the utility-maximizing \( D_i \), the investor can simply solve the following maximization problem:

\[
\max_{D_i \in \mathbb{R}} W(D_i) \equiv D_i \mu + (1 - D_i)P(1 + r_f) - \frac{1}{2}\rho_i D_i^2 \sigma^2.
\]

In finance literature, \( W(D_i) \) is referred to as the certainty equivalent of \( W \) for the investor, induced by the investment strategy \( D_i \).

The last maximization problem involves only a quadratic objective function, and hence is easy to solve. The necessary and sufficient first-order condition gives

\[
D_i(P) = \frac{\mu - P(1 + r_f)}{\rho_i \sigma^2}, \quad i = 1, 2.
\]

This finishes part (i).

Now, for part (ii), the markets-clearing condition requires that at the equilibrium price \( P^* \),

\[
D_1(P^*) + D_2(P^*) = 2 \Rightarrow P^* = \frac{\mu - \overline{\rho} \sigma^2}{1 + r_f},
\]

where

\[
\overline{\rho} = \frac{2}{\frac{1}{\rho_1} + \frac{1}{\rho_2}}
\]

is the harmonic mean of \( \rho_1 \) and \( \rho_2 \). Note that \( P^* \) equals \( \frac{\mu}{1 + r_f} \) (the martingale pricing formula for risk neutral investors!) if the riskiness of the common stock, measured by \( \sigma^2 \), vanishes, or if \( \overline{\sigma} \) vanishes (which happens if either \( \rho_1 \) or \( \rho_2 \) is equal to zero). Thus the term \( \overline{\rho} \sigma^2 \) represents
a risk discount: the two investors are actually risk averse, not risk neutral.

Next, consider part (iii). It is easy to see that

\[ D_i(P^*) = \frac{P}{\rho_i}, \quad i = 1, 2, \]

and hence

\[ D_1(P^*) > 1 > D_2(P^*). \]

This is a very intuitive result. Investor 2 is more risk averse than investor 1, since \( \rho_2 > \rho_1 \), and hence in equilibrium investor 1 must hold more of the risky common stock than investor 2 does. Note also that this result verifies our earlier conjecture that \( i - D_i(P^*) \neq 0 \) for \( i = 1, 2 \).

For part (iv), recall that the two investors start with the same initial wealth, and so the investor holding more of the risk common stock must also be the one that is borrowing in equilibrium. Hence investor 1 is borrowing and investor 2 is lending in equilibrium.

7. There are 3 agents in a two-period frictionless economy with a single consumption good. Agent 0 owns a firm, which generates 1 unit of consumption at date 1, and he wants to consume at date 0 only. Agents 1 and 2 seek to maximize

\[ u(c_0) + E[u(c_1)] \]

where \( u(c) = \log(c) \). There are two equally likely states at date 1. In state \( i \), agent \( i \) is endowed with \( X > 0 \) units of consumption, but agent \( j \) is endowed with nothing, where \( i, j = 1, 2 \). Both agents are also endowed with one unit of consumption at date 0.

(i) First suppose that the entrepreneur issues one share of common stock to sell the firm. Show that the equilibrium firm value is \( \frac{2 + 2X}{2 + 3X} \), and both agents 1 and 2 hold \( \frac{1}{2} \) shares of the common stock.

(ii) Next, suppose instead that the entrepreneur issues the two Arrow-Debreu securities to the market, with prices \( p_1 \) and \( p_2 \). Let \( a_{ij} \) be the number of shares of the \( i \)-th Arrow-Debreu security held by agent \( j \). Show that in equilibrium

\[ a_{11} = a_{22} = \frac{1 - X}{2}, \quad a_{12} = a_{21} = \frac{1 + X}{2}, \quad p_1 = p_2 = \frac{1}{2 + X} \]
so that the firm value in this case is

\[ \frac{2}{2 + X}, \]

and verify that

\[ \frac{2}{2 + X} < \frac{2 + 2X}{2 + 3X}. \]

(iii) Conclude that a firm’s changing its capital structure may change its value in equilibrium and that a firm’s value may be higher when markets are incomplete than complete. Interpret.

(iv) Now suppose instead that \( X < 5 \) and \( u(c) = 100c - \frac{1}{3}c^3 \), for \( c \in [0, 10] \). Determine if the firm value is higher when the firm issues the two Arrow-Debreu securities than when it issues common stock only.

**Solution.**

Note that the firm’s cash flow is 1 (like a pure discount bond, hence our result applies to riskless rate of interest as well), which is common knowledge, but there are two equally likely states, and agent \( i \) is endowed with \( X \) units of consumption in state \( i \), together with 1 unit at date 0.

When the firm issues only equity, with price \( S \), the first-order conditions to both investors’ maximization problems are the same, which is

\[ Su'(1 - Sa) = Su'(c_0) = E[u'(c_1)]. \]

This happens because we have assumed enough symmetry between agents 1 and 2. By symmetry, we know that in equilibrium

\[ E[u'(c_1)] = 0.5[u'(X + a) + u'(a)], \]

where it must be that \( a = 0.5 \) in equilibrium. This finishes part (i).

Now, for part (ii), if markets are complete with \( p_i \) being the price of the \( i \)-th Arrow-Debreu security (which pays one dollar if and only if state \( i \) occurs at date 1), then the first-order conditions become

\[ p_i u'(c_0) = 0.5u'(c_{1i}), \quad i = 1, 2, \]
where \( c_{1i} \) is the investor’s equilibrium date-1 consumption in state \( i \). Recall that in part (i), \( c_{11} \) and \( c_{12} \) are different; one of them is equal to \( X + 0.5 \) and the other is 0.5. Here, in part (ii), thanks to complete markets, \( c_{11} = c_{12} = 0.5(X + 1) \). By symmetry, agents 1 and 2 have the same date-0 consumption in equilibrium, which is

\[
c_0 = 1 - \left[ \frac{1 - X}{2} + \frac{1 + X}{2} \right] \frac{S^*}{2} = 1 - 0.5S^*,
\]

where \( S^* = p_1 + p_2 \) is the equilibrium firm value in part (ii). Now observe that

\[
S^*u'(1 - p_1 0.5S^*) = E[u'(c_1)] = u'(0.5(X + 1)) < 0.5[u'(X + 0.5) + u'(0.5)] = Su'(1 - 0.5S),
\]

where the inequality follows from Jensen’s inequality and the fact that \( u'(\cdot) \) is a convex function given that \( u(\cdot) = \log(\cdot) \), and the last equality is the first-order condition in part (i), where recall that the markets are incomplete. Since the function \( h(z) \equiv zu'(1 - 0.5z) \) is strictly increasing, we conclude that the firm value is lower in the complete-markets case than in the incomplete-markets case. This finishes part (ii).

We learn two things from this example. First, the value of a firm may change when it changes its capital structure by issuing different corporate securities to public investors. This happens even if financial markets are perfect. This fact does not violate Modigliani and Miller’s Proposition 1, which says that at any point in time, two firms possessing the same assets in perfect financial markets must have the same equilibrium firm value. Second, a firm need not have an incentive to engage in financial innovation. As this example shows, the value of the firm may become lower if an all-equity firm creates and issues more securities to public investors. This finishes part (iii).

If instead \( u'(\cdot) \) is concave, as in part (iv), then the firm can attain a higher value if it issues two linearly independent securities to complete the markets. This exercise is adapted from a working paper of mine.
8. An investor is endowed with a VNM utility function \( u : \mathbb{R} \rightarrow \mathbb{R} \), with \( u' > 0 > u'' \). Moreover, it is known that for some constant \( x > 0 \),

\[
\frac{u''(z)}{u'(z)} > \frac{u''(x + z)}{u'(x + z)}, \quad \forall z \in \mathbb{R}.
\]

Show that there exists \( f' > 0 > f'' \) such that

\[
u(z) = f(u(x + z)), \quad \forall z \in \mathbb{R}.
\]

**Solution.** Given \( x > 0 \), define

\[
f(\cdot) \equiv u(u^{-1}(\cdot) - x).
\]

It can be easily seen that

\[
f(u(x + z)) = u(u^{-1}(u(x + z)) - x) = u((x + z) - x) = u(z), \quad \forall z.
\]

It remains to verify that \( f' > 0 > f'' \). Note that given \( x > 0 \),

\[
u'(z) = f'(u(x + z))u'(x + z), \quad \forall z,
\]

and since \( u' > 0 \), we have \( f' > 0 \). Differentiating one more time, we have

\[
u''(z) = f''(u(x + z))[u'(x + z)]^2 + f'(u(x + z))u''(x + z).
\]

Dividing the above equation by \(-u'(z) = -f'(u(x + z))u'(x + z) < 0\), we have

\[
\frac{u''(z)}{u'(z)} = \frac{f''(u(x + z))[u'(x + z)]^2 + f'(u(x + z))u''(x + z)}{-u'(z)}.
\]

Recall that by assumption, given the constant \( x > 0 \), we have

\[
\frac{u''(z)}{u'(z)} > \frac{u''(x + z)}{u'(x + z)}, \quad \forall z.
\]

It follows that, given the constant \( x > 0 \),

\[
\frac{f''(u(x + z))}{f'(u(x + z))} \times u'(x + z) > 0, \quad \forall z.
\]

Since \( f', u' > 0 \), we must have \( f'' < 0 \). This concludes the proof.

In this exercise we have assumed that for a given \( x > 0 \), the \( R_A \) function associated with \( u(\cdot) \) lies everywhere above the \( R_A \) function associated with \( u(\cdot + x) \). If this is true for all \( x > 0 \), then \( u(\cdot) \) must be a DARA function. (Verify!)