Finance Theory

A Quick Review of Game Theory

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1. **Static Game with Complete Information.** A game can be described by:

   - Who are the players?
   - What strategies are available to each player?
   - What does each player get given players' choices of strategies?

   A game described by *(i) the set of players, (ii) the strategies available to each player, and (iii) the payoff of each player as a function of the vector of all players' strategic choices* is called a game depicted in *normal form.*

2. **Example 1**. The following is a two-player normal-form game.

player 1/player 2	L	R
U	0,1	-1,2
D	2,-1	-2,-2

   - Who are the players? Players 1 and 2.
   - What strategies are available to player 1? U and D. What strategies are available to player 2? L and R.
   - What does each player get given players' choices of strategies? Players 1 and 2 get respectively 0 and 1, if the vector of the two players' strategies is (U, L); that is, if player 1 plays U and player 2 plays L. Let us write

$$u_1(U, L) = 0, \; u_2(U, L) = 1. \tag{1}$$

Similarly, we have

$$u_1(U,R) = -1,\ u_2(U,R) = 2,\ u_1(D,L) = 2, \tag{2}$$

$$u_2(D,L) = -1,\ u_1(D,R) = -2,\ u_2(D,R) = -2. \tag{3}$$

The functions $u_1(\cdot,\cdot)$ and $u_2(\cdot,\cdot)$ are referred to as the two players' *payoff functions*.

3. **Example 2.** Consider the following Cournot game: firms 1 and 2 producing the same product must compete in supply quantities. The inverse demand curve is

$$P(Q) = 1 - Q,$$

where

$$Q = q_1 + q_2$$

is the total supply of the product. For simplicity, assume that firms have no costs.

- Who are the players? Firms 1 and 2.
- What strategies are available to player 1? Any non-negative real number q_1, standing for the supply quantity chosen by firm 1. What strategies are available to player 2? Any non-negative real number q_2, standing for the supply quantity chosen by firm 2.
- What does each player get given players' choices of strategies (q_1,q_2)? Firm 1 gets profit $u_1(q_1,q_2) = q_1(1-q_1-q_2)$, and firm 2 gets profit $u_2(q_1,q_2) = q_2(1-q_1-q_2)$. (We generally will write π_1 and π_2 instead of u_1 and u_2, if the latter actually represent firms' profits.)

4. **Nash Equilibrium (NE)**. A (pure strategy) Nash equilibrium (NE) for a two-player normal-form game where the set of player 1's strategies is X (hereafter referred to as player 1's *strategy space*) and the set of player 2's strategies is Y (hereafter referred to as player 2's *strategy space*) is a pair (x^*,y^*) such that $x^* \in X$, $y^* \in Y$, and

$$u_1(x^*,y^*) \geq u_1(x,y^*), \quad \forall x \in X, \tag{4}$$

$$u_2(x^*, y^*) \geq u_2(x^*, y), \quad \forall y \in Y. \tag{5}$$

(The last two inequalities are called *incentive compatibility conditions.*) More generally, given $y \in Y$, we say that $x^* \in X$ is player 1's *best response* against y if

$$u_1(x^*, y) \geq u_1(x, y), \quad \forall x \in X. \tag{6}$$

If for all $y \in Y$, there exists a unique player 1's best response against y, then we can write $x^* = r_1(y)$, and refer to $r_1(\cdot)$ as player 1's *reaction function*. Similarly, given $x \in X$, we say that $y^* \in Y$ is player 2's *best response* against x if

$$u_2(x, y^*) \geq u_1(x, y), \quad \forall y \in Y. \tag{7}$$

If for all $x \in X$, there exists a unique player 2's best response against x, then we can write $y^* = r_2(x)$, and refer to $r_2(\cdot)$ as player 2's *reaction function*. Apparently, if the two-player game has a unique Nash equilibrium (x^*, y^*), then it must be that

$$r_1(y^*) = x^*, \quad y^* = r_2(x^*). \tag{8}$$

That is, the unique NE must appear at the intersection of the two reaction functions.

5. **Example 3**. (**How to solve a mixed-strategy NE?**) Now we look for the NE of the game described in Example 2. By definition, it is a pair (q_1^*, q_2^*), such that given q_2^*, q_1^* is profit maximizing for firm 1, and given q_1^*, q_2^* is profit maximizing for firm 2. The procedure is first to find the best response r_i for firm i given *any possible* q_j, for $i, j = 1, 2$, $i \neq j$. Then, the NE can be obtained by finding the intersection of the two reaction functions. So, consider step 1. To solve for $r_i(\cdot)$, given any q_j, consider firm i's problem of finding its profit-maximizing supply quantity:

$$\max_{q_i} q_i(1 - q_i - q_j),$$

and the (necessary and sufficient) first-order condition gives

$$r_i(q_j) = \frac{1 - q_j}{2}.$$

(The reaction function is well-defined because given each q_j, there exists exactly 1 optimal q_i for firm i.) Now, consider step 2. By definition, an NE (q_1^*, q_2^*) is located at the intersection of the two firms' reaction functions. Thus (q_1^*, q_2^*) must satisfy

$$r_1(q_2^*) = q_1^*, \; r_2(q_1^*) = q_2^*. \tag{9}$$

Solving, we obtain the Nash equilibrium for the above Cournot game:

$$(q_1^*, q_2^*) = (\frac{1}{3}, \frac{1}{3}).$$

6. Recall Example 1, where player 1 can play U or D, and player 2 can play L or R. A *mixed strategy* for player 1 is a probability distribution over U and D, and a mixed strategy for player 2 is a probability distribution over L and R. For example, playing U and D with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ is one mixed strategy for player 1, and playing U and D with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ is another mixed strategy for player 1. Apparently, each player has an infinite number of different mixed strategies. A *mixed strategy Nash equilibrium* (p, q) of this game is a pair of the two players' mixed strategies, such that player 1 plays U with probability $p \in (0, 1)$ and player 2 plays L with probability $q \in (0, 1)$, and such that player 2's mixed strategy makes player 1 feel indifferent about U and D, and player 1's mixed strategy makes player 2 feels indifferent about L and R; that is, the following *incentive compatibility conditions* hold:

$$qu_1(U, L) + (1 - q)u_1(U, R) = qu_1(D, L) + (1 - q)u_1(D, R), \tag{10}$$

$$pu_2(U, L) + (1 - p)u_2(D, L) = pu_2(U, R) + (1 - p)u_2(D, R). \tag{11}$$

There is an obvious reason for the above two equations: if given his rival's mixed strategy, a player strictly prefers one pure strategy to the other, then he will assign zero probability to the latter; that is, a mixed strategy can never be his best response. Thus in a mixed strategy Nash equilibrium, where each player assigns a positive probability to every pure strategy, a player has to feel *indifferent* about his two pure strategies.[1]

[1] Consider a two-player game, where the two must pick an integer from the set $\{1, 2, \cdots, 100\}$ at the same time. If they pick the same number, then they each get 1; or else, they each get zero. Find the pure strategy NE's. Find the mixed strategy NE's.

7. (**Dynamic Game with Complete Information**.) The games appearing in Examples 1 and 2 are called *simultaneous games*, because players move at the same time in those games. In a *sequential game*, on the other hand, players take turns to move. Sequential games are usually described in extensive form and represented by a *game tree.* A game tree is composed of a collection of decision nodes (with one player being assigned to each decision node) and a set of branches connecting those nodes, where the branches leaving a decision node represent the actions available to the decision maker residing at that decision node. For example, consider the following game tree:

$$(1)\text{—}\left[\begin{array}{lll} Up\text{—} & (2)\text{—} & \left[\begin{array}{ll} Right\text{—} & (0,1)\\ Left\text{—} & (-1,2)\end{array}\right. \\ & \vdots & \\ Down\text{—} & (2)\text{—} & \left[\begin{array}{ll} Right\text{—} & (2,-1)\\ Left\text{—} & (-2,-2)\end{array}\right.\end{array}\right.$$

In this game tree, the first mover's (player 1's) decision node is the root of the tree, and each of the two pure strategies available to the first mover is represented by a branch emanating from that decision node. These branches, labeled up and down respectively, lead to the second mover's decision nodes. Note that player 2's two decision nodes are connected by a dotted line, which says that player 2, when determining her own moves, does not know whether player 1 has moved up or down. Thus these two decision nodes define player 2's *information set* at the time player 2 must choose between right and left. Formally, an *information set* is a set of decision nodes for a player, who, while knowing that he is sitting on one of those nodes contained in the information set, cannot tell which node in the information set he is exactly sitting on.

The remaining game tree starting from a *singleton* information set is called a **subgame** of the original game. In the above game tree, only the root of the tree is a singleton information set, and hence the only subgame in this game tree is the original game itself. If we modify that game tree by removing the dotted line connecting player 2's two decision nodes, then we obtain two more singleton information sets, and hence the game would contain three subgames (including the original game) following the modification.

From each player's perspective, a feasible pure strategy must prescribe one feasible action at each information set where the player is called upon to make a move. Thus a pure strategy can be very complex when a player must move more than once in a sequential game. Again, the set of feasible pure strategies for a player is then called the player's strategy space, and a mixed strategy for the player is simply a probability distribution over the player's strategy space.

A **subgame-perfect Nash equilibrium** (SPNE) is an NE for a game in extensive form, which specifies NE strategies in each and every subgame. The following example clarifies the difference between an NE and an SPNE.

8. (**Difference between NE and SPNE**). M is the owner-manager of a firm which is protected by limited liability against its creditor(s). The debt due one year from now has a face value equal to \$10. There is a single debtholder, referred to as C. The total assets in place are worth only \$8 in one year. Just now, a new investment opportunity with NPV=$x > 1 + e > 1$ became available, which requires that M make an unobservable effort but no addition investment. Making the effort would incur a disutility $e > 0$ to M. M has told C that he will make the effort for the new investment project only if C agrees to reduce the face value of debt by \$1. The extensive game proceeds as follows. First C can accept (A) or reject (R) M's request. Then, M can choose to (I) or not to (N) make the effort. Both M and C are risk-neutral without time preferences.
(i) Suppose $x > 2 + e$. Show that there is an NE in which the creditor agrees to reduce the face value of debt and M makes the investment.
(ii) Show that the NE in (i) is not an SPNE. Find an SPNE.
(iii) How may your conclusion about (ii) change if $x \in (1 + e, 2 + e]$?
(iv) Define bankruptcy as a state where the firm's equity value drops to zero. Explain why bankruptcy does not take place in (iii).

Solution. Note that M can choose one action following A and another action following R. Hence C has 2 pure strategies, A and R, but M has 4 pure strategies

$$\begin{pmatrix} A \to I \\ R \to I \end{pmatrix},$$

$$\begin{pmatrix} A \to N \\ R \to N \end{pmatrix},$$

$$\begin{pmatrix} A \to I \\ R \to N \end{pmatrix},$$

and

$$\begin{pmatrix} A \to N \\ R \to I \end{pmatrix}.$$

The normal-form bimatrix is as follows.

M/C	A	R
$\begin{pmatrix} A \to I \\ R \to I \end{pmatrix}$	$(x-1-e, 9)$	$(\max(x-2,0)-e, \min(8+x,10))$
$\begin{pmatrix} A \to N \\ R \to N \end{pmatrix}$	$(0,8)$	$(0,8)$
$\begin{pmatrix} A \to I \\ R \to N \end{pmatrix}$	$(x-1-e, 9)$	$(0,8)$
$\begin{pmatrix} A \to N \\ R \to I \end{pmatrix}$	$(0,8)$	$(\max(x-2,0)-e, \min(8+x,10))$

In part (i), the strategy profile

$$(\begin{pmatrix} A \to I \\ R \to N \end{pmatrix}, A)$$

is indeed a pure strategy Nash equilibrium. However, it is not an SPNE: given that C has chosen R, M would be better off choosing I over N. Things are different in part (iii), where the above strategy profile becomes an SPNE.

9. (**Repeated Game and a Folk Theorem.**) Consider the following strategic game $G(1)$, which is referred to as the Prisoner's Dilemma:

player 1/player 2	C	D
C	1,1	-3,2
D	2,-3	0,0

This game has a unique NE, where both players choose to play D. Note that the outcome (D,D) is Pareto dominated by the outcome (C,C). Suppose that $G(1)$ is repeated for an infinite number of times, and let us call this dynamic game $G(\infty)$. A player's payoff in $G(\infty)$ is $\sum_{k=1}^{\infty} \rho^{k-1} u(k)$, where $u(k)$ is the player's payoff from playing $G(1)$ for the k-th time in $G(\infty)$, and $\rho \in (0,1)$ is a constant representing the player's discount factor.

Our main point here is that when the two players get to play $G(1)$ for an infinite number of times, then it is possible that in an SPNE the two players play (C,C) in each and every stage of $G(\infty)$, as long as ρ is sufficiently close to 1. Indeed, imagine that they will continue to play (C,C) until one of them has played D in the past, and after the latter event, they then play the NE (D,D) in G(1) forever (the latter is referred to as the trigger strategy). It can be verified easily that this does constitute an NE for $G(\infty)$, and since it specifies NE strategies for the two players in each and every subgme in $G(\infty)$, it is not only an NE, but also an SPNE!

To get a lower bound for ρ so that (C,C) can be sustained in an SPNE, let π denote a player's equilibrium payoff $\sum_{k=1}^{\infty} \rho^{k-1} u(k)$. Then, we have

$$\pi = 1 + \rho\pi \Rightarrow \pi = \frac{1}{1-\rho}.$$

In order that a player does not wish to deviate, we need

$$\pi > 2 + \frac{\rho \cdot 0}{1-\rho} \Rightarrow \rho \geq \frac{1}{2}.$$

The idea is that, as long as ρ is big enough, the two players will take future retaliation into serious account when they are considering unilateral deviations from the tacit agreement that they should play (C,C) in each and every stage. This idea has been formalized and given the name *Folk Theorem* by game theorists, and the folk theorem will be useful for us to understand Dinc (2000).

10. The preceding section has considered infinitely repeated games. Now, let us consider finitely repeated games. Call the game in Example 1 $G(1)$, and consider $G(2)$ and $G(3)$, where for simplicity let us assume that $\rho = 1$. It can be verified that (U,L) cannot be sustained as an equilibrium outcome in any stage of $G(2)$, but it can appear in the first-stage equilibrium of $G(3)$. Indeed, suppose that in an SPNE of G(3), the two players will play (U,L) in stage 1, and then (U,R) in stage 2, and then (D,L) in stage 3. If instead (U,R) was played in stage 1, then (D,L) would be played in the remaining 2 stages; and if instead (D,L) was played in stage 1, then (U,R) would be played in the remaining 2 stages. It can be verified that neither player 1 nor player 2 wishes to deviate in the first stage from playing (U,L). The lesson here is that, in order for the threat of future retaliation to influence the two players' current behavior, the two players must expect to interact in the future for a sufficiently large number of times.

11. (**Static Game with Incomplete Information**.) In the above we have assumed that players know the payoff functions of each other. Such a game is a game with *complete (or symmetric) information*. What if at least one player in the game does not know for sure another player's payoff function? We call it a game with *information asymmetry*, or a game with *incomplete information*, or simply a *Bayesian game*. In a Bayesian game, at least one player Z has more than one possible payoff function. We say that this player Z has more than one *type*. At least one other player W cannot be sure which type player Z has. In this case, we shall look for an equilibrium called **Bayesian equilibrium (BE)**. This is nothing but a Nash equilibrium of an enlarged version of the original game, where each different type of Z is treated as a distinct player.

    To give a concrete example, suppose that we have a two-player game where player 1's strategy space is X and player 2's strategy space is Y, and player 2 has two possible types (or two possible payoff functions), θ_1 and θ_2, which, from player 1's perspective, may occur with probabilities π_1 and π_2 respectively. Certainly, player 2 knows his own type. For all $x \in X$, $y \in Y$, and $\theta \in \{\theta_1, \theta_2\}$, let $u_1(x, y)$ be player 1's payoff, and $u_2(x, y; \theta)$ the type-θ player 2's payoff. (This is referred to as a private-value model. If u_1 also depends on θ, then this is a common-value model.) A Bayesian equilibrium for this two-player game is nothing but the Nash equilibrium of the three-player game where the two types of player 2 are treated as two different players. Thus a Bayesian equilibrium is a triple (x^*, y_1^*, y_2^*) such that $x^* \in X$, $y_1^* \in Y$, $y_2^* \in Y$, and the following three *incentive compatibility* conditions hold:

$$\pi_1 u_1(x^*, y_1^*) + \pi_2 u_1(x^*, y_2^*) \geq \pi_1 u_1(x, y_1^*) + \pi_2 u_1(x, y_2^*), \quad \forall x \in X; \tag{12}$$

$$u_2(x^*, y_1^*; \theta_1) \geq u_2(x^*, y; \theta_1), \quad \forall y \in Y; \tag{13}$$

$$u_2(x^*, y_2^*; \theta_2) \geq u_2(x^*, y; \theta_2), \quad \forall y \in Y. \tag{14}$$

In words, x^* is player 1's best response, which is *on average* the optimal strategic choice of player 1. It is not really player 1's best response against player 2 if player 1 is sure that player 2 will use y_1^*. Neither is it player 1's best response against player 2 if player 1 is sure that player 2 will use y_2^*. Since player 1 can only choose one x in X to play against two possible types of player 2, given his conjecture of (y_1^*, y_2^*), the choice x^* must be *on average optimal*. On the other hand, player 2 knows his own type, and his best response against player 1's average optimal choice x^* depends on his type. Note that θ denotes player 2's type, and it determines $u_2(x, y)$! This is why we say that incomplete information in this game is equivalent to player 1 not knowing player 2's payoff function. Again, player 1's average optimal choice x^*, player 2's best response y_1^* when his type is θ_1, and player 2's best response y_2^* when his type is θ_2, must altogether form a Nash equilibrium. This three-player Nash equilibrium is what we defined as the *Bayesian equilibrium*.

12. **Example 4**. (**How to solve for a BE?**) Let us modify Example 2 by assuming a random demand curve. More precisely, let the inverse demand curve be

$$P(q_1 + q_2) = \tilde{a} - q_1 - q_2, \tag{15}$$

where only firm 1 knows the outcome of the random variable $\tilde{a}$. Firm 2 only knows that $\tilde{a}$ may be 2 with prob. $\frac{1}{3}$ or 4 with prob. $\frac{2}{3}$. Find a BE.

Solution. First observe that firm 1 has two possible types. Firm 2 has only one type (no private information). So, we should consider a three-firm game, where the two types of firm 1 will be regarded as two different players, and then we look for the NE of the new 3-player game. By definition, we must find three strategies $q_1^*(2)$, $q_1^*(4)$, and q_2^*. These 3 strategies are such that, given any two of them, the third one is the corresponding player's best response!

In other words, for firm 2,

$$q_2^* = arg\max_{q_2} \frac{1}{3}[q_2(2 - q_1^*(2) - q_2)] + \frac{2}{3}[q_2(4 - q_1^*(4) - q_2)].$$

Similarly, for the firm 1 that has seen $\tilde{a} = 2$,

$$q_1^*(2) = arg\max_{q_1} q_1(2 - q_1 - q_2^*);$$

and for the firm 1 that has seen $\tilde{a} = 4$,

$$q_1^*(4) = arg\max_{q_1} q_1(4 - q_1 - q_2^*).$$

Each of the above three maximization problems is concave, and so the 3 first-order conditions are necessary and sufficient. Each first-order condition gives a reaction function for one of the player. Again, the NE must appear at the intersection of the 3 reaction functions. Solving, the NE of the 3-player game, or the BE of the original game with information asymmetry, is the following:

$$(q_1^*(2), q_1^*(4), q_2^*) = (\frac{4}{9}, \frac{13}{9}, \frac{10}{9}).$$

13. **Example 5**. Two players, 1 and 2, are playing the following Bayesian game. Player 1 knows which normal form game he is playing, but player 2 thinks that both normal forms are equally likely. Find a BE.

1/2	a	b
A	0,0	1,2
B	2,1	0,0

1/2	a	b
A	0,0	0,0
B	2,1	0,0

Solution Let us look for pure strategy BE's first. Call the player 1 knowing for sure that he is playing the first normal form game the "type-1" player 1. Similarly, call the player 1 knowing that he is playing the second normal form game the "type-2" player 1.

First we ask, "Is there a BE where player 2 plays a with probability one?" Suppose that such a BE exists. Then in the BE, player 2 plays a, and given a, it can be easily verified that player 1's best response is B regardless of his type. On the other hand, given that both types of player 1 will play B, it can be easily verified that a is indeed player 2's best response. Thus such a BE does exist, where both types of player 1 play B and player 2 plays a.

Next we ask, "Is there a BE where player 2 plays b with probability one?" If such a BE exists, then in equilibrium player 1 plays A if he is of type 1 and he feels indifferent about A and B if he is of type 2. One can check that given the two types of player 1's strategies, playing b is indeed a best response for player 2. Thus such a BE also exists, where player 2 plays b, the type-1 player 1 plays A, and the type-2 randomizes in any way over A and B.

Finally, let us determine if there are BE's where player 2 randomizes over a and b. Suppose that player 2 plays a with prob. $\pi \in (0, 1)$. Then the type-2 player 1's best response is B for sure, but the type-1 player 1's best response is A if $\pi < \frac{1}{3}$; B if $\pi > \frac{1}{3}$; and A and B if $\pi = \frac{1}{3}$. On the other hand, player 2 will not feel indifferent about a and b unless the type-1 player 1 also randomizes over A and B. Let η be the prob. that the type-1 player 1 chooses A. It can be easily shown that $\pi = \frac{1}{3}$ and $\eta = \frac{2}{3}$ together with the type-2 player 1's playing B constitutes the unique BE in this remaining case.

14. Now we give a formal defintion of Bayesian game.
(**Common Knowledge and Private Information.**) Given a game, an event is *mutual knowledge* if every player knows it. Given a game, an event is called the players' *common knowledge* if every player knows it, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it, and so on. (When there is uncertainty, then an event may be mutual or common knowledge in one state but not in another state, and hence we must specify the state of nature when we talk about mutual knowledge or common knowledge.) Anything

which is not common knowledge is some player's *private information*. A player's private information is also called his *type*. A game where no players have private information (everything relevant is common knowledge) is *a game with complete information*. Otherwise, the game is one with *incomplete information*, or one with *information asymmetry*.

15. **Dynamic Games with Incomplete Information.** An incomplete-information game where the uninformed players can move after observing the informed players' moves is called a *dynamic game*, and otherwise a *static game*.

16. **Signaling Game.** A dynamic Bayesian game is called a *signalling game* if there are only two players, informed and uninformed, and the uninformed moves after seeing the informed's move, and the game ends once the uninformed makes his move.

17. (**The Signaling Game of Beer and Quiche**.) Two cowboys A and B meet in a bar, and A may be weak (w) or strong (s), which is A's private information. The game proceeds as follows. A first decides to order either a beer (b) or a quiche (q), and upon observing A's order, B decides to or not to fight A. We assume that in the absence of B, A prefers beer (b) to quiche (q) if he is (s), otherwise he prefers (q) to (b). The prior beliefs of B are such that A is (s) with probability 0.9. Now the payoffs: if A orders and eats something he dislikes, he gets 0, or else he gets 1, if B does not fight A, A gets an additional payoff of 2. On the other hand, B gets 1 if he has no chance to fight, gets 2 if he fights A and A is of the weak type, and gets zero if he fights A and A is of the strong type.

    This game has two pooling PBE's:
    (1) Equilibrium (B): Both types of A order a beer and B's strategy is to fight A if and only if he sees A order a quiche. What are the supporting beliefs? Let $f(s)$ =pro.(A is strong| A orders s), for all $s \in \{b, q\}$. Then of course $f(b) = 0.9$. Note that $s = q$ is a zero probability event. Recall that Bayes Law says

    $$P(E|F)P(F) = P(E \bigcap F),$$

where E and F are two random events. From the probability theory, we know that for any two events C and D,

$$C \subset D \Rightarrow P(C) \leq P(D).$$

Thus we have

$$P(F) = 0 \Rightarrow P(E \bigcap F) = 0,$$

since $E \bigcap F \subset F$. Thus given $P(F) = 0$, Bayes Law requires

$$P(E|F) \cdot 0 = 0,$$

and hence $P(E|F)$ can be anything contained in $[0, 1]$. Let E be the event that A is of the strong type, and F the event that B has observed that A ordered a quiche. We conclude that any $f(q) \in [0, 1]$ will be consistent with Bayes Law in this case. We must find at least <u>one</u> $f(q) \in [0, 1]$ so that the above strategy profile does constitute the two players' best responses against each other. Note that for B to fight A after seeing A order a quiche, it is necessary that

$$1 \leq f(q) \cdot 0 + [1 - f(q)] \cdot 2 \Rightarrow f(q) \leq \frac{1}{2}.$$

Now we show that given the beliefs $f(b) = 0.9$ and $f(q)$ being anything in $[0, \frac{1}{2}]$, the aforementioned A's and B's strategies are respectively the two players best responses. For A, if his type is (s), he gets $1 + 2 = 3$ if he orders a beer, and if he deviates and orders a quiche, then not only he eats something he hates but he also must fight B, yielding a payoff of $0 + 0 = 0$. Thus A will not deviate if he is of type (s). What if A is of type (w)? If he orders a beer, then he must eat something he hates, but the good news is that he can avoid fighting B, so that his payoff is $0 + 2 = 2$; and if he deviates and orders a quiche, then he will have to fight B, so that his payoff is $1 + 0 = 1$. We conclude that the weak type of A does not want to deviate either. What about B? We have shown that given $f(q) \leq \frac{1}{2}$, fighting A if A dares to order the quiche is really optimal for B. On the other hand, if A orders a beer, then since B expects both types of A to do so, ordering the beer really does not tell B anything new, and B's posterior beliefs are identical to his prior beliefs (A is of the strong type with prob. 0.9), and so not to fight A

is optimal for B.
To sum up, we have shown that the following is a PBE (check if it corresponds to our definition of a PBE!):
(i) The strong type of A orders a beer;
(ii) The weak type of A also orders a beer;
(iii) B's strategy must describe what he will do in every possible contingency: B will fight A if A ordered a quiche, and B will not fight A if A ordered a beer;
(iv) The supporting beliefs fully describe what B thinks of A *in every possible contingency*: B thinks that A is of the strong type with prob. 0.9 if he sees A order a beer; and B thinks that A is of the strong type with prob. $f(q)$ if he sees A order a quiche, where $f(q)$ is any real number contained in $[0, \frac{1}{2}]$.
(2) Equilibrium (Q): Both types of A order a quiche and B's strategy is to fight A if and only if he sees A order a beer. What are the supporting beliefs? Let $f(s)$ =pro.(A is strong| A orders s), for all $s \in \{b, q\}$. Then of course $f(q) = 0.9$. Now for $f(b)$ to induce B to fight A after seeing A order a beer, it is necessary that

$$1 \leq f(b) \cdot 0 + [1 - f(b)] \cdot 2 \Rightarrow f(b) \leq \frac{1}{2}.$$

Now we show that given the beliefs $f(q) = 0.9$ and $f(b)$ being anything in $[0, \frac{1}{2}]$, the aforementioned A's and B's strategies are respectively the two players best responses. For A, if his type is (w), he gets $1+2 = 3$ if he orders a quiche, and if he deviates and orders a beer, then not only he eats something he hates but he also must fight B, yielding a payoff of $0 + 0 = 0$. Thus A will not deviate if he is of type (w). What if A is of type (s)? If he orders a quiche, then he must eat something he hates, but the good news is that he can avoid fighting with B, so that his payoff is $0+2 = 2$; and if he deviates and orders a beer, then he will have to fight B, so that his payoff is $1 + 0 = 1$. We conclude that the strong type of A does not want to deviate either. What about B? We have shown that given $f(b) \leq \frac{1}{2}$, fighting A if A dares to order the beer is really optimal for B. On the other hand, if A orders a quiche, then since B expects both types of A to do so in equilibrium, ordering the quiche really does not tell B anything new about A, and B's posterior beliefs are identical to his prior beliefs (A is of the strong type with

prob. 0.9), and so not to fight A is optimal for B.
To sum up, we have shown that the following is a PBE (check if it corresponds to our definition of a PBE!):
(i) The strong type of A orders a quiche;
(ii) The weak type of A also orders a quiche;
(iii) B's strategy must describe what he will do in every possible contingency: B will fight A if A ordered a beer, and B will not fight A if A ordered a quiche;
(iv) The supporting beliefs fully describe what B thinks of A *in every possible contingency*: B thinks that A is of the strong type with prob. 0.9 if he sees A order a quiche; and B thinks that A is of the strong type with prob. $f(b)$ if he sees A order a beer, where $f(b)$ is any real number contained in $[0, \frac{1}{2}]$.

18. According to Cho and Kreps, some PBEs may involve *implausible* supporting beliefs and should be disregarded. To demonstrate Kreps' idea, consider Equilibrium (Q) in the previous section. There, B knows that ex-ante A may be strong with prob. 0.9 and A hates the quiche if he is strong, and yet B still thinks that A is more likely to be the weak type when he sees A deviate by ordering the beer. Consider the speech that the strong-type of A would have made to B if he were allowed to: I am having beer, so I am the strong type. To see this, note that if I were the weak type, I would have got 3 by having the quiche, and a weak type could never get a payoff of 3 by having beer, which is so *no matter how you may respond after the beer is ordered*! Moreover, if this speech can convince you that I am strong, then I expect you to not fight me, so that, as a strong type, I have the beer that I like and I do not have to fight you. In fact, I expect to get 3 if this communication works, and that is why I am having beer.....

    These two suppositions
    (i) the weak type of A is absolutely better off by not deviating; and
    (ii) if supposition (i) is accepted then the strong type of A is expected to be treated in a better way by B that justifies the strong type's deviation in the first place,
    comprises the so-called *intuitive criterion*.

19. **Cho-Kreps Equilibrium**. Those PBE's survive the intuitive criterion

are called **intuitive equilibria**. (A formal treatment will be given below when we introduce the *divine equilibria*.)

20. We now show that in the game of beer and quiche, only Equilibrium (B) is intuitive. By definition, a PBE is intuitive, if either (i) we cannot find a deviation which certain types of the informed player would never make; or (ii) we can find a deviation which certain types of the informed player would never make, but by restricting to the supporting beliefs that assign zero prob. to these types, we still cannot find a type of the informed player that strictly prefers to deviate.
    Take equilibrium (Q) for example. The equilibrium, by assumption, involves both types of A playing the strategy (q), but as we stated above, by having beer, the weak type of A can at best get the payoff 2 (which occurs if B decides not to fight A following an order of beer). Thus the weak type of A strictly prefer his equilibrium action to strategy (b). Now all reasonable beliefs should assign zero prob.'s to the weak type of A following an order of beer from A. That means that there is only one reasonable belief, the belief that assigns prob. 1 to the strong type of A after beer is ordered. Given this belief, B is expected to behave optimally, which is not to fight A. But then the strong type of A can get the payoff 3 by deviating from (q), while he gets 2 by ordering quiche. Thus the strong type of A would strictly prefer to deviate from (q), proving that equilibrium (Q) is not intuitive. This shows that both of the suppositions defined above hold for this equilibrium, so that this PBE fails the intuitive criterion, and it is not an intuitive equilibrium.

21. Next let us ask if equilibrium (B) is intuitive. Having observed the deviation (q), can we conclude that at least one type of A would never have done this? Apparently, the strong type has obtained a payoff of 3 on the equilibrium path, and by deviating and ordering (q), he could get no more than 2. Thus the strong type of A strictly prefers his equilibrium payoff to what he could get by deviating and ordering quiche. In this case, any reasonable beliefs after quiche is ordered should assign prob. 1 to the weak type of A. Now what is the best response of B given this reasonable belief? Of course B should fight A! But then, even the weak type of A could not gain by deviating from (b) to order (q)! To sum up, Cho-Kreps' first supposition holds but

the second supposition fails for this PBE, and hence this PBE survives from Cho-Kreps' intuitive criterion. Thus this PBE is an intuitive equilibrium.

22. **Classification of games.** So far, we have been able to classify games into 4 groups according to whether they are one-shot (static) or dynamic games, and whether there are privately informed players in the games. The following table summarizes the appropriate equilibrium concepts:

information/time horizon	static	dynamic
complete	NE	SPNE
incomplete	BE	PBE

23. Let us now practice PBE and intuitive equilibrium. In each game that follows, player 1 has two equally likely types, denoted by t_1 and t_2, and given his type, player 1 must send a signal. There are three possible signals that player 1 can choose, which are m_1, m_2, and m_3. Upon seeing the signal selected by player 1, player 2 must then form a posterior belief about player 1's type, and given her belief, player 2 must take an action. There are three feasible actions for player 2, which are a_1, a_2, or a_3. The game ends after player 2 chooses her action.

    Each signalling game below is depicted by three tables. The k-th table gives the two players' payoffs in the event that player 1 chooses to send signal m_k; $k = 1, 2, 3$. As you can see, player 1's payoff not only depends on the two players' actions, it also depends on player 1's type. For example, in the first table appearing in Problem 1 below, player 1 gets 2 and player 2 gets 1 if player 1 is of type t_1 and he sends signal m_1, and player 2 responds by taking action a_1; and in the second table, player 1 gets 0 and player 2 gets 6 if player 1 is of type t_2 and he sends signal m_2, and player 2 responds by taking action a_3.[2]

[2]Recall that a PBE is defined as:
(A) *a strategy for player 1*, which specifies one m_k for each type t_i;
(B) *a strategy for player 2*, which specifies one a_j for each m_k; and
(C) *a posterior belief for player 2*, which specifies one probability distribution on the set

(a) Find the PBEs:

m_1	a_1	a_2	a_3
t_1	$(2,1)$	$(2,0)$	$(0,2)$
t_2	$(1,3)$	$(2,0)$	$(2,1)$

m_2	a_1	a_2	a_3
t_1	$(3,1)$	$(1,0)$	$(0,0)$
t_2	$(2,1)$	$(0,0)$	$(0,6)$

m_3	a_1	a_2	a_3
t_1	$(1,2)$	$(1,1)$	$(3,0)$
t_2	$(0,2)$	$(3,1)$	$(1,1)$

(b) Find the PBEs:

m_1	a_1	a_2	a_3
t_1	$(1,2)$	$(2,2)$	$(0,3)$
t_2	$(2,2)$	$(1,4)$	$(3,2)$

m_2	a_1	a_2	a_3
t_1	$(1,2)$	$(1,1)$	$(2,1)$
t_2	$(2,2)$	$(0,4)$	$(3,1)$

m_3	a_1	a_2	a_3
t_1	$(3,1)$	$(0,0)$	$(2,1)$
t_2	$(2,2)$	$(0,0)$	$(2,1)$

(c) Find intuitive equilibria:

m_1	a_1	a_2	a_3
t_1	$(0,3)$	$(2,2)$	$(2,1)$
t_2	$(1,0)$	$(3,2)$	$(2,1)$

$\{t_1, t_2\}$ for each m_k, and moreover, (A), (B) and (C) must also satisfy:
(1) the a_j specified in (B) after player 2 sees m_k must be expected-utility-maximizing for player 2 given player 2's posterior belief specified in (C); and
(2) the m_k specified in (A) for type t_i must be expected-utility-maximizing for player 1 of type t_i, given that player 2's strategy is specified in (B).
This is a complicated definition. However, a definition is a definition. So, when you report that a PBE is found, you must make sure that you report (A),(B) and (C), and you must also verify that (A), (B) and (C) satisfy (1) and (2)!

m_2	a_1	a_2	a_3
t_1	$(1,2)$	$(2,1)$	$(3,0)$
t_2	$(0,1)$	$(3,1)$	$(2,6)$

m_3	a_1	a_2	a_3
t_1	$(1,6)$	$(4,1)$	$(2,0)$
t_2	$(0,0)$	$(4,1)$	$(0,6)$

24. Now we consider a sequence of applications in corporate finance and security trading.

25. (**Seasoned Equity Offering**.) Firm A has a single owner-manager Mr. A, who needs to raise \$100 in order to take a positive NPV project. There are two possible states, called G and B. In state G, the assets in place of firm A worth \$150 and the new project's NPV equals \$20. In state B, the assets in place worth only \$50 and the NPV is accordingly \$10. The state is Mr. A's private information. The public investors think that the state is G with prob. a, and they are Bertrand competitive. The game proceeds as follows. The firm chooses to or not to issue new equity. If new equity is issued, the public investors ask for a proportion of ownership with value equal to \$100. Find all the pure strategy PBE's of this signaling game.
Solution. We first look for separating equilibria. Suppose there were a separating PBE where only type G issues new equity. Then in exchange of the \$100 raised, the outsiders ask for a share $\alpha = \frac{100}{150+20+100}$ of the ownership. The type B firm will deviate: By issuing, the insider gets
$$(1 - \frac{100}{150+20+100})(50+10+100) = 100.74,$$
which is greater than 50, the value of type B firm if passing on the new project. Hence, there is no such separating equilibrium. Now, suppose there were a separating PBE where only type B issues new equity. Then, the public investors would ask for a share of ownership equal to $\frac{100}{10+50+100}$. Type B insider would indeed want to issue: By issuing, he gets
$$(1 - \frac{100}{10+50+100})(100+50+10) = 60,$$

greater than 50. On the other hand, type G insider would not issue if and only if

$$(1 - \frac{100}{10 + 50 + 100})(100 + 150 + 20) = 101.25 < 150,$$

which indeed is true. Thus this separating equilibrium does exist.

Next, we look for pooling equilibria. Suppose that in equilibrium neither type issues new equity. But then type B wants to deviate: By issuing, type B cannot do worse than being identified, but even in that case, issuing is preferred to not issuing. Therefore there is no such pooling equilibrium. Suppose now that in equilibrium both types issue. The fair share of ownership that outsiders would ask for (assuming risk neutrality of outsiders), α, solves

$$\alpha[a(150 + 20 + 100) + (1 - a)(50 + 10 + 100)] = 100,$$

and hence

$$\alpha = \frac{100}{160 + 110a}.$$

Type G insider must be willing to issue in equilibrium:

$$(1 - \frac{100}{160 + 110a})(100 + 150 + 20) > 150;$$

and so must type B insider:

$$(1 - \frac{100}{160 + 110a})(100 + 50 + 10) > 50.$$

(Note that the outsiders' beliefs following the off-equilibrium signal "not issuing" is irrelevant here.) Thus the pooling equilibrium exists if $a > \frac{13}{22}$. In this game, both pure strategy PBE's are robust against Cho and Kreps' intuitive criterion.

Remark. This example shows several things. First, financing does interfere with investment. If the firm is financed by retained earnings (internal equity), then it will accept the new project for sure. External financing involves the problem of adverse selection, and the high-quality firm is hurt by the presence of low-quality firm. In equilibrium, issuing new equity either has no information content about firm quality, or it is interpreted as bad news. Second, note that in a separating equilibrium, the efficiency is not the same as the full-information equilibrium. Although information is symmetrized in a separating equilibrium, the high-quality firm has to go through a painful signaling process, so painful that it may have to give up the new investment opportunity completely.

26. (**Share Repurchase.**) Firm YGO is all-equity financed, and it has three shareholders, the manager M and two outside investors O and Z, each holding one share of the equity. All investors are risk neutral without time preference.

    It is common knowledge that the firm has 20-dollar cash at date 0, and the firm will generate cash $\tilde{x}$ at date 1, which is either 40 dollars (with probability π) or 25 dollars. At date 0, M privately learns the realization of $\tilde{x}$.

    At date 0, O and Z both know that M is considering buying back one share from O and Z. The game proceeds as follows. First, M decides to or not to repurchase 1 share from either O or Z (signaling with two possible signals!). If M announces the share repurchase program, then O and Z compete in price to tender one share to M. Competition between O and Z ensures that the stock price P equals one-third of the expected value of the firm (expectation is taken using the information revealed by M's announcement). Clearly, in any PBE, $P \leq 20$ (why?).
    (i) Show that before this signaling game gets started at date 0, the stock price is $P_{-1} = 15 + 5\pi$.
    (ii) Show that this game has a *unique* separating PBE. Find the date-0 stock price $P_0(j)$ in the separating equilibrium, where $j = r$ (denoting the event that M announces a share repurchase program) or $j = nr$ (denoting the event that M does not announce the share repurchase program). Show that $P_0(r) > P_{-1} > P_0(nr)$.
    (iii) Show that this game has a *unique* pooling PBE where M does not repurchase shares regardless of M's type.
    (iv) Explain why $P_{-1} > P_0(nr)$, when apparently no public information arrives at the stock market at date 0. (The finance literature has raised the question why no bad news was found to arrive at the U.S. stock market before the 1987 crash.)

Solution.
Since the total cash at date 1 cannot exceed 20+40=60, we know that $P \leq 20$. Consider part (i). At date 0, when all three investors are equally uninformed, the stock price is

$$\frac{1}{3}[\pi(20+40)+(1-\pi)(20+25)] = 15+5\pi.$$

Consider part (ii). First consider the separating PBE where the type-40 firm announces share repurchase but the type-25 firm chooses not to. In such a PBE, announcing share repurchase is taken by O and Z as direct evidence that the firm is of type 40, and hence the share price is 20. Since the share is fairly priced for the type-40 M, he has no incentive not to announce share repurchase: paying 20 to either O and Z, and dividing the date-1 cash flow 40 between the remaining two shares, M will get 20, which is also his payoff if he just sits back and does nothing. Will the type-25 M want to announce share repurchase also? If he does not, then he waits to get his share of cash flow at date 1, which is $\frac{1}{3}[20+25] = 15$; but if he announces share repurchase and gets accepted by either O or Z, then he gets

$$\frac{1}{2}[20+25-20] = 12.5 < 15.$$

Thus such a separating PBE does exist.

Next consider the other type of separating PBE. If share repurchase announcement is taken as evidence that the firm is of type 25, then the transaction price will be $\frac{1}{3}[20+25] = 15$, following O and Z's Bertrand competition. But then the type-40 firm has an incentive to deviate: M's payoff would become

$$\frac{1}{2}[20+40-15] = 22.5 > 20,$$

where 20 is what the type-40 M would get if M follows his supposed equilibrium strategy of announcing nothing. We conclude that the separating PBE of this game is unique.

Consider part (iii). Consider the pooling PBE where M never announces share repurchase. If deviation occurs, let μ be the probability that O and Z assign to the type-40 M. Then the transaction price following O and Z's Bertrand competition would be

$$P = \frac{1}{3}[60\mu + 45(1-\mu)] = 15 + 5\mu.$$

Apparently, the type-40 M would deviate unless $\mu = 1$. This defines a pooling PBE. Is there the other PBE where M always announces share repurchase? The answer is negative, for if otherwise the equilibrium transaction price will again be $15 + 5\pi < 20$, so that the type-40 M will deviate.

Finally, consider part (iv). When "trapped" in the above separating PBE, no news is taken as bad news by the stock market, and hence the date-0 stock price $P_0(nr)$ drops below P_{-1}. The investors (O and Z in our example) believe that the firm will repurchase shares if and only if M receives good news. Game theory and information economics have offered one useful explanation to the documented phenomenon that no bad news was known to arrive in the U.S. stock market during the week before the 1987 crash.

27. (**Screening Game.**) A monopolistic commercial bank is facing two types of borrowers: with probability π_i, the borrower is of type θ_i, and by investing q_i dollars today, that borrower will generate $\theta_i V(q_i)$ dollars (for sure) tomorrow.

    The bank's problem can be stated as

$$\text{(P)} \quad \max_{(q_i,T_i),\ i=1,2,\cdots,n} \sum_{i=1}^{n} \pi_i[T_i - cq_i]$$

    subject to

$$\begin{cases} \text{(IC)} & \forall i,j \in \{1,2\}, \quad \theta_i V(q_i) - T_i \geq \theta_i V(q_j) - T_j; \\ \text{(IR)} & \forall i \in \{1,2\}, \quad \theta_i V(q_i) - T_i \geq 0. \end{cases}$$

    In plain words, the bank will lend q_i dollars to the type-θ_i borrower, and require a repayment T_i tomorrow; we are assuming that nobody has a time preference.

It can be shown that, if

$$V'(0) > \frac{c}{\theta_i} > V'(+\infty) \equiv \lim_{y \uparrow +\infty} V'(y),$$

then the *socially efficient* level of lending q_i^* for the type-θ_i borrower, which maximizes $\theta_i V(q) - cq$, is such that

$$\theta_i V'(q_i^*) = c.$$

If the bank has full information about the borrower's parameter θ_i, then the bank's optimal (q_i^*, T_i^*), which is termed *the first-best scheme*, is such that $T_i^* = \theta_i V(q_i^*)$, so that the bank extracts all the surplus from the borrower. In this case, q_i^* must maximize the *social benefit* from serving the type-θ_i borrower, which is $\theta_i V(q) - cq$. It follows that $\theta_i V'(q_i^*) = c$ if

$$V'(0) > \frac{c}{\theta_i} > V'(+\infty);$$

and if instead $V'(0) \leq \frac{c}{\theta_i}$, then $q_i^* = T_i^* = 0$.

It is easy to show that under information asymmetry, the bank's optimal scheme, which is termed *the second-best scheme*, is such that $q_2^{**} = q_2^*$ but $q_1^{**} < q_1^*$. The former called the property of *efficiency at the top*. On the other hand, we have $\theta_1 V'(q_1^{**}) > c$, implying the bank lends too little to the type-θ_1 borrower.

Note that The θ_2-type borrower's surplus is

$$(\theta_2 - \theta_1) V(q_1^{**}),$$

which increases with the difference between θ_2 and θ_1 and with the amount borrowed by the θ_1-type borrower. This creates an incentive for the bank to reduce q_1 to below q_1^*. If $q_1^{**} = 0$, then the θ_2-type borrower's surplus is also zero. Note that the latter remains positive whenever $q_1^{**} > 0$.

Recall that the bank chooses to fulfill social efficiency in the full information case. Here, with information asymmetry, the bank in choosing $q_1^{**} > 0$ cannot exhaust the θ_2-type borrower's surplus, and this implies that the bank's producer surplus is less than the social benefit. From this perspective, it is not surprising that the bank wants to distort the lending scheme in the presence of information asymmetry.

There is another implication from the above analysis. Note that the type-θ_2 borrower has rent exactly because he has private information. Thus the borrower has an incentive to over-invest in activities that help maintain his information advantage.

28. Now we review the classic agency theories developed in Jensen and Meckling (1976) and Myers (1977).

29. (Jensen and Meckling, 1976, *JFE*) Suppose A is the owner-manager of a firm whose value is
$$V = 1 - L,$$
where $L \in [0,1]$ is A's on the job leisure. A has utility function $U(V,L) = V^{\frac{4}{5}} L^{\frac{1}{5}}$.
(i) Compute the optimal leisure for A. What is the corresponding value of the firm?
(ii) Now suppose A wants to sell $\frac{1}{3}$ of his ownership to outsiders. The game proceeds as follows. A first sells his partial ownership to outsiders in exchange for money M. Then, after the transaction, A chooses his leisure L. Assume that outside investors are competitive and have perfect foresight, so that M is exactly the worth of the partial ownership they obtain in equilibrium. What is the equilibrium value of the firm? Suppose there is no portfolio effect between ownership and money for A, determine if A should make this ownership transaction in the first place. What if there is a portfolio effect?
Solution. First, part (i). Recall the following consumption problem: with constants $a, b > 0$, $a + b = 1$, and $p_x, p_y, I > 0$ given, the solution to
$$\max_{x,y} U(x,y) = x^a y^b,$$
$$s.t.\ p_x x + p_y y \leq I$$
is simply
$$x^*(p_x, p_y, I) = \frac{aI}{p_x}, \quad y^*(p_x, p_y, I) = \frac{bI}{p_y}.$$

(The above utility function is called a Cobb-Douglas utility function.) Using this fact, we have for part (i),

$$V^* = \frac{4}{5}, \quad L^* = \frac{1}{5}.$$

That is, the firm value is $\frac{4}{5}$. In the following, we continue to denote the manager's monetary wealth by V. Consider part (ii). In the subgame where M has been given, the manager's problem is to choose L to maximize her utility. Let V be the manager's monetary wealth including the cash M. Then, the value of the firm will be

$$\frac{V-M}{\frac{2}{3}}.$$

Thus, the manager seeks to

$$\max_{V,L} V^{\frac{4}{5}} L^{\frac{1}{5}}$$

$$s.t.\ \frac{V-M}{\frac{2}{3}} = 1 - L.$$

Using the above result for the Cobb-Douglas utility function, we have

$$L^* = \frac{1}{5}(1 + \frac{3}{2}M).$$

Observe that two things happen here. First, the price of ownership (V) relative to leisure (L) has increased from 1 to $\frac{3}{2}$. Second, before choosing the optimal L, the manager has received M (as part of his V), which implies by the concavity of U in V that L has become more desirable than in part (i). Thus it is not surprising that $L^* > \frac{1}{5}$, where $\frac{1}{5}$ is the optimal leisure in part (i), and moreover, the difference $L^* - \frac{1}{5}$ increases with M and the fraction of ownership held by the outside investors.

Now, using backward induction, we can infer what M must be in equilibrium: With rational expectations, the M outsiders are willing to pay to the manager is exactly $\frac{1}{3}$ the value of the firm:

$$M = \frac{1}{3}(1 - L^*) = \frac{1}{3}(1 - \frac{1}{5}(1 + \frac{3}{2}M)).$$

Solving, we have

$$M^* = \frac{8}{33},$$

which implies that $L^* = \frac{3}{11}$, and the value of the firm becomes $\frac{8}{11}$ (which was originally $\frac{8}{10}$). How about the manager's utility in equilibrium? It is

$$(\frac{2}{3} \times \frac{8}{11} + \frac{8}{33})^{\frac{4}{5}}(\frac{3}{11})^{\frac{1}{5}} < (\frac{4}{5})^{\frac{4}{5}}(\frac{1}{5})^{\frac{1}{5}}.$$

This has assumed that there is no portfolio effect in the manager's utility function, and we conclude that in this case the manager will not sell the partial ownership willingly in the first place. On the other hand, if the manager considers cash different from ownership of the firm, then selling the ownership may still enhance her satisfaction. In other words, an owner-manager may be hit by a liquidity shock and must sell a fraction of equity to get cash. This however raises the following question, "why can't the owner-manager simply borrow some money, if after all getting some cash will resolve his problem?" A likely answer is that borrowing also creates agency costs; see the following two sections.

30. (Jensen and Meckling, 1976, *JFE*) At date 0, Mr. B is the owner-manager of a firm protected by limited liability. The firm is endowed with \$50 in cash. There are two mutually exclusive investment projects available to B at date 0. Alternative 1 is a riskless project which incurs an immediate \$100 cash outflow and generates \$105 at date 1. Alternative 2 is a risky project which incurs an immediate cash outflow of \$100 and generates a random cash inflow $\$\tilde{X}$, where $\tilde{X}$ has two equally likely outcomes, 0 and 180. Note that alternative 1 generates a positive NPV of \$5, but alternative 2 leads to an expected loss of \$10. Since taking alternative 1 is productively efficient, we assume that Mr. B will take alternative 1 whenever he feels indifferent about the two investment alternatives.

    Mr. B decides to come to Mr. C for a loan of \$50. The game proceeds as follows. B offers a debt contract with face value F to C, which C can either accept or reject. If C rejects the contract, no investment is made and both B and C get zero payoffs. If C accepts the contract, then B must choose between alternative 1 and alternative 2. After the investment decision is made, the state of nature is realized, and B and C get paid according to the debt contract. Find the subgame perfect Nash equilibrium of the game.

Solution. First consider the subgame where F is given and the loan is made (or else the game has ended). If B chooses alternative 1, she is sure that she will get $\max(0, 105 - F)$. If B chooses alternative 2, then her payoff is random: with prob. $\frac{1}{2}$, she gets 0; and with prob. $\frac{1}{2}$, she gets $\max(180 - F, 0)$. Thus, B chooses alternative 2 over alternative 1 if and only if

$$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \max(180 - F, 0) \geq \max(105 - F, 0). \tag{16}$$

The following table summarizes the investment behavior of B given different values of F:

F	alternative chosen
$\in [0, 30]$	1
$\in (30, 105]$	2
$\in (105, 180)$	2
$\in [180, +\infty)$	1

Now we consider the subgame where C must decide whether to accept B's debt contract. According to the above table, B would subsequently invest in alternative 1 if and only if $F \leq 30$ or $F \geq 180$, but C is sure to lose money if she accepts any offer with $F \leq 30$ or with B choosing alternative 2. Thus, C accepts B's offer if and only if $F \geq 180$. Now consider B's problem of making an offer to C. Given the above analysis, B can expect her offer to be accepted by C only if $F \geq 180$, but B would be better off giving up the new investment and keeping her 50 dollars at hand.

Our conclusion is that, in the unique subgame perfect Nash equilibrium of this game, B does not make any offers to C in the first place, and the game ends at the very beginning with the firm passing on the good investment opportunity (alternative 1).

This kind of shareholders' incentive problems is referred to as *risk shifting* or *asset substitution* in the finance literature. There are other kinds of incentive problems involving shareholders or creditors which we shall review later on. These incentive problems lead to investment inefficiencies and hence reductions in firm value.

Remarks. Implicitly assumed in the above extensive game is that B's investment decision cannot be *observed* by C, or it can be observed by C but cannot be *verified* by the contract enforer (usually the court of law). For if the choice of the investment alternative is both *observable and verifiable* (which will be referred to as *contractible*), then B can sign a contract with C saying that B will choose the riskless project, or else C can, say, break B's arms. This is called a *forcing contract*, which apprently removes the agency problem, as long as B cares enough about his arms. The problem is then, "Why can't C observe B's investment decision?" One may argue that, B, as the CEO, makes the decision in his office, and may not be observed by C. The problem is more delicate than that! Note that if ex-post cash flows are contractible, then by observing the cash flows C can prove whether B has invested the riskless project or not, and hence a forcing contract seems possible. (Of course, breaking somebody's arms may not be legal, and hence itself unenforceable; this could create a new problem: penalizing B in a monetary manner may not work as well as breaking arms, for B may not have enough money to implement a monetary penalty on him!) Therefore, it seems necessary to assume that the ex-post cash flows are not contractible. Alas! This is again not the end of the problem. One must then ask, "Why can't cash flows be observable?" Is it a reasonable assumption? As we shall see, a large body of literature in optimal design of financial contracts has assumed that cash flows can be costlessly observed only by the insiders of the firm (here, B). However, it has also been assumed that by spending some money, C may be able to verify the true cash flows. Of course this money, paid to an accountant for example, is a deadweight loss, and should be by all means avoided or minimized in an optimal contract, but allowing C an opportunity to verify is indeed a more reasonable assumption. The bottom line here is that, the above conclusion that external financing leads to the asset substitution problem actually stems from the somewhat arbitrary assumption that Mr. B can only use a standard debt contract when raising funds from outside investors. If B and C are rational, they should be able to use Pareto optimal contracts, and one of them is clearly equity contract.

31. (Myers, 1977, *JFE*) A growth firm may be more vulnerable to an agency problem (known as *debt overhang*) than a firm with no growth opportunities. The following is an example. A firm finances the date-0 cost $g > 0$ for its search for a valuable investment opportunity by borrowing, and it promises to repay the debtholder $F > 0$ at date 2. (The cost g can be viewed as an R&D expense.) It is known at date 0 that , some public information will arrive at date 1, which will reveal how much cash inflow the project will generate at date 2. Suppose that it is investors' common knowledge that the date-2 cash inflow is equally likely to be either 20 or 10. To generate that cash flow, an additional I dollars must be spent at date 1. However, The firm has no cash at date 1, and must issue junior debt at date 1 to raise the I dollars. Now, assume that competitive investors are all risk neutral without time preferences (recall that this implies that asset prices are all marginales). At date 1, if the state is that the date-2 cash inflow is C, then the new investor (debtholder) will get $\min(C - F, F')$ at date 2, where F' is the face value of junior debt. Thus the new investor will lend I to the firm if and only if $I \leq \min(C - F, F')$, and since $F' \geq I$, this equivalent to $C \geq F + I$. In case

$$20 > F + I > 10 > I,$$

the new investor will refuse to lend to the firm, if $C = 10$ at date 1. Since $I < 10$, this creates a deadweight loss, and is referred to as an *agency cost* pertaining to debt.

Thus, solving the SPNE of this game, we conclude that when $20 > 2g + I > 10 > I$, then in equilibrium, $F = 2g$, so that the date-0 firm value is $\frac{1}{2}(20 - I) > g$, which justifies the firm's inital R&D effort. Note that if the firm were to have enough cash earnings at date 1, the date-0 firm value would be $\frac{1}{2}(20 - I) + \frac{1}{2}(10 - I)$.

It is not surprising that the standard debt contract is Pareto suboptimal in this example. Let us derive a Pareto optimal financial contract for the initially raised g dollars, assuming more generally that $C = 20$ and 10 with probability π and $1 - \pi$ (in the above we have assumed that

$\pi = \frac{1}{2}$). Assume correspondingly

$$20 > I + \frac{g}{\pi} > I + g > 10 > I > 0. \quad (\Theta)$$

Before solving the optimal financial contract at date 0, let us first consider the equilibrium F associated with the (asserted suboptimal) standard debt contract written at date 0. Quickly deduce that $F \leq 20 - I$. (Why?) Similarly, we claim that $F \geq 10 - I$. Thus given F, the firm can raise I at date 1 if and only if $C = 20$. Rationally expecting this, the F can be obtained by solving the zero expected profit condition of the senior debtholder:

$$\pi F = g \Rightarrow F = \frac{g}{\pi}.$$

This result is consistent with assumption (Θ). Thus at date 0, the value of debt is exactly g, showing that trading financial assets yields zero NPV at date 0. The date-0 value of equity is then $\pi(20 - I)$. What happens at date 1? It depends on C. In case $C = 20$, then the date-1 equity value is $20 - F - I$, and the date-1 value of the senior (old) debt is F (the junior debt is fairly priced, and hence is worth I); and in case $C = 10$, then all securities are worthless.

Now we consider the Pareto optimal financial contracts at date 0. Such a contract must allow the firm to maximize its date-0 value (allowing the firm to adopt as many positive-NPV projects as possible) while allowing all investors to at least break even of the time financing is granted. Let $f(C)$ be the initial investor's payoff at date 2 when C is the date-2 cash inflow. We must look for $f(10)$ and $f(20)$ such that

$$10 - f(10) \geq I; \; 20 - f(20) \geq I;$$

$$0 \leq f(20) \leq 20 - I; \; 0 \leq f(10) \leq 10 - I.$$

We shall maintain the assumption that

$$0 < g \leq 20\pi + 10(1 - \pi) - I,$$

so that establishing the firm by spending g in the first place makes sense to the entrepreneur. Note that this assumption implies that

$$g - 10\pi \leq 10 - I.$$

It is easy to show that (i) if $g < 10\pi$, then

$$f(10) = 0,\ f(20) = \frac{g}{\pi};$$

and (ii) if $g \geq 10\pi$, then

$$f(10) = g - 10\pi,\ f(20) = 10 + g - 10\pi,$$

are optimal contracts. (There are other optimal contracts, all leading to the same date-0 value.) Compared to the long-term debt maturing at date 2, these contracts allow the date-2 repayments to the date-0 investor to be indexed by the net present value of the date-1 project. Recall that the above *debt overhang* problem occurs because the firm promises to repay the senior debtholder too much in the poor state $C = 10$. Thus, by indexing the face value of debt to the realization of C, the problem is solved; see a profound analysis based on this idea in Froot, K., D. Scharfstein, and J. Stein, 1989, LDC Debt: Forgiveness, Indexation, and Investment Incentives, Journal of Finance, 44, 1335-1350.

There is also a second resolution to the above debt overhang problem: at date 0, issuing a properly designed convertible bond instead of the straight bond. This will give the initial bondholder has an option to convert the senior debt into a fraction α of equity right before the firm tries to raise I at date 1.

How does this work? Note that the firm fails to raise I at date 1 if and only if the senior debt was not converted and the face value of the senior debt is $F > C - I$. In this event the senior debt will also be worthless, while by converting and holding a fraction α of the equity, the senior debtholder's payoff will be strictly positive: the new investor will be happy to lend I, as he will be the sole debtholder at date 2, and will hence be sure to get back the I dollars he invests at date 1. Thus investment efficiency is attained at date 1.[3]

[3] The point here is to make sure that the new investor holds the senior claim at date 2, if it is known that $C = 10$ at date 1. Thus, one even simpler solution is to issue outside equity at date 0. That is, in exchange of g dollars raised at date 0, the firm gives the initial investor a fraction $\frac{g}{20\pi+10(1-\pi)-I+g}$ of equity. Can you give a story that justifies the seniority of the financial claim issued at date 0?

It remains to compute the pair (F, α), which completely describes the CB issued at date 0. The zero expected profit condition requires only that

$$\pi F + (1 - \pi)\alpha(10 - I) = g,$$

and hence there is more than one solution. For the bondholder to optimally convert in the date-1 subgame where $C = 10$, we need $\alpha > 0$, so that $F < g$. If we do not want the initial investor to convert the bond in the state $C = 20$, then we should choose $F > \alpha(20 - I)$.

Finally, we must re-consider Myers' reasoning that leads to the debt overhang problem. At date 1, when $C = 10$, what prevents the equityholder (assuming there is only one) and the senior debtholder from renegotiating the inefficient debt contract? This is a legitimate question, for both of them will get zero if they choose to do nothing. On the other hand, imagine that the equityholder says to the senior debtholder that, "if you can just reduce the face value to $x < 20 - I$, then you know that you will receive $x > 0$ for sure at date 2 instead of getting nothing." Of course, any $x \leq 20 - I$ will do, and which $x \in [0, 20 - I]$ will actually prevail at date 1 must depend on the relative bargaining power between the equityholder and the senior debtholder, but as you can see, renegotiation *should* occur, as long as renegotiation is costless (Coase, 1937, *Economica*).

Can renegotiation be costly anyway? Imagine that the senior debt is a corporate bond diffusely held by a large number of small investors. Renegotiation can be costly, although the equityholder may have more bargaining power, in this case. On the other hand, if the initial investor is a commercial bank, then renegotiation may not be very costly, although the equityholder may not enjoy as much bargaining power as when he is faced with a large number of small creditors. Thus the type of the debt instrument (bond or bank loan) and the ownership structure of the debt (diffuse or concentrated) may both affect the possibility of ex-post renegotiation.

Notice that unlike issuing CB or outside equity, the outcome of ex-post renegotiation is not guaranteed. Both the equityholder and the initial investor must form expectations about how much they may respectively get in the stage of renegotiation, and based on these expectations, the terms of the initial debt can be determined at date 0 (via backward induction). Although we are assuming risk neutrality for everyone in this model, it is important to notice the risk involved in the ex-post renegotiation.