Capital Market Theory, II

Lecture M2: Continuous-Time Calculus of Variations

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1. (**Euler Equation**) Consider the following *simplest problem in calculus of variations*:

$$\max_{x(t)} \int_0^T F(t, x(t), x'(t))dt$$

$$s.t. \quad x(0) = x_0, \quad x(T) = x_T.$$

We assume F to be continous in t and continuously differentiable in $x(t)$ and $x'(t)$ as separate arguments. An 'admissible' $x(t)$ must be continuously differentiable and satisfy the end-point conditions. Let $x^*(t)$ be optimal to the above problem. An admissible deviation is a continuously differentiable function $h(t)$ such that $x(t) \equiv x^*(t) + h(t)$ is admissible. It follows that $h(0) = h(T) = 0$. Then, given $h(t)$ is an admissible deviation, so are $ah(t)$ for all constants a. With $x^*(t)$ and $h(t)$ fixed, define

$$g(a) = \int_0^T F(t, y(t), y'(t))dt,$$

where $y(t) = x^*(t) + ah(t)$. This function must assume its maximum at $a = 0$. A necessary condition is

$$g'(0) = 0.$$

($g'(0)$ is called the *first variation* of the objective function at $x^*(t)$ in the direction of $h(t)$.) I leave you to show that this condition can be rewritten[1]

$$\int_0^T [F_x - \frac{dF_{x'}}{dt}]h(t)dt = 0,$$

where functions in the square bracket are evaluated at $(t, x^*(t), x^{*\prime}(t))$. We state a lemma:

Lemma 1 Suppose $g(\tau)$ is a given continuous function defined on $[t, T]$ which is such that

$$\int_t^T g(\tau)h(\tau)d\tau = 0$$

[1]This relies on the supposition that $\frac{dF_{x'}}{dt}$ exists at optimum. But this can be proved separately. This fact implies that the optimal policy function is twice differentiable whenever $F_{x'x'} \neq 0$.

for every $h(\tau)$ continuous on $[t, T]$ with $h(t) = h(T) = 0$, then $g \equiv 0$ on $[t, T]$.
Using lemma 1, we get the following necessary condition:

$$F_x(t, x^*(t), x^{*\prime}(t)) = \frac{dF_{x'}(t, x^*(t), x^{*\prime}(t))}{dt}, \quad \forall t \in [0, T].$$

This is known as the *Euler equation*. Solutions satisfying the Euler equation are called *extremals*. It is seen that the optimal policy functions must be extremals but extremals may not be optimal. There are several equivalent forms for the Euler equation:

$$F_x = F_{x't} + F_{x'x}x' + F_{x'x'}x'', \quad \forall t \in [0, T]$$

and

$$\frac{d(F - x'F_{x'})}{dt} = F_t, \quad \forall t \in [0, T],$$

where the last one is in particular useful when F does not depend on t explicitly.

2. (**Legendre Condition**) The above developed Euler equation is the first-order necessary condition for optimality of $x^*(t)$. There are other necessary conditions required. These include the second-order Legendre condition and the transversality conditions. The latter arises from concerns for optimality. We now derive the Legendre condition. Consider the following simplest problem in calculus of variation:

$$\max_{x(\cdot)} \int_0^1 F(t, x, x')dt, \quad s.t. \quad x(0) = b, \quad x(1) = B.$$

Recall the function

$$g(a) = \int_0^1 F(t, x^* + ah, x^{*\prime} + ah')dt,$$

where $x^*(t)$ and $h(t)$ are respectively optimal policy function and some fixed admissible deviation function. Define the *second variation* as

$$g''(0) = \int_0^1 [F_{xx}h^2 + 2F_{xx'}hh' + F_{x'x'}(h')^2]dt,$$

which must be negative at $x^*(t)$ for all admissible h.
Fact: The second variation is automatically negative, if F is concave in (x, x').
In fact, we have
Theorem 1 (Sufficiency of Euler equation) If F is concave in (x, x'), then the Euler equation is necessary and sufficient for optimality.

To prove theorem 2, use the following mathematic fact: If $f(x, y)$ is concave in (x, y), then

$$f(x_2, y_2) - f(x_1, y_1) \leq (x_2 - x_1) f_x(x_1, y_1) + (y_2 - y_1) f_y(x_1, y_1).$$

For most of our applications to finance and economics, F will not be concave in (x, x'). The second-order Legendre condition requires that F be locally concave in x' along $x^*(t)$. Using integration by parts to the second term of $g''(0)$, we have

$$g''(0) = \int_0^1 [(F_{xx} - \frac{dF_{xx'}}{dt}) h^2 + F_{x'x'} h'^2] dt.$$

Here we need another lemma:
Lemma 2 Let $P(t)$ and $Q(t)$ be continuous on $[0, 1]$ and the following quadratic functional

$$\int_0^1 [P(h')^2 + Qh^2] dt$$

be defined on $[0, 1]$ for all $h(t)$ such that $h(0) = h(1) = 0$. A necessary condition for this functional to be negative is that $P(t)$ is negative on $[0, 1]$.
It follows from lemma 2 that

$$F_{x'x'}(t, x^*(t), x^{*\prime}(t)) \leq 0,$$

which is the *Legendre condition*.

3. (**Transversality Condition for Free End Value Problems**) Consider

$$\max_{x(\cdot)} \int_0^1 F(t, x, x') dt, \quad s.t. \;\; x(0) = b.$$

Since $h(1)$ need not be zero for h to be admissible, the first variation becomes

$$g'(0) = \int_0^1 h(F_x - \frac{dF_{x'}}{dt}) dt + (F_{x'} h)|_{t=1} = 0.$$

But, this must hold for all h with $h(1) = 0$, and hence the Euler equation is still necessary. It follows that, for h with $h(1) \neq 0$, we need

$$F_{x'}(1, x^*(1), x^{*\prime}(1)) = 0.$$

This end-point condition arises solely from the concern for optimality, and is called a *transversality condition*.

4. (**Transversality Condition for Free-Horizon Problems**) Consider

$$\max_{T,x(\cdot)} \int_0^T F(t,x,x')dt, \quad s.t. \;\; x(0)=b.$$

Suppose $(T^*, x^*(t))$ solves the above problem, and consider a local deviation $(\delta T, h(t))$. Define

$$g(a) = \int_0^{T^*+a\delta T} F(t, x^*(t)+ah(t), x^{*\prime}(t)+ah'(t))dt.$$

Again, we require $g'(0)=0$. Using Leibnitz's rule and integration by parts, we have

$$g'(0) = F\delta T + (F_{x'}h)|_{T^*} + \int_0^{T^*} h(F_x - \frac{dF_{x'}}{dt})dt.$$

Define

$$\delta x_{END} \equiv x(T^*+\delta T) - x^*(T^*).$$

When δT is infinitesimal, we have approximately $x(T^*+\delta T) = x(T^*) + x^{*\prime}(T^*)\delta T$, and hence

$$\delta x_{END} = h(T^*) + x^{*\prime}(T^*)\delta T.$$

Substituting $h(T^*)$ from the last expression into $g'(0)$, we can rewrite the latter as

$$g'(0) = \int_0^{T^*} h(F_x - \frac{dF_{x'}}{dt})dt + F_{x'}|_{T^*}\delta x_{END} + (F - x'F_{x'})|_{T^*}\delta T = 0.$$

Since the compared $x(t)$ <u>could</u> terminate at the same time as $x^*(t)$ does with the same end value, the Euler equation holds again. But then,

$$F_{x'}|_{T^*}\delta x_{END} + (F - x'F_{x'})|_{T^*}\delta T = 0$$

which implies that both

$$F_{x'}(T^*, x^*(T^*), x^{*\prime}(T^*)) = 0$$

and

$$F(T^*, x^*(T^*), x^{*\prime}(T^*)) = x^{*\prime}(T^*)F_{x'}(T^*, x^*(T^*), x^{*\prime}(T^*))$$

must hold. These are the two transversality conditions. I leave you to show that, in case $x(T)$ is fixed and T is free, the transversality condition in this case becomes

$$F(T^*, x^*(T^*), x^{*\prime}(T^*)) = x^{*\prime}(T^*)F_{x'}(T^*, x^*(T^*), x^{*\prime}(T^*))$$

only. Also, if the only requirement for $(T, x(T))$ is that for some differentiable function $R(\cdot)$, $x(T) = R(T)$, then the transversality condition becomes, at $(T^*, x(T^*)$,

$$F + (R' - x')F_{x'} = 0.$$

5. (**Autonomous Problems and Phase Diagrams**) There is a special problem arising in finance and economics literature very often, called the *autonomous* problem. That is a problem where the integrand of the objective function does not explicitly depend on time. In the following problem

$$\max_{x(\cdot)} \int_0^\infty e^{-rt} F(x(t), x'(t))dt, \quad s.t. \ \ x(0) = b,$$

F is time homogeneous but $e^{-rt}F$ depends on time. Most economists still consider this an autonomous problem, because the Euler equation will be independent of time, and that is what matters. For autonomous problems with infinite time horizon, developing transversality conditions may not be of much help. Usually, however, one would like to look for steady state solutions. This makes sense, considering the impacts from time are only through the discount term. If there is a steady state solution, then $x' = x'' = 0$. Note that the Euler equation is

$$F_x = -rF_{x'} + F_{x'x}x' + F_{x'x'}x''.$$

Hence, the steady state solution x_s solves

$$F_x(x_s, 0) + rF_{x'}(x_s, 0) = 0.$$

A second boundary condition is then to require

$$\lim_{t\to\infty} x(t) = x_s.$$

For autonomous problems, usually a diagrammatic analysis is very illuminating. This is especially useful, when we do not want to specify the functional form of F. Consider the following autonomous problem with finite time horizon:

$$\min_{x(\cdot)} \int_0^T e^{-rt}[f(x'(t)) + g(x(t))]dt \ \ s.t. \ \ x(0) = b, \ \ x(T) = B,$$

with $f'', g'' > 0$. Apparently, theorem 1 says that Euler equation together with boundary conditions are necessary and sufficient for optimality. The Euler equation is

$$x'' = \frac{rf'(x') + g'(x)}{f''(x')}.$$

Unfortunately, not much can we say about the solution $x(t)$. In this case, however, a diagrammatic analysis on the (x, x') plane shows that the optimal $x(t)$ must be monotonic, single-peaked or single-troughed. Consider the loci of points (x, x') such that respectively $x' = 0$ and $x'' = 0$ hold.

6. (**Bolza-Type Problems**) Consider modifying the objective function by adding a bequest function:

$$\max_{x(\cdot)} \int_0^T F(t, x, x')dt + G(T, x(T))$$

$$s.t. \;\; x(0) = x_0.$$

I leave you to show that the first variation in this case is

$$g'(0) = \int_0^T (F_x h + F_{x'} h')dt + [(F + G_t)\delta T + G_x \delta x(T)]|_{t=T} = 0.$$

One can show that Euler equation must hold again. It follows that, using integration by parts,

$$(F - x' F_{x'} + G_t)\delta T + (F_{x'} + G_x)\delta x(T) = 0.$$

We conclude: (i) If $x(T)$ is free, then $F_{x'} + G_x = 0$ at T; (ii) if T is free, then $F - x' F_{x'} + G_t = 0$ at T; and (iii) if at T, for some differentiable $R(\cdot)$, we need $R(T) = x(T)$, then at T the transversality condition is

$$F + F_{x'}(R' - x') + G_x R' + G_t = 0.$$

7. (**Non-simple Problems**) Now we consider problems with constaints other than the boundary constraints appearing in the simplest problems. First consider the following *isoperimetric problems*:

$$\max \int_0^T F(t, x, x')dt$$

$$s.t. \;\; \int_0^T G(t, x, x')dt = B, \; x(0) = x_0, \; x(T) = x_T,$$

where G is twice continuously differentiable and B a constant. As usual, we form the Lagrangian:

$$\int_0^T [F(t, x, x') - \pi G(t, x, x')]dt + \pi B.$$

As long as x^* is not an extremal for $\int_0^T Gdt$, the first variation implies that the following Euler equation must hold for x^* and some constant π:

$$F_x - \pi G_x = \frac{F_{x'} - \pi G_{x'}}{dt}.$$

Next, we consider inequality constraints. First we consider inequality constraints at the terminal date, say

$$T \leq \hat{T}, \quad \text{or} \quad x(T) \geq a.$$

Let $x^*(t)$ be optimal for $t \in [0, T]$. Let $F^*(t) = F(t, x^*(t), x^{*\prime}(t))$ and J^* be the value function to the maximization problem below:

$$\max_{T, x(\cdot)} \int_0^T F(t, x, x')dt$$

$$s.t. \; x(0) = x_0, \quad T \leq \hat{T}, \quad x(T) \geq a.$$

Let $x(t)$ be a comparison function close to $x^*(t)$ on $[0, T + \delta T]$. Extend either $x(t)$ or $x^*(t)$ so that they have common domain. Let J be the value function corresponding to $x(t)$. We have

$$J - J^* = \int_0^{T+\delta T} F(t, x, x')dt - \int_0^T F(t, x^*, x^{*\prime})dt$$

$$= \int_T^{T+\delta T} F(t, x, x')dt + \int_0^T [F(t, x, x')dt - F(t, x^*, x^{*\prime})]dt$$

$$\sim F^*(T)\delta T + \int_0^T [(x - x^*)F_x^* + (x' - x^{*\prime})F_{x'}^*]dt$$

$$\equiv F^*(T)\delta T + \int_0^T [hF_x^* + h'F_{x'}^*]dt.$$

Using integration by parts and approximating $h(T)$ by $\delta x(T) - x^{*\prime}(T)\delta T$, we have

$$\delta J \equiv J - J^* = (F - x'F_{x'})|_T \delta T + F_{x'}|_T \delta x(T) + \int_0^T (F_x - \frac{dF_{x'}}{dt})hdt.$$

Since x^* is optimal, we require $\delta J \leq 0$ always. Once again, it is seen that Euler equation must hold. It then follows that

$$(F - x'F_{x'})|_T \delta T + F_{x'}|_T \delta x(T) \leq 0.$$

For comparison path with $x(T) = x^*(T)$, we need the above first term to be ≤ 0. If T were fixed, this would give no restrictions whatsoever.

But, now we are requiring $T \leq \hat{T}$. In case the constraint binds at optimal T, then δT can only be ≤ 0, which implies that $(F - x'F_{x'})|_T \geq 0$. If it does not bind at optimum, then since δT can take either sign, we require $(F - x'F_{x'})|_T = 0$. Stating it compactly, we have the transversality condition arising from the optimality of T:

$$T \leq \hat{T}, \quad F - x'F_{x'} \geq 0, \quad (\hat{T} - T)(F - x'F_{x'}) = 0 \ \text{ at } \ T.$$

Similar reasonings yield the transversality condition arising from the optimality of $x(T)$:

$$x(T) \geq a, \quad F_{x'}(T) \leq 0, \quad (x(T) - a)F_{x'}(T) = 0.$$

We next consider inequality constraints on the pair (t, x), say

$$R(t) \geq x(t), \ t \in [0, T].$$

Note that this constraint either binds or does not at optimum. If at t it binds, then $x(t)$ is simply $R(t)$; or else, the Euler equation must hold. Let t_0 be some time point at which the solution switches from binding area to non-binding area or vice versa. Then, if at t_0 $F_{x'x'} \neq 0$, it is necessary that

$$R(t_0) = x(t_0) \ \text{ and } \ R'(t_0) = x'(t_0).$$

Finally, we consider inequality constraints on x' in a very special problem known as *most rapid approach path* (MRAP):

$$\max \int_0^T e^{-rt}[M(x) + N(x)x']dt$$

$$s.t. \ \ x(0) = x_0, \quad A(x) \leq x' \leq B(x).$$

The Euler equation depends only on x:

$$M'(x) + rN(x) = 0.$$

If this equation has a unique solution $\hat{x}$, then the optimal solution to the MRAP is to approach $\hat{x}$ as soon as possible and then stay there forever. (The proof is lengthy and I will spare it.)

8. (**Corners**) So far, we have restricted ourselves to continuously differentiable policy functions. Before we end this set of notes, we broaden the set of admissible policy functions by considering piecewise smooth policy functions. These are functions for which there are at most a finite number of corners (points at which derivatives are discontinuous). Except at these corners, Euler equation must hold. Hence our major task is for search of these corner points. Suppose for simplicity that $\hat{t} \in (0, T)$ is the only

corner point and $x^*(t)$ is an optimal piecewise smooth function. Then, $x^*(t)$ must remain optimal on respectively $[0,\hat{t}]$ and $[\hat{t},T]$. In particular, at optimal $(\hat{t},x(\hat{t}))$, any admissible deviations $(\delta\hat{t},\delta x(\hat{t}))$ cannot not improve the value. This can be shown to imply that both $F_{x'}$ and $(F - x'F_{x'})$ be continuous functions at optimum. The latter fact is referred to as the *Weierstrass-Erdmann corner conditions*. These conditions together with Euler equation and boundary conditions can be used to solve the optimal piecewise smooth policy function.

9. (**Examples**) Now let us consider some examples.

   (a) Find the shortest length between the points $(0,0)$ and $(1,0)$. (**Solution**: $x(t) = 0, \ \forall t \in [0,1]$.)

   (b) $\max \int_1^e (3x' - tx'^2)dt \quad s.t. \quad x(1) = x_1, \quad x(e) = 1.$ (**Solution**: $x(t) = x_1 + (1 - x_1)\ln(t)$.)

   (c) Find extremals with corners, if any, for

   $$\min \int_0^T (cx'^2 + Cx)dt \quad s.t. \ x(0) = 0, \ \ x(T) = B.$$

   (**Solution**: No corners.)

   (d) $\min \int_0^1 (x')^2 dt \ \ s.t. \ \ \int_0^1 x^2 dt = 2, \ \ x(0) = x_0, \ x(1) = 0.$
   Solution The augmented integrand is

   $$(x')^2 + \pi x^2,$$

   of which the corresponding Euler equation is

   $$x'' = \pi x.$$

   Let us conjecture that $\pi > 0$. Then, for some constants a and b,

   $$x(t) = ae^{\sqrt{\pi}t} + be^{-\sqrt{\pi}t}.$$

   The boundary conditions require that

   $$a + b = x_0,$$
   $$ae^{\sqrt{\pi}} + be^{-\sqrt{\pi}} = 0.$$

   The isoperimetric constraint requires

   $$a^2 \int_0^1 e^{2\sqrt{\pi}t}dt + 2ab + b^2 \int_0^1 e^{-2\sqrt{\pi}t}dt = 2.$$

   The constants a, b and π are obtained by solving the above system of simultaneous equations.

More Exercises

1. Solve

$$\min_{x(t)} \int_0^1 [x'(t)]^2 dt$$

$$s.t. \quad \int_0^1 x(t)dt = B, \quad x(0) = 0, \quad x(1) = 2.$$

2. Solve

$$\max \int_0^T x dt$$

$$s.t. \quad \int_0^T \sqrt{1 + (x')^2} dt = B, \; x(0) = x(T) = 0.$$

3. Solve

$$\min \int_a^b \sqrt{1 + [x'(t)]^2} dt$$

$$s.t. \; x(a) = A.$$

4. Solve

$$\min_{T, x(\cdot)} \int_0^T [c_1[x'(t)]^2 + c_2 x(t)] dt$$

$$s.t. \quad x(0) = 0, \quad x(T) = B.$$

5. Two agents A and B form a partnership for a project that yields random payoff t next period, with t having support $[0, T]$. Suppose the two agents have von Neumann-Morgenstern utility functions

$$u_A(x) = -e^{-Rx}, \quad u_B(x) = \frac{x^{1-b}}{1-b},$$

where $R > 0, b \in (0, 1)$ are constants. A sharing rule $x^*(t)$ is Pareto optimal, if for some constant U, $x^*(t)$ solves

$$\max_{x(\cdot)} E[u_A(t - x(t))]$$

$$s.t. \; E[u_B(x(t))] = U.$$

Suppose that t is uniform on $[0, T]$. Characterize the set of optimal sharing rules.

6. You are endowed with K_0 units of capital with i being its instantaneous rate of return. You seek to

$$\max_{K(t)} \int_0^T e^{-rt} \ln(C(t))$$

where $C(t)$ is your consumption at time t and $\ln(\cdot)$ is your VNM utility function for time-t consumption. Note that the increment of capital at time t is the interest earned on capital minus the consumption at time t. Suppose T is your life span and you do not want to leave any capital after T. Find your optimal consumption policy.

7. Show that the shortest path from $(0,0)$ to a differentiable curve $R(t) = x$ is a straight line from $(0,0)$ to $(T, R(T))$ perpendicular to the tangent to the curve at $(T, R(T))$ for some T.

8. Prove theorem 1.