Game Theory with Applications to Finance and Marketing

Lecture 5: Some Applications in Marketing

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1. This note will consider several game-theoretic marketing models. Example 0 reviews the so-called Schmalensee effect and Nelson effect found in the literature of product price signaling. Example 1 considers the impact of coupon resale on a monopolistic manufacturer's optimal product-line, pricing and promotion strategies. Examples 2-4 analyze how the issuance of a giftcard may alter duopolistic retailers' price competition via a complicated effect of advance selling. Examples 5 and 6 consider transaction-based or behavior-based discrimination in respectively a duopolistic market and a monoplistic market. Example 7 considers coupon competition between a retailer and a manufacturer. Example 8 considers a monoplistic seller's optimal design of a product line and an associated return policy.

2. (**Example 0**.) (**The Schmalensee effect and the Nelson effect.**)

   - (The Schmalensee effect) Consider a monopolist M trying to sell an experience good E to two buyers with unit demand. The experience good E may be of quality H or quality L, where $H > L > 0$. M can produce L costlessly, but must incur a unit cost $c > 0$ in producing H. M has chosen the quality of E, which is unobservable to the buyers. Buyer $j \in \{1, 2\}$ is willing to pay V_j for H, with $V_2 > V_1 > v$, where v is the two buyers' common valuation for L. The game proceeds as follows. First, the seller chooses a unit price p. Upon seeing p, the two buyers can simultaneously tell the seller whether they want to purchase E.

     We claim that this game has a separating PBE where a high price signals high product quality if

$$2v > V_2 > V_2 - c > 2(V_1 - c).$$

These inequalities state the basic fact that the type-H seller is more willing to let go of buyer 1 than the type-L seller; after all, the former seller must incur a unit cost c to produce the premium product. This is referred to as the *Schmalensee effect*. One can show that the above inequalities are only sufficient; such a separating PBE may exist even if these inequalities do not hold.

- (The Nelson effect.) The above example does not consider repeat purchase. To incorporate repeat purchase, let us modify the preceding example by assuming that only buyer 1 exists, with $V_1 = \theta$, $v = 0$, and buyer 1 has unit demand for E at both dates 0 and 1. Assume that buyer 1 will not come back at date 1 if she does not make a purchase at date 0, and that if she does make a purchase at date 0, then buyer 1 can find out the true quality of E right after she consumes E at date 0. Buyer 1 seeks to maximize the sum of discounted expected consumer surpluses and M seeks to maximize the sum of discounted expected profits over dates 0 and 1, and we let $\delta \in (0, 1]$ denote the common discount factor for buyer 1 and the seller M.. At each date $t = 0, 1$, the game proceeds just like in Example 3.

  We claim that a PBE exists in which a low date-0 price signals high product quality if the following condition holds:

$$(1 + \delta)c \leq \delta\theta.$$

  In this equilibrium, at date 0, $p_H(0) = 0 < p_L(0)$, and at date 1, $p_H(1) = \theta > 0 = p_L(1)$. Buyer 1's posterior belief is such that M is of type L if the date-0 price is greater than zero and that M is of type H if the date-0 price is zero. Note that the type-H M does not deviate at date 0, because a positive date-0 price will drive away buyer 1, who will not come back at date 1, whereas M can obtain a positive equilibrium payoff by setting $p_H(0) = 0$ (recall that $-c + \delta(\theta - c) \geq 0$). The type-L M does not deviate because it will get a zero payoff no matter how it chooses its date-0 price. This separating PBE exists because the type-H seller is more optimistic than its type-L counterpart about the prospect of making profits at date 1. This is referred to as the *Nelson effect*.

The Nelson effect does not necessarily take the form of setting a low introductory price for E. For the sake of demonstration, assume that the price has been fixed at $\overline{p} = \theta$ at both dates 0 and 1. Thus M cannot signal product quality via choosing price levels. Assume however that M can choose to or not to spend $K > 0$ on wasteful advertisements.[1]

It is easy to see that this modified game has a separating PBE where spending on wasteful advertisements is taken as evidence that E is of high quality if the following condition holds:

$$\theta - K = \overline{p} - K \leq 0, \ (\theta - c)(1 + \delta) - K \geq 0.$$

Again, the idea here is that spending on wasteful advertisements (or other wasteful promotions) is a way to convince buyers that the firm will stay in business for a long time (because its product is of high quality). Such an expenditure may lower its short-term profits, but it helps enhance its long-term profits.

3. **Example 1**. (**Coupon Resale and Product Line Design**)
   In this example, I shall solve the optimal product, pricing and promotion strategies for an integrated channel selling a single product to two segments of consumers, for both the case where coupon resale is prohibited and the case where coupon resale is allowed.

4. A monopolistic firm must first spend a cost $\frac{cq^2}{2}$ to develop a product item with quality $q \in \Re_+$, and must then announce a unit price p and a coupon with face value $R \in \Re_+$. Other than the above product development cost, I shall assume no production costs.

   Consumers can each buy either 1 or 0 unit of the product, and upon seeing (q, p, R), they each must decide whether to make the purchase, and whether to incur a redemption cost to acquire and carry the coupon before making the purchase. A consumer pays $p - R$ instead of p if she can present a coupon when making the purchase.

   There are two segments of consumers, indexed by respectively θ_2 and θ_1. The populations of these two segments are respectively α and $1-\alpha$, where $0 < \alpha < 1$. A θ_j consumer will incur a fixed cost $T_j \in \Re_+$ if she decides to acquire and carry the coupon.

[1] Advertising need not be wasteful in reality; it can be persuasive or informative, or both.

To incorporate coupon resale into the analysis, assume that there exists a coupon reseller who is not interested in making any purchase but can spend a redemption cost $T' \in \Re_+$ to acquire the coupon and then sell to consumers in the aforementioned two segments. Assume that this coupon reseller has all bargaining power against the consumers from the aforementioned two segments in coupon resale. Assume that trading with the coupon reseller incurs a cost $t_j \in \Re_+$ for a θ_j consumer, which includes also the cost of carrying the coupon till the time the θ_j consumer makes a purchase.

5. A *marketing environment* in this model is a tuple

$$(\theta_1, \theta_2, \alpha, c, T_1, T_2, t_1, t_2, T').$$

A *marketing plan* in this model is a tuple (p, q, R). I shall consider the marketing environments that satisfy the following regularity conditions:

Assumption 1

$$\theta_2 > \theta_1 > \alpha\theta_2 > 0, \tag{1}$$

$$T_2 - t_2 < T' < T_1 - t_1 < T_1 < \alpha T_2. \tag{2}$$

6. Let me recapitulate the timing of the game. The firm first chooses a marketing plan (p, q, R). In case coupon resale is allowed, then upon seeing (p, q, R) the coupon reseller can set a retail price r for the coupon. Then, consumers simultaneously decide whether to make a purchase from the firm, and if (and only if) a consumer θ_j decides to make a purchase, she then must decide whether to spend T_j to acquire the coupon on her own, or, if coupon resale is allowed, whether to incur a cost t_j and pay r to the coupon reseller to get the coupon, or not to obtain the coupon at all. A consumer without (respectively, with) a coupon pays p (respectively, $p - R$) when purchasing from the firm.

7. First consider the optimal marketing plan when coupon resale is prohibited. We can classify the marketing plans into two categories, those intending to serve only θ_2 consumers, and those intending to serve all consumers.

Under Assumption 1, the optimal marketing plan falls in the latter category. Too see this, note that if the firm intends to serve only θ_2 consumers, then given q it is optimal to set $R = 0$ and $p = \theta_2 q$. The firm's optimal q in this case must then maximize $\alpha\theta_2 q - \frac{cq^2}{2}$, yielding the optimal $q = \frac{\alpha\theta_2}{c}$. By Assumption 1, however, since $\alpha\theta_2 \leq \theta_1$,even at the product quality $q = \frac{\alpha\theta_2}{c}$ that best fits the purpose of serving θ_2 consumers alone, lowering the price to the level $\frac{\alpha\theta_1\theta_2}{c}$ and serving all consumers is still a dominant choice than pricing at $\frac{\alpha\theta_2^2}{c}$ and abandoning θ_1 consumers.

Thus we can confine our attention to marketing plans that intend to serve all consumers. At first, given q, we solve for the optimal pricing and promotion strategies. Given q, the firm seeks to

$$\text{(P)} \quad \max_{p,R} p - (1-\alpha)R$$

subject to

$$\theta_2 q \geq p,$$

$$T_2 \geq R,$$

$$\theta_1 q - p + R - T_1 \geq 0.$$

Since given R the objective function is increasing in p, either the above first constraint or the third (last) constraint must be binding at optimum. When the latter happens, by replacing p in the objective function by $\theta_1 q + R - T_1$, we see that the objective function is strictly increasing in R, and hence the above second constraint will bind at optimum. That is, whenever the last constraint is binding, $R = T_2$; and given that $R = T_2$, the last constraint will bind whenever $p = \theta_1 q + R - T_1 = \theta_1 q + T_2 - T_1 \leq \theta_2 q$, or equivalently, whenever

$$q \geq \overline{q} \equiv \frac{T_2 - T_1}{\theta_2 - \theta_1}.$$

When $q \leq \overline{q}$, on the other hand, the p obtained from the binding last constraint would violate the first constraint, so that all θ_2 consumers would drop out of the market, violating optimality. Thus at optimum the first constraint will bind if $q \leq \overline{q}$. In this case the objective function becomes strictly decreasing in R, so that R must make the last constraint binding. Thus we can conclude that, given q, the optimal (p, R) is such that

$$p = \begin{cases} \theta_1 q + T_2 - T_1, & \text{if } \; q \geq \overline{q} \equiv \frac{T_2 - T_1}{\theta_2 - \theta_1}; \\ \theta_2 q, & \text{if otherwise.} \end{cases}$$

and

$$R = \begin{cases} T_2, & \text{if } \; q \geq \overline{q} \equiv \frac{T_2 - T_1}{\theta_2 - \theta_1}; \\ (\theta_2 - \theta_1) q + T_1, & \text{if otherwise.} \end{cases}$$

Now, we solve for the optimal product quality. First we find the optimal q lying in the interval $[0, \overline{q}]$. Given $q \in [0, \overline{q}]$, our preceding analysis shows that the firm should optimally choose $(p, R) = (\theta_2 q, (\theta_2 - \theta_1) q + T_1)$, so that the firm's profit as a function of q is

$$\theta_2 q - (1 - \alpha)[(\theta_2 - \theta_1) q + T_1] - \frac{cq^2}{2},$$

which has an unconstrained maximum appearing at

$$q_2 \equiv \frac{\alpha \theta_2 + (1 - \alpha) \theta_1}{c}.$$

The optimal q lying in the interval $[0, \overline{q}]$ is therefore equal to $\min(q_2, \overline{q})$.

Next, we find the optimal q contained in the interval $[\overline{q}, +\infty)$. Given q lying in this region, our preceding analysis shows that the firm should optimally choose $(p, R) = (\theta_1 q + T_2 - T_1, T_2)$, so that the firm's profit as a function of q is

$$\theta_1 q + \alpha T_2 - T_1 - \frac{cq^2}{2},$$

which has an unconstrained maximum appearing at

$$q_1 \equiv \frac{\theta_1}{c}.$$

Thus the optimal q lying in the interval $[\overline{q}, +\infty)$ is equal to $\max(q_1, \overline{q})$. Observe that $q_2 > q_1$. Thus we summarize the firm's optimal marketing plan in the absence of coupon resale as follows.

Proposition 1 *Suppose that Assumption 1 holds. Then the optimal marketing plan without coupon resale depends on which among the following three conditions holds.*

- *(***Condition 1***.)* $q_2 > q_1 > \overline{q}$. *In this case, q_1 is optimal, and hence we have*
$$q^* = \frac{\theta_1}{c},$$
implying that
$$p^* = \frac{\theta_1^2}{c} + T_2 - T_1, \quad R^* = T_2.$$
The firm's equilibrium payoff is
$$\Pi^* = \frac{\theta_1^2}{c} + \alpha T_2 - T_1 - \frac{\theta_1^2}{2c} = \frac{\theta_1^2}{2c} + \alpha T_2 - T_1.$$

- *(***Condition 2***.)* $q_2 > \overline{q} > q_1$. *In this case, $\overline{q}$ is optimal, and hence we have*
$$q^* = \frac{T_2 - T_1}{\theta_2 - \theta_1},$$
implying that
$$p^* = \frac{\theta_1(T_2 - T_1)}{\theta_2 - \theta_1} + T_2 - T_1, \quad R^* = T_2.$$
The firm's equilibrium payoff is
$$\Pi^* = \frac{\theta_1(T_2 - T_1)}{\theta_2 - \theta_1} + \alpha T_2 - T_1 - \frac{c(T_2 - T_1)^2}{(2(\theta_2 - \theta_1)^2}.$$

- *(***Condition 3***.)* $\overline{q} > q_2 > q_1$. *In this case, q_2 is optimal, and hence we have*
$$q^* = \frac{\alpha\theta_2 + (1 - \alpha)\theta_1}{c} \equiv \frac{\overline{\theta}}{c},$$

implying that

$$p^* = \frac{\theta_2 \overline{\theta}}{c}, \quad R^* = \frac{(\theta_2 - \theta_1)\overline{\theta}}{c} + T_1.$$

The firm's equilibrium payoff is

$$\Pi^* = \frac{\overline{\theta}^2}{2c} - (1-\alpha)T_1.$$

Proposition 1 provides a benchmark with which we shall derive implications of coupon resale on the firm's optimal pricing, promotion, and product strategies, and on firm's profit and consumers' welfare.

8. Now consider the case with coupon resale.

Lemma 1 *Facing a marketing plan* (p, q, R) *that induces* θ_1 *consumers to make a purchase, the coupon reseller prices the coupon that he acquires at* $\min(T_1, \theta_1 q - p + R) - t_1$.

Proof. In this case the maximum amount of money that a θ_j consumer is willing to pay the coupon reseller in order to obtain the coupon is

$$\min(T_j, \theta_j q - p + R) - t_j,$$

and this implies that no trade can ever take place between a type θ_2 consumer and the coupon reseller. To see this, note that

$$\min(T_2, \theta_2 q - p + R) - t_2 \leq T_2 - t_2 < T',$$

and hence by spending T' to acquire the coupon and then selling the coupon to a type θ_2 consumer will result in a net loss to the coupon reseller.

Thus the coupon reseller will price optimally to trade only with θ_1 consumers. It is optimal to price in such a manner that θ_1 consumers are left with zero surplus. ‖

Lemma 2 *Suppose that Assumption 1 holds. When coupon resale is allowed, all consumers are served in equilibrium under the firm's optimal marketing plan* (p, q, R).

Proof. If (p, q, R) is the optimal marketing plan that induces all consumers to make a purchase in equilibrium, then given q, (p, R) must solve the following maximization problem:

$$\max_{p,R} p - (1 - \alpha)R$$

subject to

$$\theta_2 q \geq p,$$

$$T_2 \geq R,$$

$$\min(T_1, \theta_1 q - p + R) - t_1 \geq T'.$$

In the above, the first two constraints are respectively the θ_2 consumers' IR condition (ensuring that they will make a purchase) and IC condition (ensuring that they do not redeem the coupon), and the last constraint compactly gives the θ_1 consumers' IC condition (ensuring that they will acquire the coupon from the coupon reseller) and the coupon reseller's IR condition (ensuring that his profit is non-negative; cf. Lemma 1).

Assumption 1 implies that $T_1 - t_1 \geq T'$, and hence the last constraint is equivalent to

$$\theta_1 q - p + R - t_1 \geq T'.$$

Thus the optimal marketing plan that induces all consumers to make a purchase in equilibrium must be such that, given q, (p, R) solves the following maximization problem

$$(\text{P}') \quad \max_{p,R} p - (1 - \alpha)R$$

subject to

$$\theta_2 q \geq p,$$

$$T_2 \geq R,$$

$$\theta_1 q - p + R - t_1 \geq T'.$$

Comparing program (P$'$) to program (P), we see that the only difference between the two programs is that, by Assumption 1, the former program has a larger feasible set; that is, (p, R) satisfies the last constraint in (P$'$) if it satisfies the last constraint in (P). Thus the optimal marketing plan that induces all consumers to make a purchase in equilibrium generates a higher payoff for the firm with than without coupon resale. Since whether coupon resale is allowed does not affect the profitability of the optimal marketing plan that serves only θ_2 consumers, we conclude that under Assumption 1, when coupon resale is allowed, the firm's optimal marketing plan must again induce all consumers to make a purchase in equilibrium. $\|$

Because of Lemma 2, we continue to solve for the optimal marketing plan that induces all consumers to make a purchase in equilibrium. We shall take cases. Throughout this note, I shall assume that Condition 1 holds, and leave the other two cases to the reader.

Define

$$\hat{q} \equiv \frac{T_2 - (T' + t_1)}{\theta_2 - \theta_1}, \tag{3}$$

which is strictly positive because $T_2 > T_1 > T' + t_1$. In fact, we have by Assumption 1,

$$\hat{q} > \overline{q}.$$

Lemma 3 *Suppose that Assumption 1 and Condition 1 both hold. Then with coupon resale, given q, the firm's optimal pricing and promotion strategies are*

$$(p, R) = \begin{cases} (\theta_2 q, (\theta_2 - \theta_1)q + (T' + t_1)), & \text{if } \hat{q} \geq q; \\ (\theta_1 q + T_2 - (T' + t_1), T_2), & \text{if } \hat{q} \leq q. \end{cases} \tag{4}$$

Proof. Given q, (p, R) solves the firm's problem of finding the optimal pricing and promotion strategies:

$$\max_{p,R} p - (1 - \alpha)R \tag{5}$$

subject to

$$\theta_2 q \geq p, \tag{6}$$

$$T_2 \geq R, \tag{7}$$

$$\theta_1 q - p + R - t_1 \geq T'. \tag{8}$$

Since the objective function is increasing in p, given R, p must satisfy either $\theta_2 q = p$ or $\theta_1 q - p + R - t_1 = T'$. The same reasoning as that we used to solve problem (P) leads to

$$R = \min(T_2, (\theta_2 - \theta_1)q + (T' + t_1))), \tag{9}$$

and

$$p = \begin{cases} \theta_2 q, & \text{if } \theta_1 q + T_2 - (T' + t_1) > \theta_2 q; \\ \theta_1 q + T_2 - (T' + t_1), & \text{if } \theta_1 q + T_2 - (T' + t_1) \leq \theta_2 q. \end{cases} \tag{10}$$

This completes the proof. ‖

Note that in equilibrium the θ_1 consumers and the coupon reseller have no rent. The θ_2 consumers may or may not have rent, depending on whether R is equal to or strictly less than T_2. Coupon resale does not affect the θ_2 consumers directly, but via its effect on the firm's changing strategies, it might nonetheless change the θ_2 consumers' welfare.

Now we are ready to characterize the optimal product strategy with coupon resale.

Lemma 4 *The optimal marketing plan* (p, q, R) *must be such that*

$$q = \begin{cases} q_1, & \textit{if } q_2 > q_1 > \hat{q}; \\ q_2, & \textit{if } q_1 < q_2 < \hat{q}; \\ \hat{q}, & \textit{if } q_2 \geq \hat{q} \geq q_1. \end{cases} \tag{11}$$

Proof. The preceding lemma shows that under the optimal marketing plan (p, q, R), given q, the firm's payoff under the corresponding optimal (p, R) is

$$\Pi(q) = \begin{cases} \underline{\Pi}(q) \equiv \theta_2 q - (1-\alpha)[(\theta_2 - \theta_1)q + (T' + t_1)] - \frac{cq^2}{2}, & \text{if } q \leq \hat{q}; \\ \overline{\Pi}(q) \equiv \theta_1 q + \alpha T_2 - (T' + t_1), & \text{if } q \geq \hat{q}. \end{cases} \tag{12}$$

The unique unconstrained maximum of $\underline{\Pi}(q)$ on $\Re_+$ is

$$\frac{\alpha\theta_2 + (1-\alpha)\theta_1}{c},$$

and the unique unconstrained maximum of $\overline{\Pi}(q)$ on $\Re_+$ is

$$\frac{\theta_1}{c}.$$

Since both functions $\underline{\Pi}(q)$ and $\overline{\Pi}(q)$ are strictly concave, the lemma follows from a straightforward comparison between the unconstrained maxima to $\overline{\Pi}(\hat{q}) = \underline{\Pi}(\hat{q})$. ‖

The following propositions follow directly from the preceding lemmas.

Proposition 2 *Suppose that Assumption 1 and Condition 1 hold. Suppose also that* $q_1 > \hat{q}$*. Then with coupon resale nothing changes in the firm's optimal marketing plan except that the product price is increased by* $T_1 - (T' + t_1)$*.*

Proposition 3 *Suppose that Assumption 1 and Condition 1 hold. Suppose that $q_2 < \hat{q}$. Then with coupon resale the firm optimally raises product quality by $q_2 - q_1$ and product price by $\theta_2 q_2 - \theta_1 q_1 - (T_2 - T_1)$. The firm also optimally reduces the face value of the coupon by $T_2 - (T' + t_1) - (\theta_2 - \theta_1) q_2$.*

A few remarks for the preceding proposition are in order.

- Note that the increase in the product price caused by coupon resale can be decomposed into three terms:

$$\theta_2 q_2 - \theta_1 q_1 - (T_2 - T_1)$$

$$= \theta_2(q_2 - q_1) + [\theta_2 q_1 - (\theta_1 q_1 + T_2 - T' - t_1)] + [(\theta_1 q_1 + T_2 - T' - t_1) - (\theta_1 q_1 + T_2 - T_1)],$$

  where the last term reflects an intention to extract rent from the coupon reseller and the θ_1 consumers, following a reduction in this group's coupon-redemption cost; the second term is the price concession that the firm must make in order to prevent the θ_2 consumers from leaving the market; and the first term reflects the benefit from raising the product quality given that the new marginal consumers become the θ_2 consumers.

- Note that an increase in θ_1 may lead to more or less increase in the product price when coupon resale is allowed. This happens because a higher θ_1 encourages the firm to raise both the product quality (which equals $\frac{\theta_1}{c}$) and the product price (which, given q, is equal to $\theta_1 q + T_2 - T_1$) when coupon resale is prohibited, but it also raises the marginal benefit from raising the product quality when coupon resale becomes allowed. The latter can be understood as follows. An increase in θ_1 implies a higher consumption utility for the θ_1 consumers, so that the firm can reduce more the face value R of the coupon (recall that $R = (\theta_2 - \theta_1)q + (T' + t_1)$) and still make the coupon reseller's IR condition satisfied. This latter benefit increases with q, which then encourages the firm to raise the product quality more (recall that the firm's revenue with coupon resale is $\theta_2 q - (1 - \alpha)R$), thereby allowing the firm to raise the product price more (recall that the product price is $\theta_2 q$). With the price without coupon resale and the price with coupon resale both increasing in θ_1, an increase in θ_1 may or may not lead to a higher increase in the product price caused by coupon resale. (Given Assumption 1 and Condition 1, a sufficient condition ensuring that an increase in θ_1 leads to a higher increase in the product price caused by coupon resale is $(1-\alpha)\theta_2 \geq 2\theta_1$.)

- Next, observe that the increase in product quality caused by coupon resale,
$$q_2 - q_1 = \frac{\overline{\theta} - \theta_1}{c} = \frac{\alpha(\theta_2 - \theta_1)}{c},$$
is increasing in α and $\theta_2 - \theta_1$ and decreasing in c. The latter is self-evident. Given θ_1, the former says that the more important the θ_2 consumers become, the more the firm would like to raise the product quality when coupon resale is allowed. This happens because coupon resale makes the θ_2 consumers the new marginal consumers.

- Finally, observe that coupon resale does not imply a reduction in the face value of the coupon that equals the difference in the redemption costs of the coupon reseller and of the θ_1 consumers. The reduction is actually smaller, so that the coupon reseller would enjoy a rent if the product price were to remain unchanged. In equilibrium the reseller enjoys no rent, because the product price does rise, although not by an amount to extract all the coupon resellers' surplus—the firm must make sure that the θ_2 consumers are willing to stay in the market. Essentially, the firm would like to extract the coupon reseller's rent by raising the prodoct price, but when raising the price alone cannot do it, the firm resorts to lowering the face value of the coupon.

Proposition 4 *Suppose that Assumption 1 and Condition 1 hold, and that $q_2 \geq \hat{q} \geq q_1$. Then with coupon resale the firm optimally (i) raises the product quality from q_1 to $\hat{q}$; and (ii) raises the product price by $\theta_1(\hat{q} - q_1)$. However, the face value of coupon remains unchanged.*

9. **Example 2**. (**Giftcard and Price Competition, 1.**)
Consider the following duopoly model where all buyers have unit demand. Two retailers R1 and R2 are competing in price at date 1. For

simplicity, retailers have no productoin costs. Retailer Rj is faced with $a > 0$ loyal date-1 shoppers and 1 loyal gift-buyer, and these customers' valuation for Rj's product is V. There are also $c > 0$ switchers, who are date-1 shoppers who regard the two retailers' products as perfect substitutes. Let v denote switchers' valuation for either retailer's product. Assume that $V > v > 0$. Let p_1, p_2 denote the two retailers' date-1 product prices.

A date-1 shopper will never visit the retailers at date 0. The two gift-buyers, on the other hand, can visit the retailers at date 0 if they want to. The following events occur at date 0.

- At date 0, the two retailers R1 and R2 can simultaneously decide whether to spend $f \geq 0$ to issue a giftcard. One unit of Rj's giftcard will allow its holder to pick up one unit of Rj's product at date 1.
- Upon seeing the two retailers' decisions in the previous stage, the two retailers simultaneously announce the date-0 prices of their own giftcards. Let q_1, q_2 denote the date-0 prices of the giftcards. Let $q_j = +\infty$ if Rj has announced in the previous stage that no giftcards will be issued.
- Upon seeing the two retailers' decisions in the above two stages, giftbuyer 1 and giftbuyer 2 must simultaneously decide whether to buy a giftcard. (Because of loyalty to Rj, giftbuyer j has no reason to buy the giftcard issued by Ri.) Here, the two giftbuyers must play a simultaneous game. That is, each giftbuyer must guess (correctly) whether the other giftbuyer decides to buy a giftcard or not when making her own giftcard-purchasing decision. We assume that the two giftbuyers are fully strategic: giftbuyer j knows that she is the only giftbuyer interested in Rj's giftcard.

Following the above date-0 events, at date 1, there are therefore 4 possible demand states facing the two retailers, and we assume that the two retailers know which demand state has been realized before engaging in date-1 price competition. We represent the 4 demand states at date 1 by $\{(i, j) : \ i = 0, 1; j = 0, 1\}$. In demand state (i, j), giftbuyer 1 decides to make a purchase (of either a giftcard or a product) at time

i, and giftbuyer 2 decides to make a purchase at time j. For example, in state $(1, 0)$ R1 is faced with $a+1$ loyals and R2 is faced with a loyals before competing in price at date 1. Assume that

$$(\Theta) \quad (1 + a + c)v > (1 + a)V > (1 + a + c),$$

so that in each demand state at date 1 there exist only mixed-strategy NE's. (Note that under condition (Θ), we have $v > 1$.) Denote the date-1 mixed-strategy Nash equilibrium in state (i, j) by $(\tilde{p}_1(i, j), \tilde{p}_2(i, j))$.
(i) Derive $(\tilde{p}_1(i, j), \tilde{p}_2(i, j))$, for all $i, j = 0, 1$.
(ii) Now, consider the simultaneous game played by the two giftbuyers at date 0, when q_1, q_2 are already given. Denote

$$\mu_{11} \equiv E[\tilde{p}_1(1, 1)] = E[\tilde{p}_2(1, 1)],$$

$$\mu_{00} \equiv E[\tilde{p}_1(0, 0)] = E[\tilde{p}_2(0, 0)],$$

$$\mu^* \equiv E[\tilde{p}_1(1, 0)] = E[\tilde{p}_2(0, 1)],$$

$$\mu_* \equiv E[\tilde{p}_1(0, 1)] = E[\tilde{p}_2(1, 0)].$$

Now we characterize partially the relationship among these expected date-1 prices. Show that under condition (Θ), we have $\mu_{11} > \mu^*$, $\mu_{11} > \mu_*$, and $\mu_{11} > \mu_{00}$.
(iii) Show that if $q_1, q_2 > \mu_{11}$ or if $q_1, q_2 < \mu^*$ then this simultaneous game has a symmetric pure-strategy NE.[2] Show that if $q_i > \mu_{11}$ while $q_j < \mu_{11}$ or if $q_i < \mu^*$ whereas $q_j > \mu^*$ then this game has an asymmetric NE. Show that in the remaining case, this simultaneous game has a mixed strategy NE, where giftbuyer i's mixed strategy makes giftbuyer j indifferent about accepting or rejecting q_j.
(iv) Now, consider the date-0 simultaneous game where R1 and R2 must choose q_1 and q_2. Recall that Rj can avoid spending f only if $q_j = +\infty$. Find the NE of this simultaneous game. (Here each retailer must maximize the sum of expected profits over dates 0 and 1.)

[2]For example, if $q_1 > \mu_{11}$, then giftbuyer 2 rationally expects giftbuyer 1 to reject q_1. Being strategic, giftbuyer 2 knows that she will face the expected date-1 price μ_{11} unless she accepts q_2: she knows that she is the only one interested in buying R2's giftcard at date 0. Consequently, she will accept q_2 if $q_2 < \mu_{11}$ and she will reject q_2 if $q_2 > \mu_{11}$. If $q_2 = \mu_{11}$, she feels indifferent about accepting and rejecting, and she is ready to adopt any mixed strategy in this situation.

Solution.
Consider part (i). We shall focus on the symmetric date-1 pricing equilibrium whenever $i = j$.

- If $i = j = 0$, then both R1 and R2 adopt the following mixed strategy (described by the distribution function of the random product price) in equilibrium:

$$F(p) = \begin{cases} 0, & p < \frac{aV}{a+c}; \\ 1 + \frac{a}{c}[1 - \frac{V}{p}], & p \in [\frac{aV}{a+c}, v); \\ 1 + \frac{a}{c}[1 - \frac{V}{v}], & p \in [v, V); \\ 1, & p \geq V. \end{cases}$$

- If $i = j = 1$, then both R1 and R2 adopt the following mixed strategy in equilibrium:

$$F(p) = \begin{cases} 0, & p < \frac{(1+a)V}{1+a+c}; \\ 1 + \frac{1+a}{c}[1 - \frac{V}{p}], & p \in [\frac{(1+a)V}{1+a+c}, v); \\ 1 + \frac{1+a}{c}[1 - \frac{V}{v}], & p \in [v, V); \\ 1, & p \geq V. \end{cases}$$

- If $i = 1, j = 0$, then R1 and R2 adopt respectively the following mixed strategies in equilibrium:

$$F_2(p) = \begin{cases} 0, & p < \frac{(1+a)V}{1+a+c}; \\ 1 + \frac{1+a}{c}[1 - \frac{V}{p}], & p \in [\frac{(1+a)V}{1+a+c}, v); \\ 1, & p \geq v, \end{cases}$$

and

$$F_1(p) = \begin{cases} 0, & p < \frac{(1+a)V}{1+a+c}; \\ (1+\frac{a}{c})[1-\frac{(1+a)V}{(1+a+c)p}], & p \in [\frac{(1+a)V}{1+a+c}, v); \\ (1+\frac{a}{c})[1-\frac{(1+a)V}{(1+a+c)v}], & p \in [v, V); \\ 1, & p \geq V. \end{cases}$$

- If $i = 0, j = 1$, then R1 and R2 adopt respectively the following mixed strategies in equilibrium:

$$F_1(p) = \begin{cases} 0, & p < \frac{(1+a)V}{1+a+c}; \\ 1+\frac{1+a}{c}[1-\frac{V}{p}], & p \in [\frac{(1+a)V}{1+a+c}, v); \\ 1, & p \geq v, \end{cases}$$

and

$$F_2(p) = \begin{cases} 0, & p < \frac{(1+a)V}{1+a+c}; \\ (1+\frac{a}{c})[1-\frac{(1+a)V}{(1+a+c)p}], & p \in [\frac{(1+a)V}{1+a+c}, v); \\ (1+\frac{a}{c})[1-\frac{(1+a)V}{(1+a+c)v}], & p \in [v, V); \\ 1, & p \geq V. \end{cases}$$

Next, consider part (ii). Let us first compute μ_{00} and then prove that $\mu_{11} > \mu_{00}$. It can be shown that

$$\mu_{00} = \int_{\frac{aV}{a+c}}^{v} p dF(p) + V \cdot \frac{a}{c}[\frac{V}{v} - 1],$$

where

$$F(p) = \begin{cases} 0, & p < \frac{aV}{a+c}; \\ 1 + \frac{a}{c}[1 - \frac{V}{p}], & p \in [\frac{aV}{a+c}, v); \\ 1 + \frac{a}{c}[1 - \frac{V}{v}], & p \in [v, V); \\ 1, & p \geq V. \end{cases}$$

By integration by parts, we have

$$\int_{\frac{aV}{a+c}}^{v} p dF(p) = vF(v-) - \int_{\frac{aV}{a+c}}^{v} F(p)dp$$

$$= -(1 + \frac{a}{c})[v - \frac{aV}{a+c}] + \frac{aV}{c}\log(\frac{v}{\frac{aV}{a+c}}).$$

It follows that

$$\mu_{00} = H(x) = -xV + x\frac{V^2}{v} + xV\log(\frac{v(1+x)}{xV}),$$

where

$$x = \frac{a}{c}.$$

Hence we have

$$H'(x) = V[\log(\frac{v(1+x)}{Vx}) - 1 + \frac{Vx}{v(1+x)}].$$

Recall that by assumption $\frac{1+x}{x} \geq \frac{V}{v} > 1$. Consider $G : [1, +\infty) \to \Re$ defined by $G(y) = \log(y) + \frac{1}{y} - 1$. Note that

$$\lim_{y \downarrow 1} G(y) = G(1) = 0,$$

and

$$G'(y) = \frac{1}{y}[1 - \frac{1}{y}] > 0 = G'(1), \ \forall y > 1.$$

It follows that $G(y) > 0$ for all $y > 1$, and hence

$$H'(x) = VG(\frac{v(1+x)}{Vx}) > 0.$$

This proves that

$$\mu_{11} = \mu_{00} + \int_{\frac{a}{c}}^{\frac{a+1}{c}} H'(x)dx > \mu_{00}.$$

Next, note that

$$\mu_* = \frac{(1+a)V}{c}[\log(\frac{v(1+a+c)}{V(1+a)} + 1] - \frac{(1+a)v}{c},$$

$$\mu^* = (1+\frac{a}{c})[\frac{(1+a)V}{(1+a+c)}][\frac{V}{v} + \log(\frac{v(1+a+c)}{V(1+a)})] - \frac{aV}{c},$$

$$\mu_{11} = \frac{(1+a)V}{c}[-1 + \frac{V}{v} + \log(\frac{v(1+a+c)}{V(1+a)})].$$

We have

$$\mu_{11} - \mu^*$$
$$= [\frac{V}{v} + \log(\frac{v(1+a+c)}{V(1+a)})][\frac{(1+a)V}{c}][\frac{1}{1+a+c}] - \frac{1}{c}$$
$$= \frac{1}{c}\{[-(\log(\frac{V}{v}) - \frac{V}{v}) + \log(\frac{1+a+c}{1+a})][\frac{(1+a)V}{1+a+c}] - 1\} > 0,$$

where the inequality follows from the fact that $\max_{z>0} \log(z) - z = -1$ (and hence $-(\log(\frac{V}{v}) - \frac{V}{v}) \geq 1$) and condition ($\Theta$). Moreover, note that

$$\mu_{11} - \mu_* = \frac{(1+a)}{cv}[V-v]^2 > 0.$$

This finishes part (ii).

Now consider part (iii).

- Suppose that both $q_1, q_2 > \mu_{11}$. In this case, according to part (ii), it is a dominant strategy for giftbuyer i to reject q_i. Hence there is a symmetric equilibrium in which both giftbuyers choose to stay till date 1.
- In case $q_1, q_2 < \mu^*$, then giftbuyer i should accept q_i if she expects giftbuyer j to accept q_j: by rejecting q_i, giftbuyer i will be faced with the higher expected price μ^*. Thus there is a symmetric equilibrium where both giftbuyers choose not to show up at date 1.

- In case $q_i > \mu_{11} > q_j$, then we claim that there exists an equilibrium where giftbuyer i stays till date 1 but giftbuyer j chooses not to. Again, staying till date 1 is giftbuyer i's dominant strategy, and in anticipation of this, giftbuyer j understands that she will be faced with the expected date-1 price μ_{11} if she rejects the lower price q_j. Hence no giftbuyer can benefit from a unilateral deviation from the supposed equilibrium strategy.
- In case $q_j > \mu^* > q_i$, then we claim that there exists an equilibrium where giftbuyer i accepts q_i whereas giftbuyer j rejects q_j. To see that giftbuyer i will not deviate unilaterally, recall from part (ii) that $\mu^* < \mu_{11}$, and hence $q_i < \mu_{11}$, which implies that giftbuyer i should accept the lower price q_i instead of waiting for the date-1 price, which, given that giftbuyer j will stay till date 1, has an expected value equal to μ_{11}. Now, given that giftbuyer i is expected to accept q_i, giftbuyer j must compare q_j to μ^*. Clearly, rejecting q_j is her best response.
- Now, suppose that $\mu_{11} \geq q_1, q_2 \geq \mu^*$. Define λ_1 and λ_2 as such that

  $$\lambda_1\mu_{11} + (1 - \lambda_1)\mu^* = q_2, \quad \lambda_2\mu_{11} + (1 - \lambda_2)\mu^* = q_1.$$

  Since $\mu_{11} > \mu^*$, there exist unique solutions for λ_1 and λ_2 in the unit interval $[0, 1]$. We claim that there exists a mixed-strategy equilibrium in which giftbuyer i chooses to stay till date 1 with probability λ_i, $i = 1, 2$. To see that this indeed defines an equilibrium, note that by construction, rationally expecting λ_i, giftbuyer j feels indifferent about rejecting or accepting q_j, and is ready to adopt any mixed strategy λ_j, and we pick *the* λ_j that makes giftbuyer i feel indifferent about rejecting or accepting q_i (which justifies the λ_i adopted by giftbuyer i in the first place).

Now we consider part (iv). We shall focus on the case where $f = 0$ and demonstrate an equilibrium where giftcard issuance improves the retailers' welfare. Consider the strategy profile where $q_1^* = \mu_{11} > \mu^* = q_2^*$. We claim that this pair (q_1^*, q_2^*) forms an equilibrium. The subgame equilibria following respectively (q_1^*, q_2^*) and some unilateral deviation from (q_1^*, q_2^*) are now summarized as follows.

- Giftbuyers 1 and 2 will reject q_1^* and accept q_2^* respectively with probability one.
- Facing (q_1^*, q_2) with $q_2 \in [\mu^*, \mu_{11}]$, giftbuyer 2 will rejct q_2 with probability one, so that giftbuyer 1 remains indifferent about q_1^*, and she decides to accept q_1^* with probability one, which justifies giftbuyer 2's decision to reject q_2.
- Facing (q_1^*, q_2) with $q_2 > \mu_{11}$, it is a dominant strategy for giftbuyer 2 to reject q_2, and hence giftbuyers 1 becomes indifferent about rejecting or accepting q_1^*, and her decision is irrelevant to giftbuyer 2's decision to reject q_2.
- Facing (q_1^*, q_2) with $q_2 < \mu^*$, again, giftbuyers 1 and 2 will reject q_1^* and accept q_2 respectively with probability one.
- Facing (q_1, q_2^*) with $q_1 \geq \mu^*, q_1 \neq \mu_{11}$, giftbuyer 1 will reject q_1 (cf. part (iii)), and hence giftbuyer 2 decides to accept q_2^* with probability one.
- Facing (q_1, q_2^*) with $q_1 < \mu^*$, both q_1 and q_2^* will be accepted (cf. part (iii)).

According to the above subgame equilibria, R2 has no incentive to deviate unilaterally from the equilibrium (q_1^*, q_2^*). R1, on the other hand, cannot benefit from adopting a different $q_1 \geq \mu^*$. If R1 adopts some $q_1 < \mu^*$, her payoff becomes

$$q_1 + aV < \mu^* + aV < (a+1)V,$$

where the right-hand side is R1's payoff by sticking to q_1^*.

Thus we have shown that the pair

$$(q_1^*, q_2^*) = (\mu_{11}, \mu^*)$$

does form a Nash equilibrium at date 0. Now we claim that this equilibrium outcome Pareto dominates the equilibrium outcome in the absence of giftcards for the two retailers. Recall that in the latter case, both retailers obtain $(1+a)V$. In the above (q_1^*, q_2^*) equilibrium, R1 gets $(1+a)V$ again, but R2's payoff becomes

$$\mu^* + \frac{(1+a)V}{1+a+c} \cdot (a+c) > \frac{(1+a)V}{1+a+c} + \frac{(1+a)V}{1+a+c} \cdot (a+c) = (1+a)V,$$

where the inequality follows from the fact that $\mu^* > \frac{(1+a)V}{1+a+c}$. That is, R2 is better off with giftcard issuance.

Note that by symmetry

$$(q_1^*, q_2^*) = (\mu^*, \mu_{11})$$

also defines an equilibrium for the subgame where R1 and R2 price their giftcards. Ex-ante, before the game gets started, each of these two equilibria is likely to arise.[3] Hence, under condition (Θ), allowing the retailers to issue giftcards enhances both retailers' ex-ante welfare. The assumption that giftcard-buyers can act collectively at date 0 is crucial here. This actually weakens the giftcard-buyers' bargaining power against the giftcard-issuing retailer.

10. **Example 3**. (**Giftcard and Price Competition, 2.**)
Reconsider Example 2, but with the following modifications. First, only R1 has giftbuyers. Second, R1 has a continuum of giftbuyers with a population of 1. Let α be the population of the giftbuyers that decide to ignore the giftcard (issued by R1) and remain in the date-1 market. Hence we have a continuum of possible date-1 demand states, denoted by $\alpha \in [0, 1]$. Assume that $(1+a+c)v > (1+a)V$, so that there will be only mixed-strategy NE's at date 1. Denote the date-1 mixed-strategy Nash equilibrium in state α by $(\tilde{p}_1(\alpha), \tilde{p}_2(\alpha))$.
(i) Derive $(\tilde{p}_1(\alpha), \tilde{p}_2(\alpha))$, for all α.
(ii) Now, consider the date-0 subgame where the giftbuyers must simultaneous decide whether to accept q_1. Denote $\mu_1 \equiv E[\tilde{p}_1(1)]$ and $\mu_0 \equiv E[\tilde{p}_1(0)]$. Show that if $q_1 > \mu_1$ or if $q_1 < \mu_0$ then there exists a symmetric NE (where in equilibrium either $\alpha = 1$ or $\alpha = 0$). Show that if $q_1 \in [\mu_0, \mu_1]$, then there exists a NE where for some $\alpha \in (0, 1)$, exactly α giftbuyers decide to remain in the date-1 market. (Here let us forget about the problem of how to fulfill an asymmetric equilibrium in an anonymous game.)
(iii) Now, consider R1's decision of choosing q_1. Again, R1 can avoid

[3]For example, if there is a binomial sunspot taking values 1 and 2 with probability θ and $1-\theta$ respectively, then a *sunspot equilibrium* or a *correlated equilibrium* will result where both retailers believe (correctly) that $(q_i^*, q_j^*) = (\mu_{11}, \mu^*)$ in the event that the realization of the sunspot is i.

spending f only if $q_1 = +\infty$. Find R1's optimal q_1. (Here R1 must maximize the sum of expected profits over dates 0 and 1.)
Solution. Consider part (i). It can be shown that the following pair of distribution functions forms a Nash equilibrium at date 1 given that the demand state is α:

$$F_2(p) = \begin{cases} 0, & p < \frac{(\alpha+a)V}{\alpha+a+c}; \\ 1 + \frac{\alpha+a}{c} - \frac{(a+\alpha)V}{cp}], & p \in [\frac{(\alpha+a)V}{\alpha+a+c}, v); \\ 1, & p \geq v, \end{cases}$$

and

$$F_1(p) = \begin{cases} 0, & p < \frac{(\alpha+a)V}{\alpha+a+c}; \\ 1 + \frac{a}{c} - \frac{(\alpha+a)(a+c)V}{c(\alpha+a+c)p}, & p \in [\frac{(\alpha+a)V}{\alpha+a+c}, v); \\ 1 + \frac{a}{c} - \frac{(\alpha+a)(a+c)V}{c(\alpha+a+c)v}, & p \in [v, V); \\ 1, & p \geq V. \end{cases}$$

Note that we have chosen the asymmetric equilibrium for the case $\alpha = 0$.

Now, consider part (i). Define

$$\mu(\alpha) \equiv E[\tilde{p}_1(\alpha)],$$

so that

$$\mu(\alpha) \equiv H(x(\alpha)) = V\{\frac{V}{v}\frac{(1+\frac{a}{c})}{x(\alpha)} - \frac{a}{c} + (1+\frac{a}{c})\frac{V}{x(\alpha)}[\log(x(\alpha)) - \log(V)v]\},$$

where

$$x(\alpha) \equiv \frac{a+\alpha+c}{a+\alpha},$$

implying that

$$x'(\alpha) = \frac{-c}{(a+\alpha)^2} < 0.$$

Note that

$$H'(x) = \frac{V(1+\frac{a}{c})}{x^2}[\log(\frac{V}{v}) - \frac{V}{v} + 1] < 0,$$

where the inequality follows from the fact that

$$\log(\frac{V}{v}) - \frac{V}{v} < \max_{z>0} \log(z) - z = -1 = \log(1) - 1.$$

Hence we conclude that

$$\mu'(\alpha) > 0.$$

Now, if $q_1 > \mu(1) \equiv \mu_1$, then it is an equilibrium where all giftbuyers reject q_1: a single giftbuyer by assumption cannot alter the date-1 expected equilibrium price by changing her own giftcard-purchasing decision. If a single giftbuyer believes that all her fellow giftbuyers are rejecting q_1, then she knows that she will be faced with the expected date-1 price $\mu_1 < q_1$, and hence rejecting q_1 is the best response for her. This proves our assertion that it is an equilibrium where all giftbuyers reject q_1.

Similarly, if $q_1 < \mu(0) \equiv \mu_0$, then it is an equilibrium where all giftbuyers accept q_1. Again, a single giftbuyer knows that she will be faced with the expected date-1 price μ_0 if she rejects q_1 alone, which is not a wise decision.

Finally, for each $q_1 \in [\mu_0, \mu_1]$, by the fact that $\mu(\alpha)$ is continuous and strictly increasing on the interval $[0, 1]$, there exists $\alpha(q_1)$ such that

$$q_1 = \mu(\alpha(q_1)),$$

and hence expecting a population $\alpha(q_1)$ of giftbuyers to reject q_1, each and every single giftbuyer finds rejecting and accepting q_1 equally good, and hence we can assume that exactly a population $1 - \alpha(q_1)$ of giftbuyers purchase the giftcard at date 0 at the price q_1. This is an equilibrium. This finishes part (ii).

Now consider part (iii). Since retailer 1 by assumption has always a larger loyal base than retailer 2 does at date 1, and since by assumption there can exist only a a mixed-strategy Nash equilibrium at date 1, we can now summarize retailer 1's payoff of choosing q_1 at date 0 as follows:

$$\begin{cases} (a+1)V, & q_1 > \mu_1; \\ q_1[1-\alpha(q_1)] + V[a+\alpha(q_1)], & q_1 \in [\mu_0, \mu_1]; \\ q_1 + aV, & q_1 < \mu_0. \end{cases}$$

Since $\mu_1 < V$, we see that retailer 1's optimal decision is not to issue the giftcard at date 0 (even in the case $f = 0$). This result should be constrasted with our conclusion in part (ii), where giftcard buyers act strategically and collectively, which makes giftcard issuance benefit the retailers (at the expense of the date-1 switchers).

11. **Example 4**. (**Giftcard and Price Competition, 3.**)
Reconsider Example 3, but now assume that the continuum of giftbuyers are loyal to R2, and moreover, assume that the population of R1's loyal date-1 shoppers is $A > 1 + a$, with $AV < (A + c)v$. Thus only R2 has a date-0 decision about the issuance of a giftcard. Use backward induction to determine R2's equilibrium q_2.
Solution. The crucial observation here is that $F_2(\cdot)$ will be independent of α, the population of the giftbuyers waiting to purchase at date 1. Indeed, $F_2(\cdot)$ can be shown to satisfy, for all $p \in [\frac{AV}{(A+c)}, v)$,

$$AV = p[AF_2(p) + (A + c)(1 - F_2(p))] \Rightarrow F_2(p) = 1 + \frac{A}{c} - \frac{AV}{cp},$$

and

$$\Delta F_2(v) = \frac{A}{c}[\frac{V}{v} - 1].$$

This implies that $E[\tilde{p}_2]$ is independent of the giftbuyers' giftcard-purchasing decisions at date 0. Hence it is optimal for retailer 2 to choose either $q_2 > E[\tilde{p}_2]$ or $q_2 = E[\tilde{p}_2]$. The payoff to retailer 2 using the former strategy is

$$\frac{(1 + a + c)AV}{(A + c)};$$

and the payoff to retailer 2 using the latter strategy is

$$E[\tilde{p}_2] + \frac{(a + c)AV}{A + c} - f.$$

Since $\frac{AV}{A+c}$ is the lower bound of the support of $\tilde{p}_2$, we see that retailer 2 always wants to issue the giftcard if $f = 0$. The same remains true if $f > 0$ but is small; indeed, retailer 2 will issue the giftcard if and only if

$$f \leq E[\tilde{p}_2] - \frac{AV}{A+c}.$$

Examples 2-4 intend to deliver some insights regarding giftcard issuance. Example 2 shows that, seemingly contrary to our intuition, the retailers are better off facing giftbuyers that can act collectively. Examples 3 and 4 show that when there are many small giftbuyers, the retailer with a smaller loyal base is more likely to issue the giftcard. These preliminary analyses have assumed that there only exist mixed-strategy equilibria at date 1. More interesting results can arise when we allow pure-strategy equilibria in the presence or in the absence of giftcards.[4]

While we have interpreted the above scenario as one where the retailers consider issuing giftcards, these exercises actually relate to the recent literature in advanced selling. Unlike in the current exercises, where regular shoppers are assumed to be around only at date 1, a formal model of advanced selling must allow different segments of consumers to consider advanced buying. This creates new complexity in the analysis.

12. **Example 5**. (**Transaction-based Discrimination and Poaching with Demand Uncertainty**)
Firms A and B are located at the left and right endpoints of the Hotelling main street, denoted by the closed interval $[0, 1]$, and they

[4] For example, suppose that all giftbuyers are switchers that regard the two retailers' products as perfect subsitutes, and that only the retailer with a smaller loyal base (a) can issue the giftcard. In this case, giftcard issuance may reduce the population of date-1 switchers by so much that the other retailer (with loyal base A) chooses to serve only her loyals. (More precisely, this is true if $(A+c)v > AV > (A+\alpha c)v$ and $(a+\alpha c)v > aV$, where α is the fraction of switchers that are date-1 regular shoppers.) That is, the giftcard-issuing retailer can price at v at date 1 if all giftbuyers purchase the giftcard at date 0. With a continuum of acting-alone giftbuyers, this game has an equilibrium where the giftcard-issuing retailer induces all giftbuyers to purchase the giftcard at the price $q = v - \epsilon$. This leads to the shocking conclusion that with giftcard issuance, the two retailers can essentially obtain the profits of a perfectly colluding cartel; that is, one retailer gets the profit $(a + c)v$ and the other gets AV.

engage in price competition for two periods ($t = 1$ and $t = 2$). At the beginning of $t = 1$, firm j privately learns its loyal base $\tilde{a}_j$, which is the population of consumers residing at the same location as firm j, while the other firm believes that $\tilde{a}_j = 0$ with probability $1 - \pi$ and $\tilde{a}_j = a > 0$ with probability π. It is the two firms' common knowledge at the beginning of $t = 1$ that there are c consumers at each point $x \in (0, 1)$, and these consumers will be referred to as the *switchers*. A consumer located at $x \in [0, 1]$ must spend x dollars for round-trip transportation if she decides to make a purchase from firm A, and similarly she must spend $1-x$ dollars if she chooses to purchase from firm B instead. (From now on, we identify a consumer with her location on the Hotelling maing street.) The two firms produce a homogeneous good. Each consumer may buy either zero or one unit of the good, and we assume that the gross utility $v > 0$ from consuming the good is sufficiently high so that it plays no role in the subsequent price equilibria.

The game proceeds as follows. First the two firms simultaneously choose their first-period prices upon privately seeing their own loyal bases. Then, at the beginning of the second period, the loyal bases of the two firms become common knowledge, and the firms choose their second-period prices at the same time. Here we distinguish two cases: either a firm can offer two different prices to its new and old customers at $t = 2$, or it can only offer one price. We refer with Fudenberg and Tirole to the former case "price competition with poaching."

13. We now look for a symmetric Bayesian equilibrium of the game described above. First consider the second period with poaching. Suppose that in state (a_A, a_B), consumers purchased from firm A at $t = 1$ if and only if their locations $x \in [0, x^*]$, and consumers purchased from firm B at $t = 1$ if and only if their locations $x \in (x^*, 1]$. (In equilibrium, x^* varies with (a_A, a_B), but at the beginning of $t = 2$, we can take x^*, a_A, a_B as three separate state variables.)

    First consider the market segment $[0, x^*]$ (firm A's turf). Let firm A's and firm B's prices in firm A's turf be p_1 and p_2, and given x^*, let $\hat{t}$ be those consumers who feel indifferent about buying from firm A or from firm B. That is,

$$p_1 + \hat{t} = (1 - \hat{t}) + p_2.$$

Thus given x^*, p_1, p_2, firm A's sales volume in its own turf is $a_A + c\hat{t}$, and firm B's sales volume in firm A's turf is $c(x^* - \hat{t})$. Given x^*, p_2, firm A's optimal price in its own turf must then solve the following maximization problem:

$$\max_{p_1} p_1 a_A + p_1 c[\frac{1 + p_2 - p_1}{2}],$$

yielding

$$p_1 = \frac{1 + p_2}{2} + \frac{a_A}{c}.$$

Similarly, given p_1 and x^*, firm B seeks to

$$\max_{p_2} p_2 c[\frac{2x^* - 1 + p_1 - p_2}{2}],$$

yielding

$$p_2 = \frac{2x^* - 1 + p_1}{2}.$$

Note that given p_1, p_2, an increase in x^* raises firm B's sales volume, and hence it induces firm B to optimally raises price. Solving the above two reaction functions simultaneously, we obtain

$$p_1^* = \frac{1}{3} + \frac{2x^*}{3} + \frac{4a_A}{3c}, \quad p_2^* = -\frac{1}{3} + \frac{4x^*}{3} + \frac{2a_A}{3c}.$$

It follows that

$$\hat{t} = \frac{1}{6} + \frac{x^*}{3} - \frac{a_A}{3c}.$$

Note that an increase in a_A shifts up firm A's reaction function, which in turn raises both firms' equilibrium prices via strategic complementarity. (An increase in a_A raises p_1^* more than it raises p_2^*, because one unit of increase in p_1 raises p_2 by only $\frac{1}{2}$ units.) It is straightforward to verify that the two firms' second-period profits in firm A's turf are respectively

$$\pi_1 = \frac{c}{2}[p_1^*]^2 = \frac{c}{2}[\frac{1}{3} + \frac{2x^*}{3} + \frac{4a_A}{3c}]^2, \quad \pi_2 = \frac{c}{2}[p_2^*]^2 = \frac{c}{2}[-\frac{1}{3} + \frac{4x^*}{3} + \frac{2a_A}{3c}]^2.$$

Note that these profits are increasing in x^*.

Next, we consider the market segment $[x^*, 1]$ (firm B's turf). Let firm A's and firm B's prices in firm A's turf be p_A and p_B, and given x^*, let $\bar{t}$ be those consumers who feel indifferent about buying from firm A or from firm B. That is,

$$\bar{t} = \frac{1 + p_B - p_A}{2}.$$

Thus given x^*, p_A, p_B, firm B's sales volume in its own turf is $a_B + c(1 - \bar{t})$, and firm A's sales volume in firm B's turf is $c(\bar{t} - x^*)$. Given x^*, p_B, firm A's optimal price in its own turf must then solve the following maximization problem:

$$\max_{p_A} p_A c[\bar{t} - x^*] = p_A c[\frac{1 - 2x^* + p_B - p_A}{2}],$$

yielding

$$p_A = \frac{1 - 2x^* + p_B}{2}.$$

Similarly, given p_A and x^*, firm B seeks to

$$\max_{p_B} p_B a_B + p_B c[1 - \bar{t}] = p_B a_B + p_B c[\frac{1 - p_B + p_A}{2}],$$

yielding

$$p_B = \frac{a_B}{c} + \frac{1 + p_A}{2}.$$

Note that given p_A, p_B, an increase in x^* reduces both firms' sales volumes. Solving the above two reaction functions simultaneously, we obtain

$$p_A^* = 1 - \frac{4x^*}{3} + \frac{2a_B}{3c}, \; p_B^* = 1 - \frac{2x^*}{3} + \frac{4a_B}{3c}.$$

Note that an increase in a_B shifts up firm B's reaction function, which in turn raises both firms' equilibrium prices via strategic complementarity. (An increase in a_B raises p_B^* more than it raises p_A^*, because one unit of increase in p_B raises p_A by only $\frac{1}{2}$ units.)

It is straightforward to verify that the two firms' second-period profits in firm A's turf are respectively

$$\pi_A = \frac{c}{2}[p_A^*]^2 = \frac{c}{2}[1 - \frac{4x^*}{3} + \frac{2a_B}{3c}]^2, \; \pi_B = \frac{c}{2}[p_B^*]^2 = \frac{c}{2}[1 - \frac{2x^*}{3} + \frac{4a_B}{3c}]^2.$$

Note that these profits are decreasing in x^*.

Remark. An increase in x^*, other things (such as p_1, p_2, p_A, p_B) being equal, raises firm B's sales volume in firm A's turf, so that firm B wants to raise its price above p_2, which by strategic complementarity leads to a rise in firm A's price also. To see what happens, let $D(p_2; p_1, x^*)$ be the residual demand in firm A's turf that is faced by firm B, and recall that the optimal p_2 must satisfy the following first-order condition to firm B's profit maximization problem:

$$p_2 D'(p_2) + D(p_2) = 0;$$

that is, at the optimal p_2, the marginal profit produced by an infinitesimal change in p_2 must equal zero. Now if x^* rises to the level of $x^* + \epsilon$, then since $\hat{t}$ does not depend on x^*, firm B's sales volume would rise from $D(p_2)$ to $D(p_2) + \epsilon$. This means that raising the price above the original p_2 by an infinitesimal amount now becomes profitable:

$$p_2 D'(p_2) + [D(p_2) + \epsilon] > 0,$$

so that firm B would optimally raise p_2 to a higher level. The latter implies a rise in the equilibrium p_1 also via firm A's second-period reaction function in firm A's turf; that is, $p_1 = \frac{1+p_2}{2} + \frac{a_A}{c}$.

Similarly, an increase in x^*, other things equal, reduces firm A's sales volume in firm B's turf immediately, which makes lowering p_A profitable. To see what happens, let $D(p_A; p_B, x^*)$ be the residual demand in firm B's turf that is faced by firm A, and recall that the optimal p_A must satisfy the following first-order condition to firm B's profit maximization problem:

$$p_A D'(p_A) + D(p_A) = 0;$$

that is, at the optimal p_A, the marginal profit produced by an infinitesimal change in p_A must equal zero. Now if x^* rises to the level of $x^* + \epsilon$, then since $\bar{t}$ does not depend on x^*, firm A's sales volume would drop from $D(p_A)$ to $D(p_A) - \epsilon$. This means that lowering the price below the original p_A by an infinitesimal amount now becomes profitable:

$$p_A D'(p_A) + [D(p_A) - \epsilon] < 0,$$

and the latter fact also implies a drop in the equilibrium p_B via firm B's second-period reaction function in firm B's turf; that is, $p_B = \frac{a_B}{c} + \frac{1+p_A}{2}$.

14. For future reference, firm A's second-period equilibrium profit given (x^*, a_B, a_A) is

$$F(x^*, a_B, a_A) \equiv \pi_1 + \pi_A = \frac{c}{2}[\frac{1}{3} + \frac{2x^*}{3} + \frac{4a_A}{3c}]^2 + \frac{c}{2}[1 - \frac{4x^*}{3} + \frac{2a_B}{3c}]^2.$$

How does an increase in x^* affect firm A's second-period profit in state (a_B, a_A)? Apparently, it raises firm A's second-period profit from its own turf, but it also reduces firm A's second-period profit from firm B's turf. The following lemma shows that, starting from a very low x^*, reducing x^* by an infinitesimal amount enhances firm A's second-period profit, but starting from a sufficiently high x^*, raising x^* by an infinitesimal amount enhances firm A's second-period profit.

Lemma 5 *F is increasing (respectively, decreasing) in x^* if and only if $x^* \geq$ (respectively, $\leq$) $\frac{1}{2} + \frac{2(a_B - a_A)}{5c}$.*

Also, we have the following lemma.

Lemma 6 *The maximum of $F(x^*; a_A, a_B)$ on $[0, 1]$ appears at*

$$\begin{cases} 0, & \text{if } a_B \geq a_A; \\ 1, & \text{if } a_B \leq a_A. \end{cases}$$

15. Does poaching shift up or shift down firm A's first-period reaction function? From firm A's perspective, either $a_B = a$ (with probability π) or $a_B = 0$ (with probability $1 - \pi$). At $t = 1$, expecting firm B's first-period strategy $(P_B(0), P_B(a))$, firm A given a_A seeks to

$$\max_{P_A} G(P_A; a_A, P_B(0), P_B(a), \rho) \equiv \pi\{[x^*(a)c + a_A]P_A$$

$$+\rho(\frac{c}{2}[\frac{1}{3} + \frac{2x^*(a)}{3} + \frac{4a_A}{3c}]^2 + \frac{c}{2}[1 - \frac{4x^*(a)}{3} + \frac{2a}{3c}]^2)\}$$

$$+(1-\pi)\{[x^*(0)c + a_A]P_A + \rho(\frac{c}{2}[\frac{1}{3} + \frac{2x^*(0)}{3} + \frac{4a_A}{3c}]^2 + \frac{c}{2}[1 - \frac{4x^*(0)}{3}]^2)\},$$

subject to

$$x^*(a) = \frac{P_B(a) + 1 - P_A}{2}, \quad x^*(0) = \frac{P_B(0) + 1 - P_A}{2}.$$

In the above maximization program, $\rho = 1$ if the firms must compete in price with poaching at $t = 2$, $\rho = 0$ if they must compete in price without poaching at $t = 2$, and $G(P_A; a_A, P_B(0), P_B(a), 1)$ is the (undiscounted) sum of firm A's first-period and second-period profits.

Lemma 7 *The objective function in the above maximization problem is concave in P_A, regardless of $\rho = 0$ or $\rho = 1$.*

Proof. The assertion is self-evident when $\rho = 0$. When $\rho = 1$, G is a quadratic function of P_A, in which the coefficient of P_A^2 is

$$\frac{c}{2}[-1 + \pi \cdot \frac{5}{9} + (1-\pi) \cdot \frac{5}{9}] < 0. \parallel$$

Define

$$\overline{P}_B \equiv \pi P_B(a) + (1-\pi) P_B(0).$$

It is easy to see that when $\rho = 0$, firm A's first-period reaction function is

$$P_A(P_B(0), P_B(a); a_A) = \frac{a_A}{c} + \frac{1 + \overline{P}_B}{2}.$$

Now we solve for firm A's first-period reaction function for the case $\rho = 1$. Just like in the case $\rho = 0$, we must, given a_A, express P_A as a function of $P_B(0)$ and $P_B(a)$. The reaction function $P_A(P_B(0), P_B(a); a_A)$ is the implicit function defined by the following first-order condition:

$$0 = G'(P_A; a_A, P_B(0), P_B(a), 1) = \frac{\overline{P}_B + 1}{2} \cdot c + a_A - P_A c + \pi \frac{\partial x^*(a)}{\partial P_A} \frac{\partial F}{\partial x^*}(x^*(a), a_A, a)$$

$$+ (1-\pi) \frac{\partial x^*(0)}{\partial P_A} \frac{\partial F}{\partial x^*}(x^*(0), a_A, 0)$$

$$= \frac{\overline{P}_B + 1}{2} \cdot c + a_A - P_A c + \pi \cdot \frac{-1}{2} \cdot \frac{\partial F}{\partial x^*}(x^*(a), a_A, a) + (1-\pi) \cdot \frac{-1}{2} \cdot \frac{\partial F}{\partial x^*}(x^*(0), a_A, 0).$$

Note that

$$\pi \frac{\partial F}{\partial x^*}(x^*(a), a_A, a) + (1-\pi) \frac{\partial F}{\partial x^*}(x^*(0), a_A, 0)$$

$$= c\pi[\frac{2}{3}(\frac{1}{3} + \frac{2}{3}x^*(a) + \frac{4a_A}{3c}) - \frac{4}{3}(1 - \frac{4x^*(a)}{3} + \frac{2a}{3c})]$$
$$+c(1-\pi)[\frac{2}{3}(\frac{1}{3} + \frac{2}{3}x^*(0) + \frac{4a_A}{3c}) - \frac{4}{3}(1 - \frac{4x^*(0)}{3})]$$
$$= c[-\frac{10}{9} + \frac{20\overline{x}^*}{9} + \frac{8a_A}{9c}] - \frac{8ac\pi}{9},$$

where

$$\overline{x}^* = \pi x^*(a) + (1-\pi)x^*(0) = \frac{\overline{P}_B + 1 - P_A}{2}.$$

Thus we obtain the following lemma.[5]

[5]These computations remain valid when we replace the binomial distribution of $\tilde{a}_A$ (and $\tilde{a}_B$) by general distributions. Let $\overline{a} = E[\tilde{a}_A] = E[\tilde{a}_B]$. Note that in the binomial case, $\overline{a} = \pi a$. Given a_A, recall that firm A's expected second-period profit is

$$\frac{c}{2}E\{[\frac{1}{3} + \frac{2x^*(\tilde{a}_B)}{3} + \frac{4a_A}{3c}]^2 + [1 - \frac{4x^*(\tilde{a}_B)}{3} + \frac{2\tilde{a}_B}{3c}]^2\},$$

and one can show that its partial derivative with respect to P_A is

$$-\frac{c}{2}[-\frac{10}{9} + \frac{20\overline{x}^*}{9} + \frac{8(a_A - \overline{a})}{9c}],$$

where

$$\overline{x}^* = E[x^*(\tilde{a}_B)],$$
$$x^*(\tilde{a}_B) = \frac{P_B(\tilde{a}_B) + 1 - P_A}{2},$$

and firm B is expected to price at $P_B(\tilde{a}_B)$ at $t = 1$, upon seeing the realization of $\tilde{a}_B$. With poaching the first-period best response of firm A is therefore the P_A that solves the following first-order condition:

$$\frac{(1+\overline{P}_B)c}{2} + a_A - P_Ac = \frac{c}{2}[-\frac{10}{9} + \frac{20\overline{x}^*}{9} + \frac{8(a_A - \overline{a})}{9c}],$$

where again,

$$\overline{P}_B = E[P_B(\tilde{a}_B)],$$

so that we have

$$P_A^1(\overline{P}_B; a_A) = \frac{9}{8} - \frac{\overline{P}_B}{8} + \frac{5a_A}{4c} + \frac{\overline{a}}{c},$$

which is the same formula as the one appearing in the next lemma.

Lemma 8 *With $\rho = 0$ or $\rho = 1$, given a_A, firm A's first-period reaction function is a function of $\overline{P}_B$ only; that is, it does not depend on $P_B(0)$ and $P_B(a)$ separately. Denote it by $P_A^\rho(\overline{P}_B; a_A)$, and we have*

$$P_A^\rho(\overline{P}_B; a_A) = \begin{cases} \frac{a_A}{c} + \frac{1+\overline{P}_B}{2}, & \text{if } \rho = 0; \\ \\ \frac{9}{8} - \frac{\overline{P}_B}{8} + \frac{5a_A}{4c} + \frac{a\pi}{c}, & \text{if } \rho = 1. \end{cases}$$

An important fact is that poaching may alter the strategic complementarity in period 1. A higher $\overline{P}_B$ implies that both $x^*(a)$ and $x^*(0)$ are higher, and given a higher $\overline{P}_B$ an decrease in P_A raises the expected second-period profit more than the resulting loss in the first-period profit from firm A's perspective. This happens because F is convex in both $x^*(a)$ and $x^*(0)$, so that at higher and higher $x^*(a)$ and $x^*(0)$, a reduction in P_A becomes more and more profitable from the second-period perspective.

Letting $P_A^1 \geq P_A^0$, we obtain the following lemma.

Lemma 9 *Given a_A, with poaching firm A reacts by pricing higher in the first period if and only if it expects $\overline{P}_B \leq \frac{2a_A}{5c} + \frac{8a\pi}{5} + 1$.*

This lemma tells us that poaching is more likely to induce a firm with a loyal base than a firm without a loyal base to price higher in the first period. Equivalently, poaching is more likely to induce a firm without a loyal base than a firm with loyal base to lower its first-period price in order to gain the market share. The intuition can be captured by comparing the first-order condition with $\rho = 0$,

$$\frac{(1 + \overline{P}_B)c}{2} + a_A - P_A c = 0,$$

and the first-order condition with $\rho = 1$,

$$\frac{(1 + \overline{P}_B)c}{2} + a_A - P_A c = \frac{c}{2}[-\frac{10}{9} + \frac{20\overline{x}^*}{9} + \frac{8(a_A - \overline{a})}{9c}]$$

$$= \frac{c}{2}[-\frac{10}{9} + \frac{10(\overline{P}_B + 1 - P_A)}{9} + \frac{8(a_A - \overline{a})}{9c}].$$

Intuitively, given $\overline{P}_B$, with poaching an increase in P_A affects not only firm A's first-period expected profit, as in the case without poaching, but also firm A's second-period expected profit. An increase in P_A reduces both $x^*(0)$ and $x^*(a)$, and reduces firm A's second-period expected profit in firm A's turf and increases firm A's second-period expected profit in firm B's turf. The net effect of an increase in P_A on firm A's second-period expected profit is more likely to be positive when P_A gets larger but is more likely to be negative when a_A gets larger. This happens because a larger P_A implies that $x^*(0)$ and $x^*(a)$ are both smaller, and reducing $x^*(0)$ and $x^*(a)$ at smaller levels result in smaller changes in firm A's second-period expected profit, thanks to the convexity of F in x^*. A larger a_A, on the other hand, implies a larger loss in firm A's second-period expected profit in firm A's own turf, because firm A's second-period profit in firm B's turf no longer increases with a_A. These observations imply that $G'(P_A;\rho)$ when $\rho = 1$ is less decreasing in P_A and less increasing in a_A compared to $G'(P_A;\rho)$ when $\rho = 0$. Whether or not the optimal P_A becomes more responsive to an increase in a_A then depends on parameters. In our model, firm A's concern about its second-period expected profit leads to the optimal P_A becoming more responsive to a_A. In fact, one can show that one unit of increase in a_A leads to $\frac{1}{c}$ units of increase in P_A in the case $\rho = 0$, and it leads to $\frac{5}{4}\frac{1}{c}$ units of increase in P_A in the case $\rho = 1$.

16. Now, we are ready to solve explicitly the first-period symmetric Bayesian equilibrium. Let me adopt the general distribution described in footnote 1, and assume that $\tilde{a}_A$ and $\tilde{a}_B$ have a common support, of which the least upper bound is $a > 0$ and the greatest lower bound is zero.

   - Suppose that $\rho = 0$. In this case, by symmetry (i.e, $\overline{P}_A = \overline{P}_B = \overline{P}$), we have

$$\overline{P}_A = E[\frac{\tilde{a}_A}{c} + \frac{1 + \overline{P}_B}{2}]$$

$$\Rightarrow \overline{P} = 1 + \frac{2\overline{a}}{c}$$

$$\Rightarrow P_j(a_j) = \frac{a_j}{c} + \frac{1 + 1 + \frac{2\overline{a}}{c}}{2}$$

$$= 1 + \frac{a_j + \overline{a}}{c},$$

for all $j = A, B$ and for all realizations a_j of the random variable $\tilde{a}_j$.

- Suppose that $\rho = 1$. In this case, by symmetry (i.e, $\overline{P}_A = \overline{P}_B = \overline{P}$), we have

$$\overline{P}_A = E[\frac{9}{8} - \frac{\overline{P}_B}{8} + \frac{5\tilde{a}_A}{4c} + \frac{\overline{a}}{c}]$$
$$\Rightarrow \overline{P} = 1 + \frac{2\overline{a}}{c},$$

implying that the average first-period price remains the same with or without poaching. It follows that

$$P_j(a_j) = \frac{9}{8} - \frac{1 + \frac{2\overline{a}}{c}}{8} + \frac{5a_j}{4c} + \frac{\overline{a}}{c}$$

$$= 1 + \frac{3\overline{a}}{4c} + \frac{5a_j}{4c},$$

for all $j = A, B$ and for all realizations a_j of the random variable $\tilde{a}_j$.

Correspondingly, we can obtain the equilibrium payoff for each firm, which is the un-discounted sum of profits over $t = 1$ and $t = 2$. We only compute the equilibrium payoffs for the case $\rho = 1$, and leave the case $\rho = 0$ to the reader.

In case $\rho = 1$, in state (a_A, a_B), we have

$$x^*(a_A, a_B) = \frac{5(a_B - a_A)}{8c} + \frac{1}{2},$$

so that, by letting G_j denote firm j's equilibrium payoff, we have

$$G_j = [1 + \frac{3a_i + 5a_j}{4c}][a_j + \frac{5(a_i - a_j)}{8} + \frac{c}{2}]$$

$$+\frac{c}{2}[\frac{1}{3}+\frac{2}{3}(\frac{5(a_i - a_j)}{8c}+\frac{1}{2})+\frac{4a_j}{3c}]^2+\frac{c}{2}[1-\frac{4}{3}(\frac{5(a_i - a_j)}{8c}+\frac{1}{2})+\frac{2a_i}{3c}]^2, \quad j = A, B.$$

17. Now we look for a set of regularity conditions that will ensure that the above analysis is valid. In particular, we need to make sure that in equilibrium, (i) for all realizations (a_A, a_B),

$$0 \leq x^*(a_A, a_B), \bar{t}, \hat{t} \leq 1;$$

and (ii) at both $t = 1$ and $t = 2$, all consumers can get a non-negative consumer surplus by purchasing from either of the two firms.

Using the equilibrium pricing strategies derived above, we can obtain the following conditions. To ensure participation of all consumers in the first period, we impose

$$v \geq 1 + \max_{a_j} 1 + \frac{3\overline{a}}{4c} + \frac{5a_j}{4c} \Rightarrow v \geq 2 + \frac{3\overline{a} + 5a}{4c}.$$

This condition implies that even the consumers located at the right endpoint of the Hotelling main street can obtain non-negative surplus by purchasing from firm A, and even the consumers located at the left endpoint of the Hotelling main street can obtain non-negative surplus by purchasing from firm B.

Next, using

$$x^*(a_A, a_B) = \frac{5(a_B - a_A)}{8c} + \frac{1}{2},$$

we obtain

$$p_1^* = \frac{1}{3} + \frac{2[\frac{5(a_B-a_A)}{8c} + \frac{1}{2}]}{3} + \frac{4a_A}{3c} = \frac{2}{3} + \frac{5a_B}{12c} + \frac{11a_A}{12c},$$

$$p_2^* = -\frac{1}{3} + \frac{4[\frac{5(a_B-a_A)}{8c} + \frac{1}{2}]}{3} + \frac{2a_A}{3c} = \frac{1}{3} + \frac{5a_B}{6c} - \frac{a_A}{6c},$$

$$\hat{t} = \frac{1}{6} + \frac{[\frac{5(a_B-a_A)}{8c} + \frac{1}{2}]}{3} - \frac{a_A}{3c} = \frac{1}{3} + \frac{5a_B}{24c} - \frac{13a_A}{24c},$$

$$p_A^* = 1 - \frac{4[\frac{5(a_B-a_A)}{8c} + \frac{1}{2}]}{3} + \frac{2a_B}{3c} = \frac{1}{3} - \frac{a_B}{6c} + \frac{5a_A}{6c},$$

$$p_B^* = 1 - \frac{2[\frac{5(a_B-a_A)}{8c} + \frac{1}{2}]}{3} + \frac{4a_B}{3c} = \frac{2}{3} + \frac{11a_B}{12c} + \frac{5a_A}{12c},$$

and

$$\bar{t} = \frac{1 + p_B^* - p_A^*}{2} = \frac{2}{3} - \frac{5a_A}{24c} + \frac{13a_B}{24c}.$$

Hence to ensure that all consumers' IR are satisfied if they purchase in either of the two markets at the second period from either of the two firms, we impose

$$v \geq 1 + \max_{a_A, a_B} p_1^* = \frac{5 + 4a}{3c};$$

$$v \geq 1 + \max_{a_A, a_B} p_2^* = \frac{5a + 8c}{6c};$$

$$v \geq 1 + \max_{a_A, a_B} p_A^* = \frac{5a + 8c}{6c};$$

$$v \geq 1 + \max_{a_A, a_B} p_B^* = \frac{5 + 4a}{3c}.$$

Next, we must ensure that for all realized (a_A, a_B), $\hat{t} \leq x^*(a_A, a_B) \leq \bar{t}$. For the former, we require

$$\frac{1}{3} + \frac{5a_B}{24c} - \frac{13a_A}{24c} \leq \frac{5(a_B - a_A)}{8c} + \frac{1}{2},$$

and hence we need

$$\frac{10a_B - 2a_A}{24c} + \frac{1}{6} \geq 0,$$

which holds for all realized (a_A, a_B) if

$$\min_{a_A, a_B} \frac{10a_B - 2a_A}{24c} + \frac{1}{6} \geq 0,$$

so that we need

$$\frac{-2a}{24c} + \frac{1}{6} \geq 0.$$

For the latter, similarly, we impose

$$\frac{2}{3} - \frac{5a_A}{24c} + \frac{13a_B}{24c} \geq \frac{5(a_B - a_A)}{8c} + \frac{1}{2},$$

and hence we need

$$\frac{2a_B - 10a_A}{24c} - \frac{1}{6} \leq 0,$$

or equivalently,

$$\frac{10a_A - 2a_B}{24c} + \frac{1}{6} \geq 0,$$

which leads to the same requirement, namely

$$\frac{-2a}{24c} + \frac{1}{6} \geq 0.$$

Finally, we need to ensure that $0 \leq x^*(a_A, a_B) \leq 1$ for all realized (a_A, a_B). That is, we impose

$$|\frac{5(a_B - a_A)}{8c}| \leq \frac{1}{2}.$$

Hence we need

$$\frac{5a}{8c} \leq \frac{1}{2}.$$

Putting all the requirements derived above together, we have obtained a set of sufficient conditions ensuring the validity of our analysis, which consists of

$$4c \geq 5a,$$

and

$$v \geq \max(2 + \frac{3\overline{a} + 5a}{4c}, \frac{5 + 4a}{3c}, \frac{5a + 8c}{6c}).$$

18. **Example 6**. (**Transaction-based Discrimination with Non-linear Pricing**)

In Example 5, the duopolists are assumed to be able to use only linear pricing schemes. In this example, we consider a monoplist that can use non-linear pricing schemes.

A monopolistic firm M is trying to sell two products A and B to 1 buyer. The buyer, with her private information (α, β), seeks to maximize the expected value of

$$\alpha q_a + \beta\sqrt{q_b} - T_a - T_b,$$

where $q_a \in \{0, 1\}$ and $q_b \in \Re_+$ are respectively the amounts of products A and B consumed by the buyer, and T_j is the amount of money paid to M by the buyer for the purchase of product j, $j = A, B$. It is common knowledge that the buyer's reservation value α may equal a_2

or a_1 with respectively probability γ and $1-\gamma$, where $a_2 > a_1 > 1$, with 1 being the unit production cost for product A. Conditional on $\alpha = a_j$, $\beta = 1$ with prob. π_j and $\beta = 2$ with prob. $1-\pi_j$, where $\pi_2 = 0$ and $\pi_1 = \pi$. Let c be the unit cost for producing product B. The buyer may be myopic with probability $1-z$, or fully rational with probability z. A myopic buyer does not know her need of trading product j when the seller presents to her only the opportunity of trading product i. However, if the seller presents to the buyer the terms of transaction for both products A and B, then there is no difference between a myopic buyer and a fully rational buyer.

19. First consider the case where the seller offers the buyer the terms of trade about both products. In this case, the buyer is fully rational with probability one, and she has 3 possible types. We say that the buyer is of type 3 if $(\alpha, \beta) = (a_2, 2)$, of type 2 if $(\alpha, \beta) = (a_1, 2)$, and of type 1 if $(\alpha, \beta) = (a_1, 1)$. Let $Q_j \in \{0, 1\}$ be the quantity of product A sold to type-j buyer. Theorem AS-1 of Lecture 4 shows that the seller can restrict attention to incentive feasible direct contract; that is, the seller seeks to

$$\max_{\{(Q_j,q_j,T_j)\}} \gamma(T_3-Q_3-cq_3)+(1-\gamma)(1-\pi)(T_2-Q_2-cq_2)+(1-\gamma)\pi(T_1-Q_1-cq_1)$$

subject to the three types of the buyer's IC and IR conditions. We shall refer to the solution to this problem as the *optimal bundling contract*.

20. The seller can alternatively choose to sell A and then B, or to sell B and then A, to the buyer. We shall refer to these transaction modes the *sequential transaction contracts*. Intuitively, by adopting a sequential transaction scheme the seller can extract information from the buyer during the first-stage transaction, and then practice *transaction-based discrimination* in the second-stage transaction. However, the revelation principle introduced in Lecture 4 leads to the following result.

Lemma 10 *No sequential transaction schemes can outperform the optimal bundling scheme if $z = 1$.*

To see that this is true, suppose that $\{q_j(k), T_j(k); j = A, B, k = 1, 2, 3\}$ is the equilibrium outcome after the seller implements a fixed sequential transaction scheme $\mathcal{S}$. Then in equilibrium, a type-k buyer can mimic her type-k' counterpart's equilibrium behavior and obtain $\{q_j(k'), T_j(k'); j = A, B\}$ but would rather not to. This implies that $\mathcal{B} \equiv \{q_j(k), T_j(k); j = A, B, k = 1, 2, 3\}$ is one incentive feasible bundling scheme, which by definition cannot outperform the optimal bundling scheme!

21. The preceding lemma shows that, for a sequential transaction scheme to outperform the optimal bundling scheme, it is necessary (but in general insufficient) that $z < 1$. Note that with $z < 1$, each of the 3 types of the buyer may be myopic or fully rational, and hence the buyer has 6 possible types in total. We shall denote a type-k myopic buyer by Nk, and a type-k rational buyer by Rk. Indeed, applying the revelation principle again, we obtain the following result.

    Lemma 11 *Any sequential transaction scheme resulting in, for all $k = 1, 2, 3$, Nk and Rk behaving identically in equilibrium, cannot outperform the optimal bundling scheme.*

    To see that this is true, consider a sequential transaction scheme $\mathcal{S}$ resulting in, for all $k = 1, 2, 3$, Nk and Rk making the same payment to the seller and

    $$q_j(Nk) = q_j(Rk), \ \forall j = A, B.$$

    Since this is an equilibrium outcome, a rational type-k buyer can mimic her rational type-k' counterpart's equilibrium behavior but would rather not to. This implies that $\mathcal{B} \equiv \{q_j(k), T_j(k); j = A, B, k = 1, 2, 3\}$ is one incentive feasible bundling scheme for the rational buyer, and since with $\mathcal{S}$ in equilibrium the myopic buyer consumes exactly the same amounts of A and B and pays the seller the same amount of money, the sequential transaction scheme $\mathcal{S}$ must yield for the seller a payoff which coincides with the payoff generated by $\mathcal{B}$, which by definition cannot exceed the payoff generated by the <u>optimal</u> bundling scheme!

22. The upshot of the preceding lemma is that, in search of the optimal transaction scheme, we can ignore all sequential transaction schemes except those resulting in, for at least one $k \in \{1, 2, 3\}$, Nk and Rk behaving differently in equilibrium.

23. We shall first derive a series of lemmas for the sequential transaction schemes involving selling A and then B. The game proceeds as follows. The seller first posts the price T_A for product A. Then the buyer can decide to or not to buy one unit of product A. Then the seller forms his posterior belief about the buyer's type based on whether the buyer has purchased product A, and then the seller offers to the buyer an optimal screening contract for product B based on her posterior belief (transaction-based discrimination!). The seller cannot commit to the screening contract for selling B when he sells product A.

Lemma 12 *Given T_A, R1 decides to purchase product A if and only if N1 does.*

Proof. Whatever the seller's posterior belief may be, R1 will gain no surplus from consuming product B. Thus when deciding whether to purchase product A at the price T_A, R1 and N1 have the same payoff function, where recall that N1 totally disregard his need of subsequently purchasing product B when he is faced with T_A. ‖

Lemma 13 *Given T_A, N3 will purchase product A if N2 will, and N2 will purchase product A if and only if N1 will.*

Proof. Obvious.‖

Lemma 14 *Given T_A, R3 will purchase product A if R2 will.*

Proof. R2 and R3 will obtain the same surplus from consuming B as long as they make the same decision about purchasing or not purchasing product A at the price T_A. Since R3 obtains a higher gross utility than R2 in consuming product A, R3 will purchase product A whenever R2 will. ‖

Lemma 15 *Given T_A, if N3 decides not to purchase product A in equilibrium, neither does R2 or R3.*

Proof. Suppose that at the price T_A N3 decides not to purchase product A. This implies that $T_A > a_2 > a_1$, and by the preceding lemmas, N2, N1 and R1 must choose not to purchase product A either. Suppose that either R2 alone or R3 alone or both R2 and R3 choose to purchase product A at the price T_A. Then the seller's posterior belief upon seeing the buyer purchase product A must be such that with probability one the buyer will get the surplus $2\sqrt{q} - T$ when consuming q units of product B and paying the price T subsequently. Thus the seller will offer the first-best contract for the latter buyer, resulting in zero surplus from consuming product B for that buyer. Thus the type of buyer that purchases product A at the price T_A, which by assumption is either R2 or R3, must have an equilibrium payoff $a_2 - T_A < 0$, which is a contradiction because this type can choose not to buy product A and obtain a non-negative payoff. $\|$

Lemma 16 *Given T_A, if N1 decides to purchase product A in equilibrium, then so do R2 and R3.*

Proof. Suppose that at the price T_A, N1 decides to purchase product A in equilibrium. This implies that $a_2 > a_1 \geq T_A$, and hence by the preceding lemmas, R1, N2, and N3 will also purchase product A in equilibrium. Suppose that either R2 alone or R3 alone or both R2 and R3 choose not to purchase product A. Then the seller's posterior belief upon seeing the buyer refuse to purchase product A must be such that with probability one the buyer will get the surplus $2\sqrt{q} - T$ when consuming q units of product B and paying the price T subsequently. Thus the seller will offer the first-best contract for the latter buyer, resulting in zero surplus from consuming product B for that buyer. Thus the type of buyer that refuse to purchase product A at the price T_A, which by assumption is either R2 or R3, must have an equilibrium payoff equal to zero, which is a contradiction because this type can buy product A and obtain a payoff which is greater than or equal to $a_2 - T_A > 0$. $\|$

Lemma 17 *In order to generate a positive sales from selling product A, the seller will optimally choose a T_A lying in the interval $[a_1, a_2]$.*

Proof. This follows directly from the preceding lemmas. ‖

Because of this lemma, we can safely assume that the seller will set $T_A > a_2$ if he chooses to give up selling product A and to directly offer an optimal screening contract for product B based on his prior belief; our earlier remark says that this sequential transaction scheme can never outperform the optimal bundling scheme.

Now we classify the equilibria in which the seller decides to generate a positive sales volume for product A.

Lemma 18 *The optimal T_A that results in N1 and R1 refusing to buy product A but some other type buying product A must result in N2 and R2 also refusing to buy product A.*

Proof. Apparently that other type cannot be N2. Buying product A in this case results in the seller offering the first best contract for the type of buyer that will obtain surplus $2\sqrt{q} - T$ from accepting contract (q, T) for product B. Thus buying product A will generate a payoff $a_2 - T_A$ for N3 and R3, and a payoff $a_1 - T_A$ for R2, implying that R2 should not purchase product A in equilibrium either. This says that if there does exist such an equilibrium, N1, N2, R1, and R2 do not purchase product A, although N3 and R3 may still do. ‖

Thus there are only three classes of sequential schemes involving selling A and then B to consider.

The first class of such schemes lead to an equilibrium where N1 and R1 accept T_A and buy product A, which implies, by the preceding lemmas, that all other types do also. Since this class of sequential schemes lead to a pooling outcome at the stage of selling product A, it implies that for all $k = 1, 2, 3$, Nk and Rk must consume the same amounts of A and B and make the same payment to the seller. An earlier lemma shows that we can ignore this sequential scheme, because it cannot outperform the optimal bundling scheme.

With the second class of such schemes, the resulting equilibrium involves N1, R1, N2, R2, and R3 refusing to buy product A, and only N3 deciding to buy product A.

With the third class of such schemes, in the resulting equilibrium N1, R1, N2, and R2 refuse to buy product A, and only R3 and N3 decide to buy product A. Again, this class of schemes can be safely ignored when we search for the optimal scheme.

Proposition 5 *In search of the optimal scheme, we can ignore all sequential schemes asking the seller to sell A and then B but those that result in an equilibrium in which N1, R1, N2, R2, and R3 refuse to buy product A, and only N3 decides to buy product A.*

24. Now we consider the sequential transaction schemes under which the seller first sells B and then A. Again, we shall develop a series of useful lemmas.

Lemma 19 *With such a scheme* $N1$ *and* $R1$ *must behave identically when the seller sells product B.*

This lemma follows from the fact that $R1$ will receive no rent when the seller sells A regardless of $R1$'s behavior when the seller sells B.

Lemma 20 *With such a scheme,* $N2$*,* $N3$ *and* $R2$ *must behave identically when the seller sells product B.*

This lemma follows from the fact that $N2$ and $N3$ have the same payoff function when the seller sells product B, and the fact that $R2$, like $R1$, will receive no rent when the seller sells product A.

Lemma 21 *With such a scheme,* $R3$ *cannot be separated when the seller sells product B.*

This lemma calls for some explanations. If instead $R3$ can be identified after the seller sells product B, then $R3$ will receive no rent from A (the seller will set $T_A = a_2$). In equilibrium, $R3$ can mimic $N3$ but would rather not to, and since $N3$'s rent from A is non-negative, $R3$ must obtain a rent from B which is higher than or equal to the rent that $N3$ obtains from B. Since $N3$ can mimic $R3$'s purchase decision about B, $N3$ must obtain a rent from B which is higher than or equal to the rent that $R3$ obtains from B. It follows that these two must obtain the same rent from B, and to prevent $R3$ from mimicking $N3$, both of them (and $N2$ and $R2$) must obtain no rent from A. Thus this sequential scheme actually offers some (q_b, T_b) for $N1$ and $R1$ and some (q'_b, T'_b) for $N2, N3$, and $R2$, and some (q''_b, T''_b) to $R3$, where (q'_b, T'_b) and (q''_b, T''_b) must yield the same rent for $R3$, and to be an optimal choice for the seller, offering (q'_b, T'_b) or offering (q''_b, T''_b) for $R3$ must be equally profitable from the seller's perspective, which is a contradiction. Hence, $R3$ must pool with either $N1$ and $R1$, or with $N2, N3$, and $R2$. The latter involves for all $k = 1, 2, 3$, Nk and Rk behaving identically when the seller sells B, and hence can be safely ignored. Thus we come to the following conclusion.

Proposition 6 *In search of the optimal scheme, we can ignore all sequential schemes asking the seller to sell B and then A but those that result in an equilibrium in which the seller offers two distinct pairs* (T, q) *and* (T', q') *when selling B, such that N1, R1, and R3 take* (T, q) *and the rest 3 types take* (T', q') *in equilibrium.*

25. Now we consider a numerical example. Suppose that

$$z = 0,\ \gamma = c = \frac{1}{2},\ \pi = \frac{3}{4},$$

and that

$$\frac{1}{2}(a_2 - 1) > a_1 - 1 \Leftrightarrow \frac{a_2 + 1}{2} > a_1.$$

It can be shown that there are two undominated bundling schemes; in the first one, $\mathcal{B}_1$,

$$(Q_1, q_1) = (0, 0),\ (Q_2, q_2) = (0, 4),\ (Q_3, q_3) = (1, 4),$$

and in the second one, $\mathcal{B}_2$,

$$(Q_1, q_1) = (1, 0),\ (Q_2, q_2) = (1, 4),\ (Q_3, q_3) = (1, 4).$$

Correspondingly these two bundling schemes generate for the seller the payoffs

$$\frac{1}{2}(a_2-1)+\frac{5}{8}\cdot 2=\frac{4a_2+6}{8},$$

and

$$(a_1-1)+\frac{5}{8}\cdot 2=a_1+\frac{1}{4},$$

where note that $\frac{5}{8}=\gamma+(1-\gamma)(1-\pi)$,

$$4=\arg\max_q 2\sqrt{q}-cq,$$

and

$$2\sqrt{4}-c\cdot 4=2.$$

26. Let us denote the optimal sequential scheme asking the seller to sell A and then B by $\mathcal{S}_A$, and the optimal sequential scheme asking the seller to sell B and then A by $\mathcal{S}_B$. First let us find $\mathcal{S}_A$. By backward induction, we should consider the seller's optimal selling scheme for B and then the seller's optimal selling scheme for A, but since $z=0$, we can simplify the analysis by first consider the optimal selling scheme for A. It is obvious that the seller should set $T_a=a_2$, and hence the seller's payoff function can be written as

$$\gamma[(a_2-1)+\max_{q_2'}2\sqrt{q_2'}-cq_2']+(1-\gamma)\{0+\max_{q_1,q_2}\pi[\sqrt{q_1}-cq_1]+(1-\pi)[2\sqrt{q_2}-\sqrt{q_1}-cq_2]\},$$

It follows that at optimum,

$$q_2'=\frac{1}{c^2}=4=q_2,\quad q_1=\frac{(2\pi-1)^2}{4c^2}=\frac{1}{4}.$$

The seller's payoff from implementing $\mathcal{S}_A$ is therefore

$$\frac{a_2}{2}+\frac{53}{64}.$$

27. Now, we solve for $\mathcal{S}_B$. The seller's problem is to

$$\max_{q_1,q_2}\pi(1-\gamma)[\sqrt{q_1'}-cq_1'+(a_1-1)]+[1-\pi(1-\gamma)]\{2\sqrt{q_2}-\sqrt{q_1}-cq_2+\frac{4}{5}(a_2-1)\},$$

where

$$\frac{4}{5} = \frac{\gamma}{\gamma + (1-\gamma)(1-\pi)}.$$

At optimum, we have

$$q_1' = 0 = q_1, \quad q_2 = \frac{1}{c^2} = 4.$$

The seller's payoff from implementing $\mathcal{S}_B$ is therefore

$$\frac{4a_2 + 3a_1 + 3}{8}.$$

Note that $\mathcal{S}_A$ dominates $\mathcal{S}_B$ if and only if

$$\frac{29}{24} > a_1 > 1.$$

28. Now, the seller's optimal scheme is either $\mathcal{B}_1$, or $\mathcal{B}_2$, or $\mathcal{S}_A$, or $\mathcal{S}_B$. It is easy to see that $\mathcal{B}_1$ is dominated by $\mathcal{S}_A$, and that $\mathcal{B}_2$ is dominated by $\mathcal{S}_B$ if and only if

$$\frac{4a_2 + 1}{5} > a_1.$$

Summing up the above analysis, we have the following result.

- Suppose $a_2 < \frac{121}{96}$. In this case, we have

$$1 < \frac{a_2 + 1}{2} < \frac{4a_2 + 1}{5} < \frac{29}{24},$$

so that $\mathcal{S}_A \succ \mathcal{S}_B \succ \mathcal{B}_2$, and the optimal scheme is $\mathcal{S}_A$.

- Suppose $a_2 > \frac{136}{96}$. In this case, we have

$$1 < \frac{29}{24} < \frac{a_2 + 1}{2} < \frac{4a_2 + 1}{5},$$

so that $\mathcal{S}_B \succ \mathcal{B}_2$, and the optimal scheme is $\mathcal{S}_A$ if $a_1 < \frac{29}{24}$ and $\mathcal{S}_B$ if otherwise.

- Suppose $\frac{136}{96} > a_2 > \frac{121}{96}$. In this case, we have

$$1 < \frac{a_2 + 1}{2} < \frac{29}{24} < \frac{4a_2 + 1}{5},$$

so that $\mathcal{S}_A \succ \mathcal{S}_B \succ \mathcal{B}_2$, and the optimal scheme is again $\mathcal{S}_A$.

Remark. Note that $\mathcal{S}_A$ allows $N1$ and $R1$ to consume a positive amount of product B, and $\mathcal{S}_B$ allows $N1$ and $R1$ to consume a positive amount (1 unit) of product A. This happens because for the second-stage transaction these types become important once $N3$ and $R3$ are removed from the picture. This difference from the optimal bundling scheme $\mathcal{B}^*$ can be good or bad. With the current parameter values, we have $\mathcal{S}_A \succ \mathcal{B}_1$. Comparing $\mathcal{S}_B$ to $\mathcal{B}_2$, we see that with the former, $N2$ and $R2$ fail to consume product A, whereas $\mathcal{B}_2$ allows the seller to sell A to all types of the buyer. Again, this can be good or bad. With the latter the seller ensures that all types of the buyer pays a_1 in buying product A, but with the former the seller cannot commit not to give up $N2$ and $R2$ when selling A to $N2, N3, R2$ and $R3$. It should not be surprising that with $z = 0$ the seller's action in the first-stage transaction is always profit-efficient; a bundling strategy can at best attain the same profit-efficiency regarding the first-stage traded commodity. With the current parameter values, $\mathcal{B}_2$ fails to extract $N3$ and $R3$'s rent from A, because rent concession for A is necessary for the bundling strategy to reduce rent concession for B.

The above has assumed $z = 0$. With $0 < z < 1$, sequential schemes generally lead to profit losses in the first-stage transactions, because of the so-called *ratchet effect*: the rational buyer knows that revealing her type in the first-stage transaction will reduce her rent from the second-stage transaction, and hence the seller must concede more rents in the first-stage transactions than with a bundling scheme. The rent concession typically depends on z as well as other parameters.

29. **Example 7**. (**Coupon Competition within a Distribution Channel**)
Consider a non-integrated distribution channel composed of a manufacturer (M) and a retailer (R), facing two segments of consumers (the *highs* and the *lows*). M produces costlessly a single product and is restricted to use linear pricing schemes. We shall allow M and R to offer coupons to consumers. The populations of the highs and the lows are α and $1 - \alpha$ respectively, and each consumer has unit demand. Let V_H and V_L be the highs' and the lows' reservation prices for M's product. The highs would incur a fixed redemption cost H if they want

to carry and redeem at least one coupon.[6] Similarly, L represents the fixed redemption cost that the lows must incur to carry and redeem at least one coupon. Assume that consumers seek to maximize consumer surplus and both M and R seek to maximize expected profits.

The interactions among M, R and the two segments of consumers will be modelled as an extensive game, and the timing of the sequence of relevant events is as follows.

(a) R must first announce a retail coupon $c_R > 0$.

(b) Given c_R, M must announce its price promotion (w, c_M), where w is the wholesale price for R (the push promotion) and $c_M \geq 0$ is a manufacturer coupon for consumers (the pull promotion).

(c) Given (w, c_M), R can decide whether to buy M's product at the price w. If R chooses not to, then the game ends and everyone gets a zero payoff. If R accepts M's offer w, then R must choose a retail price p.

(d) Finally, upon seeing (p, c_M, c_R), consumers simultaneously decide whether to spend a redemption cost to carry the coupons, and whether to buy the product at the price p.

We shall denote the manufacturer's and the retailer's equilibrium expected profits by π_M and π_R respectively. Assume that[7]

$$1 > \alpha > 0, \ H > L \geq 0, \ \Delta \equiv (V_H - H) - (V_L - L) > 0. \qquad (13)$$

We shall look for the subgame-perfect Nash equilibria for the above extensive game.

Consider the subgame where retailer has chosen c_R. We can divide M's feasible contracts (w, c_M) into four classes, according to whether the

[6]Note that with at least one carried coupon, carrying an additional coupon incurs no marginal cost.

[7]The inequality $\Delta > 0$ ensures that the definitions for the highs and for the lows are independent of the manufacturer's and retailer's pull promotion strategies. In other words, even if all consumers choose to redeem coupons in equilibrium, the consumers with reservation price V_H still constitute the segment with a higher valuation. This assumption greatly simplifies our analysis.

lows will be served and whether the highs will redeem coupons. More specifically, we have

- Class 1: M wants R to serve both the highs and the lows, and to allow all buyers to redeem coupon(s). In this case, M's problem is:

$$\max_{w,c_M} \pi_M = w - c_M \tag{14}$$

subject to

$$\begin{array}{ll} & c_R + c_M > H \\ (IC) & (V_L + c_R + c_M - L - w) - c_R \geq \alpha(V_H + c_R + c_M - H - w) - \alpha c_R, \\ (IR) & (V_L + c_R + c_M - L - w) - c_R \geq 0. \end{array} \tag{15}$$

Note that the first constraint makes sure that the highs would want to redeem the coupons, if they decide to make a purchase. The IC constraint ensures that R would rather charge a low price $V_L + c_R + c_M - L$ and serve both the highs and the lows than give up the lows by charging a high price $V_H + c_R + c_M - H$. The IR constraint, on the other hand, ensures that by accepting M's offer (w, c_M), R can make a non-negative profit.

- Class 2: M wants R to serve only the highs, and $c_R + c_M > H$. In this case, M's problem is:

$$\max_{w,c_M} \pi_M = \alpha w - \alpha c_M \tag{16}$$

subject to

$$\begin{array}{ll} & c_R + c_M > H \\ (IC) & (V_L + c_R + c_M - L - w) - c_R \leq \alpha(V_H + c_R + c_M - H - w) - \alpha c_R, \\ (IR) & \alpha(V_H + c_R + c_M - H - w) - \alpha c_R \geq 0. \end{array} \tag{17}$$

Note that in this case the IC constraint ensures that R would rather serve the lows only than serve all consumers, and the IR constraint makes sure that R would get a non-negative profit by doing so.

- Class 3: M wants R to serve both the highs and the lows, but to allow only the lows to redeem the coupons; i.e. $L < c_R + c_M \leq H$.

In this case, M's problem is:

$$\max_{w,c_M} \pi_M = w - (1-\alpha)c_M \tag{18}$$

subject to

$$\begin{array}{ll} & L < c_R + c_M \leq H \\ (IC) & (V_L + c_R + c_M - L - w) - (1-\alpha)c_R \geq \alpha(V_H - w), \\ (IR) & (V_L + c_R + c_M - L - w) - (1-\alpha)c_R \geq 0. \end{array} \tag{19}$$

- Class 4: M wants R to serve only the highs, but to still require $L < c_R + c_M \leq H$. In this case, M's problem is:

$$\max_{w,c_M} \pi_M = \alpha w \tag{20}$$

subject to

$$\begin{array}{ll} & L < c_R + c_M \leq H \\ (IC) & (V_L + c_R + c_M - L - w) - (1-\alpha)c_R \leq \alpha(V_H - w), \\ (IR) & \alpha(V_H - w) \geq 0. \end{array} \tag{21}$$

Derive the optimal contract in each of the 4 classes above. Then, move backwards to consider R's problem of choosing c_R. Show that R will announce $c_R = H$ in equilibrium, which forces M to choose $(w, c_M) = (\hat{w}, 0)$, where

$$\hat{w} = \frac{V_L - L - \alpha(V_H - H)}{1-\alpha}. \tag{22}$$

Solution. I shall solve only the class-1 problem in detail, and leave the rest 3 problems to the reader.[8]

[8] First make the observation that under the optimal class-2 or class-4 contract, R must receive zero profits. This happens because in the two maximizatoin problems an increase in w only relaxes R's IC constraints, and hence the optimal w must make R's IR constraints binding, implying zero profits for R.

Next, consider M's maximization problem of finding the optimal class-1 or class-3 contract with $c_R > H$. Apparently, no class-3 contracts can be consistent with $c_R > H$, Formally, the class-1 problem can be stated as

$$\max_{w,c_M} \pi_M = w - c_M$$

subject to

$$\begin{array}{ll} & c_R + c_M > H \\ (IC) & (V_L + c_R + c_M - L - w) - c_R \geq \alpha(V_H + c_R + c_M - H - w) - \alpha c_R, \\ (IR) & (V_L + c_R + c_M - L - w) - c_R \geq 0. \end{array}$$

Note that we have computed the retailer's optimal pricing strategy before writing down the IR and IC constraints. If all consumers will be redeeming the coupons, then the valuation of the highs becomes

$$V_H + c_R + c_M - H > V_L + c_R + c_M - L,$$

because $c_R + c_M \geq c_R > H$. On the other hand, it can be easily seen that for a class-1 contract satisfying $c_R > H$ only $w - c_M$ matters to M, and hence we can assume that M optimally chooses $c_M = 0$. It follows that the optimal w must make R's IC constraint binding, so that $w = \hat{w}$ is optimal to M. It follows that R's profit is $\alpha(V_H - H - \hat{w})$ in this case, which is independent of c_R (as long as $c_R > H$).

Finally, consider M's maximization problem of finding the optimal class-1 or class-3 contract with $c_R \leq H$. In this case, any class-1 (w, c_M) must satisfy $c_M > H - c_R \geq 0$; and any class-3 (w, c_M) must satisfy $0 \leq c_M \leq H - c_R$.

- Consider first a class-1 (w, c_M) satisfying $c_R \leq H$ and $c_M > H - c_R \geq 0$. This maximization problem does not depend on c_R, and although it depends on $w - c_M$, it does not depend on c_M separately. Thus it is optimal for M to choose any $c_M > H - c_R$, and set

$$w - c_M = w_0 \equiv \frac{V_L - \alpha V_H + H - L}{1 - \alpha} = \hat{w} + H > \hat{w}.$$

  This implies that R's profit would be $\alpha(V_H - H - w_0) < \alpha(V_H - H - \hat{w})$.

- Next, consider a class-3 (w, c_M) satisfying $c_R \leq H$ and $0 \leq c_M \leq H - c_R$. We can re-write π_M as $(w - c_M) + \alpha c_M$, and now it becomes obvious that only $(w - c_M)$ appears in R's IC and IR constraints. Since given $w - c_M$, π_M is strictly increasing in c_M, implying that $c_M = H - c_R$, indicating the fact that R's and M's pull promotions are strategic substitutes. Given $c_M = H - c_R$, R's IC constraint must be binding at the optimal w, which yields

$$w + c_R = w_0, \tag{23}$$

  so that R's profit is $\alpha(V_H - w) = \alpha(V_H - w_0 + c_R)$.

Summing up the above findings, we can conclude that it is optimal for R to announce in the first stage $c_R = H$, which then induces M to choose $c_M = 0$ and $w = \hat{w}$ accordingly.

where the right-hand side is the lows' valuation after carrying the coupons. Thus the retailer can choose to price at $V_L + c_R + c_M - L$ or $V_H + c_R + c_M - H$. (Other prices are obviously dominated.) Pricing at the former level means that the retailer chooses to serve all consumers (every consumer can get a non-negative consumer surplus by buying the product from the retailer), and pricing at the latter means that the retailer chooses to serve only the highs.

In the class-1 problem, we have assumed that the manufacturer wants to choose (w, c_M) to make sure that the retailer would rather serve all consumers, and hence we require that (w, c_M) must be such that the retailer can get a non-negative profit by doing so (IR constraint), and doing so is better than dropping the lows (the IC constraint).[9] The first constraint is to make sure that the highs will also redeem the coupons (which implies that the lows will too).

Now, to solve this problem, the first step is to determine if the objective function is increasing or decreasing in respectively w and c_M. A look at the functional form tells us that it is increasing in w but decreasing in c_M. Next, by rewriting the IC as

$$g(w, c_M) = (V_L + c_R + c_M - L - w) - c_R - \{\alpha(V_H + c_R + c_M - H - w) - \alpha c_R\} \geq 0,$$

we see that g is decreasing in w. Similarly, the IR constraint is

$$h(w, c_M) = (V_L + c_R + c_M - L - w) - c_R \geq 0,$$

where h is decreasing in w also. In order to maximize the objective function, we should make w as high as possible, given any feasible value of c_M. However, the fact that both g and h are decreasing in w means that we cannot keep raising w without limits. Given a fixed c_M, a continual increase in w will ultimately lead to either $g(w, c_M) < 0$ or $h(w, c_M) < 0$. This implies that given c_M, there must *exist* an optimal solution for w, and that optimal w must make either $g(w, c_M) = 0$ or $h(w, c_M) = 0$; that is, either the IC constraint or the IR constraint will be *binding* at the optimal (w, c_M). The question is which constraint.

[9] Here notice that if the retailer chooses to serve only the highs, then only the highs will redeem the coupons in equilibrium, and hence the retailer has to pay αc_R. A consumer will not redeem the coupon if he does not purchase the product.

We claim that given c_M, the optimal w will make the IC constraint but not the IR constraint binding. To prove our claim, we only need to show that a contradiction will arise if the IR constraint is binding, since either the IR or the IC constraint must be binding at the optimal w.

Note that if $h(w, c_M) = 0$, then

$$w = V_L + c_M - L.$$

Replacing this result into $g(w, c_M) \geq 0$, we have

$$0 \geq \alpha\{(V_H + c_R + c_M - H - w)\} - \alpha c_R = \alpha\{V_H + c_M - H - w\} = \alpha\{(V_H - H) - (V_L - L)\} > 0,$$

which is a contradiction.

Thus we conclude that given c_M, the optimal w must make the IC but not the IR constraint binding. From here we can express w in terms of c_M:

$$g(w, c_M) = 0 \Rightarrow w = \frac{V_L - L + \alpha V_H - \alpha H + (1-\alpha)c_M}{1-\alpha}.$$

Hence the optimal c_M must solve the following maximization problem:

$$\max_{c_M} \pi_M = \frac{V_L - L + \alpha V_H - \alpha H + (1-\alpha)c_M}{1-\alpha} - c_M \tag{24}$$

subject to

$$\begin{array}{ll} & c_R + c_M > H \\ (IR) & (V_L + c_R + c_M - L - \frac{V_L - L + \alpha V_H - \alpha H + (1-\alpha)c_M}{1-\alpha}) - c_R \geq 0. \end{array} \tag{25}$$

Note that the IC constraint disappears in this new maximization problem, since we have used it to express w as a function of c_M!

Now we go through the process again: first check if π_M as a function of c_M is increasing or decreasing. A close look at the functional form reveals that π_M is independent of c_M! Moreover, h is independent of c_M also! It follows that any non-negative c_M that satisfies the first constraint $c_M \geq H - c_R$ is equally good, which does not affect the value of the objective function. Thus, for $c_R > H$ already, we can pick $c_M = 0$; and for $c_R \leq H$, we can pick any $c_M > H - c_R$, say $H - c_R + 1$.

Having solved the above reduced maximization problem, we can now determine the optimal w. Using the relation between w and c_M, we conclude that (asterisk indicates the *optimal* or *equilibrium* value)

$$(w^*, c_M^*) = \begin{cases} (\frac{V_L - L + \alpha V_H - \alpha H}{1-\alpha}, 0), & c_R > H; \\ (\frac{V_L - L + \alpha V_H - \alpha H + (1-\alpha)(H - c_R + 1)}{1-\alpha}, H - c_R + 1), & c_R \leq H. \end{cases}$$

Note that these arrangements all give rise to the same π_M^*, which is

$$\pi_M^* = \frac{V_L - L + \alpha V_H - \alpha H}{1-\alpha}.$$

We can also obtain π_R^*. We have

$$\pi_R^* = h(w^*, c_M^*) = V_L - L - \frac{V_L - L + \alpha V_H - \alpha H}{1-\alpha} = \frac{\alpha}{1-\alpha}[(V_H - H) - (V_L - L)] > 0.$$

This finishes my derivation for the optimal class-1 contract.

Remark. In the current model, the distribution channel as a whole can best benefit from pull promotions provided by the two channel members if and only if the lows redeem the coupons but the highs do not. This imposes an upper bound on the total amount of R's and M's pull promotions (i.e., $c_R + c_M \leq H$), leading to a crowding-out relationship between c_R and c_M. Another important observation is that whenever M wants to induce R to serve all consumers, in equilibrium R's IC constraint will be binding, implying that R's equilibrium profit is strictly decreasing in w. Hence R wishes to induce M to pick a low w. To this end, R can benefit from committing to the largest possible c_R, because by doing so, it induces M to reduce c_M, and therefore w.

30. **Example 8**. (**Optimal Design of a Product Line and a Return Policy**)
A seller wants to design two products to serve two segments of consumers. Segment i has population $\pi_i \in (0,1)$ with $\pi_1 + \pi_2 = 1$. A product is featured by its quality, and producing a product with quality $\alpha \geq 0$ will cost the seller $C(\alpha) = \frac{c\alpha^2}{2}$ per unit, where $c > 0$. A

consumer in segment i has gross valuation $\theta_i\alpha$ for a product with quality α. A consumer can buy at most one product from the seller, and the quantity is, for simplicity, 1 unit. Assume that $\theta_2 > \theta_1 > 0$, and consumers all seek to maximize consumer surplus. Assume also that $c > \theta_2$. The seller moves first by designing a product line and choosing the prices $\{(\alpha_i, p_i); i = 1, 2\}$, and then seeing the two products and their prices offered by the seller, consumers decide to or not to buy a product.[10]

(i) Assume that the seller can distinguish the two segments of consumers. Show that under the first best contract,

$$\alpha_i^{FB} = \frac{\theta_i}{c},\ p_i^{FB} = \frac{\theta_i^2}{c}.$$

Explain why the seller's decisions about product quality are "socially efficient."

(ii) Now assume instead that the seller cannot tell a θ_1-consumer from a θ_2-consumer. Show that under the second-best contract,

$$\alpha_1^{SB} = \max(0, \alpha_1^{FB} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}).$$

(iii) Now we assume that the seller must choose a return policy, in addition to the above product line decision. Here we confine $\alpha_i \in [0, 1]$, and interpret α_i as the probability that product i may be working. A consumer in segment j obtains a gross utility θ_j from using a product that works. A product that fails generates zero utility for all consumers. Under these new interpretations, $\theta_j\alpha$ becomes the "expected gross utility" for a consumer from segment j who purchases a product that may work with probability α. Here we are assuming risk-neutral consumers, who cannot resell what they purchase from the seller to other consumers. Again, the product is a search good; that is, consumers learn α_i when they see product i on the selling spot.

The seller's contract now becomes $\{(\alpha_i, p_i, p_i'); i = 1, 2\}$, where p_i' is the re-imbursement the seller promises to make to a consumer who returns

[10]Here we are assuming that consumers cannot resell what they purchase from the seller to other consumers. The product is referred to as a *search good*, in the sense that its quality can be ascertained by the buyer at the selling point.

product i when it fails subsequently. (Assume that neither the seller nor consumers have time preferences; that is, there is no discounting.) Note that $p'_i = 0$ is the same as a *no returns* policy for product i. For simplicity, assume that (i) a product that fails has no salvage value for the seller; and (ii) a consumer from segment j has to incur a cost $K_j \geq 0$ to return a product to the seller, and moreover $K_2 > K_1$ (rich people have higher θ's, and their opportunity costs of time are also higher). We shall focus on the case where

$$\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1} > 0. \quad (\Theta)$$

This condition implies that the optimal product line with a no-returns policy is such that

$$\alpha_1 = \frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}, \ \alpha_2 = \frac{\theta_2}{c},$$

$$p_1 = \theta_1[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}],$$

and

$$p_2 = \frac{\theta_2^2}{c} - (\theta_2 - \theta_1)[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}],$$

and the seller's payoff under this no-returns contract is

$$\pi_1\{\theta_1[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - (\theta_2 - \theta_1)[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_2}{c}]^2\}.$$

Solve for the optimal contract $\{\alpha_i, p_i, p'_i; i = 1, 2\}$ for the seller.

Solution. The seller seeks to

$$\max_{\{\alpha_i, p_i, p'_i; i=1,2\}} \sum_{i=1}^{2} \pi_i\{p_i - \frac{c\alpha_i^2}{2} - (1 - \alpha_i)1_{\{p'_i \geq K_i\}} p'_i\}$$

subject to

$$\theta_i\alpha_i - p_i + (1 - \alpha_i)\max(0, p'_i - K_i)$$

$$\geq \max[0, \theta_i\alpha_j - p_j + (1-\alpha_j)\max(0, p'_j - K_i)],\ \forall i, j \in \{1, 2\}.$$

Note that in the seller's objective function, 1_A is the indicator function for event A, which equals 1 if event A occurs and 0 if otherwise. The constraints compactly give IR_1, IR_2, IC_1, and IC_2. A contract $\{\alpha_i, p_i, p'_i; i = 1, 2\}$ is feasible if it satisfies the above contraints. We shall divide the set of feasible contracts into 4 subsets: (1) $p'_1 < K_1,\ p'_2 < K_2$; (2) $p'_1 \geq K_1, p'_2 < K_2$; (3) $p'_1 < K_1, p'_2 \geq K_2$; and (4) $p'_1 \geq K_1, p'_2 \geq K_2$.

Class 1. The seller seeks to

$$\max_{\{\alpha_i, p_i, p'_i < K_i; i=1,2\}} \pi_1[p_1 - \frac{c\alpha_1^2}{2}] + \pi_2[p_2 - \frac{c\alpha_2^2}{2}]$$

subject to, without loss of generality, $p'_1 = p'_2 = 0$ and

$$(\text{IR}_1)\ \ \theta_1\alpha_1 - p_1 \geq 0,$$

$$(\text{IC}_1)\ \ \theta_1\alpha_1 - p_1 \geq \theta_1\alpha_2 - p_2,$$

$$(\text{IR}_2)\ \ \theta_2\alpha_2 - p_2 \geq 0,$$

$$(\text{IC}_2)\ \ \theta_2\alpha_2 - p_2 \geq \theta_2\alpha_1 - p_1.$$

Note that the set of class-1 contracts coincides with the feasible contracts in part (ii). By Theorem AS-1 of Lecture 4, the maximization problem can be re-written as

$$\max_{\{\alpha_i; i=1,2\}} \pi_2[\theta_2\alpha_2 - \frac{c\alpha_2^2}{2} - (\theta_2 - \theta_1)\alpha_1] + \pi_1(\theta_1\alpha_1 - \frac{c\alpha_1^2}{2}). \tag{26}$$

Under condition (Θ), the seller's optimal choice is

$$\alpha_1 = \frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1},\ \ \alpha_2 = \frac{\theta_2}{c},$$

which generates for the seller the payoff

$$\pi_1\{\theta_1[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - (\theta_2 - \theta_1)[\frac{\theta_1}{c} - \frac{(\theta_2 - \theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_2}{c}]^2\}.$$

Class 2. The seller seeks to

$$\max_{\{\alpha_i,p_i,p'_i;i=1,2\}} \pi_1[p_1 - \frac{c\alpha_1^2}{2} - (1-\alpha_1)p'_1] + \pi_2(p_2 - \frac{c\alpha_2^2}{2})$$

subject to $p'_2 = 0$ and[11]

$$(\mathrm{IR}_1) \quad \theta_1\alpha_1 - p_1 + (1-\alpha_1)(p'_1 - K_1) \geq 0, \quad p'_1 \geq K_1,$$

$$(\mathrm{IC}_1) \quad \theta_1\alpha_1 - p_1 + (1-\alpha_1)(p'_1 - K_1) \geq \theta_1\alpha_2 - p_2,$$

$$(\mathrm{IR}_2) \quad \theta_2\alpha_2 - p_2 \geq 0,$$

$$(\mathrm{IC}_2) \quad \theta_2\alpha_2 - p_2 \geq \theta_2\alpha_1 - p_1 + (1-\alpha_1)(p'_1 - K_2).$$

At optimum either IR_2 is or is not binding. We claim that IR_2 is binding at optimum, and the proof is contained in the footnote.[12]

[11] Adding up the two IC's yields

$$(K_2 - K_1)(1-\alpha_1) + (\theta_2 - \theta_1)(\alpha_2 - \alpha_1) \geq 0,$$

and hence we do not obtain monotonicity as we did in Theorem AS-1 of Lecture 4.

[12] Suppose instead that IR_2 is not binding. Then IR_1 must be binding at optimum: otherwise raising slightly both p_1 and p_2 can enhance the seller's payoff, a contradiction to the assumed optimality. It follows that

$$p'_1 = K_1 + \frac{p_1 - \theta_1\alpha_1}{1-\alpha_1}.$$

Moreover, IC_2 must also be binding: otherwise raising slightly p_2 can enhance the seller's payoff, generating another contradiction. Hence the maximization problem can be re-written as

$$\max_{\{\alpha_i,p_i,p'_i;i=1,2\}} \pi_1\{p_1 - \frac{c\alpha_1^2}{2} - [\theta_1\alpha_1 - p_1 - (1-\alpha_1)K_1]\}$$

$$+\pi_2\{\theta_2(\alpha_2 - \alpha_1) + p_1 + (1-\alpha_1)K_2 - p_1 + \theta_1\alpha_1 - (1-\alpha_1)K_1 - \frac{c\alpha_2^2}{2}\}$$

subject to

$$0 \geq \theta_1\alpha_2 - p_2,$$

$$p'_1 \geq K_1 \Leftrightarrow p_1 - \theta_1\alpha_1 \geq 0,$$

$$\theta_2\alpha_2 - p_2 \geq 0.$$

The maximization problem can be furhter re-written as

$$\max_{\{\alpha_i,p_i,p'_i;i=1,2\}} \pi_1\{-\frac{c\alpha_1^2}{2} - \theta_1\alpha_1 + (1-\alpha_1)K_1\}$$

Given that IR_2 must be binding at optimum, we deduce that IR_1 implies IC_1, and hence we can remove IC_1. The seller's maximization problem is thus re-written as

$$\max_{\alpha_1,\alpha_2,p_1,p_1'} \pi_1[p_1 - \frac{c\alpha_1^2}{2} - (1-\alpha_1)p_1'] + \pi_2(\theta_2\alpha_2 - \frac{c\alpha_2^2}{2})$$

subject to

$$(\text{IR}_1) \quad \theta_1\alpha_1 - p_1 + (1-\alpha_1)(p_1' - K_1) \geq 0, \quad p_1' \geq K_1,$$

$$(\text{IC}_2) \quad 0 \geq \theta_2\alpha_1 - p_1 + (1-\alpha_1)(p_1' - K_2).$$

Now that the objective function is increasing in $p_1 - (1-\alpha_1)p_1'$, at optimum IR_1 must be binding also, and the maximization problem can be further re-written as

$$(\text{P2}) \quad \max_{\alpha_1,\alpha_2,p_1,p_1'} \pi_1[\theta_1\alpha_1 - (1-\alpha_1)K_1 - \frac{c\alpha_1^2}{2}] + \pi_2(\theta_2\alpha_2 - \frac{c\alpha_2^2}{2})$$

subject to

$$p_1' \geq K_1,$$

$$+\pi_2\{(1-\alpha_1)(K_2 - K_1) - (\theta_2 - \theta_1)\alpha_1 + \theta_2\alpha_2 - \frac{c\alpha_2^2}{2}\}$$

subject to

$$\theta_1\alpha_2 \leq p_2 = (1-\alpha_1)(K_2 - K_1) - (\theta_2 - \theta_1)\alpha_1 + \theta_2\alpha_2 \leq \theta_2\alpha_2,$$

$$p_1 - \theta_1\alpha_1 \geq 0.$$

The latter maximization problem can be further simplified as

$$\max_{\{\alpha_i,p_i,p_i';i=1,2\}} \pi_1\{-\frac{c\alpha_1^2}{2} - \theta_1\alpha_1 + (1-\alpha_1)K_1\}$$

$$+\pi_2\{(1-\alpha_1)(K_2 - K_1) - (\theta_2 - \theta_1)\alpha_1 + \theta_2\alpha_2 - \frac{c\alpha_2^2}{2}\}$$

subject to

$$0 \leq (1-\alpha_1)(K_2 - K_1) + (\theta_2 - \theta_1)(\alpha_2 - \alpha_1),$$

$$(1-\alpha_1)(K_2 - K_1) - (\theta_2 - \theta_1)\alpha_1 \leq 0.$$

It is clear that for $\alpha_1 \in [0,1]$, the second constraint can never be satisfied, and hence we have a contradiction.

$$(\text{IC}_2)\ \ 0 \geq \theta_2\alpha_1 - \theta_1\alpha_1 - (1-\alpha_1)(K_2 - K_1).$$

It follows that the optimal class-2 contract is such that (without loss of generality letting $p_1' = K_1$)

$$\alpha_1 = \min(1, \frac{\theta_1 + K_1}{c}, \frac{K_2 - K_1}{K_2 - K_1 + \theta_2 - \theta_1}), \quad \alpha_2 = \frac{\theta_2}{c},$$

$$p_1 = \theta_1 \min(1, \frac{\theta_1 + K_1}{c}, \frac{K_2 - K_1}{K_2 - K_1 + \theta_2 - \theta_1}), \quad p_2 = \frac{\theta_2^2}{c},$$

$$p_1' = K_1,\ p_2' = 0.$$

The optimal class-2 payoff for the seller is

$$\pi_1\{\theta_1 \min(1, \frac{\theta_1 + K_1}{c}, \frac{K_2 - K_1}{K_2 - K_1 + \theta_2 - \theta_1})$$

$$-(1-\alpha_1)K_1 - \frac{c}{2}[\min(1, \frac{\theta_1 + K_1}{c}, \frac{K_2 - K_1}{K_2 - K_1 + \theta_2 - \theta_1})]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - \frac{c}{2}[\frac{\theta_2}{c}]^2\}.$$

Class 3. Here, only α_2 is allowed for returns; i.e. $p_1' = 0$. Formally, the seller seeks to

$$\max_{\{\alpha_i, p_i, p_i'; i=1,2\}} \pi_2[p_2 - \frac{c\alpha_2^2}{2} - (1-\alpha_2)p_2'] + \pi_1(p_1 - \frac{c\alpha_1^2}{2})$$

subject to $p_1' = 0$ and

$$(\text{IR}_2)\ \ \theta_2\alpha_2 - p_2 + (1-\alpha_2)(p_2' - K_2) \geq 0, \ \ p_2' \geq K_2,$$

$$(\text{IC}_2)\ \ \theta_2\alpha_2 - p_2 + (1-\alpha_2)(p_2' - K_2) \geq \theta_2\alpha_1 - p_1,$$

$$(\text{IR}_1)\ \ \theta_1\alpha_1 - p_1 \geq 0,$$

$$(\text{IC}_1)\ \ \theta_1\alpha_1 - p_1 \geq \theta_1\alpha_2 - p_2 + (1-\alpha_2)(p_2' - K_1).$$

Now at optimum the IR_1 constraint must be binding, for if not, then by IC_2, we know that IR_2 will not be binding either, and a tiny increase in both p_1 and p_2 will be feasible and enhance the seller's payoff, which is a contradiction. Thus the maximization problem can be re-written as

$$\max_{\{\alpha_i,p_i,p_i';i=1,2\}} \pi_2[p_2 - \frac{c\alpha_2^2}{2} - (1-\alpha_2)p_2'] + \pi_1(\theta_1\alpha_1 - \frac{c\alpha_1^2}{2})$$

subject to

$$(\mathrm{IR}_2)\ \ \theta_2\alpha_2 - p_2 + (1-\alpha_2)(p_2' - K_2) \geq 0, \quad p_2' \geq K_2,$$

$$(\mathrm{IC}_2)\ \ \theta_2\alpha_2 - p_2 + (1-\alpha_2)(p_2' - K_2) \geq (\theta_2 - \theta_1)\alpha_1,$$

$$(\mathrm{IC}_1)\ \ 0 \geq \theta_1\alpha_2 - p_2 + (1-\alpha_2)(p_2' - K_1).$$

Now that the objective function is increasing in $p_2 - (1-\alpha_2)p_2'$, either IR_2 or IC_2 must be binding. Since $(\theta_2 - \theta_1)\alpha_1 \geq 0$, we conclude that it is IC_2 that has to be binding, and IR_2 will follow from IC_2. It follows that we can further re-write the maximization problem as

$$\max_{\{\alpha_i,p_i,p_i';i=1,2\}} \pi_2[\theta_2\alpha_2 - (1-\alpha_2)K_2 - \frac{c\alpha_2^2}{2} - (\theta_2-\theta_1)\alpha_1] + \pi_1(\theta_1\alpha_1 - \frac{c\alpha_1^2}{2}) \tag{27}$$

subject to

$$p_2' \geq K_2,$$

$$(\mathrm{IC}_1)\ \ (\theta_2 - \theta_1)(\alpha_2 - \alpha_1) - (1-\alpha_2)(K_2 - K_1) \geq 0.$$

At this point, we are able to prove that the optimal class-3 contract is dominated by the optimal class-1 contract. Call the maximization problem of finding the optimal class-3 contract "the constrained version of problem 3," and call the same maximization problem with IC_1 removed "the unconstrained version of problem 3." Apparently, the seller is better off if allowed to implement the latter than to implement the former. Because $(1-\alpha_2)K_2 \geq 0$, it is clear that the optimal value of the objective function in finding the optimal class-1 contract must be higher than the optimal value of the objective function in the unconstrained version of problem 3. By transitivity, the optimal value of the objective function in finding the optimal class-1 contract must also be higher than the optimal value of the objective function in the constrained version of problem 3. Thus the optimal class-3 contract is dominated by the optimal class-1 contract.

Class 4. This class of feasible contracts allows returns of both products. The seller seeks to

$$\max_{\{\alpha_i, p_i, p'_i; i=1,2\}} \pi_1[p_1 - \frac{c\alpha_1^2}{2} - (1-\alpha_1)p'_1] + \pi_2[p_2 - \frac{c\alpha_2^2}{2} - (1-\alpha_2)p'_2]$$

subject to

$$(\mathrm{IR}_1) \quad \theta_1\alpha_1 - p_1 + (1-\alpha_1)(p'_1 - K_1) \geq 0, \quad p'_1 \geq K_1,$$

$$(\mathrm{IC}_1) \quad \theta_1\alpha_1 - p_1 + (1-\alpha_1)(p'_1 - K_1) \geq \theta_1\alpha_2 - p_2 + (1-\alpha_2)(p'_2 - K_1), \quad p'_2 \geq K_2,$$

$$(\mathrm{IR}_2) \quad \theta_2\alpha_2 - p_2 + (1-\alpha_2)(p'_2 - K_2) \geq 0,$$

$$(\mathrm{IC}_2) \quad \theta_2\alpha_2 - p_2 + (1-\alpha_2)(p'_2 - K_2) \geq \theta_2\alpha_1 - p_1 + (1-\alpha_1)(p'_1 - K_2).$$

We claim that $\alpha_2 \geq \alpha_1$; that is, monotonicity is restored in this case. Again, this can be easily proved by adding up the two IC's and rearranging:

$$(\theta_2 - \theta_1 + K_2 - K_1)(\alpha_2 - \alpha_1) \geq 0.$$

We shall show that the optimal class-4 contract is dominated by the optimal class-2 contract. Note that if we define

$$P_1 = p_1 - p'_1(1-\alpha_1), \; P_2 = p_2$$

we can restate the seller's problem of finding the optimal class-2 contract as

$$(\mathrm{P2}') \qquad \max_{\{\alpha_i, p_i, p'_i; i=1,2\}} \pi_1[P_1 - \frac{c\alpha_1^2}{2}] + \pi_2(P_2 - \frac{c\alpha_2^2}{2})$$

subject to $p'_2 = 0$ and

$$(\mathrm{IR}_1) \quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq 0, \quad p'_1 \geq K_1,$$

$$(\mathrm{IC}_1) \quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq \theta_1\alpha_2 - P_2,$$

$$(\mathrm{IR}_2) \quad \theta_2\alpha_2 - P_2 \geq 0,$$

$$(\mathrm{IC}_2) \quad \theta_2\alpha_2 - P_2 \geq \theta_2\alpha_1 - P_1 - (1-\alpha_1)p'_1.$$

Moreover, since our earlier analysis shows that IC_1 will be redundant, we can replace it by

$$(\text{IC}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq -\infty.$$

Now, to best relax IC_2, we should optimally choose $p_1' \in [K_1, K_2]$, and if we define

$$P_1 = p_1 - p_1'(1-\alpha_1), \quad P_2 = p_2 - p_2'(1-\alpha_2),$$

then we can rewrite the resulting seller's maximization problem as

$$(\text{P4})\qquad \max_{\{\alpha_i, p_i, p_i'; i=1,2\}} \pi_1[P_1 - \frac{c\alpha_1^2}{2}] + \pi_2[P_2 - \frac{c\alpha_2^2}{2}]$$

subject to

$$(\text{IR}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq 0, \quad p_1' \in [K_1, K_2],$$

$$(\text{IC}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq \theta_1\alpha_2 - P_2 - (1-\alpha_2)K_1, \quad p_2' \geq K_2,$$

$$(\text{IR}_2)\quad \theta_2\alpha_2 - P_2 - (1-\alpha_2)K_2 \geq 0,$$

$$(\text{IC}_2)\quad \theta_2\alpha_2 - P_2 - (1-\alpha_2)K_2 \geq \theta_2\alpha_1 - P_1 - (1-\alpha_1)p_1'.$$

It follows that to best relax IC_2, we should set $p_1' = K_2$, so that the last constraint becomes

$$(\text{IC}_2)\quad \theta_2\alpha_2 - P_2 \geq \theta_2\alpha_1 - P_1 - (\alpha_2 - \alpha_1)K_2.$$

Now take a closer look at the following two maximization problems:

$$(\text{P2}')\qquad \max_{\{\alpha_i, p_i, p_i'; i=1,2\}} \pi_1[P_1 - \frac{c\alpha_1^2}{2}] + \pi_2(P_2 - \frac{c\alpha_2^2}{2})$$

subject to $p_2' = 0$ and

$$(\text{IR}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq 0, \quad p_1' \geq K_1,$$

$$(\text{IC}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq -\infty,$$

$$(\text{IR}_2)\quad \theta_2\alpha_2 - P_2 \geq 0,$$

$$(\text{IC}_2)\quad \theta_2\alpha_2 - P_2 \geq \theta_2\alpha_1 - P_1 - (1-\alpha_1)p_1'.$$

$$(\text{P4})\qquad \max_{\{\alpha_i,p_i,p_i';i=1,2\}} \pi_1[P_1 - \frac{c\alpha_1^2}{2}] + \pi_2[P_2 - \frac{c\alpha_2^2}{2}]$$

subject to

$$(\text{IR}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq 0, \quad p_1' \in [K_1, K_2],$$

$$(\text{IC}_1)\quad \theta_1\alpha_1 - P_1 - (1-\alpha_1)K_1 \geq \theta_1\alpha_2 - P_2 - (1-\alpha_2)K_1, \quad p_2' \geq K_2,$$

$$(\text{IR}_2)\quad \theta_2\alpha_2 - P_2 - (1-\alpha_2)K_2 \geq 0,$$

$$(\text{IC}_2)\quad \theta_2\alpha_2 - P_2 \geq \theta_2\alpha_1 - P_1 - (\alpha_2-\alpha_1)K_2.$$

Observe that if we set $p_1' \geq \frac{\alpha_2-\alpha_1}{1-\alpha_1}K_2$ in problem (P2′), then the feasible set defined by the constraints in (P4) becomes a subset of the feasible set defined by the constraints in (P2′).[13] This shows that the optimal class-4 contract is indeed dominated by the optimal class-2 contract.

The upshot of the above findings is that the seller's optimal contract is either the optimal class-1 contract or the optimal class-2 contract. We record this result as a proposition:

Proposition 7 *The seller's optimal contract is as follows.*

- *If the following condition holds,*

$$\pi_1\{\theta_1[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - (\theta_2-\theta_1)[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_2}{c}]^2\}$$

$$\geq \pi_1\{\theta_1 \min(1, \frac{\theta_1+K_1}{c}, \frac{K_2-K_1}{K_2-K_1+\theta_2-\theta_1})$$

[13] This happens because each and every constraint in (P2′) is implied by the corresponding constraint in (P4).

$$-(1-\alpha_1)K_1 - \frac{c}{2}[\min(1, \frac{\theta_1+K_1}{c}, \frac{K_2-K_1}{K_2-K_1+\theta_2-\theta_1})]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - \frac{c}{2}[\frac{\theta_2}{c}]^2\},$$

then

$$\alpha_1 = \frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1},\ \alpha_2 = \frac{\theta_2}{c},$$

$$p_1 = \theta_1[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}],$$

$$p_2 = \frac{\theta_2^2}{c} - (\theta_2-\theta_1)[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}],$$

and

$$p_1' = p_2' = 0.$$

- *If the following condition holds,*

$$\pi_1\{\theta_1[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - (\theta_2-\theta_1)[\frac{\theta_1}{c} - \frac{(\theta_2-\theta_1)\pi_2}{c\pi_1}] - \frac{c}{2}[\frac{\theta_2}{c}]^2\}$$

$$< \pi_1\{\theta_1 \min(1, \frac{\theta_1+K_1}{c}, \frac{K_2-K_1}{K_2-K_1+\theta_2-\theta_1})$$

$$-(1-\alpha_1)K_1 - \frac{c}{2}[\min(1, \frac{\theta_1+K_1}{c}, \frac{K_2-K_1}{K_2-K_1+\theta_2-\theta_1})]^2\}$$

$$+\pi_2\{\frac{\theta_2^2}{c} - \frac{c}{2}[\frac{\theta_2}{c}]^2\},$$

then

$$\alpha_1 = \min(1, \frac{\theta_1+K_1}{c}, \frac{K_2-K_1}{K_2-K_1+\theta_2-\theta_1}),\quad \alpha_2 = \frac{\theta_2}{c},$$

$$p_1 = \theta_1 \min(1, \frac{\theta_1+K_1}{c}, \frac{K_2-K_1}{K_2-K_1+\theta_2-\theta_1}),\quad p_2 = \frac{\theta_2^2}{c},$$

$$p_1' = K_1,\ p_2' = 0.$$