Game Theory with Applications to Finance and Marketing, I

Homework 4, due in recitation on 11/29.

1. (**Bayesian Equilibrium**) Two workers can each choose to or not to make an effort for their joint project. The project generates one unit of utility to each worker if at least one worker chooses to make the effort. Making effort incurs a disutility c_i to worker i, where c_i is worker i's private information, and worker j believes that c_i is uniformly distributed over $[0, 2]$. Exante it is common knowledge that c_1 and c_2 are independent random variables. Find a symmetric pure-strategy Bayesian equilibrium.[1]

2. In the following two signaling games, player 1 is equally likely to be of type t_1 and type t_2, and can send signal m_1 or m_2 or m_3, and player 2 can respond by taking action a_1 or a_2 or a_3. The three tables indicate their payoffs following each of the 3 signals sent by player 1.

   - There is a separating PBE for the following game, where m_3 is not an equilibrium signal. Find this PBE. Is this PBE an intuitive equilibrium?

m_1	a_1	a_2	a_3
t_1	$(1, 0)$	$(4, 3)$	$(2, 4)$
t_2	$(10, 5)$	$(4, 4)$	$(4, 1)$

m_2	a_1	a_2	a_3
t_1	$(2, 2)$	$(6, 0)$	$(8, 1)$
t_2	$(2, 2)$	$(2, 3)$	$(6, 2)$

m_3	a_1	a_2	a_3
t_1	$(6, 1)$	$(4, -2)$	$(1, 2)$
t_2	$(6, 2)$	$(2, 3)$	$(0, -1)$

   - There is a pooling PBE for the game below, where player 1's equilibrium signal is not m_1. Find this PBE. Is this PBE an intuitive equilibrium?

[1] **Hint**: There should be a cut-off level of c_i, say c_i^*, such that a type-c_i chooses to make an effort if and only if $c_i \leq c_i^*$.

m_1	a_1	a_2	a_3
t_1	$(8, 0)$	$(4, 3)$	$(2, 4)$
t_2	$(10, 5)$	$(4, 4)$	$(4, 1)$

m_2	a_1	a_2	a_3
t_1	$(2, 2)$	$(6, 0)$	$(8, 1)$
t_2	$(2, 2)$	$(2, 3)$	$(6, 2)$

m_3	a_1	a_2	a_3
t_1	$(6, 1)$	$(4, -2)$	$(2, 2)$
t_2	$(6, 2)$	$(2, 3)$	$(0, -1)$

3. Recall the game of beer and quiche discussed in Lecture 4, and consider a modified version of that game as follows. Here we assume that everything is the same as described in section 22 of Lecture 4 except that the strong type of A prefers to fight B. More precisely, let F denote the event that there is a fight between A and B, and NF the event of no fight. Let b and q denote respectively the signals of ordering beer and quiche respectively. Let s and w denote A's two possible types. Let u_A and u_B denote A's and B's payoffs respectively. Then the payoff functions of A and B in this modified game of beer and quiche can be summarized as follows.

$$u_A(F, b, s) = 3,\ u_A(F, q, s) = 2,\ u_A(NF, b, s) = 1,\ u_A(NF, q, s) = 0,$$

$$u_A(F, b, w) = 0,\ u_A(F, q, w) = 1,\ u_A(NF, b, w) = 2,\ u_A(NF, q, w) = 3,$$

$$u_B(F, w) = 2,\ u_B(F, s) = 0,\ u_B(NF, w) = u_B(NF, s) = 1.$$

Let x be B's prior probability assigned to the event that A is of the strong type. In this exercise, we shall assume that $x > \frac{1}{2}$.

Find all PBEs of this game. Determine if these PBEs are intuitive or not.[2]

[2]**Hint**: This game has a continuum of PBEs where both types of A adopt mixed strategies. Besides, this game also has two PBEs in which at least one type of A uses a pure strategy, and the latter two PBEs have the following features: In one PBE, the weak type of A orders quiche with probability one, the strong type of A orders beer and quiche with respectively probability $2 - \frac{1}{x}$ and $\frac{1}{x} - 1$, and B chooses to not fight A after seeing the signal b, and B chooses to fight A with probability $\frac{1}{2}$ after seeing the signal q. In another PBE, both types of A order b, and B chooses to not fight A after seeing b, but B chooses to fight A with probability $\frac{1}{2}$ after seeing q.

4. In the following dynamic game with incomplete information, player 1 has two equally likely types, denoted by t_1 and t_2, and given his type, the informed player 1 must choose either strategic game A or strategic game B. After observing player 1's choice, the informed player 1 and the uninformed player 2 must simultaneously take actions in the chosen strategic game. In each strategic game, player 1 can choose either U or D, and player 2 can choose either L or R. The resulting payoff x for the type-t_1 player 1, y for the type-t_2 player 1, and z for player 2, are written as a row vector (x, y, z). The following two tables summarize the players' type-and-action-contingent payoffs. For example, if player 1 chooses game A and then action U, and if player 2 chooses action L in game A, then $x = 2$, $y = 1$, and $z = 3$.

Strategic Game A

	L	R
U	$(2, 1, 3)$	$(1, 2, 5)$
D	$(1, 2, 0)$	$(0, 12, 10)$

Strategic Game B

	L	R
U	$(\frac{3}{2}, 21, 3)$	$(\frac{3}{4}, 2, 1)$
D	$(0, 0, 0)$	$(0, 10, 4)$

We shall only consider PBEs in which players use pure strategies in each and every subgame. For supporting beliefs, let us define $\mu_A \equiv \text{prob}(t_1|A)$ and $\mu_B \equiv \text{prob}(t_1|B)$, where A and B stand for "strategic game A" and "strategic game B" respectively.
(i) Find all separating and pooling PBEs of this game.[3]

[3]**Hint**: For each PBE, you must write down explicitly player 1's and player 2's strategies, together with μ_A and μ_B. In particular, for player 1's strategy, you must state clearly

$$\begin{pmatrix} t_1 \rightarrow \text{(A,U) or (A,D) or (B,U) or (B,D)} \\ \\ t_2 \rightarrow \text{(A,U) or (A,D) or (B,U) or (B,D)} \end{pmatrix},$$

(ii) For each PBE obtained in part (i), determine whether it is a Cho-Kreps intuitive equilibrium or not.[4]

5. Consider the following stock trading model with one traded common stock and three classes of traders: one insider (or informed speculator), one noise trader, and several Bertrand-competitive market makers. Everyone is risk-neutral without time preferences. Stock trading takes place at date 0, and the true value of the stock, denoted v, will become public information at date 1. The extensive game proceeds as follows.

   - At the beginning of date 0, the insider alone learns about the realization of v, when everyone else only knows that v is equally likely to be $-2, -1, 1$ or 2.
   - Then simultaneously, the insider and the noise trader must each submit one market order. The insider's market order is denoted by X, and the noise trader's market order is denoted by u, and we assume that u is equally likely to be 1 or -1; that is, the noise trader is equally likely to buy one share or sell one share. By submitting a market order a trader commits to accepting order execution at the market-clearing price subsequently announced by the stock-trading platform.
   - At the same time when the insider and the noise trader submit their market orders, the market makers must each submit one *pricing schedule*, denoted by $P(\cdot)$. By submitting a schedule $P_i(\cdot)$, a market maker i commits to absorbing any market order $z \in \Re$ at the share price $P_i(z)$ that he specifies via $P_i(\cdot)$.
   - Then, the stock-trading platform receives X, u and the market makers' pricing schedules. The platform insists on matching X

and for player 2's strategy, you must state clearly

$$\begin{pmatrix} A \to \text{L or R} \\ \\ B \to \text{L or R} \end{pmatrix}.$$

[4]**Hint**: For part (i), show that this game has two pooling but no separating equilibria; and for part (ii), show that both pooling PBEs are intuitive.

and u first, and in case $z = X + u \neq 0$, then the platform will pick one market maker i whose $P_i(z)$ appears to be the lowest when $z > 0$ or whose $P_i(z)$ appears to be the highest in case $z < 0$. In case $z = 0$, then the platform will just pick $P(0) = E[v]$.

- Then, the date-0 stock trading session ends, and the game moves on to date 1. Then the realization of v becomes publicly known, and each stock-trading participant gets his realized gain or loss from the date-0 stock-trading.

The above is a signaling game, where v is the informed insider's type, and X is the signal he sends. This is referred to as a signaling game *with noise*, because market makers (i.e., the uninformed players) do not observe X directly; rather, what they learn from the stock-trading platform is $z = X + u$ only (not u and X separately), where we recall that u is a zero-mean random variable.

We shall look for pure strategy perfect Bayesian equilibria in which the market makers submit the same $P(\cdot)$. Let us call them symmetric PBEs. A symmetric PBE is formally a pair $\{P(z), X(v)\}$ such that (i) given $P(\cdot)$, $X(v) \in \arg\max_y E[y(v - P(y + u))|v]$; and (ii) given any $z = X + u$, either $P(z)$ would ensure that no trade would occur between the selected market maker and the traders submitting market orders, or in the opposite case, the selected market maker must break even by absorbing $z = X(v) + u$; that is, $P(z) = E[v|X(v) + u = z]$.

Show that for each $a \in (0, \frac{2}{3})$, $\{P_a(\cdot), X_a(\cdot)\}$ is one symmetric PBE, where $X_a(\cdot)$ is such that

$$X_a(2) = -X_a(-2) = 1 + a, \quad X_a(1) = -X_a(-1) = 1 - a,$$

and $P_a(\cdot)$ is such that

$$\forall z \in \{-2 - a, -2 + a, -a, a, 2 - a, 2 + a\}, \; P_a(z) = -P_a(-z),$$

$$P_a(a) = \frac{1}{2}, \; P_a(2 + a) = 2, \; P_a(2 - a) = 1,$$

$$\forall z > 0, \; z \neq 2 + a, 2 - a, a, \; P_a(z) = 2,$$

and

$$\forall z < 0, \; z \neq -2 + a, -2 - a, -a, \; P_a(z) = -2.$$

6. Firm A has a single owner-manager Mr. A, who needs to raise \$100 for a positive-NPV investment project at date 0. There are two possible date-0 states, called G and B, and the date-0 state is Mr. A's private information. In state G, the assets in place of firm A are worth \$150 and the new project's NPV equals \$20. In state B, the assets in place are worth only $\$x$ and the new project's NPV is accordingly $\$y$. The public investors (also referred to as the outsiders) believe that the state may be G with prob. a. Mr. A and public investors are all risk-neutral without time preferences.

   The game proceeds as follows. Mr. A first decides to or not to issue new equity to raise \$100 (two feasible signals!). Then, upon seeing Mr. A's decision, the public investors form posterior beliefs about the date-0 state, and they engage in Bertrand competition to determine the fraction α of equity that Mr. A must sell in order to raise \$100.

   (i) Suppose that $x = 50$ and $y = 10$. Find all the pure-strategy PBE's of this signaling game.

   (ii) Suppose that $x = 60$ and $y = -25$. Assume that the firm, after raising \$100 from new investors, can either undertake the new investment project or put \$100 in a riskless money market account. The risk-free interest rate is zero. In this case, a pooling equilibrium where both types of the firm choose to issue new equity exists if and only if the prior probability a for the good state is such that $a \geq a^*$. Compute a^*.

   (iii) Suppose that $x = 60$ and $y = -25$. Unlike in part (ii), assume instead that the firm, after raising \$100 from new investors, must spend it on the new investment project, regardless of the state. In this case, a pooling equilibrium where both types of the firm choose to issue new equity exists if and only if the prior probability a for the good state is such that $a \geq a^{**}$. Compute a^{**}.

(iv) Suppose that $x = 60$ and $y = -25$. Suppose that $a = a^{**}$. Then in the pooling equilibrium obtained in part (ii), Mr. A ends up possessing a fraction $1 - \alpha$ of firm A's equity. Compute α.

7. Let us modify the game of chain-store paradox in Lecture 4 by assuming 5 entrants instead of 3. Find as many PBE's as possible for this reputation game.

8. We shall consider here a modified version of the *Chain-store Paradox* with 3 entrants E_1, E_2 and E_3.

   As in the original version considered in Lecture 4, here there are two types of incumbent, referred to as the sane and the crazy. Again, x_j is the entrants' common probability for the event that the incumbent may be crazy at the time E_j is about to enter. Note that the prior probability x_1 is an exogenous parameter, but the posterior probabilities x_2 and x_3 must be derived in equilibrium.

   The players' payoffs in this new version are different from those in the original version, as explained below.

   - By preying following entry, the sane gets an immediate payoff of -2 and the crazy gets an immediate payoff of $\frac{3}{2}$.
   - By accomodating following entry, the sane gets an immediate payoff of 0 and the crazy gets an immediate payoff of 1.[5]
   - As in the original version discussed in Lecture 4, the incumbent gets $\frac{3}{4}$ in a period without entry, and an entrant gets 0 from staying out, 1 from entering and then being accomodated, and -1 from entering and then being preyed.

[5] That the crazy gets $1 > \frac{3}{4}$ from accomodating a current entrant may seem odd at the first glance. This, however, may be explained by a network externality. In a telecommunication industry for example, entry by a new firm may raise consumers' valuations for the incumbents' products. Alternatively, this may be due to an advertising effect when the incumbent and entrants are operating in a market for an unconventional new product. In any case, this payoff assumption implies that the crazy may have an incentive to pool with the sane and accomodate a current entrant, as an attempt to lure future entrants. The bottom line here is that whether pooling occurs because the sane mimics the crazy (as in Lecture 4) or because the crazy mimics the sane (as in the current problem) depends crucially on the nature of the product.

The following table summarizes the players' payff information in a one-entrant case.

	Entrant staying out	Entrant preyed	Entrant accomodated
Entrant	0	-1	1
The sane	$\frac{3}{4}$	-2	0
The crazy	$\frac{3}{4}$	$\frac{3}{2}$	1

Find *all* the PBEs of this modified *Chain-store Paradox* with 3 entrants.[6]

[6]**Hint**: The sane has a dominant strategy in this game. The PBE of this game depends on x_1. Summarize the PBEs for the cases of $x_1 > \frac{7}{8}$, $x_1 = \frac{7}{8}$, $\frac{7}{8} > x_1 > \frac{3}{4}$, and $x_1 \leq \frac{3}{4}$.