1. Suppose that you are an investment consultant and you want to help a client choose an optimal portfolio. Your client can invest 1 dollar in $n$ risky assets, of which the rate of return on the $j$-th risky asset is denoted by $\tilde{r}_j$. The distribution function of $(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n)$ is denoted by

$$F(r_1, r_2, \cdots, r_n) \equiv \text{prob.}(\tilde{r}_1 \leq r_1, \tilde{r}_2 \leq r_2, \cdots, \tilde{r}_n \leq r_n),$$

which is symmetric in its $n$ arguments in the sense that the functional value of $F$ remains unchanged if $(r_1, r_2, \cdots, r_n)$ is replaced by one of its permutations.

Suppose that the only information that you have about your client is that he is risk averse (i.e., his VNM utility function $u(\cdot)$ is concave), and you want to recommend a portfolio that maximizes his expected
utility. Which portfolio would you recommend?¹

**Solution.** The permutation-invariant property of $F(\cdot)$ implies that

$$E[\tilde{r}_i \mid \sum_{j=1}^n \tilde{r}_j] = E[\tilde{r}_k \mid \sum_{j=1}^n \tilde{r}_j], \quad \forall i, k = 1, 2, \ldots, n,$$

which implies that

$$nE[\tilde{r}_i \mid \sum_{j=1}^n \tilde{r}_j] = \sum_{i=1}^n E[\tilde{r}_i \mid \sum_{j=1}^n \tilde{r}_j] = \sum_{i=1}^n \tilde{r}_i,$$

so that

$$E[\tilde{r}_i \mid \sum_{j=1}^n \tilde{r}_j] = \frac{\sum_{j=1}^n \tilde{r}_j}{n},$$

and hence

¹**Hint:** A portfolio is a vector $w_{n \times 1}$ such that $w'1 = \sum_{j=1}^n w_j = 1$. Symmetry suggests that you guess the equally weighted portfolio: if you recommend $w_i \geq w_j$, then you should recommend $w_j \geq w_i$. But this is just a conjecture. You need to prove that the equally weighted portfolio is indeed the best choice. To this end, first show that by symmetry $E[\tilde{r}_j \mid \frac{1}{n} \sum_{i=1}^n \tilde{r}_i]$ is independent of $j$. Then, show that, necessarily,

$$E[\tilde{r}_j \mid \sum_{i=1}^n \tilde{r}_i] = \frac{1}{n} \sum_{i=1}^n \tilde{r}_i, \quad \forall j.$$

Conclude therefore that for any $w_{n \times 1}$ with $w'1 = 1$,

$$E[\sum_{j=1}^n w_j \tilde{r}_j \mid \sum_{i=1}^n \tilde{r}_i] = \frac{1}{n} \sum_{i=1}^n \tilde{r}_i$$

also. Now, apply the law of iterated expectations and Jensen’s inequality to show that

$$E[u(1 + \sum_{j=1}^n w_j \tilde{r}_j)] \leq E[u(1 + \sum_{j=1}^n \tilde{r}_j)],$$

so that the equally weighted portfolio is indeed the best choice for your client.
\[ E[\hat{r}_i | \frac{\sum_{j=1}^{n} \tilde{r}_j}{n}] = \frac{\sum_{j=1}^{n} \tilde{r}_j}{n}, \forall i = 1, 2, \ldots, n. \]

It follows that, given any \( w_{n \times 1} \), we must have
\[
E\left[ \sum_{i=1}^{n} w_i \hat{r}_i \middle| \frac{\sum_{j=1}^{n} \tilde{r}_j}{n} \right] = \sum_{i=1}^{n} w_i E[\hat{r}_i | \frac{\sum_{j=1}^{n} \tilde{r}_j}{n}]
\]
\[
= \sum_{i=1}^{n} w_i \left( \frac{\sum_{j=1}^{n} \tilde{r}_j}{n} \right) = \left( \frac{\sum_{j=1}^{n} \tilde{r}_j}{n} \right) \left( \sum_{i=1}^{n} w_i \right) = \frac{\sum_{j=1}^{n} \tilde{r}_j}{n}.
\]

Now, observe that
\[
E[u(1 + \sum_{j=1}^{n} w_j \hat{r}_j)] = E[E[u(1 + \sum_{j=1}^{n} w_j \hat{r}_j) | \frac{\sum_{j=1}^{n} \tilde{r}_j}{n}]]
\]
\[
\leq E[u(E[1 + \sum_{j=1}^{n} w_j \hat{r}_j | \frac{\sum_{j=1}^{n} \tilde{r}_j}{n}])]
\]
\[
= E[u(1 + \frac{\sum_{j=1}^{n} \tilde{r}_j}{n})],
\]
so that the equally weighted portfolio is indeed the best choice for your client.

**Remark.** If \( \hat{x} = \hat{y} + \hat{e} \) with \( E[\hat{e} | \hat{y} = y] = 0 \) for all realizations \( y \) of \( \hat{y} \), then we say that \( \hat{x} \) is a mean-preserving spread of \( \hat{y} \). This terminology follows from the fact that \( \hat{x} \) and \( \hat{y} \) have the same mean, and yet the distribution of \( \hat{x} \) spreads more widely than the distribution of \( \hat{y} \). Theorem 7 of Lecture 2 shows that \( \hat{y} \geq_{SSD} \hat{x} \) if \( \hat{x} \) is a mean-preserving spread of \( \hat{y} \). In this exercise, the rate of return \( \hat{r}_i \) on each ingredient asset \( i \) is a mean-preserving spread of the rate of return on the equally-weighted portfolio, and hence so is the rate of return on any portfolio differing from the equally-weighted portfolio.
2. Suppose that U is more risk averse than V in the sense of Theorem 4 of Lecture 2. Suppose that U and V both have initial wealth $W_0 > 0$, and are considering undertaking a risky investment project which will generate a random cash flow $\tilde{x}$ (so that their post-investment wealth would become $W_0 + \tilde{x}$). Show that if V does not want to undertake the project, then U does not want to undertake the project either. (Hint: To show that $E[u(W_0 + \tilde{x})] \leq u(W_0)$ if $E[v(W_0 + \tilde{x})] \leq v(W_0)$, use Jensen’s inequality and the fact that $f(v(x)) = u(x)$ with $f' > 0 > f''$.)

Solution. By Theorem 4, we can write $u(z) = f(v(z))$ where $f(\cdot)$ is such that $f' > 0 > f''$. Then, given that $E[v(W_0 + \tilde{x})] \leq v(W_0)$, we have

$$E[u(W_0 + \tilde{x})] = E[f(v(W_0 + \tilde{x}))] \leq f(E[v(W_0 + \tilde{x})]) = f(v(W_0)) = u(W_0).$$

3. Consider a frictionless two-period economy where there are $N > 1$ risky assets and one riskless asset traded at date 0, and assets generate cash flows at date 1. An investor with initial wealth $W_0$ and VNM utility function $u$ for terminal wealth is seeking to find his optimal portfolio. Call the riskless asset asset 0. Let $a_j$ be the amount of money the investor invests in the $j$-th risky asset, for all $j = 1, 2, \cdots, N$ (so that the amount of money invested in asset 0 is $W_0 - \sum_{j=1}^{N} a_j$). Define $\tilde{R}_j \equiv 1 + \tilde{r}_j$ and $R_f \equiv 1 + r_f$. We shall assume that the optimal portfolio strategy $(a_1^*, a_2^*, \cdots, a_N^*)$ satisfies the following first-order condition:

$$\forall j = 1, 2, \cdots, N, \ E[u'(\tilde{W})(\tilde{R}_j - R_f)] = 0.$$

(i) Show that

$$\tilde{W} = W_0 R_f + \sum_{j=1}^{N} a_j (\tilde{R}_j - R_f).$$

(ii) Assume that $u(x) = \frac{x^{1+\rho}}{1-\rho}$, where $1 > \rho > 0$. Prove that for all $j = 1, 2, \cdots, N$, the optimal portfolio weight $\frac{a_j^*}{W_0}$ is independent of $W_0$. Conclude that the optimal portfolio weight for asset 0, which is $1 - \sum_{j=1}^{N} \frac{a_j^*}{W_0}$, is also independent of $W_0$.

(iii) Assume that $u(x) = \log(x)$. Prove that for all $j = 1, 2, \cdots, N$,
the optimal portfolio weight \( \frac{a^*_j}{W_0} \) is independent of \( W_0 \). Conclude that the optimal portfolio weight for asset 0, which is \( 1 - \sum_{j=1}^{N} \frac{a^*_j}{W_0} \), is also independent of \( W_0 \).

(iv) Assume that \( u(x) = -e^{-\rho x} \), where \( \rho > 0 \). Prove that for all \( j = 1, 2, \cdots, N \), the ratio \( \frac{a^*_j}{\sum_{j=1}^{N} a^*_j} \) is independent of \( W_0 \). Conclude that the optimal portfolio, when \( W_0 \) varies, is always a portfolio of asset 0 and a fixed portfolio of the \( N \) risky traded assets.\(^2\)

**Solution.** Part (i) is obvious. Consider part (ii). Replacing \( u(x) = \frac{x^{1-\rho}}{1-\rho} \) into the first-order condition

\[
\forall j = 1, 2, \cdots, N, \quad E[u'(\tilde{W})(\tilde{R}_j - R_f)] = 0,
\]

we obtain

\[
\forall j = 1, 2, \cdots, N, \quad 0 = E[(\tilde{W})^{-\rho}(\tilde{R}_j - R_f)]
\]

\[
= E[(W_0R_f + \sum_{j=1}^{N} a_j(\tilde{R}_j - R_f))^{-\rho}(\tilde{R}_j - R_f)]
\]

\[
= W_0^{-\rho}E[(R_f + \sum_{j=1}^{N} \frac{a_j}{W_0}(\tilde{R}_j - R_f))^{-\rho}(\tilde{R}_j - R_f)],
\]

so that the first-order condition can be equivalently written as

\[
\forall j = 1, 2, \cdots, N, \quad 0 = E[(R_f + \sum_{j=1}^{N} \frac{a_j}{W_0}(\tilde{R}_j - R_f))^{-\rho}(\tilde{R}_j - R_f)].
\]

In solving for the \( N \) unknowns \( \frac{a_1}{W_0}, \frac{a_2}{W_0}, \cdots, \frac{a_N}{W_0} \), \( W_0 \) is not a parameter that shows up in the above system of \( N \) equations. Hence we conclude that the optimal solutions

\[
\left\{ \frac{a^*_j}{W_0}; j = 1, 2, \cdots, N \right\}
\]

\(^2\)\textbf{Hint:} You don’t need to obtain the closed-form solution for \( a^*_j \). Just replace \( u(\cdot) \) into the first-order condition, and figure out a way to remove \( W_0 \) from the first-order condition (so that either \( a^*_j \) alone or \( \frac{a^*_j}{W_0} \) is left in the first-order condition).
must be independent of $W_0$. This finishes part (ii).

Next consider part (iii). Again, replacing $u(x) = \log(x)$ into the first-order condition

$$\forall j = 1, 2, \cdots, N, \ E[u'(\tilde{W})(\tilde{R}_j - R_f)] = 0,$$

we obtain

$$\forall j = 1, 2, \cdots, N, \ 0 = E[(\tilde{W})^{-1}(\tilde{R}_j - R_f)]$$

$$= E[(W_0R_f + \sum_{j=1}^{N} a_j(\tilde{R}_j - R_f))^{-1}(\tilde{R}_j - R_f)]$$

$$= W_0^{-1}E[(R_f + \sum_{j=1}^{N} a_j(\tilde{R}_j - R_f))^{-1}(\tilde{R}_j - R_f)],$$

so that the first-order condition can be equivalently written as

$$\forall j = 1, 2, \cdots, N, \ 0 = E[(R_f + \sum_{j=1}^{N} \frac{a_j}{W_0}(\tilde{R}_j - R_f))^{-1}(\tilde{R}_j - R_f)].$$

In solving for the $N$ unknowns $\frac{a_1}{W_0}, \frac{a_2}{W_0}, \cdots, \frac{a_N}{W_0}$, $W_0$ is not a parameter that shows up in the above system of $N$ equations. Hence we conclude that the optimal solutions

$$\left\{ \frac{a_j^*}{W_0}; j = 1, 2, \cdots, N \right\}$$

must be independent of $W_0$. This finishes part (iii).

Finally, consider part (iv). Replacing $u(x) = -e^{-\rho x}$ into the first-order condition

$$\forall j = 1, 2, \cdots, N, \ E[u'(\tilde{W})(\tilde{R}_j - R_f)] = 0,$$

we obtain

$$\forall j = 1, 2, \cdots, N, \ 0 = E[\rho e^{-\rho \tilde{W}}(\tilde{R}_j - R_f)]$$

$$= E[\rho e^{-\rho W_0R_f + \sum_{j=1}^{N} a_j(\tilde{R}_j - R_f)}(\tilde{R}_j - R_f)]$$

$$= e^{-\rho W_0R_f} E[\rho e^{-\rho \sum_{j=1}^{N} a_j(\tilde{R}_j - R_f)}(\tilde{R}_j - R_f)],$$

6
so that the first-order condition can be equivalently written as

\[ \forall j = 1, 2, \cdots, N, \ 0 = E[e^{-\phi(\sum_{j=1}^{N} a_j(\tilde{R}_j - R_f))}](\tilde{R}_j - R_f)]. \]

In solving for the \( N \) unknowns \( a_1, a_2, \cdots, a_N \), \( W_0 \) is not a parameter that shows up in the above system of \( N \) equations. Hence we conclude that the optimal solutions

\[ \{a_j^*; j = 1, 2, \cdots, N\} \]

must be independent of \( W_0 \). It follows that when \( W_0 \) varies, the investor always holds the same optimal risky portfolio (that contains no riskless asset), which is

\[
\begin{pmatrix}
\frac{a_1^*}{a_1^* + a_2^* + \cdots + a_N^*} \\
\frac{a_2^*}{a_1^* + a_2^* + \cdots + a_N^*} \\
\vdots \\
\frac{a_{N-1}^*}{a_1^* + a_2^* + \cdots + a_N^*} \\
\frac{a_N^*}{a_1^* + a_2^* + \cdots + a_N^*}
\end{pmatrix}
\]

That is, when \( W_0 \) varies, this investor will re-allocate her initial wealth between the riskless asset and the above fixed risky portfolio. We say that the CARA investor’s portfolio selection behavior exhibits two-fund separation; we shall have more to say on this in Lecture 4. This finishes part (iv).

4. At date 0, Mr. A is endowed with \( W_0 = 1000 \) and the von Neumann-Morgenstern utility function \( u(\tilde{W}_2) = -e^{-\tilde{W}_2} \), where Mr. A only consumes at date 2, and his date-2 wealth is denoted by \( \tilde{W}_2 \). A risky asset (called the stock) and a riskless asset are traded at date 0 and at date 1. The per-period riskless interest rate is \( r_f = 0 \). The date-0-date-1 period rate of return on the stock is denoted by \( \tilde{r}_1 \), and the date-1-date-2 period rate of return on the stock is denoted by \( \tilde{r}_2 \). The two
random variables $\tilde{r}_1$ and $\tilde{r}_2$ are independent, and they have the same distribution function.

At date 2, the following facts become known:

- Mr. A’s date-1 wealth is $\tilde{W}_1 = 1140$;
- The realized rates of return on the stock are respectively $r_1 = \frac{1}{5}$ in the first period and $r_2 = \frac{11}{50}$ in the second period;
- Mr. A’s date-2 wealth is $\tilde{W}_2 = 1294$.

What is Mr. A’s portfolio weight for the stock in the first period? What is Mr. A’s (realized) portfolio weight for the stock in the second period?

**Solution.** By Theorem 3 of Lecture 2, independent of $\tilde{W}_1$, the same $a_2$ will solve the following maximization problem

$$\max_a E[-e^{-[\tilde{W}_1(1+r_f)+a(\tilde{r}_2-r_f)]}|\tilde{W}_1].$$

Indeed, this implies that $a_2$ is also the solution to the following maximization problem

$$\max_a E[-e^{-[W_0(1+r_f)+a(\tilde{r}_2-r_f)]}].$$

**Hint:** By Theorem 3 of Lecture 2, independent of $\tilde{W}_1$, the same $a_2$ will solve the following maximization problem

$$\max_a E[-e^{-[W_0(1+r_f)+a(\tilde{r}_2-r_f)]}|\tilde{W}_1].$$

At date 0, given the above $a_2$, Mr. A would like to find $a_1$ that maximizes

$$E[-e^{-[\tilde{W}_1(1+r_f)+a_2(\tilde{r}_2-r_f)]}] = E[-e^{-[[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)](1+r_f)+a_2(\tilde{r}_2-r_f)]}].$$

The above objective function can be further re-written as

$$E[-e^{-\rho[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)]} \cdot e^{-a_2(\tilde{r}_2-r_f)}]$$

$$= E[-e^{-\rho[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)]} \cdot E[e^{-a_2(\tilde{r}_2-r_f)}]],$$

where $\rho \equiv 1 + r_f$. 

8
Since $\tilde{r}_1$ and $\tilde{r}_2$ are equal in distribution, we conclude that $a_2$ is also the solution to the following maximization problem

$$\max_a E[-e^{-[W_0(1+r_f)+a(\tilde{r}_1-r_f)]}].$$

At date 0, given the above $a_2$, Mr. A would like to find $a_1$ that maximizes

$$E[-e^{-[\tilde{W}_1(1+r_f)+a_2(\tilde{r}_2-r_f)]}] = E[-e^{-[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)](1+r_f)+a_2(\tilde{r}_2-r_f)]}.$$

The above expression can be further re-written as

$$E[-e^{-A[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)]} \cdot e^{-a_2(\tilde{r}_2-r_f)}] = E[-e^{-A[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)]}] \cdot E[e^{-a_2(\tilde{r}_2-r_f)}],$$

where the last equality follows from the fact that $\tilde{r}_1$ and $\tilde{r}_2$ are independent,\(^4\) and where

$$A = 1 + r_f.$$  

Given $a_2$, $E[e^{-a_2(\tilde{r}_2-r_f)}]$ is a positive constant at date 0. Hence Mr. A should find $a_1$ to maximize

$$E[-e^{-A[W_0(1+r_f)+a_1(\tilde{r}_1-r_f)]}].$$

That is, in choosing $a_1$ at date 0, Mr. A acts as if he were having a CARA utility function with $R^A = 1 + r_f$.

So far, we have shown that $a_2$ is the solution to the maximization problem

$$\max_a E[-e^{-[W_0(1+r_f)+a(\tilde{r}_1-r_f)]}],$$

and $a_1$ is the solution to the following maximization problem

$$\max_a E[-e^{-A[W_0(1+r_f)+a(\tilde{r}_1-r_f)]}].$$

It follows from Proposition 1 (page 27) of Lecture 2 that

\(^4\)If $\tilde{x}$ and $\tilde{y}$ are independent, and $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are Borel functions, then $f(\tilde{x})$ and $g(\tilde{y})$ are also independent.
\[
\begin{align*}
\begin{cases}
    a_1 > a_2, & \text{if } A < 1; \\
    a_1 = a_2, & \text{if } A = 1; \\
    a_1 < a_2, & \text{if } A > 1.
\end{cases}
\]
\]

Equivalently, we have
\[
\begin{align*}
\begin{cases}
    a_1 > a_2, & \text{if } -1 < r_f < 0; \\
    a_1 = a_2, & \text{if } r_f = 0; \\
    a_1 < a_2, & \text{if } r_f > 0.
\end{cases}
\]
\]

Now, applying the above result, since \( r_f = 0 \), for some \( a \in \mathbb{R} \), we must have \( a_1 = a_2 = a \) and
\[
\begin{align*}
1140 &= \tilde{W}_1 = W_0(1 + r_f) + a(\tilde{r}_1 - r_f), \\
1294 &= \tilde{W}_2 = \tilde{W}_1(1 + r_f) + a(\tilde{r}_2 - r_f),
\end{align*}
\]

where
\[
\tilde{r}_1 = \frac{1}{5}, \quad \tilde{r}_2 = \frac{11}{50}.
\]

It follows that
\[
a = 700, \quad W_0 = 1000 \Rightarrow \frac{a_1}{W_0} = \frac{7}{10}
\]

in the first period, and Mr. A’s portfolio weight for the stock is
\[
\frac{a_2}{W_1} = \frac{700}{1140} = \frac{35}{57}
\]

in the second period.

5. This exercise concerns the design of a Pareto efficient insurance contract. We shall consider a competitive insurance industry where insurance companies are risk-neutral and make zero expected profits. Costs
may be incurred when people sign an insurance contract or when an insurance company makes re-imbursements to its clients.

Mr. D is endowed with a von Neumann-Morgenstern utility function \( u(\cdot) \), where \( u' > 0 > u'' \). He has wealth \( W > 0 \), but with probability \( p \in [0, 1] \), a financial loss \( x \in [0, W] \) may occur, and in the latter case his wealth will drop to \( W - x \). Define \( \pi \) as the maximal amount he is willing to pay for an insurance contract which promises to reimburse him completely in the event that the financial loss occurs. That is, \( \pi \) solves the equation

\[
(1-p)u(W) + pu(W-x) = (1-p)u(W-\pi) + pu(W-x-\pi+x) = u(W-\pi).
\]

Define \( \pi(p, x) = \pi - px \); that is, \( \pi(p, x) \) is the excess of \( \pi \) over the expected loss, which arises from the risk aversion of Mr. D. Show that
(a). \( \pi(0, x) = \pi(1, x) = \pi(p, 0) = 0 \);
(b). \( \pi(\cdot, \cdot) \) is continuous;
(c). \( \pi(\cdot, x) \) is a strictly concave function of \( p \), \( \forall x \in (0, W) \);
(d). \( \pi(p, \cdot) \) is a strictly increasing function of \( x \), \( \forall p \in (0, 1) \).

Now suppose that in the equilibrium of a competitive insurance industry, insurance policies are priced at their expected costs. The costs of an insurance company include (1) the losses of the insured; (2) a constant fee \( k_1 \geq 0 \) per policy signed; (3) a constant fee \( k_2 \geq 0 \) per claim paid; and (4) a constant fee \( k_3 \geq 0 \) per dollar of claim paid.

(e). Show that when \( k_1 > 0, k_2 = k_3 = 0 \), Mr. D wants to purchase a fully insured policy if and only if (e-1) with \( x \) fixed, \( x \) is greater than a threshold level \( x^*(p) \); and (e-2) with \( x \) fixed, \( p \) is neither too big nor too small; i.e. there exists \( p(x), p(x) \) such that \( p(x) < p < p(x) \).

(f). Show that when \( k_2 > 0, k_1 = k_3 = 0 \), Mr. D wants to purchase a fully insured policy if and only if with \( x \) fixed, \( p \) is less than a threshold level \( p^*(x) \).
Now suppose that there can be two possible losses, $0 < x_1 < x_2 < W$, with $x_i$ occurring with probability $p_i$, $i = 1, 2$, where $0 < p_1 + p_2 < 1$. We now derive Pareto optimal insurance contracts $(z_1, z_2)$, where $z_i$ is the re-imbursement when the loss is $x_i$.

(g). Show that if $k_1 > 0 = k_2 = k_3$ and if Mr. D is willing to accept a Pareto optimal insurance policy, then the policy must specify $z_i = x_i$, $i = 1, 2$.

(h). Show that if $k_2 > 0 = k_1 = k_3$ and if Mr. D is willing to accept a Pareto optimal insurance policy, then the policy must specify $(z_1, z_2)$ such that either $(z_1, z_2) = (x_1, x_2)$ or $z_1 z_2 = 0$.

(i). Show that if $k_3 > 0 = k_1 = k_2$ and if Mr. D is willing to accept a Pareto optimal insurance policy, then the policy must specify that for some $d > 0$, $(z_1, z_2)$ is such that $z_i > 0 \iff x_i \geq d$. (Such $d$ is referred to as a **deductible**.)

**Solution.** Part (a) is obvious. For part (b), to show that $\pi(p, x)$ is continuous in $(p, x)$, it suffices to show that $\pi$ is. Suppose that $\{(p_n, x_n); n \in \mathbb{Z}_+\}$ is a sequence in $\mathbb{R}^2$ converging to $(p, x)$, so that there exists a corresponding sequence $\pi_n$, where for each given $n \in \mathbb{Z}_+$, $\pi_n$ is such that

$$
(1 - p_n)u(W) + p_n u(W - x_n) = u(W - \pi_n).
$$

\[^5\]For parts (g), (h), (i), note that the optimal insurance contract $(z_1, z_2)$ solves

$$
\max_{z_i \in [0, x_i]; i = 1, 2} f(z_1, z_2) = \sum_{i=1}^{2} p_i u(W - x_i + z_i - k_1 - \sum_{j=1}^{2} p_j[z_j(1 + k_3) + k_2])
$$

$$
+(1 - p_1 - p_2)u(W - k_1 - \sum_{j=1}^{2} p_j[z_j(1 + k_3) + k_2]).
$$

12
If we can show that $\pi_n$ converges to $\pi$ when $\{(p_n, x_n); n \in \mathbb{Z}_+\}$ converges to $(p, x)$, then by definition $\pi$ is a continuous function of $(p, x)$. Now, by the continuity of $u(\cdot)$, we have

$$u(W - \pi) = (1 - p)u(W) + pu(W - x)$$

$$= \lim_{n \to \infty} [(1 - p_n)u(W) + p_nu(W - x_n)]$$

$$= \lim_{n \to \infty} u(W - \pi_n)$$

$$= u(W - \lim_{n \to \infty} \pi_n),$$

where the last equality follows from the fact that $u(\cdot)$ is strictly increasing and continuous. Thus, we conclude that

$$\lim_{n \to \infty} \pi_n = \pi$$

whenever $\{(p_n, x_n); n \in \mathbb{Z}_+\}$ is a sequence in $\mathbb{R}^2$ converging to $(p, x)$, and hence $\pi$ is continuous in $(p, x)$.

For parts (c) and (d), we have by definition,

$$\pi = W - u^{-1}((1 - p)u(W) + pu(W - x)),$$

and hence, given $p$,

$$\frac{\partial \pi}{\partial x} = \frac{pu'(W - x)}{u'(u^{-1}((1 - p)u(W) + pu(W - x)))} > 0.$$

(Here we have used the fact that if $g(y)$ is the inverse function of the smooth function $f(x)$, then $g'(y) = \frac{1}{f'(g(y))}$. Moreover, since $u(\cdot)$ is increasing and strictly concave, $u^{-1}(\cdot)$ is increasing and strictly convex, and hence given $x$, $u^{-1}((1 - p)u(W) + pu(W - x))$ is a strictly convex function of $p$.

Next, for part (e), observe that Mr. D wants to purchase the insurance policy if and only if

$$u(W - p[k_2 + (1 + k_3)x] - k_1) > (1 - p)u(W) + pu(W - x) = u(W - \pi).$$

This says that $u(\cdot)$ and $u^{-1}(\cdot)$ are both continuous; that is, $u(\cdot)$ is a homeomorphism.
Given that $k_1 > 0 = k_2 = k_3$, the above condition can be re-written as

$$u(W - px - k_1) > u(W - \pi) \iff \pi > px + k_1 \iff \pi(p, x) > k_1.$$  

Since by (a), $\pi(p, 0) = 0 < k_1$ and given $p$, $\pi(p, \cdot)$ is strictly increasing in $x$, assertion (e-1) follows. Moreover, since $\pi(p, x)$ is strictly concave in $p$ given $x$, and since $\pi(0, x) = \pi(1, x) = 0 < k_1$, assertion (e-2) follows also.

Consider part (f). As in part (e), Mr. D wants to buy the insurance policy if and only if

$$\pi(p, x) > pk_2 \iff \pi(p, x) > k_2.$$  

Since $\pi(\cdot, x)$ is strictly concave with $\pi(0, x) = 0$, $\pi(p, x)$ is a strictly decreasing function of $p$ given $x$. Thus assertion (f) follows.

For parts (g), (h), (i), note that the optimal insurance contract $(z_1, z_2)$ solves

$$\max_{z_i \in [0, x_i]; i = 1, 2} f(z_1, z_2) = \sum_{i=1}^2 p_i u(W - x_i + z_i - k_1 - \sum_{j=1}^2 p_j [z_j (1 + k_3) + k_2])$$

$$+ (1 - p_1 - p_2) u(W - k_1 - \sum_{j=1}^2 p_j [z_j (1 + k_3) + k_2]).$$

Given $x$, define $h(p) \equiv \pi(p, x)$. Then $h(\cdot)$ is strictly concave; $h'' < 0$. By the fundamental theorem of calculus, we have

$$h(p) = h(0) + \int_0^p h'(t)dt,$$  

where $h'(\cdot)$ is strictly decreasing because $(h')(\cdot) = h'' < 0$. Thus for all $t \in [0, p)$, we have

$$h'(t) > h'(p),$$

so that, using $h(0) = 0$,

$$h(p) > h(0) + \int_0^p h'(p)dt = ph'(p).$$

Now we have

$$\frac{d(h(p))}{dp} = \frac{h'(p)p - h(p)}{p^2} < 0;$$

that is, $\pi(p, x)$ is a strictly decreasing function of $p$ given $x$.  

14
It is easy to see that \( f(\cdot, \cdot) \) is concave in \((z_1, z_2)\) (check Hessian if you want!).

In part (g), the above optimal contract problem reduces to

\[
\max_{z_i \in [0, x_i]; i=1,2} f(z_1, z_2) \equiv p_1 u(W - x_1 + z_1 - (k_1 + p_1 z_1 + p_2 z_2)) \\
+ p_2 u(W - x_2 + z_2 - (k_1 + p_1 z_1 + p_2 z_2)) + (1 - p_1 - p_2) u(W - (k_1 + p_1 z_1 + p_2 z_2)).
\]

The necessary and sufficient first-order conditions are, for positive solutions,

\[
\frac{\partial f}{\partial z_i} = 0, \ i = 1, 2.
\]

Define

\[
W_0 \equiv W - (k_1 + p_1 z_1 + p_2 z_2), \quad W_i \equiv W - (k_1 + p_1 z_1 + p_2 z_2) - x_i + z_i, \ i = 1, 2.
\]

Then \( z_1 z_2 \neq 0 \) implies that

\[
\frac{\partial f}{\partial z_1} = p_1 u'(W_1)(1 - p_1) + p_2 u'(W_2)(-p_1) + (1 - p_1 - p_2) u'(W_0)(-p_1) = 0,
\]

\[
\frac{\partial f}{\partial z_2} = p_1 u'(W_1)(-p_2) + p_2 u'(W_2)(1 - p_2) + (1 - p_1 - p_2) u'(W_0)(-p_2) = 0.
\]

It follows from

\[
0 = p_2 \frac{\partial f}{\partial z_1} - p_1 \frac{\partial f}{\partial z_2}
\]

that

\[
0 = u'(W_1) - u'(W_2) \Rightarrow z_1 - x_1 = z_2 - x_2.
\]

Letting \( T = u'(W_1) = u'(W_2) \), we have from \( \frac{\partial f}{\partial z_2} = 0 \) that \( u'(W_0) = T \).

We conclude that if \( z_1 z_2 \neq 0 \), then

\[
0 = z_1 - x_1 = z_2 - x_2.
\]

Apparently, this solution is better than either \( z_1 = 0 \) or \( z_2 = 0 \) if \( k_1 \) is small enough; or else, no insurance \((z_1 = z_2 = 0)\) is optimal.

Now in part (h), if we simply redefine \( k_1 = (p_1 + p_2) k_2 \), then every argument in part (g) carries over. Thus we conclude that if \( k_1 = k_3 = \ldots \)
0 < k_2, the optimal contract with z_1z_2 \neq 0 is still the one with full insurance; and of course, if k_2 is large enough, it may happen that z_1z_2 = 0 is optimal.

Finally, for part (i), the above optimal contract problem becomes

$$\max_{z_i \in [0, x_i]; i = 1, 2} f(z_1, z_2) = \sum_{i=1}^{2} p_i u(W - x_i + z_i - \sum_{j=1}^{2} p_j z_j (1 + k_3))$$

$$+ (1 - p_1 - p_2) u(W - \sum_{j=1}^{2} p_j z_j (1 + k_3)).$$

Let us show part (i) by contraposition. Suppose instead that at the optimum, z_1 > 0 = z_2 (in this case no such d > 0 can be found as x_2 > x_1 by assumption). For this to happen, it must be that

$$\frac{\partial f}{\partial z_1} = 0 \geq \frac{\partial f}{\partial z_2};$$

that is,

$$p_1 u'(W_1) [1 - p_1 (1 + k_3)] + p_2 u'(W_2) [-p_1 (1 + k_3)] + (1 - p_1 - p_2) u'(W_0) [-p_1 (1 + k_3)] = 0,$$

and

$$p_1 u'(W_1) [-p_2 (1 + k_3)] + p_2 u'(W_2) [1 - p_2 (1 + k_3)] + (1 - p_1 - p_2) u'(W_0) [-p_2 (1 + k_3)] \leq 0.$$

Note that 0 < x_2 < z_1 - x_1, and hence W_2 < W_1, and by strict concavity of u(\cdot), we must have u'(W_1) < u'(W_2). However, by $$\frac{\partial f}{\partial z_1} = 0 \geq \frac{\partial f}{\partial z_2}$$, we have

$$p_2 \frac{\partial f}{\partial z_1} - p_1 \frac{\partial f}{\partial z_2} \geq 0,$$

which implies that

$$u'(W_1) \geq u'(W_2),$$

a contradiction. This finishes the proof for part (i).\(^8\)

\(^8\)This exercise is adapted from Shavell, Steven, 1979, On Moral Hazard and Insurance, *The Quarterly Journal of Economics*, 93, 541-562.
Remark. In this exercise, an insurance company is risk-neutral, but the policyholder (Mr. D) is risk-averse. In the absence of transaction costs (i.e., \( k_1, k_2, k_3 \)), the most efficient way for the two parties to share a risk is to let the risk-neutral insurance company alone bears the risk; that is, Mr. D should then get full insurance and be left with a riskless wealth. In the presence of transaction costs, however, the concern of reducing costs must be taken into account, and hence Mr. D may not obtain full insurance under a Pareto optimal contract. Note that \( k_1 \) and \( k_2 \) are essentially fixed costs—when they are incurred, these costs are independent of the amounts of reimbursement \( z_1 \) and \( z_2 \). Consequently, in the presence of \( k_1 > 0 \) or \( k_2 > 0 \) alone, the insurance contract specifies either no reimbursements or, again, full insurance for Mr. D. On the other hand, \( k_3 \) is a variable cost. When \( k_3 \) alone is positive, it is wise to not reimburse small losses, which gives rise to the so-called deductible.

We have assumed away Mr. D’s incentive problems in this exercise. In reality, the insurance company may be faced with a moral hazard or adverse selection problem on the part of Mr. D. In these cases, deductibles arise naturally as a feature of the Pareto optimal insurance contract. We shall talk about this point later on.

6. At date 0, Mr. A is endowed with initial wealth \( W_0 > 0 \) and a VNM utility function
\[
u(W) = -e^{-AW},
\]
where \( A > 0 \) is a constant, and \( \tilde{W} \) is Mr. A’s date-1 wealth. He can trade three assets: a riskless asset with zero rate of return, asset 1 with rate of return \( \tilde{r}_1 \), and asset 2 with rate of return \( \tilde{r}_2 \). To state the statistical relationship between \( \tilde{r}_1 \) and \( \tilde{r}_2 \), assume that \( \mu > 0, \eta_1 > 0, \eta_2 > 0 \) are three constants, and
\[
\tilde{\epsilon}_1 - \eta_1, \tilde{\epsilon}_2 - \eta_2, \tilde{m} - \mu, \tilde{\epsilon}
\]
are four totally independent standard normal random variables. It is known that for two non-zero constants \( \beta_1 \) and \( \beta_2 \),
\[
\tilde{r}_1 = \beta_1 \tilde{m} + \tilde{\epsilon}_1,
\]

17
and

\[ \hat{r}_2 = \beta_2 \hat{m} + \hat{c}_2. \]

Mr. A's wishes to optimally invest \( a_1 \) dollars in asset 1, \( a_2 \) dollars in asset 2, and \( W_0 - a_1 - a_2 \) dollars in the riskless asset.

(i) Show that Mr. A's optimal investment plan \((a_1^*, a_2^*)\) is such that

\[ a_1^* = \frac{\beta_1 \mu + (\beta_2^2 + 1) \eta_1 - \beta_1 \beta_2 \eta_2}{A(1 + \beta_1^2 + \beta_2^2)}, \]

and

\[ a_2^* = \frac{\beta_2 \mu + (\beta_1^2 + 1) \eta_2 - \beta_1 \beta_2 \eta_1}{A(1 + \beta_1^2 + \beta_2^2)}. \]

Verify that, when \( \eta_1 = \eta_2 = 0 \), the \( a_i^* \) and \( \beta_i \) have the same sign. Verify that when \( |\beta_j| \) increases, \( |a_i^*| \) decreases.

(ii) Verify that when \( \beta_1 = \beta_2 = 0 < \eta_1, \eta_2 \),

\[ a_i^* = \frac{\eta_i}{A}, \quad \forall i = 1, 2. \]

(iii) Now, assume instead that before trading gets started at date 0, Mr. A obtains some pre-trade information. More precisely, assume that Mr. A can observe the realization of the following signal \( \hat{s} \),

\[ \hat{s} = \hat{c}_1 + \hat{c}, \]

and Mr. A then updates his belief about \( \hat{r}_1 \) using this new information. Verify that, conditional on \( \hat{s}, \hat{c}_1 \) is again normally distributed, but with mean \( \frac{m + \hat{s}}{2} \) and variance \( \frac{1}{2} \). Show that Mr. A's optimal investment plan now depends on this new information, with

\[ a_1^* = \frac{\beta_1 \mu - \beta_1 \beta_2 \eta_2 + (\beta_2^2 + 1)(\frac{m + \hat{s}}{2})}{A(\beta_1^2 + \frac{1}{2} + \beta_2^2)}, \]

18
and

\[ a_2^* = \frac{1}{2} \left( \eta_2 + \beta_2 \mu \right) + \frac{\beta_1^2 \eta_2 - \beta_1 \beta_2 \left( \frac{\eta_2}{2} + \frac{\beta_2}{2} \right)}{A \left( \frac{\beta_1}{2} + 1 \right) + \frac{\beta_2}{2}}. \]

Verify that a piece of good news about \( \tilde{r}_1 \) (i.e. a large positive realization of \( s \)) raises \( a_1^* \), and it raises (respectively, reduces) \( a_2^* \) when \( \tilde{r}_1 \) and \( \tilde{r}_2 \) are negatively (respectively, positively) correlated. Explain!

**Solution.** It is easy to show that

\[ \tilde{W} = W_0 + a_1 \tilde{r}_1 + a_2 \tilde{r}_2 \]

is normally distributed with

\[ E[\tilde{W}] = W_0 + a_1 (\beta_1 \mu + \eta_1) + a_2 (\beta_2 \mu + \eta_2), \]

and

\[ \text{var}[\tilde{W}] = a_1^2 [\beta_1^2 + 1] + a_2^2 [\beta_2^2 + 1] + 2a_1 a_2 \beta_1 \beta_2. \]

Hence Mr. A’s objective function

\[ E[u(\tilde{W})] = u(E[\tilde{W}] - \frac{A}{2} \text{var}[\tilde{W}]), \]

and since \( u(\cdot) \) is a strictly increasing function, Mr. A can equivalently choose investment plan \((a_1, a_2)\) to maximize

\[ E[\tilde{W}] - \frac{A}{2} \text{var}[\tilde{W}]. \]

Hence Mr. A seeks to

\[ \max_{a_1, a_2} F(a_1, a_2) \equiv W_0 + a_1 (\beta_1 \mu + \eta_1) + a_2 (\beta_2 \mu + \eta_2) - \frac{A}{2} \{a_1^2 [\beta_1^2 + 1] + a_2^2 [\beta_2^2 + 1] + 2a_1 a_2 \beta_1 \beta_2 \}. \]

It is easy to verify that the Hessian matrix of \( F(\cdot, \cdot) \) is always negative definite, and hence \( F(\cdot, \cdot) \) is strictly concave. Thus Mr. A’s optimal investment plan \((a_1^*, a_2^*)\) must solve

\[
\begin{align*}
\frac{\partial F}{\partial a_1} &= \beta_1 \mu + \eta_1 - \frac{A}{2} [2a_1 (\beta_1^2 + 1) + 2a_2 \beta_1 \beta_2] = 0, \\
\frac{\partial F}{\partial a_2} &= \beta_2 \mu + \eta_2 - \frac{A}{2} [2a_2 (\beta_2^2 + 1) + 2a_1 \beta_1 \beta_2] = 0.
\end{align*}
\]
Thus we have

\[ a_2^* = \frac{\beta_1 \mu + \eta_1 - a_1^*(\beta_1^2 + 1)}{\beta_1 \beta_2} \].

In words, if \( a_1^* > 0 \) is so large that the numerator on the above right-hand side becomes negative, then the hedging motive dictates that \( a_2^* > 0 \) (respectively, \( a_2^* < 0 \)) if \( \beta_1 \beta_2 < 0 \) (respectively, \( \beta_1 \beta_2 > 0 \)). When \( \eta_1 = \eta_2 = 0 \), however, Mr. A, being rational and risk-averse, will make sure that \( a_1^* \) will not get that large,\(^9\) and in that case, \( a_1^* \) and \( a_2^* \) have opposite signs even if \( \hat{r}_1 \) and \( \hat{r}_2 \) are negatively correlated.

Solving the preceding system of equations for \((a_1^*, a_2^*)\), we have

\[ a_1^* = \frac{\beta_1 \mu + (\beta_1^2 + 1)\eta_1 - \beta_1 \beta_2 \eta_2}{A(1 + \beta_1^2 + \beta_2^2)}. \]

and

\[ a_2^* = \frac{\beta_2 \mu + (\beta_1^2 + 1)\eta_2 - \beta_1 \beta_2 \eta_1}{A(1 + \beta_1^2 + \beta_2^2)}. \]

It is clear that when \( \eta_1 = \eta_2 = 0 \), we have

\[ a_1^* = \frac{\beta_1 \mu}{A(1 + \beta_1^2 + \beta_2^2)}, \]

and

\[ a_2^* = \frac{\beta_2 \mu}{A(1 + \beta_1^2 + \beta_2^2)}, \]

so that Mr. A takes a long (respectively, short) position in asset \( i \) if \( \beta_i \) is positive (respectively, negative). In the latter case, as asserted, when \( |\beta_j| \) increases, to limit the overall risk exposure, Mr. A will optimally reduce \(|a_i^*|\). This is finishes part (i).

Next, consider part (ii). If \( \beta_1 = \beta_2 = 0 \), then the returns on assets 1 and 2 no longer contain systematic components. Since in this case \( \hat{r}_1 \) and \( \hat{r}_2 \) are statistically independent, Mr. A’s demand for risky asset \( j \) coincides with his demand for risky asset \( j \) when the other risky asset

\(^9\)Note that the expected return from investing in asset 1 increases linearly in \( a_1 \), but the return variance increases in \( a_1 \) at an increasing rate!
is not available for trading. We learn from Lecture 2 that the latter is simply $\frac{\eta}{A}$.

Finally, consider part (iii). Since Mr. A can choose his investment plan $(a_1, a_2)$ after seeing the realization of $\tilde{s}$, he now adopts the conditional distribution for $\tilde{\epsilon}_1$. From Lecture 0, we know that, conditional on $\tilde{s}$, $\tilde{\epsilon}_1$ is normally distributed with conditional mean

$$E[\tilde{\epsilon}_1|\tilde{s}] = \eta_1 + \frac{\text{cov}[\tilde{\epsilon}_1, \tilde{s}]}{\text{var}[\tilde{s}]}(\tilde{s} - E[\tilde{s}]) = \frac{\eta_1 + \tilde{s}}{2}$$

and conditional variance

$$\text{var}[\tilde{\epsilon}_1|\tilde{s}] = \text{var}[\tilde{\epsilon}_1] - \text{var}[E[\tilde{\epsilon}_1|\tilde{s}]] = 1 - \text{var}\left[\frac{\eta_1 + \tilde{s}}{2}\right] = \frac{1}{2}.$$ 

Thus we have

$$E[\tilde{W}|\tilde{s}] = W_0 + a_1(\beta_1\mu + \frac{\eta_1 + \tilde{s}}{2}) + a_2(\beta_2\mu + \eta_2),$$

and

$$\text{var}[\tilde{W}|\tilde{s}] = a_1^2[\beta_1^2 + \frac{1}{2}] + a_2^2[\beta_2^2 + 1] + 2a_1a_2\beta_1\beta_2.$$ 

Hence Mr. A seeks to

$$\max_{a_1, a_2} G(a_1, a_2; \tilde{s}) = W_0 + a_1(\beta_1\mu + \frac{\eta_1 + \tilde{s}}{2}) + a_2(\beta_2\mu + \eta_2)$$

$$- \frac{A}{2}\{a_1^2[\beta_1^2 + \frac{1}{2}] + a_2^2[\beta_2^2 + 1] + 2a_1a_2\beta_1\beta_2\}.$$

Since $G$ is strictly concave in $(a_1, a_2)$, the necessary and sufficient first-order condition then yields

$$a_1^* = \frac{\beta_1\mu - \beta_1\beta_2\eta_2 + (\beta_2^2 + 1)(\frac{\eta_1 + \tilde{s}}{2})}{A(\beta_1^2 + \frac{1}{2} + \frac{\beta_2^2}{2})},$$

and

$$a_2^* = \frac{\frac{1}{2}(\eta_2 + \beta_2\mu) + \beta_1^2\eta_2 - \beta_1\beta_2(\frac{\eta_1 + \tilde{s}}{2})}{A(\beta_1^2 + \frac{1}{2} + \frac{\beta_2^2}{2})}.$$ 

**Remark.** Stock prices fluctuate when new information arrives. Good news about a security moves up its price, and bad news leads to a price decline. In this exercise, we show that new information about one security can affect not only the price of that security, but also the prices of other securities, because the new information may result in each investor “rebalancing” her portfolio.

We have shown above that a piece of good news about $\ddot{r}_1$ (i.e. a large positive realization of $\ddot{s}$) raises $a_{1*}^*$, and it raises (respectively, reduces) $a_{2*}^*$ when $\ddot{r}_1$ and $\ddot{r}_2$ are negatively (respectively, positively) correlated. Indeed, if $\beta_1 \beta_2 > 0$ so that $\ddot{r}_1$ and $\ddot{r}_2$ are positively correlated, then an increase in $a_{1*}^*$ following a piece of good news without correspondingly reducing $a_{2*}^*$ will result in too much risk exposure for Mr. A. Things are different when $\beta_1 \beta_2 < 0$. In the latter case a piece of good news will induce Mr. A to raise both $a_{1*}^*$ and $a_{2*}^*$, because the two risky assets can be used to hedge each other’s risk.