Game Theory with Applications to Finance and Marketing, I

Homework 1, due in recitation on 10/18/2018.

1. Consider the following strategic game:

player 1/player 2	L	R
U	1,1	0,0
D	0,0	3,2

   Any NE can be represented by (p, q), where p is the probability that player 1 adopts U and q the probability that player 2 adopts L.

   (i) Show that this game has 3 NE's: (1,1), (0,0), and $(\frac{2}{3}, \frac{3}{4})$.

   (ii) Now, consider the following new version of the above strategic game. At the first stage, player 1 can invite either A or B to become player 2 for the above strategic game. At the second stage, player 1 and the selected player 2 then play the above strategic game. A (or B) gets the player 2's payoffs described in the above strategic game, if he accepts the invitation to play the game. Without playing the game, A can get a payoff of $\frac{1}{200}$ on his own, and B can get a payoff of $\frac{3}{2}$ on his own.

   The game proceeds as follows. First, player 1 can invite either A or B, and if the invitation is accepted, then the game moves on to the second stage; and if the invitation gets turned down, then player 1 can invite the other candidate. If both A and B turn down player 1's invitations, then the game ends with A getting $\frac{1}{200}$, B getting $\frac{3}{2}$, and player 1 getting 0.

   Which one between A and B should player 1 invite first? Compute player 1's equilibrium payoff.

2. Consider the following strategic game:

player 1/player 2	L	M	R
U	2,0	2,2	4,4
M	6,8	8,4	5,0
D	10,6	4,4	6,5

(i) Assume that players are restricted to using only pure strategies. Find the strategy profiles that survive the procedure of *iterative deletion of strictly dominated strategies*.

(ii) Assume that players are restricted to using only pure strategies. Find the strategy profiles that survive the procedure of *iterative deletion of non-best-response strategies*.

(iii) How would your solutions for parts (i) and (ii) change if players are allowed to use also mixed strategies?[1]

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[1]**Hint**: Define for part (i)

$$S_1^0 = S_1 = \{U, M, D\}, \quad S_2^0 = S_2 = \{L, M, R\},$$

and let S_j^n be the subset of S_j^{n-1} such that S_j^n contains player j's pure strategies that are not strictly dominated when player i is restricted to using only pure strategies contained in S_i^{n-1}. Then define

$$S_1^\infty \equiv \bigcap_{n=1}^{\infty} S_1^n, \quad S_2^\infty \equiv \bigcap_{n=1}^{\infty} S_2^n.$$

The strategy profiles that survive the procedure of *iterative deletion of strictly dominated strategies* are the elements of the Cartesian product $S_1^\infty \times S_2^\infty$.

Define for part (ii)

$$H_1^0 = S_1 = \{U, M, D\}, \quad H_2^0 = S_2 = \{L, M, R\},$$

and let H_j^n be the subset of H_j^{n-1} such that H_j^n contains all player j's pure-strategy best responses when player i is restricted to using only pure strategies contained in H_i^{n-1}. Then define

$$H_1^\infty \equiv \bigcap_{n=1}^{\infty} H_1^n \quad H_2^\infty \equiv \bigcap_{n=1}^{\infty} H_2^n.$$

The strategy profiles that survive the procedure of *iterative deletion of non-best-response strategies* are the elements of the Cartesian product $H_1^\infty \times H_2^\infty$.

3. Players 1 and 2 are living in a city where on each day the weather is equally likely to be sunny (S), cloudy (C), or rainy (R). Players 1 and 2 are supposed to play the following strategic game at date 1.

player 1/player 2	L	R
U	15,3	0,0
D	12,12	3,15

(i) Suppose that the above strategic game must be played before players 1 and 2 know anything about the date-1 weather. Verify that the game has two pure-strategy NE's and one mixed-strategy NE. Suppose that before playing the strategic game, players 1 and 2 both believe that they may attain each pure-strategy NE with probability $a < \frac{1}{2}$ and they may attain the mixed-strategy NE with probability $1 - 2a$. Compute the expected Nash-equilibrium payoff for player 1 given a.

(ii) Now, suppose that for $i = 1, 2$, player i receives a weather report s_i right before playing the above strategic game at date 1. The weather report s_1 tells player 1 whether the weather will or will not be sunny. The weather report s_2 tells player 2 whether the weather will or will not be rainy. That the two players will receive these two weather reports is their common knowledge at the beginning of date 1. Consider the following strategy profile:

- Player 1 uses U if the weather will be sunny, and he uses D if the weather will not be sunny.
- Player 2 uses R if the weather will be rainy, and he uses L if the weather will not be rainy.

Does this strategy profile constitute a Nash equilibrium?[2] If it does, compute player 1's equilibrium payoff. Compare this payoff to player

[2] This strategy profile is not an NE of the original strategic game without weather reports, which has been analyzed in part (i). In part (ii), with weather reports, we have a new game where players' strategies are functions that map weather information into actions.

1's expected Nash-equilibrium payoff that you obtained in part (i). Explain.[3]

4. (**Retailer's Opportunistic Pricing Behavior and Consumers' Coupon Redemption.**) There are two consumers with unit demand for the product produced by a firm. The firm has no production costs. The two consumers' valuations for the product are respectively H and L. The firm has already issued a cents-off coupon with face value v, and to redeem the coupon the two consumers must incur costs T_H and T_L respectively.[4]

   Assume that
   $$2L - v > H \geq L + v > L > 0,$$
   and that
   $$H - v \geq H - T_H > L - T_L > v - T_L > 0.$$
   The extensive game starts after the firm has alreay chosen v, and it is described as follows.

   - Seeing v, the two consumers must decide independently whether to carry the coupon and redeem it on the shopping day. A con-

[3]**Hint**: Show that

- when the state is sunny, given player 2's strategy described above it is optimal for player 1 to use U, and given player 1's strategy described above it is optimal for player 2 to use L;
- when the state is cloudy, given player 2's strategy described above it is optimal for player 1 to use D, and given player 1's strategy described above it is optimal for player 2 to use L; and
- when the state is rainy, given player 2's strategy described above it is optimal for player 1 to use D, and given player 1's strategy described above it is optimal for player 2 to use R.

[4]Therefore consumer H gets a surplus $H - (p - v) - T_H$ if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Similarly, consumer L gets a surplus $L - (p - v) - T_L$ if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Of course, a consumer can always forget about the coupon, and simply make the purchase. In the latter case, consumer H would get a surplus $H - p$ and consumer L would get a surplus $L - p$. Recall that each consumer gets zero surplus if he chooses to make no purchase.

sumer with valuation $j \in \{H, L\}$ will incur a cost T_j *before* the shopping day if he decides to carry the coupon till the shopping day. Consumers' decisions about whether to carry the coupon are unobservable to the firm.

- Then, on the shopping day, the firm must choose a retail price p before consumers arrive.
- Then, consumers walk in the store, see p, and decide whether to make a purchase, and if they have carried a coupon till the shopping day, (it is obviously a dominant strategy at this moment) to present the coupon to the firm in order to get a price reduction equal to v.

(i) Show that given that v satisfies the above conditions, this game has a unique Nash equilibrium where consumer H will never redeem the coupon while consumer L and the firm both use mixed strategies in equilibrium; that is, in equilibrium consumer L feels indifferent about redeeming and not redeeming the coupon, and the firm feels indifferent about *two* optimal prices $p_2 > p_1$.[5]

(ii) Now, suppose instead that $2L > H > M$, where

$$M = 2L - kv,$$

with

$$k = \frac{L - v}{L + v} \in (0, 1).$$

Re-consider the above extensive game. Solve for the mixed-strategy NEs.[6]

[5] Note that the redemption cost T_j is already sunk on the shopping day. If the firm expects consumer L to carry the coupon with probability one, then $p = L + v$, so that consumer L will end up with a negative consumer surplus; and if the firm expects consumer L to not carry the coupon with probability one, then $p = L$, so that consumer L actually prefers to carry the coupon before the shopping day. Show that there can be no pure strategy equilibrium. Then, argue that in a mixed strategy equilibrium, the firm randomizes over at most two prices.

[6] Verify that the solution to part (i) is still valid if $H < M$. Show that if $H = M$, then

5. (**Competitive Manufacturers May Make More Profits with Non-integrated Distribution Channels.**) Recall the Cournot game in Example 1 of Lecture 1, Part I. Assume that $c = F = 0$ and the inverse demand in the relevant range is

$$P(Q) = 1 - Q,\ 0 \leq Q = q_1 + q_2 \leq 1.$$

(i) Find the equilibrium profits for the two firms.
(ii) Now suppose that the two manufacturing firms cannot sell their products to consumers directly. Instead, firm i (also referred to as manufacturer i) must first sell its product to retailer R_i. Then retailers R_1 and R_2 then compete in the Cournot game. The extensive game is now as follows.

- The two firms first announce F_1 and F_2 simultaneously, where F_i is the franchise fee that firm i will charge retailer i, which is a fixed cost of retailer i. R_1 and R_2 simultaneously decide to or not to turn down the offers made by the firms. Assume that firm i and retailer R_i both get zero payoffs if F_i gets turned down by retailer R_i.
- Then, after knowing whether F_1 and F_2 get accepted by respectively R_1 and R_2, the two firms announce w_1 and w_2 simultaneously, where w_i is the unit whole price that firm i will charge retailer i.
- Next, in case the firms' offers are both accepted, then given (F_1, F_2, w_1, w_2), the two retailers simultaneously choose q_1 and q_2.

Show that in the unique subgame-perfect Nash equilibrium (SPNE) each manufacturing firm gets a profit of $\frac{10}{81}$. (**Hint**: Backward induction asks you to always start from the last-stage problem, which is the Nash equilibrium of the subgame where R_1 and R_2 play the Cournot game given some (F_1, F_2, w_1, w_2). You can show that the equilibrium

we have a continuum of mixed-strategy NEs, where the firm randomizes over the three prices L, $L+v$, and H, with the probability of pricing at L being $\frac{TL}{v}$, and where consumer L redeems the coupon with probability k. Show that if $2L > H > M$, then in equilibrium the firm randomizes over L and H, with the probability of pricing at L being $\frac{TL}{v}$, and with consumer L redeeming the coupon with proability $\frac{2L-H}{v}$.

(q_1^*, q_2^*) depend on (w_1, w_2) but not on (F_1, F_2), because the latter are fixed costs. Then, you should move backwards to consider the two manufacturers' competition in choosing w_1 and w_2, given some (F_1, F_2). Here assume that the two manufacturers know that different choices of w_1 and w_2 will subsequently affect R_1's and R_2's choices of q_1 and q_2. Finally, you can move to the first-stage of the game, where the two firms simultaneously choose F_1 and F_2.)[7]

6. (**Entry Deterrence by a Monopolistic Incumbent.**) Consider the following extensive game in which firms A and B may compete in quantity at date 1 and date 2. Both firms seek to maximize the sum of expected date-1 and date-2 profits. The inverse demand at date $t \in \{1, 2\}$, in the relevant region, is $P_t = 1 - Q_t$, where P_t is the date-t product price and $Q_t = q_{At} + q_{Bt}$ is the sum of the two firms' supply quantities at date t. Assume that there are no production costs for the two firms.

   - At date 1, originally firm A is the only firm in the industry. Firm A must first choose q_{A1}. Upon seeing firm A's choice q_{A1}, firm B must decide whether to spend a cost $K > 0$ to enter the industry. If K is spent, then B must choose q_{B1}. Then the two firms' date-1 profits π_{A1} and π_{B1} are realized, where $\pi_{B1} = 0$ if firm B decides not to enter the industry.
   - At date 2, if firm B did not enter at date 1, then firm A, the monopolistic firm in the industry, must choose q_{A2}. If, on the other hand, firm B has entered at date 1, then the two firms choose quantities q_{A2} and q_{B2} simultaneously. Then, the two firms' date-2 profits π_{A2} and π_{B2} are realized, where $\pi_{B2} = 0$ if firm B did not enter the industry at date 1.

   Now we solve for the subgame perfect Nash equilibrium for this game. (i) Suppose that $K = \frac{1}{5}$. Find the equilibrium q_{A1} and q_{A2}.

[7] This exercise intends to show why employing independent retailers may be a good idea even if using a firm's own outlets can be cheaper. Essentially, employing an independent retailer amounts to delegating the retailer the choice of output, knowing that the retailer, unlike the manufacturer, will be choosing output given a positive unit cost w_i! A higher unit cost credibly convinces the rival retailer that less output will be produced, and with both manufacturers producing less outputs, their profits become higher.

(ii) Suppose that $K = \frac{1}{9} + \frac{1}{25}$. Find the equilibrium q_{A1} and q_{A2}.
(iii) Suppose that $K = \frac{1}{25}$. Find the equilibrium q_{A1} and q_{A2}.

7. (**Signal Jamming and Cournot Competition**) Consider firms 1 and 2 that engage in Cournot competition at $t = 1$ and $t = 2$, facing random demand functions at both periods. The inverse demand function at $t = 1$ is

$$\tilde{p}_1 = \tilde{a} - q_1 - q_2,$$

where $\tilde{a}$ is a positive random variable with $E[\tilde{a}] = 1$ and q_j is firm j's output level at $t = 1$. The inverse demand function at $t = 2$ is

$$\tilde{p}_2 = \tilde{b} - Q_1 - Q_2,$$

where $\tilde{b}$ is a positive random variable and Q_j is firm j's output level at $t = 2$. Each firm seeks to maximize the sum of expected profits over the two periods. That is, both firms are risk-neutral without time preferences.

The game proceeds as follows.

- At the beginning of $t = 1$, both firms must simultaneously make output choices q_1 and q_2 without seeing the realization of $\tilde{a}$.
- At the beginning of $t = 2$, after knowing q_j and the realization p_1 of $\tilde{p}_1$, firm j must choose Q_j. The two firms make output choices at the same time, without seeing the realization of either $\tilde{a}$ or $\tilde{b}$. At this time, firm j does not see q_i that was chosen by its rival, firm i.

(i) First assume that $\tilde{b}$ and $\tilde{a}$ are independently and identically distributed. Solve the equilibrium output choices $(q_1^*, q_2^*, Q_1^*, Q_2^*)$ in the unique SPNE.

(ii) Ignore part (i). Now assume instead that $\tilde{b} = \lambda\tilde{a}$, where $\lambda < 2$ is a constant known to both firms. Solve the unique symmetric SPNE.

(iii) Do the two firms get higher date-1 expected profits in part (ii) or in part (i)? Why?

(iv) Suppose that $\lambda = 1$. Do the two firms get higher date-2 expected profits in part (ii) or in part (i)? Why?[8]

[8]**Hint**: Verify that $(q_1^*, q_2^*, Q_1^*, Q_2^*) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in part (i). For part (ii), let $(q^*, Q^*(p_1, q))$ denote the unique symmetric SPNE, where both firms choose q^* at $t = 1$, and both choose $Q^*(p_1, q)$ after choosing q at $t = 1$ and subsequently learning that the realization of $\tilde{p}_1$ is p_1. Then *in equilibrium*, $\tilde{p}_1 = \tilde{a} - 2q^*$, or $\tilde{a} = \tilde{p}_1 + 2q^*$. At the beginning of $t = 2$, given the realization p_1 of $\tilde{p}_1$ and its own output choice q_i at $t = 1$, and given that firm j does not deviate from its equilibrium strategy, firm i knows that $\tilde{a} = p_1 + q_i + q^*$. Moreover, firm i knows that that firm j would believe that $\tilde{a} = p_1 + 2q^*$ and seek to maximize

$$\max_Q [\lambda(p_1 + 2q^*) - Q^*(p_1, q^*) - Q]Q,$$

where note that firm j does not know firm i has chosen q_i rather than q^*. That is, firm i believes that firm j would choose the Q that satisfies

$$Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1, q^*)}{2},$$

which has to be $Q^*(p_1, q^*)$ also. Hence firm i believes that firm j would choose

$$Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.$$

Firm i, knowing that it has chosen q_i rather than q^* at $t = 1$, seeks to maximize the following date-2 profit:

$$\max_Q [\lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q]Q,$$

so that given (p_1, q_i), firm i's optimal date-2 output level is

$$Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1 + 2q^*)}{3}}{2},$$

which yields for firm i the following date-2 profit

$$\frac{1}{4}[\frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i]^2.$$

At $t = 1$, expecting firm j to choose q^*, firm i seeks to

$$\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4}E[(\frac{2\lambda \tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],$$

which is concave in q_i because $\lambda < 2$. Show that the optimal q_i must satisfy the first-order condition for this maximization problem; that is,

$$1 - q^* - 2q_i + \frac{\lambda}{6}(\frac{2\lambda E[\tilde{p}_1]}{3} + \frac{\lambda q^*}{3} + \lambda q_i) = 0,$$

or using $E[\tilde{p}_1] = 1 - q_i - q^*$, and $q_i = q^*$ in equilibrium, show that

$$q^* = \frac{1}{3} + \frac{\lambda^2}{27}.$$

Show that then $Q^*(p_1, q^*) = \frac{\lambda \tilde{a}}{3}$.