Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 1

1. Consider the following strategic game:

player 1/player 2	L	R
U	1,1	0,0
D	0,0	3,2

Any NE can be represented by (p, q), where p is the probability that player 1 adopts U and q the probability that player 2 adopts L.

(i) Show that this game has 3 NE's: (1,1), (0,0), and $(\frac{2}{3}, \frac{3}{4})$.

(ii) Now, consider the following new version of the above strategic game. At the first stage, player 1 can invite either A or B to become player 2 for the above strategic game. At the second stage, player 1 and the selected player 2 then play the above strategic game. A (or B) gets the player 2's payoffs described in the above strategic game, if he accepts the invitation to play the game. Without playing the game, A can get a payoff of $\frac{1}{200}$ on his own, and B can get a payoff of $\frac{3}{2}$ on his own.

The game proceeds as follows. First, player 1 can invite either A or B, and if the invitation is accepted, then the game moves on to the second stage; and if the invitation gets turned down, then player 1 can invite the other candidate. If both A and B turn down player 1's invitations, then the game ends with A getting $\frac{1}{200}$, B getting $\frac{3}{2}$, and player 1 getting 0.

Which one between A and B should player 1 invite first? Compute player 1's equilibrium payoff.

Solution. Part (i) is straightforward. Player 2's payoff is 1, 2, and $\frac{2}{3}$ in respectively the equilibria $(p, q) = (1, 1)$, $(0, 0)$, and $(\frac{2}{3}, \frac{3}{4})$.

Consider part (ii). If player 1 invites A first, then A will get $\frac{1}{200}$ if A turns down the invitation, and A will get at least $\frac{2}{3}$ if A accepts the invitation. Thus A will always accept player 1's invitation. Player 1 will not get the chance to invite B again. Thus player 1's payoff from inviting A first may equal 1, or 3, or $\frac{3}{4}$.

On the other hand, if player 1 invites B first, then B will get $\frac{3}{2}$ if B turns down the invitation, and B will get more than $\frac{3}{2}$ if and only if B expects to attain the equilibrium $(0,0)$ subsequently. Thus B will accept player 1's invitation if and only if B is prepared to play L with probability one in the strategic game subsequently. Thus when B turns down player 1's invitation player 1 will get the same payoff as he would when he invited A first, and when B accepts player 1's invitation player 1 would get the payoff of 3 for sure. To sum up, forward induction implies that player 1 should invite B first.

Remark. When a firm recruits new employees, it typically gives offers first to those job applicants that other firms would also like to recruit, even if all job applicants are expected to deliver similar job performances once recruited. This exercise gives an explanation to this phenomenon. A newly recruited job applicant that gives up a high salary that he or she could otherwise have by accepting another job opportunity signals that he or she intends to work hard, and that he or she expects to earn more by working hard given that his or her intention is correctly understood (via forward induction) by the employer (so that the employer is also expected to work hard accordingly).

We have assumed that A and B do not know their co-existence, as in the case of a firm recruiting new employees. In this case, A and player 1 must interact without knowing the presence of B, and similarly, B must interact with player 1 without knowing the presence of A. We show that it is a better choice for player 1 to contact B first, which would allow player 1 to use forward induction and to ensure (D,R) as the unique equilibrium outcome after B accepts the job offer (and B will because B knows that player 1 would interpret B's accepting the offer as a clear indication that B is planning to play R).

If instead it is common knowledge that A and B both exist and have the assumed reservation payoffs, then forward induction can be used by all three players. In the latter case, player 1 can ensure that the (D,R) equilibrium will prevail no matter which job applicant he is to contact first. Essentially, player 1 can ensure the (D,R) equilibrium by first contacting B, and hence when player 1 actually chooses to contact A first, A must interpret that player 1 is planning to play D in the subsequent normal form game, and in response A would then play R with probability one.

2. Consider the following strategic game:

player 1/player 2	L	M	R
U	2,0	2,2	4,4
M	6,8	8,4	5,0
D	10,6	4,4	6,5

(i) Assume that players are restricted to using only pure strategies. Find the strategy profiles that survive the procedure of *iterative deletion of strictly dominated strategies*.

(ii) Assume that players are restricted to using only pure strategies. Find the strategy profiles that survive the procedure of *iterative deletion of non-best-response strategies*.

(iii) How would your solutions for parts (i) and (ii) change if players are allowed to use also mixed strategies?[1]

[1] **Hint**: Define for part (i)

$$S_1^0 = S_1 = \{U, M, D\}, \quad S_2^0 = S_2 = \{L, M, R\},$$

and let S_j^n be the subset of S_j^{n-1} such that S_j^n contains player j's pure strategies that are not strictly dominated when player i is restricted to using only pure strategies contained in S_i^{n-1}. Then define

$$S_1^\infty \equiv \bigcap_{n=1}^{\infty} S_1^n, \quad S_2^\infty \equiv \bigcap_{n=1}^{\infty} S_2^n.$$

Solution. Consider part (i). Define

$$S_1^0 = S_1 = \{U, M, D\}, \quad S_2^0 = S_2 = \{L, M, R\}.$$

Let S_j^n be the subset of S_j^{n-1} such that S_j^n contains player j's pure strategies that are not strictly dominated when player i is restricted to using only pure strategies contained in S_i^{n-1}. Then we have

$$S_1^1 = \{M, D\}, \quad S_2^1 = \{L, M, R\},$$
$$S_1^2 = \{M, D\}, \quad S_2^2 = \{L\},$$
$$S_1^3 = \{D\}, \quad S_2^3 = \{L\},$$
$$S_1^n = \{D\}, \quad S_2^n = \{L\}, \ \forall n \geq 3,$$

and hence

$$S_1^\infty \equiv \bigcap_{n=1}^{\infty} S_1^n = \{D\}, \quad S_2^\infty \equiv \bigcap_{n=1}^{\infty} S_2^n = \{L\}.$$

That is, in this game $\{D, L\}$ is the unique strategy profile that survives the procedure of *iterative deletion of strictly dominated strategies.*

Consider part (ii). Define

$$H_1^0 = S_1 = \{U, M, D\}, \quad H_2^0 = S_2 = \{L, M, R\}.$$

The strategy profiles that survive the procedure of *iterative deletion of strictly dominated strategies* are the elements of the Cartesian product $S_1^\infty \times S_2^\infty$.

Define for part (ii)

$$H_1^0 = S_1 = \{U, M, D\}, \quad H_2^0 = S_2 = \{L, M, R\},$$

and let H_j^n be the subset of H_j^{n-1} such that H_j^n contains all player j's pure-strategy best responses when player i is restricted to using only pure strategies contained in H_i^{n-1}. Then define

$$H_1^\infty \equiv \bigcap_{n=1}^{\infty} H_1^n \quad H_2^\infty \equiv \bigcap_{n=1}^{\infty} H_2^n.$$

The strategy profiles that survive the procedure of *iterative deletion of non-best-response strategies* are the elements of the Cartesian product $H_1^\infty \times H_2^\infty$.

Let H_j^n be the subset of H_j^{n-1} such that H_j^n contains all player j's pure-strategy best responses when player i is restricted to using only pure strategies contained in H_i^{n-1}. Then we have

$$H_1^1 = \{M, D\}, \quad H_2^1 = \{L, R\},$$

$$H_1^2 = \{D\}, \quad H_2^2 = \{L\},$$

$$H_1^n = \{D\}, \quad H_2^n = \{L\}, \; \forall n \geq 2,$$

and hence

$$H_1^\infty \equiv \bigcap_{n=1}^{\infty} H_1^n = \{D\}, \quad H_2^\infty \equiv \bigcap_{n=1}^{\infty} H_2^n = \{L\}.$$

That is, in this game $\{D, L\}$ is the unique strategy profile that survives the procedure of *iterative deletion of non-best-response strategies*.

Now, consider part (iii). First consider allowing mixed strategies in part (i). Let Σ_j be the set of mixed strategies available for player j. Let $\Sigma_j^0 = \Sigma_j$, and Σ_j^n be the set of elements in Σ_j^{n-1} that are not strictly dominated mixed strategies from player j's perspective, given that player i is restricted to using mixed strategies in Σ_i^{n-1}. Now, observe that Σ_1^1 does not contain any mixed strategies that assign a positive probability to U. That is, Σ_1^1 contains only (some) mixed strategies that randomize over M or D. Observe also that against any element in Σ_1^1, L strictly dominates any other mixed strategy for player 2. Thus $\Sigma_2^n = \{L\}, \; \forall n \geq 2$, so that $\Sigma_1^n = \{D\}, \; \forall n \geq 3$. It follows that

$$\Sigma_1^\infty \equiv \bigcap_{n=1}^{\infty} \Sigma_1^n = \{D\}, \quad \Sigma_2^\infty \equiv \bigcap_{n=1}^{\infty} \Sigma_2^n = \{L\},$$

and hence $\{D, L\}$ must be the unique strategy profile that survives the procedure of *iterative deletion of strictly dominated strategies*, even if the players are allowed to use mixed strategies.

Finally, consider allowing mixed strategies in part (ii). It is clear that each mixed strategy of player 1 that can become a best response against player 2 using any mixed strategies must assign zero probability to U.

Player 2's best response (in mixed strategy) against any mixed strategy of player 1 that assigns zero probability to U is L. Player 1's best response (in mixed strategy) against L is D. It follows that, in this game, $\{D, L\}$ must be the unique strategy profile that survives the procedure of *iterative deletion of non-best-response strategies*, even if the players are allowed to use mixed strategies.

3. Players 1 and 2 are living in a city where on each day the weather is equally likely to be sunny (S), cloudy (C), or rainy (R). Players 1 and 2 are supposed to play the following strategic game at date 1.

player 1/player 2	L	R
U	15,3	0,0
D	12,12	3,15

(i) Suppose that the above strategic game must be played before players 1 and 2 know anything about the date-1 weather. Verify that the game has two pure-strategy NE's and one mixed-strategy NE. Suppose that before playing the strategic game, players 1 and 2 both believe that they may attain each pure-strategy NE with probability $a < \frac{1}{2}$ and they may attain the mixed-strategy NE with probability $1 - 2a$. Compute the expected Nash-equilibrium payoff for player 1 given a.

(ii) Now, suppose that for $i = 1, 2$, player i receives a weather report s_i right before playing the above strategic game at date 1. The weather report s_1 tells player 1 whether the weather will or will not be sunny. The weather report s_2 tells player 2 whether the weather will or will not be rainy. That the two players will receive these two weather reports is their common knowledge at the beginning of date 1. Consider the following strategy profile:

- Player 1 uses U if the weather will be sunny, and he uses D if the weather will not be sunny.
- Player 2 uses R if the weather will be rainy, and he uses L if the weather will not be rainy.

Does this strategy profile constitute a Nash equilibrium?[2] If it does, compute player 1's equilibrium payoff. Compare this payoff to player 1's expected Nash-equilibrium payoff that you obtained in part (i). Explain.[3]

Solution. Consider part (i). Let p be the probability that player 1 may use U, and q the probability that player 2 may use L. We have 3 NE's for this game, in which (p, q) equals respectively $(1, 1)$, $(0, 0)$, and $(\frac{1}{2}, \frac{1}{2})$. Given a, player 1's expected Nash-equilibrium payoff is equal to

$$a \cdot 15 + a \cdot 3 + (1 - 2a) \cdot \frac{1}{4}(15 + 0 + 12 + 3)$$
$$= 18a + \frac{15}{2} - 15a = 3a + \frac{15}{2}.$$

Consider part (ii).

- First suppose that the true weather state is sunny.

  In this event, player 1 knows that the state is sunny, and he knows that player 2 knows that the state is not rainy, and according to player 2's strategy described above, player 1 expects player 2 to

[2] This strategy profile is not an NE of the original strategic game without weather reports, which has been analyzed in part (i). In part (ii), with weather reports, we have a new game where players' strategies are functions that map weather information into actions.

[3] **Hint**: Show that

- when the state is sunny, given player 2's strategy described above it is optimal for player 1 to use U, and given player 1's strategy described above it is optimal for player 2 to use L;
- when the state is cloudy, given player 2's strategy described above it is optimal for player 1 to use D, and given player 1's strategy described above it is optimal for player 2 to use L; and
- when the state is rainy, given player 2's strategy described above it is optimal for player 1 to use D, and given player 1's strategy described above it is optimal for player 2 to use R.

use L with probability one. Player 1's best response against player 2 using L is indeed U, according to our analysis in part (i).

On the other hand, player 2 knows that the weather state is not rainy, and hence is equally likely to be sunny or cloudy, and according to player 1's strategy described above, player 2 expects player 1 to use U or D with equal probability. It is clear from our analysis in part (i) that player 2 indeed feels indifferent about using L or R, and in equilibrium player 2 uses L with probability one.

- Next, suppose that the true weather state is cloudy.

  In this event, player 1 knows that the state is not sunny, and hence is equally likely to be cloudy or rainy, and according to player 2's strategy described above, player 1 expects player 2 to use L and R with equal probability. Player 1 feels indifferent about U and D, according to our analysis in part (i), and in equilibrium player 1 uses D with probability one.

  On the other hand, player 2 knows that the weather state is not rainy, and hence is equally likely to be sunny or cloudy, and according to player 1's strategy described above, player 2 expects player 1 to use U and D with equal probability. It is clear from our analysis in part (i) that player 2 indeed feels indifferent about using L or R, and in equilibrium player 2 uses L with probability one.

- Finally, suppose that the true weather state is rainy.

  In this event, player 1 knows that the state is not sunny, and hence is equally likely to be cloudy or rainy, and according to player 2's strategy described above, player 1 expects player 2 to use L and R with equal probability. Player 1 feels indifferent about U and D, according to our analysis in part (i), and in equilibrium player 1 uses D with probability one.

  On the other hand, player 2 knows that the weather state is rainy, and according to player 1's strategy described above, player 2 expects player 1 to use D with probability one. It is clear from our analysis in part (i) that player 2's best response against player 1 using D is indeed R.

To sum up, the aforementioned strategy profile does constitute an equilibrium. In this equilibrium, player 1's payoff is

$$15 \cdot \text{prob.(sunny)} + 12 \cdot \text{prob.(cloudy)} + 3 \cdot \text{prob.(rainy)}$$

$$= 10 > 3a + \frac{15}{2}, \quad \forall a \in [0, \frac{1}{2}].$$

Remark. To see why this "correlated equilibrium" in part (ii) generates for each player an expected payoff higher than the expected Nash equilibrium payoff in part (i), note that by making their date-1 actions contingent on the date-1 (imperfect) weather reports, the two players can make sure that the undesirable outcome (U,R) never arises in equilibrium, and the pleasant outcome (D,L), which is not an NE of the original normal-form game, can now arise when the weather is cloudy. Indeed, player 1 would adopt U only when the weather state is sunny, but player 2 would adopt R only when the weather state is rainy, and hence (U,R) never arises in any weather state. On the other hand, (D,L) is now implemented when the weather is cloudy. This cannot be done in a mixed strategy Nash equilibrium without a correlated device (i.e., the two weather reports): in the mixed-strategy NE obtained in part (i), the two players must randomize over their pure strategies in a stochastically independent manner, which implies that (U,R) may arise with probability $\frac{1}{4}$!

That the weather reports do not always deliver precise information is also important in leading to the above result. To see this, suppose instead that both players' weather reports tell them the exact weather state at date 1. In this case, given a realized weather state, the two players can only attain one Nash equilibrium payoff profile in part (i), which implies, in particular, that (D,L) can never arise as an equilibrium profile when the weather is cloudy. With imprecise weather information when the weather state is cloudy, however, player 1 thinks that player 2 may adopt L or R with equal probability, and player 2 thinks that player 1 may adopt U or D with equal probability, and that is why player 1 feels indifferent about U and D and player 2 feels indifferent about L and R, and in equilibrium player 1 adopts D with probability one and player 2 adopts L with probability one. The outcome (D,L) generates 12 for each player, which, together with the fact that (U,R) never arises in equilibrium, explains why the two players expect a payoff from this correlated equilibrium which is higher than the expected Nash equilibrium payoff of the original game without any correlated device.[4]

4. (**Retailer's Opportunistic Pricing Behavior and Consumers' Coupon Redemption.**) There are two consumers with unit demand for the product produced by a firm. The firm has no production costs. The two consumers' valuations for the product are respectively H and L. The firm has already issued a cents-off coupon with face value v, and to redeem the coupon the two consumers must incur costs T_H and T_L respectively.[5]

[4]When the weather reports always deliver precise information, an attainable expected payoff profile is simply a weighted average of the 3 Nash equilibrium payoff profiles in the original normal-form game. Indeed, the following are the attainable payoff profiles:

$$(15,3),\ (3,15),\ (\frac{15}{2},\frac{15}{2}),$$

$$\frac{2}{3}(15,3)+\frac{1}{3}(3,15)=(11,7),$$

$$\frac{2}{3}(15,3)+\frac{1}{3}(\frac{15}{2},\frac{15}{2})=(\frac{9}{2},\frac{9}{2}),$$

$$\frac{1}{3}(15,3)+\frac{2}{3}(3,15)=(7,11),$$

$$\frac{1}{3}(\frac{15}{2},\frac{15}{2})+\frac{2}{3}(3,15)=(\frac{9}{2},\frac{9}{2}),$$

$$\frac{1}{3}(15,3)+\frac{2}{3}(\frac{15}{2},\frac{15}{2})=(10,6),$$

$$\frac{1}{3}(3,15)+\frac{2}{3}(\frac{15}{2},\frac{15}{2})=(6,10),$$

$$\frac{1}{3}(15,3)+\frac{1}{3}(3,15)+\frac{1}{3}(\frac{15}{2},\frac{15}{2})=(\frac{17}{2},\frac{17}{2}).$$

In the above, if payoff profiles (x,y) and (y,x) are equally likely to arise, then the expected payoff profile always falls short of 10, where recall 10 is the expected payoff that each player obtains in the correlated equilibrium of part (ii).

[5]Therefore consumer H gets a surplus $H-(p-v)-T_H$ if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Similarly, consumer

Assume that

$$2L - v > H \geq L + v > L > 0,$$

and that

$$H - v \geq H - T_H > L - T_L > v - T_L > 0.$$

The extensive game starts after the firm has alreay chosen v, and it is described as follows.

- Seeing v, the two consumers must decide independently whether to carry the coupon and redeem it on the shopping day. A consumer with valuation $j \in \{H, L\}$ will incur a cost T_j *before* the shopping day if he decides to carry the coupon till the shopping day. Consumers' decisions about whether to carry the coupon are unobservable to the firm.
- Then, on the shopping day, the firm must choose a retail price p before consumers arrive.
- Then, consumers walk in the store, see p, and decide whether to make a purchase, and if they have carried a coupon till the shopping day, (it is obviously a dominant strategy at this moment) to present the coupon to the firm in order to get a price reduction equal to v.

(i) Show that given that v satisfies the above conditions, this game has a unique Nash equilibrium where consumer H will never redeem the coupon while consumer L and the firm both use mixed strategies in equilibrium; that is, in equilibrium consumer L feels indifferent about redeeming and not redeeming the coupon, and the firm feels indifferent about *two* optimal prices $p_2 > p_1$.[6]

L gets a surplus $L - (p - v) - T_L$ if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Of course, a consumer can always forget about the coupon, and simply make the purchase. In the latter case, consumer H would get a surplus $H - p$ and consumer L would get a surplus $L - p$. Recall that each consumer gets zero surplus if he chooses to make no purchase.

[6] Note that the redemption cost T_j is already sunk on the shopping day. If the firm expects consumer L to carry the coupon with probability one, then $p = L + v$, so that consumer L will end up with a negative consumer surplus; and if the firm expects con-

(ii) Now, suppose instead that $2L > H > M$, where

$$M = 2L - kv,$$

with

$$k = \frac{L - v}{L + v} \in (0, 1).$$

Re-consider the above extensive game. Solve for the mixed-strategy NEs.[7]

Solution. We shall give a detailed analysis for part (i), and then part (ii) can be analyzed analogously.

Consider part (i). Recall that $H \geq L+v$. Because of $T_H \geq v$, redeeming the coupon would reduce consumer H's valuation. Because consumer H would not redeem the coupon, the firm would set $p = H$ if the firm wants to serve consumer H only. Because the firm can set $p = L$ to serve both consumers H and L and because $2L - v > H$, $p = H$ is dominated by $p = L$. (If $p = L$, the worst possible case facing the firm is the situation where L carries a coupon, so that the firm must reimburse an amount v to L, implying the firm gets a revenue $2L - v$, which is still greater than H, which is the revenue from serving H alone.) Note that all $p < L$ are dominated by $p = L$ (both consumers will buy the product at any such p, with or without a coupon). Note that when consumer L sees the price p, the redemption cost has been sunk, and thus all $p \in (L, L + v)$ are dominated by $p = L + v$ (at any such p, consumer L will buy the product if and only if he is carrying a coupon;

sumer L to not carry the coupon with probability one, then $p = L$, so that consumer L actually prefers to carry the coupon before the shopping day. Show that there can be no pure strategy equilibrium. Then, argue that in a mixed strategy equilibrium, the firm randomizes over at most two prices.

[7] Verify that the solution to part (i) is still valid if $H < M$. Show that if $H = M$, then we have a continuum of mixed-strategy NEs, where the firm randomizes over the three prices L, $L+v$, and H, with the probability of pricing at L being $\frac{TL}{v}$, and where consumer L redeems the coupon with probability k. Show that if $2L > H > M$, then in equilibrium the firm randomizes over L and H, with the probability of pricing at L being $\frac{TL}{v}$, and with consumer L redeeming the coupon with proability $\frac{2L-H}{v}$.

consumer H will buy the product always if we assume that $H \geq L+v$). When $p > L + v$, only consumer H may buy the product. Because of $2L > H$, all $p > L+v$ are dominated by $p = L$. We conclude that only $p = L$ and $p = L + v$ are undominated choices for the seller.

Is there a pure-strategy equilibrium where $p = L$? Given that the firm sets price at L, consumer L will redeem the coupon. But given that consumer L will redeem the coupon, the firm has an incentive to raise the price to $L + v$ (T_L will be sunk when consumer L sees p!). Thus this equilibrium cannot exist.

Is there a pure-strategy equilibrium where $p = L + v$? Given that the firm sets price at $L + v$, consumer L will not redeem the coupon, but given that consumer L does not redeem the coupon in equilibrium, the firm is better off pricing at L. Thus, this pure-strategy equilibrium cannot exist either. We conclude that there exist no pure-strategy equilibria for this game.

Consider the the mixed-strategy equilibrium where the firm sets $p_2 = L + v$ with probability x and $p_1 = L$ with probability $1 - x$, consumer L redeems the coupon with probability y, and consumer H does not redeem the coupon. Note that at both p_1 and p_2, consumer H always buy the product. Since the firm uses a mixed strategy in equilibrium, it must obtain the same payoff choosing the prices p_1 and p_2. Thus,

$$(1 + y)(L + v) - yv = 2L - yv.$$

Solving the equation above, we have $y = \frac{L-v}{L+v}$. Similarly, since consumer L uses a mixed strategy in equilibrium, consumer L is indifferent about redeeming and not redeeming the coupon. Thus,

$$x(-T_L) + (1 - x)(v - T_L) = 0.$$

Solving the equation bove, we have $x = 1 - \frac{T_L}{v}$. Therefore, there is a unique mixed-strategy equilibrium with $x = 1 - \frac{T_L}{v}$ and $y = \frac{L-v}{L+v}$.

Now, consider part (ii). Note that by assumption

$$2L > \max(H, M).$$

The equilibria can be classified as follows.

- It is easy to verify that the solution to part (i) remains valid if $L + v \leq H < M$.[8]
- If instead $H < L + v < M$, then there exists a pure-strategy equilibrium where the firm prices at H with probability one.[9] [10]
- If instead $H = M$, then we have a continuum of mixed-strategy NEs, where the firm randomizes over the three prices L, $L + v$, and H, with the probability of pricing at L being $\frac{TL}{v}$, and where consumer L redeems the coupon with probability k.
- Finally, if $H > M$,[11] then in equilibrium the firm randomizes over L and H, with the probability of pricing at L being $\frac{TL}{v}$, and with consumer L redeeming the coupon with probability $\frac{2L-H}{v}$.

5. (**Competitive Manufacturers May Make More Profits with Non-integrated Distribution Channels.**) Recall the Cournot game in Example 1 of Lecture 1, Part I. Assume that $c = F = 0$ and the inverse demand in the relevant range is

$$P(Q) = 1 - Q, \ 0 \leq Q = q_1 + q_2 \leq 1.$$

(i) Find the equilibrium profits for the two firms.
(ii) Now suppose that the two manufacturing firms cannot sell their products to consumers directly. Instead, firm i (also referred to as manufacturer i) must first sell its product to retailer R_i. Then retailers R_1 and R_2 then compete in the Cournot game. The extensive game is now as follows.

- The two firms first announce F_1 and F_2 simultaneously, where F_i is the franchise fee that firm i will charge retailer i, which is a

[8] Note that M is exactly the expected profit that the firm obtains in the mixed-strategy equilibrium obtained in part (i).

[9] Again one can verify that the firm cannot price at either L or $L+v$ in a pure-strategy equilibrium.

[10] In this pure-strategy equilibrium, the firm's expected profit is $2H - v$, because consumer L will redeem the coupon with probability one. The firm does not want to deviate and price at $L + v$, because $2H - v > (L + v) - v = L$.

[11] Verify that $M > L + v$ always!

fixed cost of retailer i. R_1 and R_2 simultaneously decide to or not to turn down the offers made by the firms. Assume that firm i and retailer R_i both get zero payoffs if F_i gets turned down by retailer R_i.

- Then, after knowing whether F_1 and F_2 get accepted by respectively R_1 and R_2, the two firms announce w_1 and w_2 simultaneously, where w_i is the unit whole price that firm i will charge retailer i.
- Next, in case the firms' offers are both accepted, then given (F_1, F_2, w_1, w_2), the two retailers simultaneously choose q_1 and q_2.

Show that in the unique subgame-perfect Nash equilibrium (SPNE) each manufacturing firm gets a profit of $\frac{10}{81}$. (**Hint**: Backward induction asks you to always start from the last-stage problem, which is the Nash equilibrium of the subgame where R_1 and R_2 play the Cournot game given some (F_1, F_2, w_1, w_2). You can show that the equilibrium (q_1^*, q_2^*) depend on (w_1, w_2) but not on (F_1, F_2), because the latter are fixed costs. Then, you should move backwards to consider the two manufacturers' competition in choosing w_1 and w_2, given some (F_1, F_2). Here assume that the two manufacturers know that different choices of w_1 and w_2 will subsequently affect R_1's and R_2's choices of q_1 and q_2. Finally, you can move to the first-stage of the game, where the two firms simultaneously choose F_1 and F_2.)[12]

Solution. Let us solve the SPNE using backward induction. First consider the subgame where (F_1, F_2, w_1, w_2) are given, and the two retailers are about to choose q_1 and q_2. Retailer i, given q_j, seeks to

$$\max_{q_i} \Pi_i^R(q_i, q_j; w_i, F_i) \equiv q_i(1 - q_i - q_j - w_i) - F_i.$$

[12]This exercise intends to show why employing independent retailers may be a good idea even if using a firm's own outlets can be cheaper. Essentially, employing an independent retailer amounts to delegating the retailer the choice of output, knowing that the retailer, unlike the manufacturer, will be choosing output given a positive unit cost w_i! A higher unit cost credibly convinces the rival retailer that less output will be produced, and with both manufacturers producing less outputs, their profits become higher.

The first-order condition gives retailer i's reaction function

$$r_i(q_j; w_i) = \frac{1 - q_j - w_i}{2}, \quad \forall i, j \in \{1,2\},\ i \neq j.$$

Thus there is a unique NE in this subgame, which is[13]

$$(q_1^*(w_1, w_2), q_2^*(w_1, w_2)) = (\frac{1 - 2w_1 + w_2}{3}, \frac{1 + w_1 - 2w_2}{3}).$$

Correspondingly, the two retailers' profits are

$$\Pi_1^R(q_1^*(w_1, w_2), q_2^*(w_1, w_2); w_1, F_1) = \frac{(1 - 2w_1 + w_2)^2}{9} - F_1$$

and

$$\Pi_2^R(q_2^*(w_1, w_2), q_1^*(w_1, w_2); w_2, F_2) = \frac{(1 - 2w_2 + w_1)^2}{9} - F_2.$$

Now, consider the stage where (F_1, F_2) are given and the two manufacturers are about to choose w_1 and w_2. Manufacturer i, given w_j, seeks to

$$\max_{w_i} F_i + w_i q_i^*(w_i, w_j), \quad \forall i, j \in \{1,2\},\ i \neq j.$$

The first-order condition gives

$$w_i = \frac{1 + w_j}{4}, \quad \forall i, j \in \{1,2\},\ i \neq j.$$

Note that w_1 and w_2 are indeed strategic complements![14] Thus there is a unique NE in this subgame where the two manufacturers both set the unit wholesale price at $\frac{1}{3}$:

$$w_1^* = w_2^* = \frac{1}{3}.$$

[13]Why does q_i^* increase with w_j? Again, this results from the fact that q_1 and q_2 are strategic substitutes. A higher w_j means that retailer j is faced with a higher unit cost, and hence q_j ought to be lower, which then implies that retailer i should optimally respond by choosing a higher q_i.

[14]When manufacturer i expects manufacturer j to choose a higher w_j, it realizes that, keeping its choice w_i unchanged, subsequently the two retailers will choose higher q_i^* and lower q_j^*, which marginally encourages manufacturer i to raise w_i in the first place: the drawback of raising w_i is that it leads to a lower q_i^*, and hence it is less costly to do this when q_i^* rises because of a higher w_j! This explains strategic complementarity between w_i and w_j.

In this equilibrium, for $i = 1, 2$, manufacturer i's profit is

$$F_i + \frac{6}{81}.$$

The correspondingly profits of the two retailers are

$$\Pi_1^R(q_1^*(\frac{1}{3}, \frac{1}{3}), q_2^*(\frac{1}{3}, \frac{1}{3}); \frac{1}{3}, F_1) = \frac{4}{81} - F_1$$

and

$$\Pi_2^R(q_2^*(\frac{1}{3}, \frac{1}{3}), q_1^*(\frac{1}{3}, \frac{1}{3}); \frac{1}{3}, F_2) = \frac{4}{81} - F_2.$$

Now, consider the stage where the two manufacturers are about to choose F_1 and F_2. Manufacturer i's problem is

$$\max_{F_i} F_i + \frac{6}{81}$$

subject to

$$\Pi_i^R(q_i^*(\frac{1}{3}, \frac{1}{3}), q_j^*(\frac{1}{3}, \frac{1}{3}); \frac{1}{3}, F_i) = \frac{4}{81} - F_i \geq 0.$$

There is a unique SPNE in this game where $F_1 = F_2 = \frac{4}{81}$, and hence the two manufacturers' equilibrium profits are both $\frac{10}{81}$.

Remark. We must emphasize here the role of the timing of the game. That the two firms are able to first offer F_1 and F_2 to respectively R1 and R2 and then to subsequently choose w_1 and w_2 is important to the above result. If instead the two manufacturers must offer (F_1, w_1) and (F_2, w_2) to R1 and R2 at the first stage of the game, then given w_j, firm i would like to choose $w_i = 0$, because a zero unit wholesale price can serve as a commitment that convinces Rj that Ri would produce more given any quantity q_j (or, simply, Ri's reaction function will be shifted upwards).[15]

This commitment is valuable, because output choices are *strategic substitutes*, which implies that Rj will reduce output q_j if Rj believes that

[15] In this case, given (F_j, w_j), manufacturer i seeks to

$$\max_{(F_i, w_i)} F_i + w_i q_i^*(w_i, w_j),$$

it is faced with a more aggressive reaction function. Consequently, choosing $w_i = 0$ can raise Ri's profit, which in turn implies that, manufacturer i in offering $w_i = 0$, can choose a higher F_i to extract Ri's profit.

In the current setting, however, given that F_1 and F_2 were offered and accepted in the preceding stage, the two firms in choosing w_1 and w_2 would never choose a zero unit wholesale price, because a zero wholesale price would result in *no* additional income for the manufacturer. Indeed, at this stage, as we have shown, regardless of F_1 and F_2, the two firms choose $w_1 = w_2 > 0$. Retailer Ri can infer this fact (as we do) when it must decide whether to accept F_i. This explains why in equilibrium the two manufacturers are able to set $F_1 = F_2 = \frac{4}{81}$.

Note that when a single manufacturer chooses a positive unit wholesale price, it induces its downstream retailer to reduce output (because the unit wholesale price is the retailer's unit cost, and a higher unit cost

---

subject to

$$q_i^*(1 - q_i^* - q_j^* - w_i) - F_i \geq 0.$$

Optimality requires that the latter constraint be binding, and hence

$$F_i = q_i^*(1 - q_i^* - q_j^* - w_i),$$

or equivalently, manufacturer i seeks to

$$\max_{w_i \geq 0} q_i^*(1 - q_i^* - q_j^*) \equiv H(w_i; w_j) = \frac{1}{9}(1 - 2w_i + w_j)(1 + w_i + w_j),$$

where the new objective function is simply the profit function facing an otherwise-identical vertically integrated channel (that is, the firm that is both manufacturer i and Ri). Since q_i^* and q_j^* are respectively decreasing and increasing in w_i, it is easy to verify that this new objective function is decreasing in w_i given w_j, and hence we obtain a corner solution $w_i = 0$. Indeed, direct differentiation yields

$$\frac{\partial H}{\partial w_i} = \frac{1}{9}(-4w_i - w_j - 1) < 0, \Rightarrow w_i^* = 0.$$

The same argument applies to manufacturer j as well, and hence when the two manufacturers must offer (F_1, w_1) and (F_2, w_2) to R1 and R2 at the first stage of the game, the latter two retailers behave just like firms 1 and 2 in Example 1 in Lecture 1, Part I (with zero production costs). As can be easily checked, in the current situation, with $w_1 = w_2 = 0$, the two retailers will choose $q_1^* = q_2^* = \frac{1}{3}$.

leads to a lower output choice), which, by the fact that output choices are strategic substitutes, in turn encourages the other retailer to expand output, which hurts the manufacturer's downstream retailer. However, with both manufacturers offering positive unit wholesale prices, the net effect of positive wholesale prices is to induce both retailers to select an output level that is lower than the output level that the two manufacturers would choose in the absence of independent retailers (or, in the case of vertically integrated distribution channels). This lower output level then leads to a higher equilibrium retail price, which raises the sum of the manufacturer's and the retailer's profits in each distribution channel. The sum of profits of the manufacturer and the retailer coincides with the manufacturer's equilibrium profit in the current case, because by assumption the manufacturer can offer a two-part tariff to its downstream dealer, leaving the latter with a zero profit.[16]

6. (**Entry Deterrence by a Monopolistic Incumbent.**) Consider the following extensive game in which firms A and B may compete in quantity at date 1 and date 2. Both firms seek to maximize the sum of expected date-1 and date-2 profits. The inverse demand at date $t \in \{1, 2\}$, in the relevant region, is $P_t = 1 - Q_t$, where P_t is the date-t product price and $Q_t = q_{At} + q_{Bt}$ is the sum of the two firms' supply quantities at date t. Assume that there are no production costs for the two firms.

   - At date 1, originally firm A is the only firm in the industry. Firm

[16] We have assumed that the two firms have homogeneous products and the demand is linear. When the two firms' products are differentiated or when the demand functions are not linear, raising the equilibrium product prices by using an independent retailer may reduce a manufacturer's sales volume by too much and hence may or may not be a good idea; see Patrick Rey and Joseph Stiglitz, 1995, The Role of Exclusive Territories in Producers' Competition, *Rand Journal of Economics*, 26, 431-451. See also T. W. McGuire and R. Staelin, 1983, An Industry Equilibrium Analysis of Downstream Vertical Integration, *Marketing Science*, 2, 161-191. Note that if a manufacturer i sells through more than one retailer in a small district, then intra-brand competition between these retailers will lead to the Bertrand outcome where all retailers hired by manufacturer i offer w_i as the retail price—the distribution channel of manufacturer i is essentially vertically integrated! This highlights the importance of hiring exactly ONE independent retailer (a practice referred to as *exclusive territory*), if manufacturer i would like to raise its retail price by hiring independent retailers.

A must first choose q_{A1}. Upon seeing firm A's choice q_{A1}, firm B must decide whether to spend a cost $K > 0$ to enter the industry. If K is spent, then B must choose q_{B1}. Then the two firms' date-1 profits π_{A1} and π_{B1} are realized, where $\pi_{B1} = 0$ if firm B decides not to enter the industry.

- At date 2, if firm B did not enter at date 1, then firm A, the monopolistic firm in the industry, must choose q_{A2}. If, on the other hand, firm B has entered at date 1, then the two firms choose quantities q_{A2} and q_{B2} simultaneously. Then, the two firms' date-2 profits π_{A2} and π_{B2} are realized, where $\pi_{B2} = 0$ if firm B did not enter the industry at date 1.

Now we solve for the subgame perfect Nash equilibrium for this game.
(i) Suppose that $K = \frac{1}{5}$. Find the equilibrium q_{A1} and q_{A2}.
(ii) Suppose that $K = \frac{1}{9} + \frac{1}{25}$. Find the equilibrium q_{A1} and q_{A2}.
(iii) Suppose that $K = \frac{1}{25}$. Find the equilibrium q_{A1} and q_{A2}.

Solution. Let us solve the game by backward induction. Consider the subgame at date 2.

- If both firms exist, it is easy to show (or recall from Lecture 1, part I) that $q_{A2} = q_{B2} = \frac{1}{3} = P_2$, and the corresponding date-2 profit is $\frac{1}{9}$ for each firm.
- If only firm A exists at date 2, then it will get the monopoly profit $\frac{1}{4}$ by producing $q_{A2} = \frac{1}{2} = P_2$.

Now, move backwards to consider the date-1 subgame where q_{A1} has been chosen, and firm B has spent K. In this case, firm B's optimal supply quantity is $\frac{1-q_{A1}}{2}$; recall Lecture 1, part I. This implies that firm B's profit over the two dates is

$$-K + \frac{(1 - q_{A1})^2}{4} + \frac{1}{9}.$$

Next, consider the date-1 subgame where q_{A1} has been chosen, and firm B is about to decide whether to spend K. From the preceding analysis, we know that firm B's optimal decision is as follows: spending K if and only if

$$K < \frac{(1 - q_{A1})^2}{4} + \frac{1}{9}.$$

Note that we have assumed that firm B will stay out if entering does not generate a positive profit for it.

Now, we can finally consider firm A's choice of q_{A1}.

- If q_{A1} is such that

$$K \geq \frac{(1 - q_{A1})^2}{4} + \frac{1}{9},$$

  then firm B will not enter at date 1, and hence firm A's profit over the two dates is

$$q_{A1}(1 - q_{A1}) + \frac{1}{4}.$$

- If q_{A1} is such that

$$K < \frac{(1 - q_{A1})^2}{4} + \frac{1}{9},$$

  then firm B will enter at date 1, and hence firm A's profit over the two dates becomes

$$q_{A1} \times \frac{1 - q_{A1}}{2} + \frac{1}{9}.$$

Note that in either of the two cases considered above, in the absence of the constraint involving K, firm A's unconstrained optimal date-1 supply quantity must maximize $q_{A1}(1-q_{A1})$; that is, the unconstrained optimal supply quantity is $\frac{1}{2}$, which is the optimal supply quantity for a monopolistic firm.

Thus we can summarize firm A's optimal date-1 output policy as follows.

(a) If

$$K \geq \frac{(1 - \frac{1}{2})^2}{4} + \frac{1}{9} = \frac{1}{16} + \frac{1}{9},$$

then firm B would not enter when firm A chooses its unconstrained optimal supply quantity $q_{A1} = \frac{1}{2}$. Thus it is indeed optimal for firm A to choose $q_{A1} = \frac{1}{2}$, and it follows that $q_{A2} = \frac{1}{2}$ also. In this case we say that firm A's date-1 output policy *blocks the entry* of firm B.

(b) If

$$K < \frac{(1-1)^2}{4} + \frac{1}{9} = \frac{1}{9},$$

then firm B will still enter even if firm A chooses $q_{A1} = 1$ (which minmaxes firm B at date 1), and in this case firm A's optimal date-1 output strategy is $q_{A1} = \frac{1}{2}$, which leads to $q_{B1} = \frac{1-q_{A1}}{2} = \frac{1}{4}$, so that firm A's profit over the two dates is $q_{A1} \times P_1 + \frac{1}{9} = \frac{1}{8} + \frac{1}{9}$. In this case we say that firm A's date-1 output policy *accomodates the entry* of firm B.

(c) If

$$\frac{1}{16} + \frac{1}{9} > K \geq \frac{1}{9},$$

then firm B will enter if and only if $K < \frac{(1-q_{A1})^2}{4} + \frac{1}{9}$, where note that the right-hand side is strictly decreasing in q_{A1} for $q_{A1} \in [0,1]$. Thus firm A's date-1 output q_{A1} determines whether firm B will enter, and the higher q_{A1} is, the less likely that the constraint $K < \frac{(1-q_{A1})^2}{4} + \frac{1}{9}$ may hold. We say in this case that firm A's date-1 output policy *deters the entry* of firm B, if firm B does not enter in equilibrium. Firm A's optimal date-1 output that results in firm B entering the industry has been solved above, which is $q_{A1} = \frac{1}{2}$, and firm A's payoff from accomodating the entry is correspondingly $\frac{1}{8} + \frac{1}{9}$. On the other hand, firm A's optimal date-1 output that induces firm B to not enter can be obtained by solving the following maximization program:

$$\text{(P)} \quad \max_{q_{A1} \in [0,1]} q_{A1}(1-q_{A1}) + \frac{1}{4}$$

subject to

$$K \geq \frac{(1-q_{A1})^2}{4} + \frac{1}{9},$$

and at optimum the above constraint must be binding: if not, then the optimal q_{A1} would equal $\frac{1}{2}$, which, by the fact that $K < \frac{1}{16}+\frac{1}{9}$, would induce rather than deter B's entry. Thus firm A's optimal date-1 output is

$$q_{A1}^* = 1 - \sqrt{4(K - \frac{1}{9})} \in (\frac{1}{2}, 1].$$

We claim that, indeed, choosing this entry-deterring output is better than choosing $q_{A1} = \frac{1}{2}$ to accomodate entry. To see this, recall that by accomodating firm A's payoff is $\frac{1}{8} + \frac{1}{9}$, which is less than $\frac{1}{4}$, the payoff that firm A would obtain by choosing $q_{A1} = 1$ to deter B's entry. Note that the date-1 output choice $q_{A1} = 1$ is feasible but is generally suboptimal; it is optimal (i.e., $q_{A1}^* = 1$) only when $K = \frac{1}{9}$. Thus we conclude that choosing q_{A1}^* to deter entry at date 1 is indeed the optimal strategy for firm A given that $\frac{1}{16} + \frac{1}{9} > K \geq \frac{1}{9}$.

To sum up, our solutions for parts (i)-(iii) are as follows.

- (i) For $K = \frac{1}{5} = \frac{25}{125} > \frac{25}{144} = \frac{1}{16} + \frac{1}{9}$, entry is blocked, and we have $q_{A1} = q_{A2} = \frac{1}{2}$.
- (ii) For $K = \frac{1}{9} + \frac{1}{25}$, which lies between $\frac{1}{9}$ and $\frac{1}{16} + \frac{1}{9}$, entry is deterred, and $q_{A1} = \frac{3}{5}$ and $q_{A2} = \frac{1}{2}$.
- (iii) For $K = \frac{1}{25} < \frac{1}{9}$, entry can only be accomodated, and hence $q_{A1} = \frac{1}{2}$ and $q_{A2} = \frac{1}{3}$.

This exercise explains why a monopolistic firm may not always produce the monopoly output stated in an economics textbook. Observationally firm A is a monopolistic firm at date 2, but this could be a consequence of its non-monopolistic output choice q_{A1}^*: if it insists on producing the monopoly output $\frac{1}{2}$, it may induce entry at date 1, which would destroy its monopolistic status at date 2. In part (ii), for example, the monopolistic firm must produce at $\frac{3}{5} > \frac{1}{2}$ in order to deter entry. In this sense, *even a monopolistic firm has potential competitors*, and the presence of potential competitors is enough to force the monopolistic firm to produce more, so that its output choice may get closer to the socially efficient output level. See the formal analysis in Dixit, A., 1980, The role of investment in entry deterrence, *Economic Journal*, 90, 95-106.

7. (**Signal Jamming and Cournot Competition**) Consider firms 1 and 2 that engage in Cournot competition at $t = 1$ and $t = 2$, facing random demand functions at both periods. The inverse demand function at $t = 1$ is

$$\tilde{p}_1 = \tilde{a} - q_1 - q_2,$$

where $\tilde{a}$ is a positive random variable with $E[\tilde{a}] = 1$ and q_j is firm j's output level at $t = 1$. The inverse demand function at $t = 2$ is

$$\tilde{p}_2 = \tilde{b} - Q_1 - Q_2,$$

where $\tilde{b}$ is a positive random variable and Q_j is firm j's output level at $t = 2$. Each firm seeks to maximize the sum of expected profits over the two periods. That is, both firms are risk-neutral without time preferences.

The game proceeds as follows.

- At the beginning of $t = 1$, both firms must simultaneously make output choices q_1 and q_2 without seeing the realization of $\tilde{a}$.
- At the beginning of $t = 2$, after knowing q_j and the realization p_1 of $\tilde{p}_1$, firm j must choose Q_j. The two firms make output choices at the same time, without seeing the realization of either $\tilde{a}$ or $\tilde{b}$. At this time, firm j does not see q_i that was chosen by its rival, firm i.

(i) First assume that $\tilde{b}$ and $\tilde{a}$ are independently and identically distributed. Solve the equilibrium output choices $(q_1^*, q_2^*, Q_1^*, Q_2^*)$ in the unique SPNE.

(ii) Ignore part (i). Now assume instead that $\tilde{b} = \lambda\tilde{a}$, where $\lambda < 2$ is a constant known to both firms. Solve the unique symmetric SPNE.

(iii) Do the two firms get higher date-1 expected profits in part (ii) or in part (i)? Why?

(iv) Suppose that $\lambda = 1$. Do the two firms get higher date-2 expected profits in part (ii) or in part (i)? Why?[17]

[17]**Hint**: Verify that $(q_1^*, q_2^*, Q_1^*, Q_2^*) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in part (i). For part (ii), let $(q^*, Q^*(p_1, q))$ denote the unique symmetric SPNE, where both firms choose q^* at $t = 1$, and both choose $Q^*(p_1, q)$ after choosing q at $t = 1$ and subsequently learning that the realization of $\tilde{p}_1$ is p_1. Then *in equilibrium*, $\tilde{p}_1 = \tilde{a} - 2q^*$, or $\tilde{a} = \tilde{p}_1 + 2q^*$. At the beginning of $t = 2$, given the realization p_1 of $\tilde{p}_1$ and its own output choice q_i at $t = 1$, and given that firm j does not deviate from its equilibrium strategy, firm i knows that $\tilde{a} = p_1 + q_i + q^*$. Moreover, firm i knows that that firm j would believe that $\tilde{a} = p_1 + 2q^*$ and seek to maximize

$$\max_Q [\lambda(p_1 + 2q^*) - Q^*(p_1, q^*) - Q]Q,$$

where note that firm j does not know firm i has chosen q_i rather than q^*. That is, firm i believes that firm j would choose the Q that satisfies

$$Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1, q^*)}{2},$$

which has to be $Q^*(p_1, q^*)$ also. Hence firm i believes that firm j would choose

$$Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.$$

Firm i, knowing that it has chosen q_i rather than q^* at $t = 1$, seeks to maximize the following date-2 profit:

$$\max_Q [\lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q]Q,$$

so that given (p_1, q_i), firm i's optimal date-2 output level is

$$Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1 + 2q^*)}{3}}{2},$$

which yields for firm i the following date-2 profit

$$\frac{1}{4}[\frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i]^2.$$

At $t = 1$, expecting firm j to choose q^*, firm i seeks to

$$\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4}E[(\frac{2\lambda \tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],$$

which is concave in q_i because $\lambda < 2$. Show that the optimal q_i must satisfy the first-order condition for this maximization problem; that is,

$$1 - q^* - 2q_i + \frac{\lambda}{6}(\frac{2\lambda E[\tilde{p}_1]}{3} + \frac{\lambda q^*}{3} + \lambda q_i) = 0,$$

or using $E[\tilde{p}_1] = 1 - q_i - q^*$, and $q_i = q^*$ in equilibrium, show that

$$q^* = \frac{1}{3} + \frac{\lambda^2}{27}.$$

Show that then $Q^*(p_1, q^*) = \frac{\lambda \tilde{a}}{3}$.

Solution. Consider part (i). Since $\tilde{b}$ and $\tilde{a}$ are independent, the two firms do not care about their date-2 decisions Q_1 and Q_2 when they engage in the date-1 Cournot competition. Being risk-neutral, given q_j, firm i seeks to

$$\max_{q_i} q_i(E[\tilde{a}] - q_i - q_j) = q_i(1 - q_i - q_j),$$

so that this game has the same equilibrium as the Cournot game presented in Example 1 of Lecture 1, Part I. That is, in equilibrium ,

$$q_1^* = q_2^* = \frac{1}{3}.$$

Similarly, at date 2, given Q_j, firm i seeks to

$$\max_{Q_i} Q_i(E[\tilde{b}] - Q_i - Q_j) = Q_i(1 - Q_i - Q_j),$$

so that this game also has the same equilibrium as the Cournot game presented in section 11 of Lecture 1, Part I. That is, in equilibrium ,

$$Q_1^* = Q_2^* = \frac{1}{3}.$$

This finishes part (i).

Now, for part (ii), let $(q^*, Q^*(p_1, q))$ denote the unique symmetric SPNE, where both firms choose q^* at $t = 1$, and both choose $Q^*(p_1, q)$ after choosing q at $t = 1$ and subsequently learning that the realization of $\tilde{p}_1$ is p_1. Then *in equilibrium*, $\tilde{p}_1 = \tilde{a} - 2q^*$, or $\tilde{a} = \tilde{p}_1 + 2q^*$. At the beginning of $t = 2$, given the realization p_1 of $\tilde{p}_1$ and its own output choice q_i at $t = 1$, and given that firm j does not deviate from its equilibrium strategy, firm i knows that $\tilde{a} = p_1 + q_i + q^*$. Moreover, firm i knows that that firm j would believe that $\tilde{a} = p_1 + 2q^*$ and seek to maximize

$$\max_{Q}[\lambda(p_1 + 2q^*) - Q^*(p_1, q^*) - Q]Q,$$

where note that firm j does not know firm i has chosen q_i rather than q^*. That is, firm i believes that firm j would choose the Q that satisfies

$$Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1, q^*)}{2},$$

which has to be $Q^*(p_1, q^*)$ also. Hence firm i believes that firm j would choose

$$Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.$$

Firm i, knowing that it has chosen q_i rather than q^* at $t = 1$, seeks to maximize the following date-2 profit:

$$\max_Q [\lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q]Q,$$

so that given (p_1, q_i), firm i's optimal date-2 output level is

$$Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1+2q^*)}{3}}{2},$$

which yields for firm i the following date-2 profit

$$\frac{1}{4}[\frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i]^2.$$

At $t = 1$, expecting firm j to choose q^*, firm i seeks to

$$\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4}E[(\frac{2\lambda \tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],$$

which is concave in q_i because $\lambda < 2$. It follows that the optimal q_i must satisfy the first-order condition for this maximization problem; that is,

$$1 - q^* - 2q_i + \frac{\lambda}{6}(\frac{2\lambda E[\tilde{p}_1]}{3} + \frac{\lambda q^*}{3} + \lambda q_i) = 0,$$

or using $E[\tilde{p}_1] = 1 - q_i - q^*$, and $q_i = q^*$ in equilibrium, we have

$$q^* = \frac{1}{3} + \frac{\lambda^2}{27}.$$

It follows that $Q^*(p_1, q^*) = \frac{\lambda \tilde{a}}{3}$.

Now, consider part (iii). Comparing part (i) to part (ii), we see that both firms make lower expected profits at date 1 in part (ii). This happens because in part (ii) firms cannot resist the temptation of expanding outputs as means of manipulating their rivals' beliefs about the realization of $\tilde{a}$. By secretly expanding its output q_i, firm i wants to make its rival j believe in a lower realization of $\tilde{a}$, which implies a lower demand (whose intercept is $\lambda\tilde{a}$) at date 2, and if firm i succeeds in making its rival believe in a lower date-2 demand, then it can benefit from choosing a higher date-2 output Q_i given that its rival will on average choose a lower output Q_j. In equilibrium this incentive is correctly recognized by its rival j, but the incentive to engage in signal-jamming still changes the two firms' date-1 profits. Both firms are worse off in part (ii), because of a lower product price resulting from output expansion ($q^* > \frac{1}{3}$).

Finally, consider part (iv). Note that in part (ii)

$$E[Q^*(p_1, q^*)] = \frac{\lambda E[\tilde{a}]}{3} = \frac{E[\tilde{a}]}{3} = \frac{1}{3},$$

where recall that $\frac{1}{3}$ is the two firms' date-2 output choice in part (i). Signal-jamming does not fool any player in equilibrium (that is, both firms can infer correctly the realized $\tilde{a}$ from the realized date-1 price), but in part (ii), since $\tilde{a} = \tilde{b}$, the two firms' common date-2 output choice depends on the realization of $\tilde{a}$. This is in sharp contrast with part (i), where $\tilde{a}$ and $\tilde{b}$ are independent, so that the firms' date-2 output choices can never depend on the realized $\tilde{a}$. Now, since in part (ii) each firm's date-2 expected profit is a *convex* function of its date-2 output $Q^*(p_1, q^*)$, and since $Q^*(p_1, q^*)$ is a mean-preserving spread of the firms' date-2 output choice (which is $\frac{1}{3}$) in part (i), the two firms actually obtain higher expected date-2 profits in part (ii) than in part (i). Indeed, each firm gets the following expected date-2 profit in part (i),

$$\frac{1}{3}(E[\tilde{b}] - \frac{1}{3} - \frac{1}{3}) = \frac{1}{9},$$

but in part (ii) its expected date-2 profit becomes

$$E[\frac{\tilde{a}}{3}(\tilde{a} - \frac{\tilde{a}}{3} - \frac{\tilde{a}}{3})] = \frac{E[\tilde{a}^2]}{9} > \frac{(E[\tilde{a}])^2}{9} = \frac{1}{9},$$

where the inequality follows from Jensen's inequality and the fact that the function $h(z) = z^2$ is strictly convex. Thus the two firms make higher expected date-2 profits in part (ii) than in part (i).