

Game Theory with Applications to Finance and Marketing

Lecture 1: Games with Complete Information, Part II

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1. In Part I, we have introduced such solution concepts as dominance equilibrium, Nash equilibrium, SPNE, backward induction, and forward induction. Now we shall briefly go over other relevant equilibrium concepts, such as strong equilibrium, coalition-proof equilibrium, rationalizable strategies, and correlated equilibrium. The remainder of this note will consider some complex applications of the above equilibrium concepts. We shall have a more formal examination of repeated games in a subsequent note.
2. Consider the following strategic game.

player 1/player 2	D	C
D	0,0	0,0
C	0,0	1,1

This game has two mixed strategy NE's. In view of Wilson's theorem (1971), this game is quite unusual. Note that (D,D) is an NE where players play weakly dominated strategies in equilibrium. This does not seem reasonable. To get rid of this type of NE's, Selten (1975) proposes the trembling-hand perfect equilibrium in normal form games, which is a refined notion of NE's, aiming at screening out better NE's. To see Selten's idea, note that the reason that (D,D) can become an NE is because players are sure that C will be played by the rival with zero probability. Therefore, if we consider only those strategy profiles which are limits of totally mixed strategy profiles, then (D,D) can be ruled out. Formally, let Σ^0 be the set of totally mixed strategy profiles,

and given any $\epsilon \in \mathfrak{R}_{++}$, $\sigma \in \Sigma^0$ is called an ϵ -perfect equilibrium if $\forall i \in \mathcal{I}, \forall s_i, s'_i \in S_i$,

$$u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) \leq \epsilon.$$

A trembling-hand perfect equilibrium is then a profile $\sigma \in \Sigma$ (which need not be totally mixed!) such that there exists a sequence $\{\epsilon_k; k \in \mathbf{Z}_+\}$ in \mathfrak{R}_{++} and a sequence $\{\sigma_k; k \in \mathbf{Z}_+\}$ in Σ^0 with (i) $\lim_{k \rightarrow \infty} \epsilon_k = 0$; (ii) σ_k is an ϵ_k -perfect equilibrium for all $k \in \mathbf{Z}_+$; and (iii) $\lim_{k \rightarrow \infty} \sigma_{i,k}(s_i) = \sigma_i(s_i), \forall i \in \mathcal{I}, \forall s_i \in S_i$. It can be shown that a trembling-hand perfect equilibrium must exist for a finite game, and the trembling-hand perfect equilibrium is itself an NE, but the reverse is not true.¹ In particular, the above profile (D,D) is not a trembling-hand perfect equilibrium.

3. Consider the extensive game with two players where player 1 first chooses between L and R, and the game ends with payoff profile (2, 2) if R is chosen, but if instead L is chosen, then player 2 can choose between l and r, with the game ending with payoff profile (1, 0) if r is chosen, and if instead player 2 chooses l, then player 1 can choose between A and B, with the game ending with respectively payoff profiles (3, 1) and (0, -5). This game has a unique SPNE, (L,l,A), but (R,r,B)

¹Let us prove that a trembling-hand perfect equilibrium σ is an NE. Recall the following definition of NE: a profile $\sigma \in \Sigma$ is an NE if and only if for all $i \in \mathcal{I}$, for all $s_i, s'_i \in S_i$,

$$u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) = 0.$$

Note that $i \in \mathcal{I}$, for all $s_i, s'_i \in S_i$ such that

$$u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i})$$

there exists $K \in \mathbf{Z}_+$ such that

$$k \geq K \Rightarrow u_i(s_i, \sigma_{-i}^k) < u_i(s'_i, \sigma_{-i}^k),$$

by the fact that $\sigma^k \rightarrow \sigma$, and hence for any such k , we have

$$\sigma_i^k(s_i) \leq \epsilon_k,$$

implying that

$$0 \leq \sigma_i(s_i) = \lim_{k \rightarrow \infty} \sigma_i^k(s_i) \leq \lim_{k \rightarrow \infty} \epsilon_k = 0.$$

This shows that a trembling-hand perfect equilibrium is an NE.

is a trembling-hand perfect equilibrium in the corresponding (reduced) strategic game: consider letting player 1 play (L,A) with probability ϵ^2 and (L,B) with probability ϵ , where notice that player 2 will optimally respond by playing r (this happens because $\text{pro.}((L, B)|L) = \frac{\epsilon}{\epsilon + \epsilon^2}$ is close to one when $\epsilon \downarrow 0$). The problem here is that at the two information sets where player 1 is called upon to take actions, player 1's trembles are correlated. Because of this problem, Selten (1975) argues that we should pay attention to the *agent-normal form*, where player 1 appearing at different information sets is treated as different agents. Then, the trembling-hand perfect equilibria are defined as the trembling-hand perfect equilibria of the agent-normal form, and will be simply referred to as the *perfect equilibria*. With this definition, it can be shown that perfect equilibria are SPNE's.²

4. Consider the following strategic game.

player 1/player 2	L	M	R
U	1,1	0,0	-9,-9
M	0,0	0,0	-7,-7
D	-9,-9	-7,-7	-7,-7

This game has three NE's, all in pure strategy. These are (U,L), (M,M), and (D,R). In this game, (M,M) becomes a trembling-hand perfect equilibrium!³ This is unreasonable, for what we did was adding two dominated strategies R and D to the preceding strategic game! Myerson (1978) proposes a remedy to this situation. Formally, let Σ^0 be the set of totally mixed strategy profiles, and given any $\epsilon \in \mathfrak{R}_{++}$, $\sigma \in \Sigma^0$ is called an ϵ -proper equilibrium if $\forall i \in \mathcal{I}, \forall s_i, s'_i \in S_i$,

$$u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) \leq \epsilon \sigma_i(s'_i).$$

²In fact, they are also *sequential equilibria* defined by Kreps and Wilson (1982).

³To see that this claim is true, given $\epsilon > 0$, let

$$\begin{aligned} \sigma_1^\epsilon(U) &= \epsilon, \quad \sigma_1^\epsilon(M) = 1 - 2\epsilon, \quad \sigma_1^\epsilon(D) = \epsilon, \\ \sigma_2^\epsilon(L) &= \epsilon, \quad \sigma_2^\epsilon(M) = 1 - 2\epsilon, \quad \sigma_2^\epsilon(R) = \epsilon. \end{aligned}$$

A proper equilibrium is then a profile $\sigma \in \Sigma$ (which need not be totally mixed!) such that there exists a sequence $\{\epsilon_k; k \in \mathbf{Z}_+\}$ in \mathfrak{R}_{++} and a sequence $\{\sigma_k; k \in \mathbf{Z}_+\}$ in Σ^0 with (i) $\lim_{k \rightarrow \infty} \epsilon_k = 0$; (ii) σ_k is an ϵ_k -proper equilibrium for all $k \in \mathbf{Z}_+$; and (iii) $\lim_{k \rightarrow \infty} \sigma_{i,k}(s_i) = \sigma_i(s_i)$, $\forall i \in \mathcal{I}$, $\forall s_i \in S_i$. It can be shown that a proper equilibrium is necessarily a trembling-hand perfect equilibrium (this is obvious; simply observe that $\sigma_i(s_i) \leq \epsilon \sigma_i(s'_i) \Rightarrow \sigma_i(s_i) \leq \epsilon$), and hence an NE, but the reverse is not true. In particular, the above game has a unique proper equilibrium (U,L). To see this, consider any ϵ -proper equilibrium σ^ϵ , which is by definition totally mixed. Since player 1 would feel indifferent about M and D only if player 2 were expected to use R with probability one, here we conclude that player 1 prefers M to D. This implies that player 1 should assign probabilities

$$(A1) \quad \sigma_1^\epsilon(D) \leq \epsilon \sigma_1^\epsilon(M),$$

which implies that, from player 2's point of view, for $\epsilon > 0$ small enough,

$$\begin{aligned} & u_2(L, \sigma_1^\epsilon) - u_2(R, \sigma_1^\epsilon) \\ &= 10\sigma_1^\epsilon(U) + 7\sigma_1^\epsilon(M) - 2\sigma_1^\epsilon(D) \\ &\geq 10\sigma_1^\epsilon(U) + (7 - 2\epsilon)\sigma_1^\epsilon(M) > 0, \end{aligned}$$

implying that

$$\sigma_2^\epsilon(R) \leq \epsilon \sigma_2^\epsilon(L),$$

which in turn implies that, from player 1's point of view, for $\epsilon > 0$ small enough,

$$\begin{aligned} & u_1(U, \sigma_2^\epsilon) - u_1(M, \sigma_2^\epsilon) \\ &= \sigma_2^\epsilon(L) - 2\sigma_2^\epsilon(R) \\ &\geq (1 - 2\epsilon)\sigma_2^\epsilon(L) > 0, \end{aligned}$$

implying further that

$$(A2) \quad \sigma_1^\epsilon(M) \leq \epsilon \sigma_1^\epsilon(U).$$

By (A1) and (A2), we conclude that $\sigma_1^\epsilon(U) \geq 1 - \epsilon - \epsilon^2$, and hence in any proper equilibrium $\sigma = \lim_{\epsilon \downarrow 0} \sigma^\epsilon$, we have

$$1 \geq \sigma_1(U) = \lim_{\epsilon \downarrow 0} \sigma_1^\epsilon(U) \geq 1.$$

A similar reasoning applies to $\sigma_2(L)$. Hence (U,L) is the unique proper equilibrium of this strategic game.

5. Myerson also proves that any finite strategic game has a proper equilibrium, and hence any finite game has a trembling-hand perfect equilibrium and an NE. Let us sketch Myerson's proof. Note that it suffices to show that for any $\epsilon_k \in (0, 1)$, an ϵ_k -proper equilibrium σ^k exists, since by the compactness of Σ , a convergent subsequence of $\{\sigma^k; k \in \mathbf{Z}_+\}$ exists. Thus fix any $\epsilon \in (0, 1)$. Define

$$m \equiv \max\{\#(S_i; i = 1, 2, \dots, I)\},$$

where recall that $\#(A)$ is the cardinality of set A (the number of elements of A). Define $d \equiv \frac{\epsilon^m}{m}$. For all $i = 1, 2, \dots, I$, define

$$\Sigma_i^d \equiv \{\sigma_i \in \Sigma_i : \sigma_i(s_i) \geq d, \forall s_i \in S_i\}.$$

Note that Σ_i^d is a non-empty compact subset of Σ_i^0 . Define

$$\Sigma^d \equiv \prod_{i=1}^I \Sigma_i^d.$$

For all $i = 1, 2, \dots, I$, consider the correspondence $F_i : \Sigma^d \rightarrow \Sigma_i^d$ defined by

$$F_i(\sigma) = \{\sigma_i \in \Sigma_i^d : u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) \leq \epsilon \sigma_i(s'_i), \forall s_i, s'_i \in S_i\}.$$

Note that given each $\sigma \in \Sigma^d$, $F_i(\sigma)$ is convex and closed. We claim that $F_i(\sigma)$ is also non-empty. To see this, for each $s_i \in S_i$, define $\rho(s_i)$ to be the number of pure strategies $s'_i \in S_i$ with

$$u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i}).$$

Define

$$\sigma'_i(s_i) \equiv \frac{\epsilon^{\rho(s_i)}}{\sum_{s''_i \in S_i} \epsilon^{\rho(s''_i)}}, \forall s_i \in S_i.$$

By construction, we have $\sigma'_i(s_i) \geq d$, and $\sum_{s_i \in S_i} \sigma'_i(s_i) = 1$. Moreover, it can be verified that $\sigma'_i \in F_i(\sigma)$, showing that $F_i(\sigma)$ is indeed non-empty. Finally, one can verify that F_i is upper hemi-continuous. Define

$$F \equiv \prod_{i=1}^I F_i : \Sigma^d \rightarrow \Sigma^d.$$

Then F , inheriting the main properties from the F_i 's, is non-empty, convex, and upper hemi-continuous, and hence F has a fixed point by Kakutani's fixed point theorem. A fixed point of F is an ϵ -proper equilibrium. Since $\epsilon \in (0, 1)$ was chosen arbitrarily, this proves that we can construct a sequence of ϵ_k -proper equilibria, $\{\sigma^k; k \in \mathbf{Z}_+\}$, and the latter must have a convergent subsequence, of which the limit is exactly a proper equilibrium. This finishes the proof for existence.

6. Now we briefly mention other useful equilibrium concepts proposed by game theorists. (Very difficult; beginners can skip.)

Definition 10: (Aumann, 1959) Given an I -person finite strategic game Γ , a profile σ is a *strong equilibrium* if for any $J \subset \{1, 2, \dots, I\}$, and any $\sigma' \in \Sigma$, there exists $j \in J$ such that $u_j(\sigma) \geq u_j(\sigma'_j, \sigma_{-j})$, where σ'_j is the profile σ' restricted on the set of players J and σ_{-j} is the profile σ restricted on the set of players not contained in J .

From now on any nonempty subset of players from the original game is referred to as a *coalition*. Immediately from the above definition, a strong equilibrium is an NE; to see this, just let J be any singleton coalition. Define the set

$$\mathcal{U} \equiv \{(u_1(\sigma), u_2(\sigma), \dots, u_I(\sigma)) : \sigma \in \Sigma\}.$$

A strong equilibrium, if it exists, must give a profile of payoffs lying on the efficient frontier of \mathcal{U} : Otherwise, we take J to be the entire set of players and obtain a contradiction.

Thus a strong NE is an NE which is robust against not only unilateral deviations but any coalitional deviations also. The problem with this solution concept is that it asks us to check all possible coalitional deviations, including those coalitional deviations which are themselves unreasonable: given a coalition that might benefit from a joint deviation from the original NE strategy profile, there may be some sub-coalition that can benefit from a joint deviation from this supposed joint deviation of the entire coalition. Thus coalitional deviations must be treated in a logically consistent way; this is where the coalition-proof equilibrium gets in the picture. Intuitively, among the solution concepts of NE, strong NE, and coalition-proof NE, the former is the weakest, and the strong NE is the strongest, so that it can happen that

given a game, there exists an NE and a coalition-proof equilibrium, but no strong equilibrium.

7. **Definition 11:** (Bernheim, Peleg, and Whinston, 1987) Suppose that we are given an I -person finite strategic game Γ . Let \mathbf{J} be the set of all feasible coalitions.

(i) If $I = 1$, then a profile σ is a coalition-proof equilibrium if and only if $u_1(\sigma) \geq u_1(\sigma')$ for all $\sigma' \in \Sigma$.

(ii) Suppose $I \geq 2$ and coalition-proof equilibrium (CPE) has been defined for all n -person finite strategic game with $n \leq I - 1$. A profile σ is *self-enforcing* if for all $J \in \mathbf{J}$, σ_J is a coalition-proof equilibrium in the game Γ/σ_{-J} ; i.e. the $\#(J)$ -person strategic game where everything is as in Γ except that players in $-J$ are restricted to play σ_{-J} . A *coalition-proof equilibrium* is a self-enforcing profile σ such that no other self-enforcing profiles σ' can simultaneously provide each and every player in Γ a strictly higher payoff than σ .

Thus when $I = 1$, CPE requires only the best response property. Following this fact, by considering all one-person coalitions, we conclude from the definition of self-enforceability that a CPE must be an NE. For two-person finite strategic games, CPE are equivalent to the set of NE's which are not Pareto strictly dominated. However, for $I \geq 3$, no inclusion relationships can be established between the two. Apparently, a strong equilibrium, if it exists, must be a CPE.

With these definitions and discussions in mind, we now consider two problems. First, consider three players A, B, and C, who are to divide one dollar, and each of them must choose a point in the two-dimensional simplex $\{(a, b, c) \in \mathbb{R}_+^3 : a + b + c = 1\}$. The three players move simultaneously, and if at least two of them pick the same point (a, b, c) , then this point will be implemented, in the sense that a, b , and c will be the payoffs of A, B, and C respectively; or else, the dollar will be destroyed. We claim that this game has no CPE's. To see this, suppose instead that there were a CPE (denoted σ) in which the players get expected payoffs (a, b, c) , where without loss of generality, $a > 0$. Given σ_1 , players 2 and 3 could jointly deviate in the game Γ/σ_1 by announcing simultaneously $(0, \frac{a}{2} + b, \frac{a}{2} + c)$, for example, thereby having the latter implemented. This arrangement strictly Pareto dominates

the original equilibrium profile in the two-person finite strategic game Γ/σ_1 for players 2 and 3, showing that σ cannot be self-enforcing. (For σ to be self-enforcing, it is necessary that (σ_2, σ_3) be a CPE in the game Γ/σ_1 , which in turn requires that (σ_2, σ_3) be a Pareto undominated equilibrium in Γ/σ_1 .) Thus by definition, σ cannot be a CPE, a contradiction.

Lemma 1: We say that an I -person finite strategic game Γ exhibits the unique-NE property if for any $J \in \mathbf{J}$ and any σ_{-J} , there exists a unique NE in the game Γ/σ_{-J} . A game exhibiting the unique-NE property has a unique CPE.

Proof It suffices to show that for a game exhibiting the unique-NE property, self-enforcing profiles and Nash equilibrium profiles are the same. This is sufficient because in this case, a profile is a CPE if and only if it is self-enforcing.

Suppose that $I = 2$ for Γ . By hypothesis, this game has a unique NE, denoted by (σ_1, σ_2) , and given σ_i , σ_j is the unique best response of player j (the unique-NE property holds for the one-person restricted games as well). Thus σ_j is coalition-proof in the game Γ/σ_i and hence σ is self-enforcing. Now suppose that self-enforcing profiles and NE's have been shown equivalent for all I -person finite strategic games exhibiting the unique-NE property, where $I \geq 2$. We now show that this equivalence continues to hold for any such games (exhibiting the unique-NE property) with $I + 1$ players. Let σ be the unique NE for Γ , where there are $I + 1$ players in Γ . Fix any $J \in \mathbf{J}$, note that the game Γ/σ_{-J} also exhibits the unique-NE property, and in fact σ_J must be the unique NE for this $\#(J)$ -person game. It follows from the inductive hypothesis, that σ_J is self-enforcing and, in this case, the unique CPE in the game Γ/σ_{-J} . This shows that σ is self-enforcing in the $I + 1$ -person game Γ .||

The second problem to be discussed here is the familiar Cournot game where there are N firms producing costlessly a homogeneous good to consumers. The inverse demand is (in the relevant region) $p = 1 - \sum_{i=1}^N q_i$. We claim that this game has a unique CPE, which is not a strong equilibrium. To see this, note that the game exhibits the unique-NE property, and hence by lemma 1 it has a unique CPE. This game has no strong equilibria, because if it had one, then the equilibrium

must be an NE lying on the efficient frontier of the set of players' payoff vectors space, which is obviously impossible (think about the profile $(\frac{1}{2n}, \frac{1}{2n}, \dots, \frac{1}{2n})$).

8. The last equilibrium concept we shall go over is *rationalizability* (Bernheim, 1984). Define $\Sigma_i^0 = \Sigma_i$. For all natural numbers n , define

$$\Sigma_i^n = \{\sigma_i \in \Sigma_i^{n-1} : \exists \sigma_{-i} \in \Pi_{j \neq i} \text{co}(\Sigma_j^{n-1}), u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \forall \sigma'_i \in \Sigma_i^{n-1}\}.$$

We call elements in $\bigcap_{n=0}^{+\infty} \Sigma_i^n$ *rationalizable strategies*. Intuitively, rational players will never use strategies which are never best responses. Rationalizability extends this idea to fully utilize the assumption that rationality of players is their common knowledge.

9. Let us now develop the notion of rationalizability in detail. Given a game Γ in normal form with I players, consider sets $H_i \subset \Sigma_i$ for all $i = 1, 2, \dots, I$. We shall adopt the following definitions.

- Let $H_i(0) \equiv H_i$ and define inductively

$$H_i(t) \equiv \{\sigma_i \in H_i(t-1) : \exists \sigma_{-i} \in \Pi_{j \neq i} \text{co}(H_j(t-1)) \\ \ni: u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \forall \sigma'_i \in H_i(t-1)\},$$

where $\text{co}(A)$ is the smallest convex set containing A , called the convex hull generated by A . Define

$$R_i(\Pi_{i=1}^I H_i) \equiv \bigcap_{t=1}^{\infty} H_i(t).$$

- A I -tuple of sets (A_1, A_2, \dots, A_I) has the *best response property* if for all i , $A_i \subset \Sigma_i$ and for all i , for all $\sigma_i \in A_i$, there exists $\sigma_{-i} \in \Pi_{j \neq i} \text{co}(A_j)$ such that σ_i is a best response for i against σ_{-i} .
- $A_i \subset \Sigma_i$ has the *pure strategy property* if for all $\sigma_i \in A_i$, for all $s_i \in S_i$ such that $\sigma_i(s_i) > 0$, $s_i \in A_i$.
- A profile $\sigma \in \Sigma$ is rationalizable, if $\sigma_i \in R_i(\Sigma)$ for all i .

With these definitions, we have

Lemma 2 Suppose that the I -tuple of sets (A_1, A_2, \dots, A_I) is such that

for all i , $A_i \subset \Sigma_i$ is nonempty, closed, and satisfies the pure strategy property. Then, (a) for all i and all $t \in \mathbf{Z}_+$, $A_i(t)$ is nonempty, closed, and satisfies the pure strategy property; and (b) for some $k \in \mathbf{Z}_+$, $A_i(t) = A_i(k)$ for all i and all $t \geq k$.

Proof By induction, to prove (a), it suffices to show that the statement will be true for t if it is true for $t - 1$. By definition, if $\sigma_i \in A_i(t)$, then each $s_i \in S_i$ with $\sigma_i(s_i) > 0$ will too, proving the pure strategy property. To show nonemptiness, note that $\text{co}(A_i(t - 1))$ is compact for all i , since $A_i(t - 1)$ is. By the induction hypothesis, $A_i(t - 1)$ is nonempty for all i . Since u_i is continuous, the Weierstrass theorem ensures the nonemptiness of $A_i(t)$. Finally, for closedness, note that any convergent sequence $\{\sigma_i^n\}$ in $A_i(t)$ must have a limit σ_i in $A_i(t - 1)$, as by the induction hypothesis, $A_i(t - 1)$ is closed. Suppose for each n , σ_i^n is a best response against σ_{-i}^n in $\Pi_{j \neq i} \text{co}(A_j(t - 1))$. Since the set $\Pi_{j \neq i} \text{co}(A_j(t - 1))$ is compact, a subsequence $\{\sigma_{-i}^{n_k}\}$ converges to some $\sigma_{-i} \in \Pi_{j \neq i} \text{co}(A_j(t - 1))$. Now σ_i must be a best response against σ_{-i} by the continuity of u_i . Thus $\sigma_i \in A_i(t)$, showing that $A_i(t)$ is closed.

Finally, consider statement (b). Note that $A_i(t) \neq A_i(t - 1)$ only if $\text{co}(A_j(t)) \neq \text{co}(A_j(t - 1))$ for some $j \neq i$. By the pure strategy property, this can happen only if some pure strategy $s_j \in A_j(t - 1)$ was deleted and was not contained in $A_j(t)$. Since there are only a finite number of pure strategies for any given j , this process must stop somewhere.

10. Now we can give the main results regarding the rationalizable set of profiles.

Proposition 1 For all i , $R_i(\Sigma)$ is nonempty and it contains at least one pure strategy.

Proof Simply let $A_i = \Sigma_i$ and apply lemma 2.

Note that by statement (b) of lemma 2, the I tuple of sets $\{R_1(\Sigma), R_2(\Sigma), \dots, R_I(\Sigma)\}$ has the best response property.

Proposition 2 Define for all i ,

$$E_i \equiv \{\sigma_i \in \Sigma_i : \text{for some } I\text{-tuple } \{A_1, A_2, \dots, A_I\} \text{ with the best response property, } \sigma_i \in A_i\}.$$

Then, $E_i = R_i(\Sigma)$ for all i .

The proof is left as an exercise.

11. Because of proposition 2, we can show that

Proposition 3 Every NE, denoted σ , is rationalizable.

Proof The I -tuple of sets $\{ \{\sigma_1\}, \{\sigma_2\}, \dots, \{\sigma_I\} \}$ satisfies the best response property and $\sigma_i \in \{\sigma_i\}$, $\forall i$, so that proposition 2 implies that for all i , $\sigma_i \in R_i(\Sigma)$.

12. An important connection between the rationalizable set of profiles and the profiles surviving the iterated strict dominance is now given. In general, the former is contained in the latter.

Proposition 4 In two-person finite games, the two concepts coincide.

Proof Suppose that σ_i is not a best response to any element of Σ_j ; i.e. for each $\sigma_j \in \Sigma_j$ there exists $b(\sigma_j) \in \Sigma_i$ such that

$$u_i(b(\sigma_j), \sigma_j) > u_i(\sigma_i, \sigma_j).$$

Call the original game Γ , and construct a zero-sum game Γ_0 as follows. The new game has the same set of players and pure strategy spaces, but the payoffs are defined as

$$u_i^0(\sigma'_i, \sigma_j) \equiv u_i(\sigma'_i, \sigma_j) - u_i(\sigma_i, \sigma_j)$$

for all $(\sigma'_i, \sigma_j) \in \Sigma$, and

$$u_j^0(\sigma'_i, \sigma_j) = -u_i^0(\sigma'_i, \sigma_j).$$

This game has an NE in mixed strategy. Let it be (σ_i^*, σ_j^*) . For any $\sigma_j \in \Sigma_j$, we have

$$\begin{aligned} u_i^0(\sigma_i^*, \sigma_j) &\geq u_i^0(\sigma_i^*, \sigma_j^*) \geq u_i^0(b(\sigma_j^*), \sigma_j^*) \\ &\geq u_i^0(\sigma_i, \sigma_j^*) = 0, \end{aligned}$$

proving that σ_i is strictly dominated by σ_i^* . Thus a strategy for player i that can never be a best response against player j 's strategy must be strictly dominated from player i 's point of view. Define for the purpose of iterated deletion of strictly dominated strategies $S_i^0 = S_i$, $\Sigma_i^0 = \Sigma_i$, and for all $t \in \mathbf{Z}_+$,

$$S_i^t \equiv \{s_i \in S_i^{t-1} : \forall \sigma_i \in \Sigma_i^{t-1}, \exists s_{-i} \in S_{-i}^{t-1}, u_i(s_i, s_{-i}) \geq u_i(\sigma_i, s_{-i})\},$$

$$\Sigma_i^t \equiv \{\sigma_i \in \Sigma_i : \sigma_i(s_i) > 0 \Rightarrow s_i \in S_i^t\},$$

$$S_i^\infty \equiv \bigcap_{t \in \{0\} \cup \mathbf{Z}_+} S_i^t,$$

and

$$\Sigma_i^\infty \equiv \{\sigma_i \in \Sigma_i : \forall \sigma'_i \in \Sigma_i, \exists s_{-i} \in S_{-i}^\infty, u_i(\sigma_i, s_{-i}) \geq u_i(\sigma'_i, s_{-i})\}.$$

In terms of these new notations, we have proved that $\Sigma_i^1 = \Sigma_i(1)$ (since a strictly dominated strategy for player i can never be a best response against player j 's strategy). However, the above argument can be repeated which shows that $\Sigma_i^\infty = \Sigma_i(\infty)$, so that the two concepts are equivalent.

13. Let us offer another proof to the above proposition. Fix $j \in \{1, 2\}$. Let $(s_j^1, s_j^2, \dots, s_j^{\#(S_j)})$ be an enumeration of player j 's pure strategies. Let $\#(S_j) = n_j$. For each $\sigma_i \in \Sigma_i$, let

$$x_i(\sigma_i) \equiv (u_i(\sigma_i, s_j^1), u_i(\sigma_i, s_j^2), \dots, u_i(\sigma_i, s_j^{n_j})),$$

and define the set

$$X_i \equiv \{x_i(\sigma_i) : \sigma_i \in \Sigma_i\}.$$

Then X_i is non-empty, convex, and compact. To see that X_i is convex, note that $x_i : \Sigma_i \rightarrow \mathcal{R}$ is linear, and linear image of convex set is convex: For any $\sigma_i, \sigma'_i \in \Sigma_i$ and any $\lambda \in [0, 1]$,

$$\begin{aligned} & \lambda x_i(\sigma_i) + (1 - \lambda)x_i(\sigma'_i) \\ &= \lambda(u_i(\sigma_i, s_j^1), u_i(\sigma_i, s_j^2), \dots, u_i(\sigma_i, s_j^{n_j})) + (1 - \lambda)(u_i(\sigma'_i, s_j^1), u_i(\sigma'_i, s_j^2), \dots, u_i(\sigma'_i, s_j^{n_j})) \\ &= x_i(\lambda\sigma_i + (1 - \lambda)\sigma'_i), \end{aligned}$$

and since $\lambda\sigma_i + (1 - \lambda)\sigma'_i \in \Sigma_i$, $x_i(\lambda\sigma_i + (1 - \lambda)\sigma'_i) \in X_i$. Also, as the linear function $x_i(\cdot)$ is continuous, X_i is compact because Σ_i is compact.

If σ_i is not strictly dominated, we claim that $x_i(\sigma_i)$ is a boundary point of X_i . (A point $x \in A \subset \mathcal{R}^m$ is a boundary point of A if for all $r > 0$, $B(x, r) \cap A \neq \emptyset \neq B(x, r) \cap A^c$.) Suppose not. Then there would exist $r > 0$ such that $B(x_i(\sigma_i), r) \subset X_i$. This would mean that, for all

$e \in (0, r)$, $x_i(\sigma_i) + (e, e, \dots, e) \in X_i$, and it strictly dominates σ_i . Next, define

$$Y_i \equiv \{y - x_i(\sigma_i) : y \in X_i\}.$$

It follows that zero is a boundary point of the nonempty, convex, compact set Y_i . Consider the nonempty $Z \subset \mathcal{R}^{n_j}$ defined by

$$Z \equiv \{z \in \mathcal{R}^{n_j} : z \gg 0_{n_j \times 1}\},$$

where $z \gg 0$ means that for all $k = 1, 2, \dots, n_j$, the k -th element of z , denoted z_k , is strictly positive. Note that $Z \cap Y_i = \emptyset$. Moreover, Z is convex. One version of the separating hyperplane theorem implies the presence of some non-zero vector $p \in \mathcal{R}^{n_j}$ such that $p'y \leq 0 \leq p'z$ for all $y \in Y_i$ and $z \in Z$. Now we claim that for all $k = 1, 2, \dots, n_j$, the k -th element of p , denoted p_k , is non-negative. To see this, suppose that $p_k < 0$ for some k . This implies that for some l , $p_l > 0$ (so that $n_j \geq 2$). Let m be the largest l with $p_l > 0$. Pick $z^* \in Z$ such that $z_k^* > (n_j - 1)p_m$ and $z_q^* = 1$ for all $q \neq k$. It follows that, for this z^* , $p'z^* < 0$, which is a contradiction.

Thus we have shown the existence of a positive vector $p \in \mathcal{R}^{n_j}$, of which not all elements are zero, such that p defines a hyperplane (or a functional) separating the sets Z and Y_i . We can normalize this functional by letting p be such that $\sum_{k=1}^{n_j} p_k = 1$, so that p is a legitimate mixed strategy for player j . Given p , since $p'y \leq 0$ for all $y \in Y_i$, we have shown that σ_i is a best response of player i to player j 's mixed strategy p . As in the first proof for proposition 4, this argument can be iterated to show that the set of profiles surviving iterated strict dominance is included in the set of rationalizable profiles, so that the two solution concepts coincide in two-player finite strategic games.

14. The above proof for proposition 4 fails if $I > 2$ because not all prob. distributions over S_{-i} are products of independent prob. distributions over S_j , for all $j \neq i$. (Recall that an NE in mixed strategy assumes independent randomization among players.) However, the equivalence between the two concepts stated in proposition 4 is restored if players' randomization can be correlated.

Definition 12 Given a game in normal form,

$$G = (\mathcal{I} \subset \mathfrak{R}; \{S_i; i \in \mathcal{I}\}; \{u_i : \Pi_{i \in \mathcal{I}} S_i \rightarrow \mathfrak{R}; i \in \mathcal{I}\}),$$

an (objective) correlated equilibrium is a prob. distribution $p(\cdot)$ over S such that for all i , for all $s_i \in S_i$ with $p(s_i) > 0$,

$$E[u_i(s_i, \tilde{s}_{-i})|s_i] \geq E[u_i(s'_i, \tilde{s}_{-i})|s_i], \quad \forall s'_i \in S_i.$$

Each $p(\cdot)$ can be thought of as a randomization device for which $s \in S$ occurs with prob. $p(s)$, and when s occurs the device suggests player i play s_i without revealing to player i what s is, such that all players find it optimal to conform to these suggestions at all times. Let \mathcal{P} be the set of all possible devices of this sort. Immediately, all NE's in mixed strategy are elements of \mathcal{P} .

Proposition 5 If s_i is not strictly dominated for player i , then it is a best response for some $p(\cdot) \in \mathcal{P}$.

15. Find all correlated equilibria for the following game:

Player 1/Player 2	L	R
U	5, 1	0, 0
D	4, 4	1, 5

Solution Let the correlated device assigns (U,L), (U,R), (D,L), and (D,R) with respectively probability a , b , c , and d . Define the following 4 inequalities (referred to as I,II,III,and IV):

$$\begin{aligned} \frac{a}{a+c} \cdot 1 + \frac{c}{a+c} \cdot 4 &\geq \frac{a}{a+c} \cdot 0 + \frac{c}{a+c} \cdot 5, \\ \frac{a}{a+b} \cdot 5 + \frac{b}{a+b} \cdot 0 &\geq \frac{a}{a+b} \cdot 4 + \frac{b}{a+b} \cdot 1, \\ \frac{c}{c+d} \cdot 4 + \frac{d}{c+d} \cdot 1 &\geq \frac{c}{c+d} \cdot 5 + \frac{d}{c+d} \cdot 0, \\ \frac{b}{b+d} \cdot 0 + \frac{d}{b+d} \cdot 5 &\geq \frac{b}{b+d} \cdot 1 + \frac{d}{b+d} \cdot 4. \end{aligned}$$

For (a, b, c, d) to define a correlated equilibrium, when players are told to play (U,L) for instance, I and II should hold. Similarly, when players are told to play (D,L), (U,R), and (D,R), [I,III], [IV, II], and [III,IV] should respectively hold. Simplifying, we have four conditions:

$$a \geq c, \quad a \geq b, \quad d \geq c, \quad d \geq b.$$

Let the set of correlated equilibria be A . Then,

$$A = \{(a, b, c, d) : a + b + c + d = 1, a, b, c, d \geq 0, a, d \geq b, c.\}.$$

Note that all NE's are contained in A , and if (a, b, c, d) is a totally mixed strategy NE, then it must satisfy

$$\frac{a}{b} = \frac{c}{d}, \quad \frac{a}{c} = \frac{b}{d}.$$

16. **Example 6:** In a duopolistic industry two risk neutral firms (i.e. expected profits maximizers) that produce respectively products A and B are faced with three segments of consumers:

Segment	Population	Valuation for A	Valuation for B
L_A	α	V	0
L_B	β	0	V
S	$1 - \alpha - \beta$	v	v

where $0 < \beta \leq \alpha < \alpha + \beta < 1$, and $0 \leq v < V$. These segments are loyal to the two firms and the switchers who regard the two products as perfect substitutes. We have assumed that a loyal is willing to pay more than the switcher to obtain the product.

For simplicity the two firms have no production costs, and they compete in price in a simultaneous game. We shall demonstrate the equilibrium dealing behavior of the two firms.

17. First, we look for a pure strategy NE. Suppose (p_A, p_B) is an equilibrium. There are 3 possibilities: (i) $p_A, p_B > v$; (ii) $p_A, p_B \leq v$; and (iii) $\max(p_A, p_B) > v \geq \min(p_A, p_B)$. For case (i), we must have $p_A = p_B = V$, and for this to be an NE, we must require

$$\beta V \geq (1 - \alpha)v, \quad \alpha V \geq (1 - \beta)v. \quad (1)$$

When (1) holds, indeed a pure strategy where $p_A = p_B = V$ exists, and in this NE the switchers are unserved.

On the other hand, if (ii) were an NE, then $p_A = p_B$. To see this, suppose that to the contrary $p_i < p_j$. But then firm j could have

done better by pricing at V ! Again, $p_A = p_B$ is not an NE unless $p_A = p_B = 0$: for otherwise the equilibrium price is dominated by a price slightly lower. It is obvious that $p_A = p_B = 0$ is still not an NE, for each firm can at least make a profit greater than or equal to βV .

Finally, for case (iii) to be an NE, we must have either (iii-a) $p_A = V$, $p_B = v$ or (iii-b) $p_A = v$, $p_B = V$. The conditions that support (iii-a) are

$$\alpha V \geq (1 - \beta)v \geq (1 - \alpha)v \geq \beta V, \quad (2)$$

and when (2) holds, indeed a pure strategy NE where $p_A = V$ and $p_B = v$ exists. One can derive analogously conditions supporting the pure strategy NE in (iii-b).

18. Of course, we observe no dealing behavior in a pure strategy NE. Now we look for mixed strategy NE's. For the ease of exposition, we assume from now on that $\alpha = \beta$. Then (2) becomes

$$\frac{\alpha}{1 - \alpha} = \frac{v}{V},$$

which cannot hold generically. Thus the only possible generic pure strategy NE of this game occurs when

$$\frac{\alpha}{1 - \alpha} \geq \frac{v}{V}.$$

Therefore, we assume that

$$\frac{\alpha}{1 - \alpha} < \frac{v}{V}. \quad (3)$$

Condition (3) says that the loyals are not important enough, and so the firms cannot commit to not compete for the switchers.

19. Now the game is symmetric, and we shall look for a mixed strategy NE $(F_A(p_A), F_B(p_B))$, where $F_i(x) = \text{prob.}(\tilde{p}_i \leq x)$ is the (cumulative) distribution function for firm i 's random price \tilde{p}_i in equilibrium.^{4 5} We

⁴A weakly increasing function $F : \mathfrak{R} \rightarrow \mathfrak{R}$ satisfying (i) (*right-continuity*) $\lim_{x < y, y \rightarrow x} F(y) = F(x)$ for all $x \in \mathfrak{R}$; (ii) $\lim_{x \rightarrow -\infty} F(x) = 0$; and (iii) $\lim_{x \rightarrow +\infty} F(x) = 1$ is called a distribution function. It can have at most a countably infinite number of discontinuity points, and each such point is referred to as a point of jump. We denote by $\Delta F(x-) \equiv F(x) - \lim_{y < x, y \rightarrow x} F(y) \equiv F(x) - F(x-)$ the probability that $F(\cdot)$ assigns to the point of jump x .

⁵Given firm i 's equilibrium mixed strategy $F_i(\cdot)$, we know that (i) if x is a point of jump

shall make use of the following lemmas.

Lemma 3: In equilibrium, $F_A(v), F_B(v) > 0$.

Lemma 3 says that both firms have a positive prob. to choose some price level equal to or less than v . To see this, suppose not. Then at least one firm chooses to not serve the switchers in the NE, and that firm must price at V with probability one, which implies that the other firm should price at v with probability one, and that a contradiction arises because the firm pricing at V can price at $v - \epsilon$ and become better off for some $\epsilon > 0$ small enough.

Lemma 4: For $i \in \{A, B\}$, $F_i(\cdot)$ is continuous on $(-\infty, v)$.

If Lemma 4 does not hold, then at some price $x < v$ either $F_A(\cdot)$ or $F_B(\cdot)$ has a jump.⁶ Thus assume that for firm i , at some $x < v$, $\Delta F_i(x) \equiv F_i(x) - F_i(x-) \equiv F_i(x) - \lim_{y \uparrow x} F_i(y) > 0$. In mathematic terms, x is a point of jump for the function $F_i(\cdot)$, which implies that x is a best response of firm i in equilibrium. Let $\Pi_j(x)$ and $\Pi_j(x - \epsilon)$ be firm j 's expected profits when adopting respectively the pure strategies x and $x - \epsilon$ against $F_i(\cdot)$. Here we assume that $\Delta F_i(x - \epsilon) = 0$.⁷ We

for $F_i(\cdot)$, then x is a (pure-strategy) best response for firm i ; (ii) if $F_i(\cdot)$ is continuous and strictly increasing on an interval $[a, b]$, then every point $x \in (a, b)$ is a (pure-strategy) best response for firm i ; and (iii) in the case of (ii), a is also a (pure-strategy) best response for firm i . Note that (i) is true because firm i should adopt a pure strategy x with a positive probability only if x is a pure-strategy best response in equilibrium. Note that (ii) is true because if $x \in (a, b)$ then $F_i(y) > F_i(x-)$ whenever $y > x$ —in a sense this indicates that firm i may choose the price x with a non-zero *likelihood*, although we know that for a continuous random variable the *probability* that a particular realization appears in equilibrium is formally zero. Finally, note that (iii) is true because the rival's $F_j(\cdot)$ is a right-continuous function; this point is explained in class. We emphasize here that in case of (ii), b need not be a pure-strategy best response for firm i ! In Example 7 below, we shall define a point of increase for a distribution function $F(\cdot)$. It is verified there that a point of increase for an equilibrium mixed strategy $F(\cdot)$ must be a pure-strategy best response.

⁶Recall that if $F : \mathcal{R} \rightarrow \mathcal{R}$ is increasing, then the only possible discontinuity points are of the first kind: where $F(\cdot)$ has well-defined left-hand and right-hand limits, but the functional values of F need not equal these limits.

⁷Since an increasing function can have at most a countably infinite number of points of jump, no matter how small $\epsilon > 0$ is, finding a point $x - \epsilon$ at which $F_i(\cdot)$ does not jump is always possible.

have

$$\Pi_j(x) = \left[\frac{1}{2}(1-2\alpha) + \alpha\right]x \cdot \Delta F_i(x) + [(1-2\alpha) + \alpha]x \cdot [1 - F_i(x)] + \alpha x F_i(x - \epsilon)$$

$$< \Pi_j(x - \epsilon) = [(1-2\alpha) + \alpha](x - \epsilon) \cdot [1 - F_i(x - \epsilon)] + \alpha(x - \epsilon)F_i(x - \epsilon)$$

when $\epsilon > 0$ is sufficiently small. In fact, as we can easily see, given x and ϵ , there exists some $\delta > 0$ small enough such that

$$\Pi_j(y) < \Pi_j(x - \epsilon), \quad \forall y \in (x, x + \delta].$$

We have just reached the conclusion that

$$F_j(x) = F_j(x + \delta);$$

that is, no pure strategies in the interval $(x, x + \delta]$ can be best responses for firm j against firm i 's equilibrium strategy $F_i(\cdot)$, and hence firm j will assign zero probability to these pure strategies in equilibrium. However, this implies that the pure strategy x cannot be a best response for firm i against firm j 's equilibrium strategy $F_j(\cdot)$: pricing at $x + \frac{\delta}{2}$ is better, for example, because the probability that firm i wins the sales to the switchers are the same under the pure strategy $x + \frac{\delta}{2}$, but when that happens firm i gets a higher revenue! This implies a contradiction, because we started out assuming that x is a best response of firm i against firm j 's equilibrium strategy $F_j(\cdot)$; otherwise, it cannot happen that $\Delta F_i(x) > 0$.

Lemma 5: In equilibrium, $F_A(v-), F_B(v-) > 0$.

Lemma 5 says that both firms must randomize at some price strictly lower than v . This is a refinement of lemma 3. To see this, suppose that $F_i(v-) = 0$, so that by the fact that $F_i(\cdot)$ is increasing, we have

$$0 = F_i(v-) \geq F_i(x) \geq 0, \Rightarrow F_i(x) = 0, \forall x < v.$$

Now, if at some $x < v$, $F_j(x) > 0$, then there must be some $y \leq x$ such that y is a pure-strategy best response for firm j against $F_i(\cdot)$, which implies a contradiction because $\frac{y+v}{2}$ is obviously a better pure-strategy response than y for firm j ! Thus if firm i has $F_i(v-) = 0$, then firm j must also have $F_j(v-) = 0$, or equivalently, $F_j(x) = 0$ for all $x < v$.

But then, lemma 3 implies that both $F_i(\cdot)$ and $F_j(\cdot)$ must jump at v , which implies another contradiction: given $\Delta F_i(v) > 0$, for tiny $\delta > 0$, pricing at $v - \delta$ is strictly better than pricing at v for firm j . Thus we conclude that both firms must randomize below v !

Lemma 6: For $i \in \{A, B\}$, if at $a < v$, $F_i(a) > 0$, then for all $b \in (a, v)$, $F_i(b) > F_i(a)$.

Lemma 6 says that in equilibrium the distribution function must be strictly increasing at all prices that are close to but lower than v . More importantly, this says that if firm i randomizes at $p_i < v$, then not only all $x \in (p_i, v)$ are best responses for firm i , in equilibrium firm i must randomize over each $x \in (p_i, v)$.

To see that lemma 6 is true, suppose to the contrary that $F_i(\cdot)$ is flat on an interval $[x, y]$, i.e. $F_i(x) = F_i(y) > 0$, $a \leq x < y < v$, and $F_i(z) < F_i(x)$ for all $z < x$. It is important to note that x is a best response for firm i .⁸ In this case, none of the prices p_j contained in the interval $[x, y)$ can be pure-strategy best responses for firm j : pricing at $\frac{x+y}{2}$ is a better response than p_j for firm j ! It follows that $F_j(\cdot)$ has to be flat on the interval $[x, y)$, and moreover, $\Delta F_j(x) = 0$. A contradiction now arises because from firm i 's perspective, x is by assumption a best response but is now a worse response than, say, $\frac{x+y}{2}$ from firm i 's perspective!

20. Equipped with the above lemmas, now we can rigorously derive the mixed strategy NE. Let Π_i be firm i 's equilibrium expected profit. By lemma 5, we have

$$\Pi_i = p_i\{[1 - \alpha][1 - F_j(p_i)] + \alpha F_j(p_i)\},$$

for some $p_i < v$, so that for $i, j \in \{A, B\}$, $i \neq j$,

$$F_j(x) = \frac{(1 - \alpha) - \frac{\Pi_i}{x}}{1 - 2\alpha},$$

⁸By Lemma 4, $F_j(\cdot)$ is continuous over the region $(-\infty, v)$. Thus firm i 's payoff function is continuous in its (pure-strategy) price p_i on $(-\infty, v)$. Note that there must exist some $z \in [a, x)$ such that $F_i(\cdot)$ is continuous and strictly increasing on the interval $[z, x)$. This means that every price contained in $[z, x)$ is a pure-strategy best response for firm i , and by the fact that firm i 's payoff function is continuous in its (pure-strategy) price p_i on $(-\infty, v)$, so must be x ! This proves that x has to be a pure-strategy best response for firm i .

for all $x \in [\underline{p}_j, v)$ ⁹ and some $\underline{p}_j \in [0, v)$. Immediately, we have

$$\underline{p}_j = \frac{\Pi_i}{1 - \alpha}. \quad (4)$$

The continuity of $F_j(\cdot)$ on $[\frac{\Pi_i}{1-\alpha}, v)$ allows us to take limit:

$$F_j(v-) = \frac{1 - \alpha - \frac{\Pi_i}{v}}{1 - 2\alpha} < 1,$$

where the inequality follows because if otherwise, then $\Pi_i = v\alpha < \alpha V$, a contradiction. Since all $p_j \in (v, V)$ are dominated by $p'_j = V$ for firm j , it follows that either $F_j(\cdot)$ has a point mass at v or at V . Note that it is impossible that both $\Delta F_i(v), \Delta F_j(v) > 0$: if this were to happen, then v would be a best response for both firms, but given firm i 's strategy, from firm j 's perspective v would be dominated by $p_j = v - \epsilon$ for $\epsilon > 0$ small enough, which is a contradiction.

21. Note that \underline{p}_i is a best response for firm i . To see this, note that all $p_i \in (\underline{p}_i, v)$ are, and they generate the same expected profits for firm i . By letting $p_i \downarrow \underline{p}_i$ and using the fact that $F_j(\cdot)$ is continuous on $(-\infty, v)$, we have that \underline{p}_i also attains Π_i and hence is a best response for firm i . Next, we claim that $\underline{p}_i = \underline{p}_j$. To see this, suppose instead that $\underline{p}_i > \underline{p}_j$, so that firm j may randomize at, say $p_j \in (\underline{p}_j, \underline{p}_i)$. Note however that p_j is dominated by $p_j + \epsilon$ for $\epsilon > 0$ small enough. From here, using (4), we conclude that $\Pi_i = \Pi_j$ in equilibrium.
22. Now we summarize the equilibria. First suppose that for one firm i , $\Delta F_i(v) > 0$. Then $\Delta F_j(v) = 0$, implying that $\Delta F_j(V) > 0$ and hence $\Pi_j = \alpha V$. It follows that $\Pi_i = \alpha V$ also. Since v is a best response for firm i , we must have

$$\Delta F_j(V)v(1-\alpha) + [1 - \Delta F_j(V)]v\alpha = \alpha V, \Rightarrow \Delta F_j(V) = \frac{\alpha(V - v)}{v(1 - 2\alpha)}. \quad (5)$$

Alternatively, we can obtain the same result from

$$\Delta F_j(V) = 1 - F_j(v-). \quad (6)$$

⁹This follows from lemma 4 which says that if $p_i < v$ is a best response for firm i then so is x , for all $x \in (p_i, v)$.

In this case, we have

$$F_i(p_i) = \begin{cases} 0, & p_i \leq \frac{\alpha V}{1-\alpha}; \\ \frac{1-\alpha-\frac{\alpha V}{p_i}}{1-2\alpha}, & p_i \in [\frac{\alpha V}{1-\alpha}, v]; \\ p^*, & p_i \in [v, V]; \\ 1, & p_i \geq V, \end{cases}$$

and

$$F_j(p_j) = \begin{cases} 0, & p_j \leq \frac{\alpha V}{1-\alpha}; \\ \frac{1-\alpha-\frac{\alpha V}{p_j}}{1-2\alpha}, & p_j \in [\frac{\alpha V}{1-\alpha}, v]; \\ \frac{1-\alpha-\frac{\alpha V}{v}}{1-2\alpha}, & p_j \in (v, V); \\ 1, & p_j \geq V, \end{cases} \quad (7)$$

where $p^* \in (\frac{1-\alpha-\frac{\alpha V}{v}}{1-2\alpha}, 1]$.

Next consider the case where $\Delta F_A(v) = \Delta F_B(v) = 0$. In this case, the equilibrium is symmetric, and we have both $F_A(\cdot)$ and $F_B(\cdot)$ characterized by the $F_j(\cdot)$ above.¹⁰

23. Let S_A and S_B be the supports of equilibrium prices p_A and p_B . Then for $i = A, B$, we can interpret $\sup S_i$ as the *regular price* of firm i , and any price strictly lower than $\sup S_i$ as a *dealing price*. Now we can compute that the dealing frequency for a firm, which is the firm's probability of selecting a dealing price. In the symmetric equilibrium, the dealing frequency for both firms is $\frac{1-\alpha-\frac{\alpha V}{v}}{1-2\alpha}$, which is decreasing in α and V and increasing in v , a result rather consistent with our intuition. The *depth of dealing* for both firms in the symmetric equilibrium is defined as

$$V - E[\tilde{p} | \tilde{p} \leq v] = V - \frac{\alpha V \log(\frac{(1-\alpha)v}{\alpha V})}{1 - \alpha - \frac{\alpha V}{v}}. \quad (8)$$

As an exercise, you can examine how the parameters V, v , and α may respectively affect the depth of dealing.

¹⁰It can be verified that in the former equilibrium v is not a pure-strategy best response for firm j , but it is in the latter equilibrium.

24. We can also consider the case where $\alpha > \beta$, $\alpha V < (1 - \beta)v$ and $\beta V < (1 - \alpha)v$. In fact, one can show that $\alpha(1 - \alpha) > (1 - \beta)\beta$,¹¹ and hence $\alpha V < (1 - \beta)v$ implies that $\beta V < (1 - \alpha)v$. In this case, one can

¹¹Since $1 - \alpha - \beta > 0$, $\frac{1}{2} - \beta > \alpha - \frac{1}{2}$, the inequality holds when $\alpha \geq \frac{1}{2} > \beta$. If instead $\alpha < \frac{1}{2}$, then the inequality holds obviously.

show that the unique equilibrium in mixed strategy is such that¹²

$$F_A(x) = \begin{cases} 0, & x \leq \underline{p} \equiv \frac{\alpha V}{1-\beta}; \\ \frac{1-\frac{p}{x}}{1-\frac{p}{\underline{p}}\alpha}, & x \in [\underline{p}, v); \\ \frac{1-\frac{v}{x}}{1-\frac{v}{\underline{p}}\alpha}, & x \in [v, V); \\ 1, & x \geq V, \end{cases} \quad (9)$$

¹²Let us demonstrate in detail how to get this equilibrium.

- At first, the supports of F_A and F_B must share the same greatest lower bound \underline{p} , because given \underline{p}_i , each $p_j < \underline{p}_i$ is strictly dominated by, say, $\frac{p_j + \underline{p}_i}{2}$, for $i \neq j$, $i, j \in \{A, B\}$. Note that $\underline{p} > 0$ because both firms can ensure a strictly positive profit by serving their own loyal only.
- Second, in equilibrium at least one firm i must price at V with a strictly positive probability. To see this, suppose instead that both firms price equal to or less than v with probability one, which implies immediately that the supports of F_A and F_B must share a common least upper bound $\bar{p} \leq v$, because any $p_j \in (\bar{p}_i, \bar{p}_j)$ would be dominated by \underline{p} if, say, $\bar{p}_i < \bar{p}_j \leq v$. In this case, neither F_A nor F_B can have a jump at \bar{p} , because that would imply that \bar{p} is a best response, which however generates only a zero profit! Now, by the fact that F_A and F_B are continuous at \bar{p} , again, \bar{p} must be a best response because for some $\delta > 0$, every price contained in $(\bar{p} - \delta, \bar{p})$ is a best response! We conclude that a contradiction will arise unless at least one firm will price at V with a strictly positive probability in equilibrium.
- Third, for both firms, the least upper bound of the set of pure-strategy best responses less than or equal to v is exactly v . To see this, suppose instead that the aforementioned least upper bound is $\bar{p}_i < v$ for firm i . By definition of the least upper bound, there exists an increasing sequence of pure-strategy best responses $\{p_i^n; n = 1, 2, \dots\}$ converging to \bar{p}_i . However, it is clear that firm j will never randomize over (\bar{p}_i, v) , and hence for n sufficiently large, the pure strategy p_i^n is strictly dominated by, say, $\frac{\bar{p}_i + v}{2}$, which is a contradiction.
- Fourth, firm i 's equilibrium payoff (expected profit) is $\Pi_i = \alpha_i V$ if with a strictly positive probability it may price at V . This pins down \underline{p} as follows. Note that for all $x \in [\underline{p}, v)$,

$$\Pi_i = \alpha_i V = x \{ \alpha_i + (1 - \alpha_i - \alpha_j) [1 - F_j(x)] \} \Rightarrow F_j(x) = 1 - \frac{\frac{\alpha_i V}{x} - \alpha_i}{1 - \alpha_i - \alpha_j},$$

which, by the fact that $F_j(\underline{p}) = 0$, implies that

$$\underline{p} = \frac{\alpha_i V}{1 - \alpha_j}.$$

The question is which firm will price at V with a strictly positive probability. Recall that the firm with a larger loyal base cannot compete as aggressively as its rival does. Hence, by the fact that $\alpha = \alpha_A > \alpha_B = \beta$, we know that firm A will be the one that must price at V with a strictly positive probability. It follows that

$$\underline{p} = \frac{23}{1 - \beta} \alpha V,$$

and that

$$F_B(x) = 1 - \frac{\frac{\alpha V}{x} - \alpha}{1 - \alpha - \beta}, \quad \forall x \in \left[\frac{\alpha V}{1 - \beta}, v\right).$$

Note that

$$F_B(v-) \equiv \lim_{x \uparrow v} F_B(x) = 1 - \frac{\alpha(V - v)}{(1 - \alpha - \beta)v} < 1.$$

Moreover, because $\alpha V < (1 - \beta)v$, we have

$$F_B(v-) > 0.$$

- Fifth, we can deduce Π_B as follows. Note that

$$F_A(x) = 1 - \frac{\frac{\Pi_B}{x} - \beta}{1 - \alpha - \beta}, \quad \forall x \in \left[\frac{\alpha V}{1 - \beta}, v\right),$$

and

$$F_A(\underline{p}) = F_A\left(\frac{\alpha V}{1 - \beta}\right) = 0.$$

From here we have

$$\Pi_B = (1 - \alpha)\underline{p} = \frac{\alpha(1 - \alpha)V}{1 - \beta} > \beta V.$$

The last inequality follows from $\alpha(1 - \alpha) > \beta(1 - \beta)$, and it also tells us that pricing at V is a dominated strategy for firm B. It follows that $F_B(v) = 1$.

- Sixth, since $F_B(v-) < 1 = F_B(v)$, we conclude that F_B has a jump at v . This implies that $p_B = v$ is a pure-strategy best response, and it also implies that $p_A = v$ is not a pure-strategy best response for firm A (which is worse than, for example, $v - \epsilon$ for sufficiently small $\epsilon > 0$), and hence $F_A(v-) = F_A(v)$. Let us verify if this is indeed so. Pricing at $p_B = v$ will generate a payoff (expected profit) for firm B that equals

$$\begin{aligned} & \beta v F_A(v) + (1 - \alpha)v[F_A(V) - F_A(V-)] = \beta v F_A(v) + (1 - \alpha)v[1 - F_A(v)] \\ & = \beta v \left[1 - \frac{\frac{\Pi_B}{v} - \beta}{1 - \alpha - \beta}\right] + \frac{(1 - \alpha)v(\frac{\Pi_B}{v} - \beta)}{1 - \alpha - \beta} \\ & = \frac{\beta v(1 - \alpha - \beta) + (\beta - \frac{\Pi_B}{v})\beta v + (1 - \alpha)v \cdot \frac{\Pi_B}{v} - \beta(1 - \alpha)v}{1 - \alpha - \beta} = \Pi_B. \end{aligned}$$

Hence, F_B having a jump at v is indeed consistent with equilibrium.

This finishes our derivation of F_A and F_B .

and

$$F_B(x) = \begin{cases} 0, & x \leq \underline{p} \frac{\alpha V}{1-\beta}; \\ \frac{1-\frac{p}{x}}{1-\frac{\underline{p}}{1-\beta}}, & x \in [\underline{p}, v); \\ 1, & x \geq v. \end{cases} \quad (10)$$

What is the effect of an increase in α , say, on the equilibrium pricing strategies?

Observe that an increase in α has two direct influences on the configuration of consumers. It implies an increase in the population of firm A's loyal, and it also implies a decrease in the population of the switchers. The former leads to an increase in the equilibrium \underline{p} (because the set of overly low prices dominated by the price V is enlarged, given that α has increased so that there are now more loyal willing to pay V), and the latter results in an increase in the density of $F_A(\cdot)$ for all prices below v that might arise in equilibrium.

The former effect is self-evident. Let us examine the latter effect. Suppose that $\alpha_2 > \alpha_1$, and for $i = 1, 2$, $\alpha_i > \beta$, $\alpha_i V < (1 - \beta)v$. Fix h, l such that $v > h > l > \underline{p}_2 > \underline{p}_1$. Note that given i , firm 2 is indifferent about h and l :

$$h\beta F_A(h, \alpha_i) + h(1 - \alpha_i)[1 - F_A(h, \alpha_i)] = l\beta F_A(l, \alpha_i) + l(1 - \alpha_i)[1 - F_A(l, \alpha_i)], \quad \forall i = 1, 2.$$

Since given i , $hF_A(h, \alpha_i) > lF_A(l, \alpha_i)$, we conclude that $h[1 - F_A(h, \alpha_i)] < l[1 - F_A(l, \alpha_i)]$ so that when α increases from α_1 to α_2 , if firm 1's strategy were still $F_A(x, \alpha_1)$, then firm 2 would strictly prefer h to l . This result is not surprising, as an increase in α also implies a reduction in $1 - \alpha - \beta$, and given that β does not change, firm B now considers lowering the price to win the switchers and giving up the chance of extracting surplus from the loyal more costly than before.

Since only a mixed-strategy equilibrium can exist given $\alpha = \alpha_2$, just like in the case of $\alpha = \alpha_1$, given the new $F_A(x, \alpha_2)$ firm B must again feel indifferent about h and l . We demonstrate below that this will require F_A to have a higher density function under $\alpha = \alpha_2$ than under $\alpha = \alpha_1$ over the prices under v that may be chosen in both the cases $\alpha = \alpha_2$ and $\alpha = \alpha_1$.

Note that the above indifference equation can be re-arranged to get

$$F_A(h, \alpha) + l \left[\frac{F_A(h, \alpha) - F_A(l, \alpha)}{h - l} \right] = \frac{1}{1 - \frac{\beta}{1-\alpha}}.$$

Taking limit on both sides by letting $l \rightarrow h$ and assuming that F_A is differentiable on (\underline{p}, v) (which can be verified independently), we have, given α ,

$$F_A(h, \alpha) + hf_A(h, \alpha) = \frac{1}{1 - \frac{\beta}{1-\alpha}}, \quad \forall h \in (\underline{p}, v),$$

where $f_A = F'_A$ is the density function of firm A's equilibrium price. From here, we see two things. Note first that F_A is strictly concave on (\underline{p}, v) :¹³ the above right-hand side is independent of h , and since $F_A(h, \alpha)$ is increasing in h , f_A has to be strictly decreasing in h . Second, note that there is a small interval $[\underline{p}, \hat{p})$ such that at every x inside that interval, $f_A(x, \alpha)$ is increasing in α . This happens because the above right-hand side, $\frac{1}{1 - \frac{\beta}{1-\alpha}}$ is increasing in α , and Leibniz rule tells us that the change in F_A at h induced by a change in α can be attributed to a change in \underline{p} (which has a negative effect) and a change in the density function.¹⁴

It is easy to verify that, indeed, given any $h \in (\underline{p}_2, v)$ we have $f_A(h, \alpha_2) > f_A(h, \alpha_1)$. In fact, by directly differentiating, we have for all $h \in (\underline{p}, v)$,

$$\frac{\partial^2 F_A(h, \alpha)}{\partial h \partial \alpha} = \frac{\partial f_A(h, \alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \frac{\frac{p}{h^2}}{1 - \frac{\beta}{1-\alpha}} > 0.$$

This reflects the need of restoring indifference for firm 2 after an increase in α from α_1 to α_2 . Following that increase, if firm 1 were to use

¹³This can be confirmed easily when $\alpha = \beta$. In fact, we have shown earlier that in the symmetric case where $\alpha = \beta$, F_A coincides with F_B on (\underline{p}, v) , and they are concave on this price region.

¹⁴Note that $F_A(h, \alpha) = \int_{\underline{p}(\alpha)}^h f_A(x, \alpha) dx$. The Leibniz rule says that

$$\frac{\partial F_A(h, \alpha)}{\partial \alpha} = -\underline{p}'(\alpha) f_A(\underline{p}(\alpha), \alpha) + \int_{\underline{p}(\alpha)}^h \frac{\partial f_A(x, \alpha)}{\partial \alpha}(x, \alpha) dx,$$

provided that f_A and $\underline{p}(\alpha)$ are both continuously differentiable in α .

$F_A(\cdot, \alpha_1)$, then firm 2 would strictly prefer h to l , and so to restore indifference, we need to make sure that under α_2 , the difference in the probabilities of losing the switchers, $F_A(h, \alpha_2) - F_A(l, \alpha_2)$, is higher than its counterpart $F_A(h, \alpha_1) - F_A(l, \alpha_1)$ under α_1 . This being true for all h and l , we conclude that f_A is higher under α_2 than under α_1 at all $h \in (\underline{p}_2, v)$.

Now, let us summarize the effect of an increase in α on F_A . By directly differentiating, we have for all $x \in (\underline{p}, v)$,

$$\frac{\partial F_A(x, \alpha)}{\partial \alpha} = \frac{1}{[1 - \frac{\beta}{1-\alpha}]^2} \left\{ \frac{\beta}{(1-\alpha)^2} \left[1 - \frac{p(\alpha)}{x} \right] - \frac{p'(\alpha)}{x} \left[1 - \frac{\beta}{1-\alpha} \right] \right\},$$

so that the sign of $\frac{\partial F_A(x, \alpha)}{\partial \alpha}$ is the same as the sign of

$$G(x) \equiv \frac{\beta}{(1-\alpha)^2} \left[1 - \frac{p(\alpha)}{x} \right] - \frac{p'(\alpha)}{x} \left[1 - \frac{\beta}{1-\alpha} \right].$$

Note that $G(\cdot)$ is strictly increasing, with $G(\underline{p}) < 0$. Letting $G(x^*) = 0$, we have

$$x^* = \underline{p} + \frac{(1-\alpha)(1-\alpha-\beta)V}{\beta(1-\beta)}.$$

Thus we can conclude that

- If $\min(1, \frac{\alpha}{1-\beta} + \frac{(1-\alpha)(1-\alpha-\beta)}{\beta(1-\beta)}) > \frac{v}{V} > \frac{\alpha}{1-\beta}$ so that the interval (\underline{p}, v) does not contain x^* , then at all $x \in (\underline{p}, v)$, we have $\frac{\partial F_A}{\partial \alpha} < 0$.¹⁵
- If instead $1 > \frac{v}{V} \geq \frac{\alpha}{1-\beta} + \frac{(1-\alpha)(1-\alpha-\beta)}{\beta(1-\beta)}$ so that $x^* \in (\underline{p}, v)$, then $\frac{\partial F_A}{\partial \alpha}(x, \alpha) \leq 0$ if and only if $x \leq x^*$.

Intuitively, as suggested by Leibniz rule, an increase in α results in a decrease in F_A at all $x \in (\underline{p}_1, \underline{p}_2]$, but to restore a mixed equilibrium, as we mentioned above, the density f_A must become higher at all $x \in (\underline{p}_2, v)$. Thus for $x \in (\underline{p}_2, v)$, either F_A becomes higher or it becomes lower under α_2 , and which one would happen depends on which between the above two opposing effects dominates.

¹⁵Recall that in the symmetric case, where $\alpha = \beta$, we have shown that $F_A(x) = \frac{1-\alpha-\frac{\alpha V}{x}}{1-2\alpha}$ for all $x \in (\underline{p}, v)$, and indeed $\frac{\partial F_A}{\partial \alpha} < 0$.

25. **Example 7:** Consider an imperfectly competitive duopoly in the cable TV industry where consumers A and B are willing to pay up to $(2, 3.5)$ and $(0, 5)$ for the goods (L, HL) , which are offered respectively by firm L and firm HL.¹⁶ The firms compete in price. Let p and q denote generic prices chosen by firm L and firm HL respectively.

Step 1: There are no pure strategy NE's for this game.

Proof. Suppose that there were a pure strategy NE. Then in this equilibrium, consumer A either (i) prefers L to H+L; or (ii) feels indifferent about H+L and L; or (iii) prefers H+L to L.

- In case (i), firm HL's equilibrium revenue comes from consumer B only, and the optimal price is $q = 5$, yielding a profit of 5 for firm HL. Given this, firm L's optimal price is $p = 2$. However, firm HL can get $2 \times (3.5 - \epsilon) > 5$ by deviating and pricing at $3.5 - \epsilon$, a contradiction.
- In case (ii), we must have $p = q - 1.5$, and each firm can serve consumer A with probability 0.5. However, firm L can price at $q - 1.5 - \epsilon$ and become better off, unless of course $0 = p = q - 1.5$. However, this implies that $q = 1.5$, and hence firm HL cannot get more than 2×1.5 in equilibrium, which is a contradiction, as firm HL can always serve consumer B only by pricing at 5 and obtain a profit of 5!
- In case (iii), since firm L has no production costs, it must be that

¹⁶As the reader can verify, here consumer A plays the role of the switchers in Example 6, and consumer B is the loyal customer of firm HL. The derivation of the mixed-strategy equilibrium for this game follows a similar procedure outlined in sections 16-24. In particular, we shall first prove that there can exist no pure-strategy equilibrium. In deriving a mixed-strategy equilibrium, we first derive the greatest lower bound for the support of the random price offered by firm HL, \underline{q} , which, like in Example 6, can be determined using the observation that by offering \underline{q} firm HL wins the switchers for sure, which generates a profit equal exactly to the profit that firm HL would get by simply serving its loyal. Then it is easy to see that the greatest lower bound for the support of the random price offered by firm L is exactly $\underline{p} = \underline{q} - [3.5 - 2]$, where $3.5 - 2$ is the valuation differential between the two firms from the perspective of the switcher. These lower-bound prices give the equilibrium profits for the two firms. As expected, firm HL does not benefit from the presence of the switcher (consumer A), since only it has a loyal customer. Firm L, on the other hand, earns a positive expected profit in equilibrium even if it has no loyal customer. In fact, the latter *is* the source of firm L's competitive advantage against firm HL.

$q < 1.5$. Again, firm HL can price at 5 and become better off, a contradiction.

Thus we conclude that no pure-strategy NE can exist. \parallel

Recall that the support of a random variable is the smallest closed set (in the usual topology on \mathcal{R}) in which the realizations of that random variable occurs with probability one. Let S_p and S_q denote respectively the supports of p and q in a mixed strategy NE.

Step 2: In any mixed-strategy NE for this game, $S_P \subset [0, 2]$ and $S_q \subset [0, 5]$.

Proof. Obvious. \parallel

From now on, $F(\cdot)$ and $G(\cdot)$ stand for respectively the equilibrium distribution functions of p and q .

Step 3: In any NE, firm L does not randomize over any $p < 1$ and firm HL does not randomize over any $q < 2.5$.

Proof. Note that firm HL will not pick a price lower than 2.5, for at such a price even if both consumers buy from firm HL, and profit is lower than 5, which firm HL can get for sure from serving consumer B only. This implies immediately that consumer A cannot get a surplus higher than 1 if she buys from firm HL. In turn, this implies that for any $p < 1$, which gives consumer A a surplus higher than 1, firm L gets consumer A for sure. Note, however, that each and every $p < 1$ is strictly dominated by $\frac{p+1}{2}$. \parallel

Thus in any NE, firm L randomizes over $[1, 2]$ with $p = 2$ being the only possible point mass of $F(\cdot)$.

Definition 13. We shall say that p is a *point of increase* of $F(\cdot)$ if for all $e > 0$, $F(p + e) > F(p-)$.

It can be verified that p is one of firm L's pure-strategy best responses in equilibrium if it is a point of increase of $F(\cdot)$. A similar definition applies to the point of increase of $G(\cdot)$.

Step 4: In any NE, $F(\cdot)$ is continuous on $p \in [1, 2)$ and $G(\cdot)$ is continuous on $q \in [2.5, 3.5)$.

Proof. Suppose to the contrary that at $p \in (0, 2)$, $F(p) - F(p-) > 0$, where $F(p-) = \lim_{x \uparrow p} F(x)$ exists because $F(\cdot)$ is increasing. Note that

this implies that p is a best response of firm L. Then, there exist $d, e > 0$ small enough such that each $q \in [p+1.5, p+1.5+e]$ is strictly dominated by $p + 1.5 - d$ from firm HL's point of view. If this is consistent with equilibrium, then in equilibrium firm HL does not randomize over any $q \in [p + 1.5, p + 1.5 + e]$. This implies that p is strictly dominated by $p + e$ from firm L's point of view, a contradiction. The proof that $G(\cdot)$ is continuous on $q \in [2.5, 3.5]$ is similar. \parallel

Then we show that F and G are strictly increasing on respectively an interval below 2 and an interval below 3.5.

Step 5: If at $p \in [1, 2)$, $F(p) > 0$, then p is a point of increase for $F(\cdot)$.¹⁷ Similarly, if at $q \in [2.5, 3.5)$, $G(q) > 0$, then q is a point of increase for $G(\cdot)$.

Proof. We only consider the first assertion; the second assertion can be analogously proved. Suppose instead that there exists $p \in (1, 2)$ and $e > 0$ such that $F(p) = F(p + e)$. Let p' be $\inf\{p'' : F(p'') = F(p)\}$. By right-continuity of $F(\cdot)$, $F(p') = F(p)$, and for all $d > 0$, $F(p' - d) < F(p')$, which implies that there exists a pure-strategy $p_d \in (p' - d, p']$ for each tiny $d > 0$. It is clear that $p' = \lim_{d \downarrow 0} p_d$, and since $G(\cdot)$ is continuous on $q \in [2.5, 3.5)$, p' itself must also be a pure-strategy best response for firm L. However, since firm L does not randomize over rival $(p', p + e]$, firm HL should not randomize over $(p' + 1.5, p + 1.5)$. This implies that p' is strictly dominated by p from firm L's perspective, a contradiction. \parallel

Step 6: In any NE, $q \in (3.5, 5)$ is not a point of increase of $G(\cdot)$.

Proof. Observe that these prices are dominated strictly by $q = 5$. \parallel

Step 7: $G(3.5-) > 0$ and $G(5) - G(5-) > 0$.

Proof. Consider the first assertion that $G(3.5-) > 0$. The non-existence of pure-strategy equilibria (step 1) shows that $G(3.5) > 0$. If $G(3.5-) = 0$, then firm HL must price at 3.5 with a strictly positive probability, which implies that firm L will always price below 2, and hence pricing at 3.5 will not attract consumer A for firm HL, proving that pricing at 3.5 is strictly dominated by pricing at 5 for firm HL, which is a contradiction. Hence we conclude that $G(3.5-) > 0$; that is, firm HL must randomize on $[2.5, 3.5)$ with a strictly positive probability.

¹⁷It follows that $F(\cdot)$ is strictly increasing on its support S_p , which is a closed interval $[p, 2]$.

Now, consider the second assertion. Suppose instead that $G(5) = G(5-)$. We shall show that this implies that $F(2) = F(2-) = 1$, and the latter leads to a contradiction.

Suppose that $G(5) = G(5-) = 1$, which implies by step 6 that $G(3.5) = 1$. There are two possibilities: either $G(3.5) - G(3.5-) > 0$ or $G(3.5) = G(3.5-)$. In the former case, $F(\cdot)$ cannot have a jump at $p = 2$: pricing at $2 - \epsilon$ will be better than pricing at 2 for firm L. Thus $F(2) = F(2-)$. In the latter case, firm HL always prices below 3.5, implying that pricing at $p = 2$ is strictly dominated by pricing at, say, $p = 0.9$. Thus again, $F(2) = F(2-)$. Now, can it really happen that $F(2) = F(2-)$ in equilibrium? Note that the above first assertion says that $G(3.5-) > 0$, and hence there exists a pure-strategy best response $q < 3.5$ for firm HL, such that, if π_j stands for firm j 's equilibrium payoff, then

$$2q[1 - F(q - 1.5)] + qF(q - 1.5) \equiv \pi_{HL} \geq 5.$$

This gives

$$F(q - 1.5) = 2 - \frac{\pi_{HL}}{q}.$$

As $q \uparrow 3.5$ so that $F(q - 1.5)$ tends to 1 (as $F(2) = F(2-)$), we need $\pi_{HL} = 3.5 < 5$, which is a contradiction. Hence we conclude that $q = 5$ must be a point of jump for $G(\cdot)$. \parallel

Step 8: The complete characterization of equilibrium.

By step 7, $\pi_{HL} = 5$ with $G(\cdot)$ having a prob. mass at $q = 5$. Steps 4,5, and 7 together imply that $G(\cdot)$ is strictly increasing and continuous on $q \in [\underline{q}, 3.5)$, and $F(\cdot)$ is strictly increasing and continuous on $p \in [\underline{p}, 2)$, where $\underline{q} = \underline{p} + 1.5$, and $\underline{p} \geq 1$. For any $q \in [\underline{q}, 3.5)$, we must have

$$F(q - 1.5) = 2 - \frac{\pi_{HL}}{q} = 2 - \frac{5}{q},$$

or equivalently,

$$F(p) = 2 - \frac{5}{p + 1.5}, \quad \forall p \in [\underline{p}, 2),$$

implying $\underline{p} = 1$. This also implies that $F(2-) < 1$, and $\Delta F(2) = \frac{4}{7}$. To

sum up, we have

$$F(p) = \begin{cases} 0, & p < 1; \\ 2 - \frac{5}{p+1.5}, & p \in [1, 2); \\ 1, & p \geq 2. \end{cases}$$

Correspondingly, for $p \in [1, 2)$, we have

$$p[1 - G(p + 1.5)] \equiv \pi_L \geq 1,$$

which gives

$$G(p + 1.5) = 1 - \frac{\pi_L}{p}.$$

What is π_L ? Since pricing at \underline{p} is optimal for firm L, we have

$$\pi_L = \underline{p} = 1.$$

It follows that

$$G(q) = 1 - \frac{1}{q - 1.5}, \quad \forall q \in [\underline{q}, 3.5).$$

It follows that

$$\underline{q} = 2.5,$$

and since this price is optimal for firm HL, it also allows us to confirm one more time that

$$\pi_{HL} = 2 \times \underline{q} = 5.$$

We now that

$$G(3.5-) = \frac{1}{2},$$

and since $\Delta F(2) > 0$, we must have $\Delta G(3.5) = 0$, implying that

$$\Delta G(5) = 1 - G(3.5) = 1 - G(3.5-) = \frac{1}{2}.$$

To sum up, we have

$$G(q) = \begin{cases} 0, & q < 2.5; \\ 1 - \frac{1}{q-1.5}, & q \in [2.5, 3.5); \\ \frac{1}{2}, & q \in [3.5, 5); \\ 1, & q \geq 5. \end{cases}$$

Thus $F(\cdot)$ and $G(\cdot)$ constitutes the unique NE of this game.

26. **Example 8:** A monopolistic firm can costlessly produce a durable good and sell it to consumers at dates 1 and 2. At each date $t = 1, 2$, the demand for the durable good's service is $D(q) = 1 - q$. The firm seeks to maximize the sum of discounted profits over the two periods. The firm and consumers have a common discount factor $\delta \in (0, 1]$. Let q_t be the quantity produced by the firm at date t . If at date 1, q_1 units are produced, the firm has two options: either the q_1 units can be sold to consumers and allow the latter to freely resell at date 2 (assume that the durable good never depreciates), or they can be leased to consumers. We shall show that leasing is generally better than selling. As we shall see, the entire problem hinges on whether the firm can internalize the date-2 price impact brought about by the newly produced q_2 .

Let us ask: If the firm can sign a full-commitment long-term contract at $t = 0$ to sell the product over the two periods, what would the optimal selling policy (q_1, q_2) be? Note that for the market to clear at $t = 2$, given any commitment (q_1, q_2) , $p_2 = 1 - q_1 - q_2$. Thus consumers are willing to pay the price

$$p_1 = (1 - q_1) + \delta p_2$$

for the quantity q_1 at $t = 1$, or after being rearranged,

$$p_1 = (1 + \delta)(1 - q_1) - \delta q_2.$$

The interpretation is that consumers at date 1, expecting that the firm will produce an additional amount of q_2 at date 2, realize that the product's date-2 resale value will be reduced by q_2 . With rational expectations, consumers' willingness to pay for the durable good at date 1 reduces by q_2 accordingly. The firm can benefit from selling q_2 at date 2 by $q_2 p_2$. The total sum of discounted profits is

$$f(q_1, q_2) = q_1[(1 + \delta)(1 - q_1) - \delta q_2] + \delta q_2[1 - q_1 - q_2].$$

Note the following partial derivatives:

$$f_1 = (1 + \delta)(1 - 2q_1) - 2\delta q_2,$$

$$f_2 = -\delta q_1 + \delta(1 - q_1 - 2q_2),$$

$$f_{11} = -2(1 + \delta), \quad f_{22} = -2\delta = f_{12},$$

which imply that

$$f_{11} < 0, \quad f_{11}f_{22} - f_{12}^2 = 4\delta > 0,$$

showing that f is strictly concave in (q_1, q_2) , so that the unique optimal selling policy solves

$$f_1 = f_2 = 0, \Rightarrow q_1 = \frac{1}{2}, \quad q_2 = 0.$$

Note that, by inspecting f_2 , producing $q_2 > 0$ at date 2 can be beneficial if and only if the marginal profit from selling q_2 is more than the loss in the resale value of the product; i.e.

$$[1 - q_1 - 2q_2]dq_2 > q_1dq_2,$$

or

$$q_1 \leq \frac{1}{2} - q_2 < \frac{1}{2}.$$

It follows that any $q_2 > 0$ is suboptimal: the firm can at least commit to the optimal static policy $q_1 = \frac{1}{2}$ and can charge $p_1 = \frac{1+\delta}{2}$. At this point, consider increasing q_2 slightly from zero. The marginal benefit is $[1 - q_1 - 2q_2]dq_2$ which is lower than q_1dq_2 (all the q_1 units sold at date 1 lose a resale value of dq_2) at $q_1 = \frac{1}{2}$. As the function f is strictly concave, this local property implies that it holds true globally (roughly, for q_2 far apart from zero, things only get worse).

What happens here is that after selling the q_1 units of the durable good, the firm will not fully internalize the price impact brought about by supplying the additional $q_2 > 0$ units at date 2: by producing the additional $q_2 > 0$, the date-2 market value of all the $q_1 + q_2$ units drops, but since the firm only possesses q_2 units of the durable good at date 2, a portion of this loss is incurred to the other suppliers (purchasers at date 1); this raises the firm's incentive to over-produce, leading to an aggregate output level higher than the monopoly case, so that the sum of discounted profits is lower than what the firm can get if it commits to maintain the supply quantity at the monopoly level $\frac{1}{2}$. Since all other date-2 suppliers are actually purchasers at date 1, and since these

purchasers have rational expectations, this loss of profits must again be born by the firm in the date-1 equilibrium. More precisely, at date 1, consumers when purchasing q_1 realize that the firm cannot commit to not raise the aggregate output level higher than $\frac{1}{2}$, and so the product they purchase today will not have a resale value as high as it would have if the date-2 supply quantity were committed to the monopolistic level, and hence they will not pay the seller as high as they would if the seller could commit to not produce $q_2 > 0$ at date 2.

27. What if the firm can commit at $t = 0$ once and for all to a two-period leasing contract. In other words, the firm commits to retain ownership of the product, and charges (possibly different) rents from users in these two periods. Immediately we claim that under the optimal precommitment leasing contract, the firm gives identical terms of trade for both periods. The idea is that, if one period yields a higher profit than the other, then the firm should have committed to a scheme that assigns the high-profit-period terms of trade to both periods, which contradicts the assumed optimality of the precommitment contract. It then follows that the optimal contract is a straightforward repetition of the static optimal contract: pricing at $\frac{1}{2}$ in each period and producing $q_1 = \frac{1}{2}$ at date 1 and nothing later on. In this case, as it turns out, the scheme can be implemented without commitment power (simply because it is subgame perfect.) This fact implies that, without being able to make long-term commitments, the firm is better off leasing than selling the product at date 1.

Compared with the case of selling, here the firm possesses all the $q_1 + q_2$ units at date 2 so that it must fully internalize the date-2 price impact brought about by any additional units of $q_2 > 0$. This implies that $q_2 = 0$ is optimal for the firm in a subgame starting at date 2 with any $q_1 \geq \frac{1}{2}$. Backward induction then implies that $q_1 = \frac{1}{2}$ is optimal.

What can a money-back-guarantee (MBG) do to improve the selling scheme when direct commitments to output levels are infeasible? With such a guarantee, at date 2, given q_1 , the firm chooses q_2 to

$$\max_{q_2} (q_1 + q_2)(1 - q_1 - q_2).$$

(We have assumed that $p_2 \leq \frac{\delta}{1+\delta}p_1$, and this can be verified to be

necessary for equilibrium with $q_2 \geq 0$.) Note that with the guarantee, *the price charged to date-1 consumers is actually contingent on the realized p_2* . But then, the seller must fully internalize the date-2 price impact brought about by any additional $q_2 > 0$. Hence it should not be surprising that an MBG can help restore efficiency.

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