Game Theory with Applications to Finance and Marketing, I

Homework 0, due in recitation on 10/4.

In this exercise, we shall review Kuhn-Tucker Theorem and then apply the theorem to solve explicitly a screening game. Submit your solutions to Problems 8-15 to the TA on October 4.

1. A symmetric matrix $\mathbf{A}_{n\times n}$ is **positive definite** (or PD), if for all $\mathbf{x}_{n\times 1} \neq \mathbf{0}_{n\times 1}$, we have $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$. A symmetric matrix $\mathbf{A}_{n\times n}$ is **negative definite** (or ND), if $-\mathbf{A}$ is positive definite. A symmetric matrix $\mathbf{A}_{n\times n}$ is **positive semi-definite** (or PSD), if for all $\mathbf{x}_{n\times 1} \in \mathcal{R}^n$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$. A symmetric matrix $\mathbf{A}_{n\times n}$ is **negative semi-definite** (or NSD), if $-\mathbf{A}$ is positive semi-definite.

2. Consider a twice differentiable function $f : \mathcal{R}^n \to \mathcal{R}$. Let the $Df : \mathcal{R}^n \to \mathcal{R}^n$ be the vector function

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix},$$

which will be referred to as the *gradient* of f. Let $D^2 f : \mathcal{R}^n \to \mathcal{R}^{n^2}$ be the matrix function

$$D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix},$$

which will be referred to as the *Hessian* of f.

3. A function $f : \mathcal{R}^n \to \mathcal{R}$ is *concave*, if for all $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ and all $\lambda \in [0,1]$, $f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$. A concave function is said to be strictly concave if the above defining inequality is always strict. A function f is (strictly) *convex* if $-f$ is (strictly) concave. A function is *affine* if it is both concave and convex. An affine function is *linear* if it passes through the origin; that is, if $f(\mathbf{0}_{n\times 1}) = 0$. Note that by definition, f is concave if f is strictly concave.

Theorem 1 *A twice differentiable function $f : \mathcal{R}^n \to \mathcal{R}$ is concave (respectively, strictly concave) if and only if $D^2 f$ is a negative semi-definite (respectively, definite) matrix at each and every $\mathbf{x} \in \mathcal{R}^n$.*

Sometimes we will consider a function f defined on an open rectangle in $\Re^n$, in which case the above theorem and the theorems that will follow remain valid.

Example 1 *Define the function*

$$f(x,y) = x^a y^{1-a}, \ \forall x, y \in (0, +\infty),$$

where a is a constant with $0 < a < 1$. Clearly $f : (0,+\infty) \times (0,+\infty) \to \Re$ is twice continuously differentiable. Let us verify that f is strictly concave. Note that

$$D^2 f = a(1-a)x^{a-2}y^{-1-a} \begin{bmatrix} -y^2 & xy \\ xy & -x^2 \end{bmatrix}.$$

Hence for any

$$\begin{bmatrix} k \\ h \end{bmatrix} \in \Re^2,$$

we have

$$\begin{bmatrix} k & h \end{bmatrix} D^2 f \begin{bmatrix} k \\ h \end{bmatrix}$$

$$= a(1-a)x^{a-2}y^{-1-a}\begin{bmatrix} k & h \end{bmatrix}\begin{bmatrix} -y^2 & xy \\ xy & -x^2 \end{bmatrix}\begin{bmatrix} k \\ h \end{bmatrix}$$
$$= -a(1-a)x^{a-2}y^{-1-a}(ky-hx)^2 \leq 0,$$

and the last inequality is strict unless $k = h = 0$. This proves that $D^2 f$ is negative definite for all $x, y \in (0, +\infty)$, and hence $f(\cdot,\cdot)$ is strictly concave.

Example 2 *Define the function*

$$f(x) = \sqrt{x}, \quad \forall x > 0.$$

Clearly $f : (0, +\infty) \to \Re$ is twice continuously differentiable. Note that

$$D^2 f = \frac{\partial^2 f}{(dx)^2} = -\frac{1}{4x^{\frac{3}{2}}},$$

so that for all $k \in \Re$, we have

$$k \times D^2 f \times k = -\frac{k^2}{4x^{\frac{3}{2}}} \leq 0,$$

and the last inequality is strict unless $k = 0$. This proves that $D^2 f$ is negative definite for all $x > 0$, and hence $f(\cdot)$ is strictly concave.

Indeed, the preceding example shows that for a twice differentiable function $f : \Re \to \Re$ to be strictly concave it is necessary and sufficient that $f'' < 0$ at all $x \in \Re$. Similarly, for a twice differentiable function $f : \Re \to \Re$ to be concave it is necessary and sufficient that $f'' \leq 0$ at all $x \in \Re$.

Example 3 *Define the function*

$$f(x, y) = g(x) + h(y),$$

where $g, h : \Re \to \Re$ are two strictly concave twice-differentiable functions. Then $f(\cdot, \cdot)$ is a strictly concave function also. Indeed, we have in this case

$$D^2 f = \begin{bmatrix} g''(x) & 0 \\ 0 & h''(y) \end{bmatrix},$$

so that given any

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \Re^2,$$

we have

$$\begin{bmatrix} a & b \end{bmatrix} D^2 f \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= a^2 g''(x) + b^2 h''(y) \leq 0,$$

and the last inequality holds strictly except in the case where $a = b = 0$. This proves that $D^2 f$ is negative definite, and hence $f(\cdot, \cdot)$ is strictly concave.

The preceding example shows that a twice-differentiable additively separable function

$$f(x_1, x_2, \cdots, x_n) = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)$$

is strictly concave if for all j, $f_j : \Re \to \Re$ is strictly concave and twice differentiable.

Theorem 2 *Suppose that $f : \Re^n \to \Re$ is twice-differentiable and concave, then for all $\mathbf{x}, \mathbf{a} \in \Re^n$, we have*

$$f(\mathbf{x}) - f(\mathbf{a}) \leq Df(\mathbf{a})'(\mathbf{x} - \mathbf{a}).$$

The above inequality becomes strict if f is strictly concave and $\mathbf{x} \neq \mathbf{a}$.

Proof. Recall the following Taylor's Theorem with Remainder: if $f : \Re^n \to \Re$ is twice continuously differentiable, then for each $\mathbf{x}, \mathbf{a} \in \Re^n$, there exists some $\mathbf{y}$ lying on the line segment connecting $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})'(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})' D^2 f(\mathbf{y})(\mathbf{x} - \mathbf{a}).$$

Now suppose that f is concave, so that $D^2 f(\mathbf{y})$ is negative semi-definite. Then we have

$$\frac{1}{2}(\mathbf{x} - \mathbf{a})' D^2 f(\mathbf{y})(\mathbf{x} - \mathbf{a}) \leq 0,$$

so that

$$f(\mathbf{x}) - f(\mathbf{a}) \leq Df(\mathbf{a})'(\mathbf{x} - \mathbf{a}).$$

If f is strictly concave, then $D^2 f(\mathbf{y})$ is negative definite, and given $\mathbf{x} \neq \mathbf{a}$, we have

$$\frac{1}{2}(\mathbf{x} - \mathbf{a})' D^2 f(\mathbf{y})(\mathbf{x} - \mathbf{a}) < 0,$$

implying that

$$f(\mathbf{x}) - f(\mathbf{a}) < Df(\mathbf{a})'(\mathbf{x} - \mathbf{a}). \qquad \text{Q.E.D.}$$

4. Consider the following maximization program (P):

$$\max_{\mathbf{x} \in \mathcal{R}^n} f(\mathbf{x})$$

subject to

$$\forall i = 1, 2, \cdots, m, \quad g_i(\mathbf{x}) \leq 0,$$

where the $m + 1$ functions $f, g_1, g_2, \cdots, g_m$ are all twice differentiable.

Theorem 3 *(Kuhn-Tucker Theorem) Given the maximization program (P), suppose that there exists some $\hat{\mathbf{x}}$ such that $g_i(\hat{\mathbf{x}}) < 0$ for all $i = 1, 2, \cdots, m$. (This is called the* **Slater Condition***.) Then the following statements are true.*

- *(***Necessary Conditions.***) If $\mathbf{x}^*$ is a solution to (P), then there must exist m non-negative constants $\mu_1, \mu_2, \cdots, \mu_m$ (called the* **Lagrange multipliers** *for the m constraints) such that (i) $Df(\mathbf{x}^*)$ is a linear combination of $Dg_1(\mathbf{x}^*), Dg_2(\mathbf{x}^*), \cdots, Dg_m(\mathbf{x}^*)$ with $\mu_1, \mu_2, \cdots, \mu_m$ being the corresponding weightings; that is,*

$$\sum_{i=1}^{m} \mu_i Dg_i(\mathbf{x}^*) = Df(\mathbf{x}^*);$$

and (ii) (complementary slackness)

$$\forall i = 1, 2, \cdots, m, \qquad \mu_i g_i(\mathbf{x}^*) = 0.$$

- *(***Sufficient Conditions.***) Conversely, if f is concave and for all $i = 1, 2, \cdots, m$, $g_i : \mathcal{R}^n \to \mathcal{R}$ is convex, and if there exist $\mathbf{x}^*$ and m non-negative constants $\mu_1, \mu_2, \cdots, \mu_m$ satisfying the above (i) and (ii), then $\mathbf{x}^*$ is a solution to the above program (P).*
- *(***Uniqueness.***) If f is strictly concave and for all $i = 1, 2, \cdots, m$, $g_i : \mathcal{R}^n \to \mathcal{R}$ is convex, then when (P) has a solution, the solution is unique.*

Let us sketch the proof for the last two assertions. Consider the sufficiency. Suppose that f is concave and for all $i = 1, 2, \cdots, m$, $g_i : \mathcal{R}^n \to \mathcal{R}$ is convex. Suppose that there exist $\mathbf{x}^*$ and m non-negative constants $\mu_1, \mu_2, \cdots, \mu_m$ satisfying the above (i) and (ii). We must show that given any $\mathbf{x}$ such that $g_i(\mathbf{x}) \leq 0$ for all $i = 1, 2, \cdots, m$, we must have

$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq 0.$$

To this end, note that

$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq Df(\mathbf{x}^*)'(\mathbf{x} - \mathbf{x}^*)$$

$$= \sum_{i=1}^{m} \mu_i Dg_i(\mathbf{x}^*)'(\mathbf{x} - \mathbf{x}^*) \leq \sum_{i=1}^{m} \mu_i [g_i(\mathbf{x}) - g_i(\mathbf{x}^*)]$$

$$= \sum_{i=1}^{m} \mu_i g_i(\mathbf{x}) \leq 0,$$

where in the above the first inequality follows from Theorem 2 and the fact that f is concave; the first equality follows from the fact that the Kuhn-Tucker condition (i) holds for $\mathbf{x}^*$ and $\mu_1, \mu_2, \cdots, \mu_m$; the second inequality follows from Theorem 2 and the fact that $\mu_i \geq 0$ and g_i is convex for all $i = 1, 2, \cdots, m$; the last equality follows from the fact that the Kuhn-Tucker condition (ii) holds for $\mathbf{x}^*$ and $\mu_1, \mu_2, \cdots, \mu_m$; and the last inequality follows from the fact that $g_i(\mathbf{x}) \leq 0$ for all $i = 1, 2, \cdots, m$.

Now, consider uniqueness. Suppose that $\mathbf{x}^*$ and $\mathbf{x}^{**}$ are two distinct solutions to (P). This means that

$$f(\mathbf{x}^*) = f(\mathbf{x}^{**}) \equiv \hat{f},$$

and

$$g_i(\mathbf{x}^{**}), g_i(\mathbf{x}^*) \leq 0, \quad \forall i = 1, 2, \cdots, m,$$

and moreover, if $\mathbf{x}$ is such that

$$g_i(\mathbf{x}) \leq 0, \quad \forall i = 1, 2, \cdots, m,$$

then it must be that

$$f(\mathbf{x}) \leq \hat{f}.$$

Define $\mathbf{x} = \frac{\mathbf{x}^* + \mathbf{x}^{**}}{2}$. By convexity of g_i we have

$$g_i(\mathbf{x}) = g_i(\frac{\mathbf{x}^* + \mathbf{x}^{**}}{2}) \leq \frac{1}{2} g_i(\mathbf{x}^*) + \frac{1}{2} g_i(\mathbf{x}^{**}) \leq 0, \quad \forall i = 1, 2, \cdots, m,$$

showing that $\frac{\mathbf{x}^* + \mathbf{x}^{**}}{2}$ is also a feasible solution. By strict concavity of f, however, we have

$$f(\mathbf{x}) = f(\frac{\mathbf{x}^* + \mathbf{x}^{**}}{2}) > \frac{1}{2} f(\mathbf{x}^*) + \frac{1}{2} f(\mathbf{x}^{**}) = \hat{f},$$

which is contradiction. Hence whenever (P) has a solution, the solution must be unique. Q.E.D.

5. We shall now solve a screening game using Theorem 3. Consider a monopolistic firm M facing heterogeneous consumers with unit demand. The final users can be classified into regular buyers and gift-receivers.

A regular buyer is one who purchases the product from M for his own consumption. On the other hand, to each gift receiver, there correspondingly exists a gift buyer, who makes the purchase for the gift receiver's consumption. We assume that the population of regular buyers is $1-\pi$, and the population of gift buyers (which is the same as the population of gift receivers) is $\pi \in (0,1)$. Regardless of whether a final user is a regular buyer or a gift receiver, his payoff from consuming a product of quality q at price p is equally likely to be

$$\theta_H q - p$$

or

$$\theta_L q - p,$$

where

$$\theta_H > \theta_L > 0, \quad q, p \geq 0.$$

It is everyone's common knowledge that the population of final users with taste parameter θ_H equals the population of final users with taste parameter θ_L. A final user knows his own taste parameter. A gift buyer is endowed with a signal $s \in \{H, L\}$ that reveals some information regarding his gift receiver's taste parameter θ_j, where we assume that

$$\text{prob.}(s = H | j = H) = \text{prob.}(s = L | j = L) = \gamma \geq \frac{1}{2}.$$

Based on his signal s, a gift-buyer expects his gift-receiver to obtain a payoff

$$E[\tilde{\theta}|s]q \equiv \theta^s q$$

from consuming a product of quality q. Thus we assume that the gift-buyer's own payoff from purchasing a product of quality q at price p is

$$\theta^s q - p, \quad \forall s = H, L.$$

To further ease notation, let us define

$$\delta \equiv \theta_H - \theta_L, \quad \theta \equiv \theta_L.$$

Let us write

$$\theta_1 = \theta_L, \ \theta_2 = \theta^L, \quad \theta_3 = \theta^H, \quad \theta_4 = \theta_H.$$

Define correspondingly,

$$\pi_1 = \pi_4 \equiv \frac{1-\pi}{2}, \ \pi_2 = \pi_3 \equiv \frac{\pi}{2}[\gamma + (1-\gamma)] = \frac{\pi}{2}.$$

Note that π_1 is the population of low-valuation regular buyers, π_2 the population of the low-valuation gift buyers, π_3 the population of the high-valuation gift buyers, and π_4 the population of the high-valuation regular buyers. Correspondingly, θ_j is the taste parameter for the π_j-segment of buyers.

(**Problem 1**.) Verify the following statements.

- $\theta_4 \geq \theta_3 \geq \theta_2 \geq \theta_1 = \theta$.
- $\theta_4 = \theta + \delta$.
- $\theta_3 = \gamma\theta_4 + (1-\gamma)\theta_1 = \theta + \gamma\delta$.
- $\theta_2 = \gamma\theta_1 + (1-\gamma)\theta_4 = \theta + (1-\gamma)\delta$.
- $\theta_1 = \theta$.
- $\theta_4 - \theta_3 = (1-\gamma)\delta$.
- $\theta_3 - \theta_2 = (2\gamma - 1)\delta$.
- $\theta_2 - \theta_1 = (1-\gamma)\delta$.

6. Two extreme cases involve respectively $\gamma = \frac{1}{2}$ and $\gamma = 1$.

   - When $\gamma = \frac{1}{2}$, s is totally uninformative regarding θ_j, and in this case the gift-buyer is essentially a stranger to the gift-receiver, so that the gift-buyer has the same information as M does regarding θ_j. In this case, gift-buyers become homogeneous (i.e., $\theta^H = \theta^L = \theta_0 \equiv \theta + \frac{1}{2}\delta$), and their common payoff from buying a product (q,p) is
   $$\theta_0 q - p.$$
   - When $\gamma = 1$, the gift-buyer has the same information as the gift-receiver (i.e., $\theta^H = \theta_H$ and $\theta^L = \theta_L$), and in this special case, the gift-buying demand becomes identical to the regular demand.

7. By the revelation principle, which we shall review later on, M will optimally design a product line that contains 4 product items intended for respectively the 4 segments of buyers, denoted by respectively

$$q_4 = q_H, \quad q_3 = q^H, \quad q_2 = q^L, \quad q_1 = q_L,$$

and M will choose corresondingly 4 prices (p_1, p_2, p_3, p_4) to maximize M's expected profit. We assume that M needs to incur a unit production cost $\frac{kq^2}{2}$ when producing an item of quality q, where $k > 0$. For simplicity, M incurs no other costs.

The timing of relevant events in this screening game is as follows.

- M first designs a product line and chooses corresondingly 4 prices, $\{(q_j, p_j); j = 1, 2, 3, 4\}$, such that these 8 control variables satisfy the constraints IR_1, LDIC, and monotonicity in q_i's. That is, M seeks to

$$\max_{p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4 \in \mathcal{R}_+} \sum_{j=1}^{4} \pi_j [p_j - \frac{kq_j^2}{2}]$$

subject to[1]

$$\begin{cases} \text{(LDIC)} & \forall j \geq 2 \quad \theta_j q_j - p_j \geq \theta_j q_{j-1} - p_{j-1}; \\ \text{(IR1)} & \qquad \theta_1 q_1 - p_1 \geq 0; \\ \text{(monotonicity)} & \forall i \geq j \quad q_i \geq q_j. \end{cases}$$

- After M announces $\{(q_j, p_j); j = 1, 2, 3, 4\}$, regular buyers and gift-buyers arrive at the store, and each buyer can purchase exactly one q_j or purchase nothing.

Note that everyone in this game is risk neutral: M seeks to maximize its expected profit, and a regular and gift buyer seeks to maximize his expected consumer surplus.

[1] LDIC stands for "Local Downward Incentive Compatibility" conditions.

8. Let us first consider the benchmark case where the gift-buying demand does not exist (i.e., $\pi = 0$). M seeks to

$$\max_{p_1,p_4,q_1,q_4\in\mathcal{R}_+} \sum_{j=1,4} \pi_j[p_j - \frac{kq_j^2}{2}]$$

subject to

$$\begin{cases} \text{(LDIC)} & \theta_4 q_4 - p_4 \geq \theta_4 q_1 - p_1; \\ \text{(IR1)} & \theta_1 q_1 - p_1 \geq 0; \\ \text{(monotonicity)} & q_4 \geq q_1. \end{cases}$$

(**Problem 2**.) Show that in the absence of gift buyers, the optimal product line is[2]

$$q_4 = \frac{\theta_H}{k}, \quad q_1 = \begin{cases} q_1^0 \equiv \frac{\theta_L - \delta}{k}, & \text{if } \theta_L \geq \delta; \\ 0, & \text{otherwise.} \end{cases}$$

Show that the corresponding optimal pricing strategy is

$$p_4 = \theta_H(q_4 - q_1) + \theta_L q_1, \quad p_1 = \theta_L q_1.$$

9. Next, consider the case where the gift-buying demand exists, and is as described above. M seeks to

$$\text{(P}'\text{)} \quad \max_{p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4\in\mathcal{R}_+} \frac{1-\pi}{2}[p_4 - \frac{kq_4^2}{2}] + \frac{\pi}{2}[p_3 - \frac{kq_3^2}{2}] + \frac{\pi}{2}[p_2 - \frac{kq_2^2}{2}] + \frac{1-\pi}{2}[p_1 - \frac{kq_1^2}{2}],$$

subject to

$$\text{(LDIC)} \quad \theta_j q_j - p_j \geq \theta_j q_{j-1} - p_{j-1}, \quad \forall j \in \{2,3,4\};$$

[2]**Hint**: Show that at optimum, given (p_4, q_1, q_4), p_1 must be such that (IR_1) is binding (i.e., it holds as an equality). Then, show that at optimum, given (q_1, q_4) and that $p_1 = \theta_1 q_1$, p_4 must be such that (LDIC) is binding (i.e., it holds as an equality). Finally, solve for the optimal q_1^* and q_4^* in the absence of monotonicity constraint, assuming that LDIC and IR_1 are both binding, and verify that these optimal q_j^*'s automatically satisfy the monotonicity constraint.

$$(\text{IR}_1) \quad \theta_1 q_1 - p_1 \geq 0;$$

$$(\text{Monotonicity}) \quad q_4 \geq q_3 \geq q_2 \geq q_1.$$

(**Problem 3**.) Show (by contraposition) that IR_1 and LDIC must be binding at optimum of program (P′),[3] and hence we can express p_1, p_2, p_3, p_4 in terms of q_1, q_2, q_3, q_4 as follows:

$$p_4 = \theta_4 q_4 - (\theta_4 - \theta_3)q_3 - (\theta_3 - \theta_2)q_2 - (\theta_2 - \theta_1)q_1,$$

$$p_3 = \theta_3 q_3 - (\theta_3 - \theta_2)q_2 - (\theta_2 - \theta_1)q_1,$$

$$p_2 = \theta_2 q_2 - (\theta_2 - \theta_1)q_1,$$

$$p_1 = \theta_1 q_1.$$

Then show that M's maximization problem can be expressed as

$$(\text{P}) \quad \max_{q_1,q_2,q_3,q_4} f(q_1,q_2,q_3,q_4) \equiv \frac{1-\pi}{2}[\theta_4 q_4 - (\theta_4-\theta_3)q_3 - (\theta_3-\theta_2)q_2 - (\theta_2-\theta_1)q_1 - \frac{kq_4^2}{2}]$$

$$+\frac{\pi}{2}[\theta_3 q_3 - (\theta_3 - \theta_2)q_2 - (\theta_2 - \theta_1)q_1 - \frac{kq_3^2}{2}]$$

$$+\frac{\pi}{2}[\theta_2 q_2 - (\theta_2 - \theta_1)q_1 - \frac{kq_2^2}{2}]$$

$$+\frac{1-\pi}{2}[\theta_1 q_1 - \frac{kq_1^2}{2}]$$

subject to the following three monotonicity constraints[4]

$$(\text{M1}) \quad q_1 \geq 0 \Leftrightarrow g_1(q_1,q_2,q_3) \equiv -q_1 \leq 0;$$

$$(\text{M2}) \quad q_2 \geq q_1 \Leftrightarrow g_2(q_1,q_2,q_3) \equiv q_1 - q_2 \leq 0;$$

$$(\text{M3}) \quad q_3 \geq q_2 \Leftrightarrow g_3(q_1,q_2,q_3) \equiv q_2 - q_3 \leq 0.$$

Note that

$$f(q_1,q_2,q_3,q_4) = f_1(q_1) + f_2(q_2) + f_3(q_3) + f_4(q_4),$$

[3]Look at the preceding footnote.

[4]We have left out $q_4 \geq q_3$ for a reason that will soon become clear.

where verify that the 4 strictly concave functions

$$f_4(q_4) = \frac{1-\pi}{2}[\theta_4 q_4 - \frac{kq_4^2}{2}],$$

$$f_3(q_3) = \frac{\pi}{2}[\theta_3 q_3 - \frac{kq_3^2}{2}] - \frac{1-\pi}{2}(\theta_4 - \theta_3)q_3,$$

$$f_2(q_2) = \frac{\pi}{2}[\theta_2 q_2 - \frac{kq_2^2}{2}] - (\frac{\pi}{2} + \frac{1-\pi}{2})(\theta_3 - \theta_2)q_2,$$

$$f_1(q_1) = \frac{1-\pi}{2}[\theta_1 q_1 - \frac{kq_1^2}{2}] - (\frac{\pi}{2} + \frac{\pi}{2} + \frac{1-\pi}{2})(\theta_2 - \theta_1)q_1.$$

10. (**Problem 4.**) Show that in program (P) f is strictly concave and for all $j = 1, 2, 3$, g_j is convex, and that the Slater condition is satisfied in (P).[5] Conclude by Theorem 3 that there exists a unqiue solution $(q_1^*, q_2^*, q_3^*, q_4^*)$ to (P).

11. (**Problem 5.**) Show that at optimum of program (P), we must have (*efficiency at the top*)

$$q_4^* = \frac{\theta_H}{k}.$$

Hence from now on we shall write

$$f(q_1, q_2, q_3, q_4^*) = f(q_1, q_2, q_3) = f_1(q_1) + f_2(q_2) + f_3(q_3) + f_4(q_4^*),$$

Show that

$$f_j'(\hat{q}_j) = 0, \quad \forall j = 1, 2, 3,$$

where

$$\hat{q}_1 \equiv \frac{1}{k(1-\pi)}[(1-\pi)\theta - (1-\gamma)\delta(1+\pi)],$$

$$\hat{q}_2 \equiv \frac{1}{\pi k}[\pi\theta + \delta[\pi(1-\gamma) - (2\gamma - 1)]],$$

$$\hat{q}_3 \equiv \frac{1}{\pi k}[\pi\theta + \delta(\pi + \gamma - 1)].$$

[5] Try $q_j = j$ for all $j \in \{1, 2, 3\}$.

12. (**Problem 6**.) Verify the following statements.

- $\hat{q}_3 \geq \hat{q}_2 \Leftrightarrow \gamma \geq \frac{2}{3+\pi}$;
- $\hat{q}_2 \geq \hat{q}_1 \Leftrightarrow \gamma \leq \frac{1+\pi}{2}$;
- $\hat{q}_1 \geq 0 \Leftrightarrow \theta \geq (\frac{1+\pi}{1-\pi})(1-\gamma)\delta$;
- $\hat{q}_2 \geq 0 \Leftrightarrow \theta \geq \frac{\delta}{\pi}[(2\gamma-1)-\pi(1-\gamma)]$;
- $\hat{q}_3 \geq 0 \Leftrightarrow \theta \geq \frac{\delta}{\pi}[1-\gamma-\pi]$.

13. (**Problem 7**.) Verify the following statements.

$$f_1'(q_1) = \frac{1-\pi}{2}(\theta - kq_1) - \frac{1+\pi}{2}(1-\gamma)\delta,$$

$$f_2'(q_2) = \frac{\pi}{2}[\theta + (1-\gamma)\delta - kq_2] - \frac{1}{2}(2\gamma-1)\delta,$$

$$f_3'(q_3) = \frac{\pi}{2}[\theta + \gamma\delta - kq_3] - \frac{1-\pi}{2}(1-\gamma)\delta,$$

$$Df = \begin{bmatrix} f_1'(q_1) \\ \\ f_2'(q_2) \\ \\ f_3'(q_3) \end{bmatrix}, \ Dg_1 = \begin{bmatrix} -1 \\ \\ 0 \\ \\ 0 \end{bmatrix}, \ Dg_2 = \begin{bmatrix} 1 \\ \\ -1 \\ \\ 0 \end{bmatrix}, \ Dg_3 = \begin{bmatrix} 0 \\ \\ 1 \\ \\ -1 \end{bmatrix}.$$

14. Now we apply Theorem 3 and solve (P). For $j = 1, 2, 3$, let μ_j denote the Lagrange multiplier associated with the constraint $g_j(q_1, q_2, q_3) \leq 0$. Since each of the three monotonicity constraints may or may not be binding at optimum, there are 8 cases for us to consider.

- (**Case 1.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) < 0$, for all $j = 1, 2, 3$.

  (**Problem 8**.) Show that if

$$\frac{1+\pi}{2} > \gamma > \frac{2}{3+\pi}, \quad \theta > (\frac{1+\pi}{1-\pi})(1-\gamma)\delta,$$

then indeed at optimum $g_j(q_1^*, q_2^*, q_3^*) < 0$, for all $j = 1, 2, 3$ and[6]

$$q_j^* = \hat{q}_j, \quad \forall j = 1, 2, 3.$$

- (**Case 2.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) < 0 = g_1(q_1^*, q_2^*, q_3^*)$, for all $j = 2, 3$.

(**Problem 9**.) Show that if

$$\frac{1+\pi}{2} > \gamma > \frac{2}{3+\pi}, \quad \frac{\delta}{\pi}[(2\gamma-1)-\pi(1-\gamma)] < \theta \leq (\frac{1+\pi}{1-\pi})(1-\gamma)\delta,$$

[6]**Hint**: If at optimum,

$$\forall j = 1, 2, 3, \quad g_j(q_1^*, q_2^*, q_3^*) < 0,$$

then by complementary slackness, we have

$$\forall j = 1, 2, 3, \quad \mu_j g_j(q_1^*, q_2^*, q_3^*) = 0, \Rightarrow \mu_j = 0, \quad \forall j = 1, 2, 3,$$

$$\Rightarrow Df(q_1^*, q_2^*, q_3^*) = \sum_{j=1}^{3} \mu_j Dg_j(q_1^*, q_2^*, q_3^*) = \mathbf{0}_{3\times 1}$$

$$\Rightarrow \begin{bmatrix} f_1'(q_1^*) \\ f_2'(q_2^*) \\ f_3'(q_3^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Rightarrow \text{(by Problem 5)} \ \ q_j^* = \hat{q}_j, \quad \forall j = 1, 2, 3.$$

For this to actually be the unique solution to (P), it is necessary that the conjecture

$$\forall j = 1, 2, 3, \quad g_j(q_1^*, q_2^*, q_3^*) < 0,$$

can be confirmed to be true. That is, we require that

$$\forall j = 1, 2, 3, \quad g_j(\hat{q}_1, \hat{q}_2, \hat{q}_3) = g_j(q_1^*, q_2^*, q_3^*) < 0,$$

$$\Leftrightarrow \hat{q}_3 > \hat{q}_2 > \hat{q}_1 > 0,$$

or equivalently, by Problem 6,

$$\frac{1+\pi}{2} > \gamma > \frac{2}{3+\pi}, \quad \theta > (\frac{1+\pi}{1-\pi})(1-\gamma)\delta.$$

then indeed at optimum

$$g_2(q_1^*, q_2^*, q_3^*),\ g_3(q_1^*, q_2^*, q_3^*) < 0 = g_1(q_1^*, q_2^*, q_3^*),$$

and[7]

$$q_1^* = 0, \quad q_j^* = \hat{q}_j, \quad \forall j = 2, 3.$$

- (**Case 3.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) < 0 = g_2(q_1^*, q_2^*, q_3^*)$, for all $j = 1, 3$.

(**Problem 10**.) Show that if

$$\gamma > \max(\frac{1+\pi}{2}, \frac{1-\pi}{1+\pi}), \qquad \theta > \gamma\delta,$$

then indeed at optimum

$$q_1^* = q_2^* < q_3^*,$$

[7]**Hint**: If at optimum,

$$g_2(q_1^*, q_2^*, q_3^*),\ g_3(q_1^*, q_2^*, q_3^*) < 0 = g_1(q_1^*, q_2^*, q_3^*),$$

then by complementary slackness, we have

$$\mu_2 = \mu_3 = 0,$$

$$\Rightarrow Df(q_1^*, q_2^*, q_3^*) = \mu_1 Dg_1(q_1^*, q_2^*, q_3^*)$$

$$\Rightarrow \begin{bmatrix} f_1'(q_1^*) \\ f_2'(q_2^*) \\ f_3'(q_3^*) \end{bmatrix} = \begin{bmatrix} f_1'(0) \\ f_2'(q_2^*) \\ f_3'(q_3^*) \end{bmatrix} = \begin{bmatrix} -\mu_1 \\ 0 \\ 0 \end{bmatrix},$$

$$\Rightarrow \quad \mu_1 = \frac{1+\pi}{2}(1-\gamma)\delta - \frac{1-\pi}{2}\theta, \quad q_1^* = 0, \quad q_j^* = \hat{q}_j, \quad \forall j = 2, 3.$$

For this to actually be the unique solution to (P), it is necessary that $\mu_1 \geq 0$ and that

$$\forall j = 2, 3, \quad g_j(0, \hat{q}_2, \hat{q}_3) < 0,$$

can be confirmed to be true, or equivalently,

$$\frac{1+\pi}{2} > \gamma > \frac{2}{3+\pi}, \quad \frac{\delta}{\pi}[(2\gamma - 1) - \pi(1-\gamma)] < \theta \leq (\frac{1+\pi}{1-\pi})(1-\gamma)\delta.$$

and[8]

$$q_1^* = q_2^* = \frac{\theta - \gamma\delta}{k} < q_3^* = \hat{q}_3.$$

- (**Case 4.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) < 0 = g_3(q_1^*, q_2^*, q_3^*)$, for all $j = 1, 2$.

(**Problem 11**.) Show that if

$$\frac{1}{2} \leq \gamma < \min(\frac{2}{3+\pi}, \frac{4\pi}{(1+\pi)^2}), \qquad \theta > (\frac{1+\pi}{1-\pi})(1-\gamma)\delta,$$

[8]**Hint**: If at optimum,

$$g_1(q_1^*, q_2^*, q_3^*),\ g_3(q_1^*, q_2^*, q_3^*) < 0 = g_2(q_1^*, q_2^*, q_3^*),$$

then $q_1^* = q_2^* = Q$ for some $Q > 0$, and by complementary slackness, we have

$$\mu_1 = \mu_3 = 0,$$

$$\Rightarrow Df(Q, Q, q_3^*) = \mu_2 Dg_2(Q, Q, q_3^*)$$

$$\Rightarrow \begin{bmatrix} f_1'(Q) \\ f_2'(Q) \\ f_3'(q_3^*) \end{bmatrix} = \begin{bmatrix} f_1'(Q) \\ f_2'(Q) \\ f_3'(q_3^*) \end{bmatrix} = \begin{bmatrix} \mu_2 \\ -\mu_2 \\ 0 \end{bmatrix},$$

$$\Rightarrow f_1'(Q) = \frac{1-\pi}{2}(\theta - kQ) - \frac{1+\pi}{2}(1-\gamma)\delta = \mu_2 = -f_2'(Q) = -\frac{\pi}{2}[\theta + (1-\gamma)\delta - kQ] + \frac{1}{2}(2\gamma - 1)\delta,$$

$$\Rightarrow \quad \mu_2 = \frac{1}{2}[(1-\pi)\gamma\delta - (1+\pi)(1-\gamma)\delta], \quad q_1^* = q_2^* = Q = \frac{\theta - \gamma\delta}{k}, \quad q_3^* = \hat{q}_3.$$

For this to actually be the unique solution to (P), it is necessary that

$$0 \leq \mu_2 = \frac{1}{2}[(1-\pi)\gamma\delta - (1+\pi)(1-\gamma)\delta] \Leftrightarrow \gamma \geq \frac{1+\pi}{2}$$

and that

$$\frac{1}{\pi k}[\pi\theta + \delta(\pi + \gamma - 1)] = \hat{q}_3 > Q = \frac{\theta - \gamma\delta}{k} > 0,$$

or equivalently,

$$\gamma > \max(\frac{1+\pi}{2}, \frac{1-\pi}{1+\pi}), \qquad \theta > \gamma\delta.$$

then indeed at optimum

$$0 < q_1^* < q_2^* = q_3^*,$$

and[9]

$$q_3^* = q_2^* = \frac{\theta + (1 - \frac{\gamma(1+\pi)}{2\pi})\delta}{k} > q_1^* = \hat{q}_1.$$

[9]**Hint**: If at optimum,

$$g_1(q_1^*, q_2^*, q_3^*),\ g_2(q_1^*, q_2^*, q_3^*) < 0 = g_3(q_1^*, q_2^*, q_3^*),$$

then $q_3^* = q_2^* = Q$ for some $Q > q_1^* > 0$, and by complementary slackness, we have

$$\mu_1 = \mu_2 = 0,$$

$$\Rightarrow Df(q_1^*, Q, Q) = \mu_3 Dg_3(q_1^*, Q, Q)$$

$$\Rightarrow \begin{bmatrix} f_1'(q_1^*) \\ f_2'(Q) \\ f_3'(Q) \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_3 \\ -\mu_3 \end{bmatrix},$$

$$\Rightarrow f_2'(Q) = \frac{\pi}{2}[\theta + (1-\gamma)\delta - kQ] - \frac{1}{2}(2\gamma - 1)\delta = \mu_3 = -f_3'(Q) = -\frac{\pi}{2}[\theta + \gamma\delta - kQ] + \frac{1-\pi}{2}(1-\gamma)\delta,$$

$$\Rightarrow \quad \mu_3 = \frac{\delta}{4}[2 - 3\gamma - \pi\gamma], \quad q_2^* = q_3^* = Q = \frac{\theta + (1 - \frac{\gamma(1+\pi)}{2\pi})\delta}{k}, \quad q_1^* = \hat{q}_1.$$

For this to actually be the unique solution to (P), it is necessary that

$$0 \le \mu_3 = \frac{\delta}{4}[2 - 3\gamma - \pi\gamma] \Leftrightarrow \gamma \le \frac{2}{3+\pi}$$

and that

$$0 < \frac{1}{k(1-\pi)}[(1-\pi)\theta - (1-\gamma)\delta(1+\pi)] = \hat{q}_1 < Q = \frac{\theta + (1 - \frac{\gamma(1+\pi)}{2\pi})\delta}{k},$$

or equivalently,

$$\frac{1}{2} \le \gamma < \min(\frac{2}{3+\pi}, \frac{4\pi}{(1+\pi)^2}), \qquad \theta > (\frac{1+\pi}{1-\pi})(1-\gamma)\delta.$$

Note that if

$$\max(\frac{2\pi}{1+\pi}, \frac{1}{2}) \le \gamma < \min(\frac{2}{3+\pi}, \frac{4\pi}{(1+\pi)^2}), \qquad \delta > \theta > (\frac{1+\pi}{1-\pi})(1-\gamma)\delta.$$

then the presence of gift-buying demand results in M starting to serve the low-valuation regular buyers, and the quality of the product item $\hat{q}_1$ assigned to these low-valuation regular buyers is decreasing in π and δ but increasing in γ. The larger γ is, the less the valuation differential between θ_1 and θ_2 buyers, and hence the less downward distortion in q_1^* is required. An increase in π or δ implies an increase in the total population of θ_2 and θ_3 buyers or an increase in the valuation differential between these gift buyers and θ_1 regular buyers, and hence the more downward quality distortion in q_1^* is required.

- (**Case 5.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) = 0 > g_3(q_1^*, q_2^*, q_3^*)$, for all $j = 1, 2$.

  (**Problem 12.**) Show that if if either

  $$\pi > \sqrt{5}-2, \quad \frac{1+\pi}{2} > \gamma > \frac{2}{3+\pi}, \quad \delta[\frac{2\gamma-1}{\pi}-(1-\gamma)] \geq \theta > \max[0, \delta(\frac{1-\gamma}{\pi}-1)]$$

  or

  $$\gamma > \max[\frac{1-\pi}{1+\pi}, \frac{1+\pi}{2}], \quad \gamma\delta \geq \theta > \max[0, \delta(\frac{1-\gamma}{\pi}-1)],$$

  then indeed at optimum[10]

  $$q_1^* = q_2^* = 0 < q_3^* = \hat{q}_3.$$

[10]**Hint**: If at optimum,

$$g_1(q_1^*, q_2^*, q_3^*) = g_2(q_1^*, q_2^*, q_3^*) = 0 > g_3(q_1^*, q_2^*, q_3^*),$$

then $q_1^* = 0 = q_2^* < q_3^*$, and by complementary slackness, we have

$$\mu_3 = 0,$$

$$\Rightarrow Df(0, 0, q_3^*) = \mu_1 Dg_1(0, 0, q_3^*) + \mu_2 Dg_2(0, 0, q_3^*)$$

$$\Rightarrow \begin{bmatrix} f_1'(0) \\ f_2'(0) \\ f_3'(q_3^*) \end{bmatrix} = \begin{bmatrix} \mu_2 - \mu_1 \\ -\mu_2 \\ 0 \end{bmatrix},$$

$$\Rightarrow q_3^* = \hat{q}_3, \quad f_2'(0) = \frac{\pi}{2}[\theta + (1-\gamma)\delta - 0] - \frac{1}{2}(2\gamma - 1)\delta = -\mu_2,$$

$$f_1'(0) = \frac{1-\pi}{2}(\theta - 0) - \frac{1+\pi}{2}(1-\gamma)\delta = \mu_2 - \mu_1,$$

$$\Rightarrow \quad \mu_1 = \frac{\gamma\delta - \theta}{2}, \quad \mu_2 = \frac{1}{2}[(2\gamma - 1)\delta - \pi\theta - \pi(1-\gamma)\delta].$$

For this to actually be the unique solution to (P), it is necessary that

$$0 \leq \mu_1 = \frac{\gamma\delta - \theta}{2} \Leftrightarrow \theta \leq \gamma\delta,$$

$$0 \leq \mu_2 = \frac{1}{2}[(2\gamma - 1)\delta - \pi\theta - \pi(1-\gamma)\delta] \Leftrightarrow \theta \leq [\frac{2\gamma - 1}{\pi} - (1-\gamma)]\delta,$$

- (**Case 6.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) = 0 > g_1(q_1^*, q_2^*, q_3^*)$, for all $j = 2, 3$.

(**Problem 13**.) Show that if

$$\frac{1-\pi}{1+\pi} \geq \gamma \geq \max(\frac{1}{2}, 1 - (\frac{1-\pi}{1+\pi})^2), \quad \theta > \frac{\delta(1-\pi)}{1+\pi},$$

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and that

$$0 < q_3^* = \hat{q}_3 \Leftrightarrow \theta > \frac{\delta}{\pi}[1 - \gamma - \pi],$$

or equivalently,

$$\delta \min[\gamma, \frac{2\gamma - 1}{\pi} - (1-\gamma)] \geq \theta > \delta[\frac{1-\gamma}{\pi} - 1].$$

The last inequalities hold if and only if either

$$\pi > \sqrt{5} - 2, \quad \frac{1+\pi}{2} > \gamma > \frac{2}{3+\pi}, \quad \delta[\frac{2\gamma - 1}{\pi} - (1-\gamma)] \geq \theta > \delta[\frac{1-\gamma}{\pi} - 1]$$

or

$$\gamma > \max[\frac{1-\pi}{1+\pi}, \frac{1+\pi}{2}], \quad \gamma\delta \geq \theta > \delta[\frac{1-\gamma}{\pi} - 1].$$

then indeed at optimum[11]

$$q_1^* = q_2^* = q_3^* = \frac{1}{k}[\theta - \frac{(1-\pi)\delta}{1+\pi}] > 0.$$

[11]**Hint**: If at optimum,

$$g_3(q_1^*, q_2^*, q_3^*) = g_2(q_1^*, q_2^*, q_3^*) = 0 > g_1(q_1^*, q_2^*, q_3^*),$$

then $q_1^* = q_2^* = q_3^* = Q > 0$ for some Q, and by complementary slackness, we have

$$\mu_1 = 0,$$
$$\Rightarrow Df(Q,Q,Q) = \mu_2 Dg_2(Q,Q,Q) + \mu_3 Dg_3(Q,Q,Q)$$
$$\Rightarrow \begin{bmatrix} f_1'(Q) \\ f_2'(Q) \\ f_3'(Q) \end{bmatrix} = \begin{bmatrix} \mu_2 \\ \mu_3 - \mu_2 \\ -\mu_3 \end{bmatrix},$$
$$\Rightarrow f_1'(Q) = \frac{1-\pi}{2}(\theta - kQ) - \frac{1+\pi}{2}(1-\gamma)\delta = \mu_2,$$
$$f_2'(Q) = \frac{\pi}{2}[\theta + (1-\gamma)\delta - kQ] - \frac{1}{2}(2\gamma - 1)\delta = \mu_3 - \mu_2,$$
$$f_3'(Q) = \frac{\pi}{2}[\theta + \gamma\delta - kQ] - \frac{1-\pi}{2}(1-\gamma)\delta = -\mu_3,$$
$$\Rightarrow Q = \frac{1}{k}[\theta - \frac{(1-\pi)\delta}{1+\pi}],$$
$$\mu_3 = \frac{\delta}{2}[1 - \gamma - \frac{2\pi}{1+\pi}], \quad \mu_2 = \frac{\delta}{2}[\frac{(1-\pi)^2}{1+\pi} - (1+\pi)(1-\gamma)],$$

For this to actually be the unique solution to (P), it is necessary that

$$0 \leq \mu_2 = \frac{\delta}{2}[\frac{(1-\pi)^2}{1+\pi} - (1+\pi)(1-\gamma)] \Leftrightarrow \gamma \geq 1 - (\frac{1-\pi}{1+\pi})^2,$$
$$0 \leq \mu_3 = \frac{\delta}{2}[1 - \gamma - \frac{2\pi}{1+\pi}] \Leftrightarrow \gamma \leq \frac{1-\pi}{1+\pi},$$

and that

$$0 < Q \Leftrightarrow \theta > \frac{\delta(1-\pi)}{1+\pi},$$

or equivalently,

$$\frac{1-\pi}{1+\pi} \geq \gamma \geq \max(\frac{1}{2}, 1 - (\frac{1-\pi}{1+\pi})^2), \quad \theta > \frac{\delta(1-\pi)}{1+\pi}.$$

Note that this solution is consistent with the solution to Problem 2 if we let $\pi = 0$ and $\gamma \uparrow 1$. Note also that if

$$\frac{1-\pi}{1+\pi} \geq \gamma \geq \max(\frac{1}{2}, 1 - (\frac{1-\pi}{1+\pi})^2), \quad \theta > \delta,$$

- (**Case 7.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) = 0 > g_2(q_1^*, q_2^*, q_3^*)$, for all $j = 1, 3$.

  (**Problem 14.**) Show that if

$$\frac{1}{2} \leq \gamma \leq \frac{2+4\pi}{5+3\pi}, \quad \frac{(1+\pi)(1-\gamma)\delta}{1-\pi} \geq \theta > \max[0, \delta(\frac{\gamma(1+\pi)}{2\pi} - 1)],$$

---

then the presence of gift-buying demand makes all regular buyers weakly worse off. To see this, observe that $Q > q_1^0 = \frac{\theta-\delta}{k}$, and since $p_4 = \theta_4 q_4^* - (\theta_4 - \theta_1)Q$ in the presence of gift-buying demand and $p_4 = p_4 = \theta_4 q_4^* - (\theta_4 - \theta_1)q_1^0$ in the absence of gift-buying demand, the presence of gift-buying demand makes the high-valuation regular buyers worse off. Since the low-valuation regular buyers always receive zero payoffs, our claim is proved. Finally, note that if

$$\frac{1-\pi}{1+\pi} \geq \gamma \geq \max(\frac{1}{2}, 1 - (\frac{1-\pi}{1+\pi})^2), \quad \delta > \theta > \frac{\delta(1-\pi)}{1+\pi},$$

then the presence of gift-buying demand makes the regular buyers weakly better off. Again, the low-valuation regular buyers' welfare remains the same whether or not there is the gift-buying demand. Note that the high-valuation regular buyers obtain zero surplus in the absence of gift buyers (because $\delta > \theta$, implying that θ_1 buyers are unserved, which makes θ_4 buyers the only type of buyers served in equilibrium). Note that these high-valuation regular buyers now receive a positive surplus in the presence of gift buyers. Our claim is thus proved.

then indeed at optimum[12]

$$q_1^* = 0, \quad q_2^* = q_3^* = \frac{1}{k}[\theta + \delta(1 - \frac{\gamma(1+\pi)}{2\pi})] > 0.$$

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[12]**Hint**: If at optimum,

$$g_3(q_1^*, q_2^*, q_3^*) = g_1(q_1^*, q_2^*, q_3^*) = 0 > g_2(q_1^*, q_2^*, q_3^*),$$

then $q_1^* = 0 < q_2^* = q_3^* = Q$ for some Q, and by complementary slackness, we have

$$\mu_2 = 0,$$

$$\Rightarrow Df(0, Q, Q) = \mu_1 Dg_1(0, Q, Q) + \mu_3 Dg_3(0, Q, Q)$$

$$\Rightarrow \begin{bmatrix} f_1'(0) \\ \\ f_2'(Q) \\ \\ f_3'(Q) \end{bmatrix} = \begin{bmatrix} -\mu_1 \\ \\ \mu_3 \\ \\ -\mu_3 \end{bmatrix},$$

$$\Rightarrow f_1'(0) = \frac{1-\pi}{2}(\theta - 0) - \frac{1+\pi}{2}(1-\gamma)\delta = -\mu_1,$$

$$f_2'(Q) = \frac{\pi}{2}[\theta + (1-\gamma)\delta - kQ] - \frac{1}{2}(2\gamma - 1)\delta = \mu_3 = -f_3'(Q) = -\frac{\pi}{2}[\theta + \gamma\delta - kQ] + \frac{1-\pi}{2}(1-\gamma)\delta,$$

$$\Rightarrow Q = \frac{1}{k}[\theta + \delta(1 - \frac{\gamma(1+\pi)}{2\pi})],$$

$$\mu_3 = \delta[\frac{1}{2} - \frac{\pi\gamma}{4} - \frac{3\gamma}{4}], \quad \mu_1 = \frac{1}{2}[(1+\pi)(1-\gamma)\delta - (1-\pi)\theta],$$

For this to actually be the unique solution to (P), it is necessary that

$$0 \le \mu_1 = \frac{1}{2}[(1+\pi)(1-\gamma)\delta - (1-\pi)\theta] \Leftrightarrow \theta \le \frac{(1+\pi)(1-\gamma)\delta}{1-\pi},$$

$$0 \le \mu_3 = \delta[\frac{1}{2} - \frac{\pi\gamma}{4} - \frac{3\gamma}{4}] \Leftrightarrow \gamma \le \frac{2}{3+\pi},$$

and that

$$0 < Q = \frac{1}{k}[\theta + \delta(1 - \frac{\gamma(1+\pi)}{2\pi})] \Leftrightarrow \theta > \delta(\frac{\gamma(1+\pi)}{2\pi} - 1),$$

or equivalently, either

$$1 > \pi > \sqrt{5} - 2, \quad \frac{1}{2} \le \gamma \le \frac{2}{3+\pi}, \quad \frac{(1+\pi)(1-\gamma)\delta}{1-\pi} \ge \theta > \max[0, \delta(\frac{\gamma(1+\pi)}{2\pi} - 1)],$$

or

$$0 < \pi \le \sqrt{5} - 2, \quad \frac{1}{2} \le \gamma < \frac{4\pi}{(1+\pi)^2}, \quad \frac{(1+\pi)(1-\gamma)\delta}{1-\pi} \ge \theta > \max[0, \delta(\frac{\gamma(1+\pi)}{2\pi} - 1)].$$

- (**Case 8.**) We shall look for conditions which support an optimum with $g_j(q_1^*, q_2^*, q_3^*) = 0$, for all $j = 1, 2, 3$.

  (**Problem 15**.) Show that if

  $$\max(\frac{1}{2}, \frac{2\pi}{1+\pi}) < \gamma < 1-\pi, \quad 0 < \theta \leq \min[\frac{(1-\pi)\gamma\delta}{1+\pi}, \frac{\delta}{2\pi}[\gamma(1+\pi)]-\delta, \frac{(1-\pi-\gamma)\delta}{\pi}],$$

Note that if

$$1 > \pi > \sqrt{5} - 2, \quad \frac{1}{2} \leq \gamma \leq \frac{2}{3+\pi}, \quad \frac{(1+\pi)(1-\gamma)\delta}{1-\pi} \geq \theta > \delta,$$

then since

$$\frac{2}{3+\pi} < \frac{4\pi}{(1+\pi)^2} < \frac{4\pi}{(1+\pi)} \Rightarrow \delta > \delta(\frac{\gamma(1+\pi)}{2\pi} - 1),$$

by Problem 2, the presence of gift-buying demand results in M stopping serving the low-valuation regular buyers, and introducing a single gift item in equilibrium.

then indeed at optimum[13]

$$q_1^* = q_2^* = q_3^* = 0.$$

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[13]**Hint**: If at optimum,

$$g_3(q_1^*, q_2^*, q_3^*) = g_1(q_1^*, q_2^*, q_3^*) = g_2(q_1^*, q_2^*, q_3^*) = 0,$$

then $q_1^* = 0 = q_2^* = q_3^*$, and we have

$$Df(0,0,0) = \mu_1 Dg_1(0,0,0) + \mu_2 Dg_2(0,0,0) + \mu_3 Dg_3(0,0,0)$$

$$\Rightarrow \begin{bmatrix} f_1'(0) \\ f_2'(0) \\ f_3'(0) \end{bmatrix} = \begin{bmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_2 \\ -\mu_3 \end{bmatrix},$$

$$\Rightarrow f_1'(0) = \frac{1-\pi}{2}(\theta - 0) - \frac{1+\pi}{2}(1-\gamma)\delta = \mu_2 - \mu_1,$$

$$f_2'(0) = \frac{\pi}{2}[\theta + (1-\gamma)\delta - 0] - \frac{1}{2}(2\gamma - 1)\delta = \mu_3 - \mu_2,$$

$$f_3'(0) = \frac{\pi}{2}[\theta + \gamma\delta - 0] - \frac{1-\pi}{2}(1-\gamma)\delta = -\mu_3,$$

$$\Rightarrow \mu_2 = \frac{\delta}{2}[\gamma(1+\pi) - 2\pi] - \pi\theta,$$

$$\mu_3 = \frac{1}{2}[(1-\pi-\gamma)\delta - \pi\theta], \quad \mu_1 = \frac{1}{2}[(1-\pi)\delta - (1+\pi)\theta],$$

For this to actually be the unique solution to (P), it is necessary that

$$0 \le \mu_1 = \frac{1}{2}[(1-\pi)\delta - (1+\pi)\theta] \Leftrightarrow 0 < \theta \le \frac{(1-\pi)\delta}{1+\pi},$$

$$0 \le \mu_2 = \frac{\delta}{2}[\gamma(1+\pi) - 2\pi] - \pi\theta \Leftrightarrow 0 < \theta \le \frac{\delta}{2\pi}[\gamma(1+\pi)] - \delta,$$

$$0 \le \mu_3 = \frac{1}{2}[(1-\pi-\gamma)\delta - \pi\theta] \Leftrightarrow 0 < \theta \le \frac{(1-\pi-\gamma)\delta}{\pi},$$

or equivalently,

$$\max(\frac{1}{2}, \frac{2\pi}{1+\pi}) < \gamma < 1-\pi, \qquad 0 < \theta \le \min[\frac{(1-\pi)\delta}{1+\pi}, \frac{\delta}{2\pi}[\gamma(1+\pi)] - \delta, \frac{(1-\pi-\gamma)\delta}{\pi}].$$

Note that this solution is consistent with the solution to Problem 2, if we let $\pi = 0$ and let $\gamma \uparrow 1$.