Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 3

1. We shall consider in this exercise two applications of the perfect Folk theorem.

   (a) Consider the following stage game $G(1)$:

1/2	L	R
U	3,3	2,4
D	4,2	0,0

   Now consider the infinitely repeated version $G(\infty)$ of the stage game $G(1)$. Assume that in $G(\infty)$ the two players have a common discount factor $\delta \in (0,1)$.

   (i) Find the worst possible SPNE for in $G(\infty)$ for player 1.

   (ii) Find the worst possible SPNE for in $G(\infty)$ for player 2.

   (iii) Show that there exists $\delta^* \in (0,1)$ such that whenever $\delta \in [\delta^*, 1)$, there exists an SPNE for $G(\infty)$ supported by the two SPNE's stated in parts (i) and (ii), such that along the equilibrium path, player 1 plays U and player 2 plays L in each and every period until a unilateral deviation occurs. Find δ^*.

   (iv) Redo part (iii), but now assume that the SPNE is supported by the trigger strategies. Denote the cutoff level of δ by δ^{**}. Compare δ^* to δ^{**}.

   Solution. The stage game $G(1)$ has 3 NEs: (U,R), (D,L) and one mixed-strategy NE where player 1 adopts U with probability $\frac{2}{3}$ and player 2 adopts R with probability $\frac{1}{3}$. The minmax value for each player is 2, which can be drived as follows. If player 2 adopts L and R with probability p and $1-p$ respectively, then player 1's best response is U if $p \geq \frac{2}{3}$ and D if otherwise, so that

$$m_2^1 = \arg\min_p 4p \cdot 1_{\{p \geq \frac{2}{3}\}} + (2+p) \cdot 1_{\{p \leq \frac{2}{3}\}} = 0,$$

proving that $\underline{v}_1 = 2$. By symmetry, we also have $\underline{v}_2 = 2$.

Now, the worst possible SPNE for player 1 in $G(\infty)$ is obviously to repeat the pure-strategy NE (U,R) forever. Similarly, the worst possible SPNE for player 2 in $G(\infty)$ is obviously to repeat the pure-strategy NE (D,L) forever. Apparently, these two SPNE's coincide with Friedman's trigger strategies designed for players 1 and 2 respectively, and hence we can conclude that $\delta^* = \delta^{**}$. Thus we will be done after we finish part (iii) and obtain δ^*.

Thus we consider part (iii). Consider the SPNE in $G(\infty)$ where player 1 plays U and player 2 plays L in each and every period until a unilateral deviation occurs. If player 1 conforms to the equilibrium strategy today, her payoff is

$$3(1 + \delta + \delta^2 + \cdots) = \frac{3}{1-\delta};$$

and her payoff becomes

$$4 + 2(\delta + \delta^2 + \cdots) = 4 + \frac{2\delta}{1-\delta}$$

if she chooses to deviate and plays D, and after that the worst possible SPNE for player 1 in $G(\infty)$ will be in effect. Hence player 1 will not deviate if and only if

$$3 - 2\delta \geq 4 - 4\delta \Leftrightarrow \delta \geq \delta^* = \delta^{**} = \frac{1}{2}.$$

(b) Consider the following stage game $G(1)$ and its infinitely repeated version $G(\infty)$, where for simplicity, we shall assume that players can use only pure strategies:

1/2	L	M	R	A
L	0,0	2,4	0,0	6,0
M	4,2	0,0	0,0	0,0
R	0,0	0,0	3,3	0,0
B	0,6	0,0	0,0	5,5

Assume that in $G(\infty)$ the two players have a common discount factor $\delta \in (0, 1)$.

(i) Find the worst possible SPNE for in $G(\infty)$ for player 1.

(ii) Find the worst possible SPNE for in $G(\infty)$ for player 2.

(iii) Show that there exists $\delta^* \in (0, 1)$ such that whenever $\delta \in [\delta^*, 1)$, there exists an SPNE for $G(\infty)$ supported by the two SPNE's stated in parts (i) and (ii), such that along the equilibrium path, player 1 plays B and player 2 plays A in each and every period until a unilateral deviation occurs. Find δ^*.

(iv) Redo part (iii), but now assume that the SPNE is supported by the trigger strategies. Denote the cutoff level of δ by δ^{**}. Compare δ^* to δ^{**}.

Solution. It is easy to show that the worst possible SPNE for player 1 in $G(\infty)$ is obviously to repeat the pure-strategy NE (L,M) forever. Similarly, the worst possible SPNE for player 2 in $G(\infty)$ is obviously to repeat the pure-strategy NE (M,L) forever. Apparently, these two SPNE's coincide with Friedman's trigger strategies designed for players 1 and 2 respectively, and hence we can conclude that $\delta^* = \delta^{**}$. Thus we will be done after we finish part (iii) and obtain δ^*.

Thus we consider part (iii). Consider the SPNE in $G(\infty)$ where player 1 plays B and player 2 plays A in each and every period until a unilateral deviation occurs. If player 1 conforms to the equilibrium strategy today, her payoff is

$$5(1+\delta+\delta^2+\cdots)=\frac{5}{1-\delta};$$

and her payoff becomes

$$6+2(\delta+\delta^2+\cdots)=6+\frac{2\delta}{1-\delta}$$

if she chooses to deviate and plays L, and after that the worst possible SPNE for player 1 in $G(\infty)$ will be in effect. Hence player 1 will not deviate if and only if

$$5-2\delta\geq 6-6\delta \Leftrightarrow \delta\geq\delta^*=\delta^{**}=\frac{1}{4}.$$

2. This exercise verifies the common sense that the outcome of a bargaining game is highly unpredictable even if players are symmetrically informed.

   Recall the Rubinstein's bargaining game that we mentioned in sections 2 and 3 of Lecture 2. Whenever consensus has not been reached by date n, player 1 can make a take-it-or-leave-it offer to player 2 at date n if n is an odd integer, and player 2 can make a take-it-or-leave-it offer to player 1 at date n where n is an even integer. Here let us modify the game by assuming that the offers must be in multiples of 0.01.

   Show that if both players' discount factor is $\delta=\frac{1}{2}$ then there are two pure-strategy SPNEs, and if δ is very close to 1 (say $\delta>0.99$) then assocated with each $x\in\{0,0.01,0.02,\cdots,0.99,1\}$ there is an SPNE where player 1 starts by offering x at date 1 and player 2 accepts it right away.[1]

[1] Here we assume (rather realistically) that there is a finite set of feasible offers, which leads to multiple equilibria for the Rubinstein bargaining game. A similar result can obtain if we assume that there are more than two people participating in the bargaining; see exercise 4.9 of Fudenberg and Tirole (1991, Game Theory).

Solution. If player 2 never offers $x > x^k$ because player 1 cannot resist accepting x^k, then player 2 will reject any offer $x > 1 - \delta(1 - x^k)$ made by player 1. (Recall that x indicates player 1's share of the dollar.) Thus player 1's payoff upon rejecting player 2's offer is at most

$$\delta[1 - \delta(1 - x^k)],$$

which implies that when player 2 offers $\delta[1 - \delta(1 - x^k)]$ to player 1, player 1 had better accept it right away, implying in turn that player 2 would never offer

$$x > x^{k+1} = \{\delta[1 - \delta(1 - x^k)] + 0.01\},$$

where $\{z\}$ is the quantity that we would obtain by rounding z downward to the nearest 0.01.

Similarly, if player 1 never offers $x < y^k$ because player 2 would accept any offer less than or equal to y^k, then player 1 will reject any offer $x < \delta y^k$ made by player 2. Thus upon rejecting player 1's offer player 2 expects to get no more than $\delta(1 - \delta y^k)$, and hence player 2 cannot reject an offer of $\delta(1 - \delta y^k)$ made by player 1, implying that player 1 never offers

$$x < y^{k+1} = \{1 - \delta(1 - \delta y^k) - 0.01\}.$$

Now, suppose $\delta = \frac{1}{2}$, and let $x^0 = 1$ and $y^0 = 1$ (observe that player 2 never offers $x > 1$ and player 1 never offers $x < 0$). We have

$$x^1 = 0.51,\ x^2 = 0.38,\ x^3 = 0.35, x^k = 0.34,\ \forall k \geq 4;$$

and

$$y^1 = 0.49,\ y^2 = 0.62,\ y^3 = 0.65,\ y^k = 0.66,\ \forall k \geq 4.$$

Thus in an SPNE player 1 never offers $x < 0.66$ and always rejects $x < 0.33$; and player 2 never offers $x > 0.34$ and always rejects $x > 0.67$.

There are thus two SPNEs. In one SPNE, player 1 offers $x = 0.66$ which is accepted by player 2 right away. (If player 1 deviates and offers more than 0.66 then player 2 would reject the offer and make a counter offer of 0.33, which player 1 must accept.) In the other SPNE, player 1 offers $x = 0.67$, which is accepted by player 2 right away. (If player 2 deviates and rejects the offer, then player 1 would reject any counter offer less than or equal 0.33.)

There is also a mixed strategy equilibrium where player 1 randomizes between 0.66 and 0.67, and player 2 accepts 0.66 with probability one but rejects 0.67 with probability $\frac{1}{67}$.

That replacing a continuum strategy space by a discrete strategy space may result in multiple equilibria should not be surprising. In a Bertrand duopoly where two firms with zero costs are competing for the patronage of a buyer with unit demand, there is a unique Nash equilibrium where both firms price at zero. Suppose now that prices must be in muitiples of 0.01, and the buyer's willingness to pay $v > 0.01$. Apparently it is an equilibrium where both firms price at zero, but observe that it is also an equilibrium where both firms price at 0.01.[2]

Now, the same procedure that we took above to deal with the case of $\delta = \frac{1}{2}$ can be used to show that if $\delta > 0.99$ then $x^k = 1$ and $y^k = 0$ for all $k \geq 1$, implying that every $x \in \{0, 0.01, 0.02, \cdots, 0.99, 1\}$ is an equilibrium outcome.

3. (**Relationship Banking**) Consider the following extensive game, called $G(1)$. In this game $G(1)$, there are two dates (date 1 and date 2), and two players (a bank B and a borrowing firm F). At date 1, F has no money, but it has two mutually exclusive projects at hand, and it needs to borrow 2 dollars from B to invest in either of them. Project S is riskless and it can generate Y dollars at the end of date 1. Project R is risky, and F must incur a personal disutility $-k < 0$ at the beginning of date 1 if F decides to invest in project R. If project R is taken, then at the end of date 1, with probability π it may generate X dollars and with probability $1 - \pi$ it may generate nothing, but when the latter unpleasant outcome occurs, if B is willing to lend another 2 dollars to F at the beginning of date 2, then project R can be continued for one more period, and it willl generate r dollars for sure at the end

[2] Price discreteness has been used to explain the dealers' collusive behavior found in the Nasdaq dealership market; see for example Anshuman and Kalay (1998, Market Making with Discrete Prices, *Review of Financial Markets*).

of date 2, together with a non-monetary private benefit $u > 0$ to F.[3] At each date $t = 1, 2$, B has exactly 2 dollars for lending. Moreover, we assume that at the beginning of date 1, B cannot commit to lending F any money at date 2; B will lend F money at date 2 only if doing so is optimal (or subgame perfect) for B at date 2. To simplify the analysis, assume that

$$X = 12 = u, \; Y = 4, \; \pi = \frac{1}{2}, \; r = 1, \; k = 5.$$

Finally, assume that F will choose project S whenever F feels indifferent about project S and project R.

The timing of $G(1)$ can be summarized as follows.

- At date 1, after F asks B to lend it 2 dollars, B must choose the face value of debt $D \geq 0$.[4]
- The game ends if F rejects B's offer D. If F accepts B's offer, then F must choose between project R and project S. Choosing project R incurs a disutility $-k$ to F at this point.
- Then at the end of date 1, the date-1 cash flow $\tilde{z}$ is realized, where $\tilde{z} = Y$ if F has chosen project S in the previous stage, and $\tilde{z}$ may equal either X or zero if F has chosen project R in the previous stage. Given $\tilde{z}$, B gets $\min(D, \tilde{z})$, and F gets $\tilde{z} - \min(D, \tilde{z}) = \max(\tilde{z} - D, 0)$.
- Then at the beginning of date 2, B can decide whether to lend another 2 dollars to F and change the face value of debt from D to $D' \geq 0$.[5] The game ends if B chooses not to lend 2 dollars the second time.

[3] Only monetary payoffs such as X, Y and r can be shared with B. The private benefit u cannot be given to B although it is also measured in monetary terms.

[4] If F accepts this offer, then F gets 2 dollars from B at the beginning of date 1, and F must repay B the minimum of D and whatever it has at the end of date 1—we are assuming that F is protected by limited liability.

[5] Obviously this will not happen if F has chosen project S, or if F has chosen project R, and project R has successfully generated cash flow X.

- If B has chosen to lend another 2 dollars to F at the previous stage, then at the end of date 2, the date-2 cash flow is realized, and B gets $\min(D', r)$. At this point, F gets $r - \min(D', r) = \max(r - D', 0)$ together with the private benefit u.

(i) Assume that over the two-date period, F and B are risk-neutral about monetary payoffs and they have no time preferences (so that the payoffs obtained at the two dates can be added together without discounting). Find the SPNE of the above extensive game. Which project is chosen in equilibrium? What is the equilibrium D? What is the bank's equilibrium payoff? What is F's equilibrium payoff?[6]

(ii) Now, consider the infinitely repeated version $G(\infty)$ of the above stage game $G(1)$, where at stage $n = 1, 2, \cdots$, F and B must play $G(1)$ repeatedly. Assume that F and B have a common discount factor $\rho \in (0, 1)$ that applies to two consecutive stages, although we still assume that within each stage (or within each $G(1)$), there is no discounting for F and B. Find a smallest ρ^* such that if ρ is greater than ρ^* then there is an SPNE sustained by the trigger strategy, where F invests in project R at the first date in each stage n, and B will lend another 2 dollars at the second date of each stage n if and only if project R generates no cash at the first date in stage n.[7]

[6]**Hint**: First consider the subgame where at the beginning of date 2, B is considering whether it should lend another 2 dollars to F, after B has lent F the first 2 dollars, and after F has chosen project R which generated the date-1 earnings $\tilde{z} = 0$. Should B lend another 2 dollars to F in this subgame? Now, move backward to consider F's investment decision at date 1, given that F has accepted a loan contract specifying a face value D. Determine F's optimal investment decision for non-negative D. Now, move backward again to consider B's choice of D at the beginning of date 1. What is B's optimal choice of D? Which project will F choose to implement given this optimal D? What is B's equilibrium payoff? What is F's equilibrium payoff?

[7]**Hint**: For part (ii), conjecture that in the best SPNE from the bank's perspective, at each stage the bank will lend F another 2 dollars in case F chooses project R in that stage and produces a date-1 cash flow $\tilde{z} = 0$. The bank may be tempted to deviate at date 2 in each stage n. Verify that the immediate gain from such a deviation is 1, and the loss in each future stage is the difference between the bank's profit obtained in part (i) and $\frac{7}{2}$. Solve for ρ^* by letting the bank's no-deviation IC constraint binding at ρ^*.

Solution. Consider part (i). Consider the subgame where at the beginning of date 2, B is considering whether it should lend another 2 dollars to F, after B has lent F the first 2 dollars, and after F has chosen project R which generated the date-1 earnings $\tilde{z} = 0$. We claim that B's optimal strategy in this subgame is not to lend another 2 dollars to F. To see this, note that by refusing to lend another 2 dollars, B's payoff is zero in this subgame; and B's payoff would become

$$-2 + 1 < 0,$$

if B chose to lend F another 2 dollars and optimally set $D' \geq r = 1$.

Now, having solved the date-2 subgame equilibrium and shown that B would never lend another dollars to F at the beginning of date 2, we can now move backward to consider F's date-1 investment decision. If F chooses project S, F's payoff will be $\max(Y - D, 0) = \max(4 - D, 0)$, and if F chooses project R instead, then F's payoff will be

$$\frac{1}{2}\max(X - D, 0) + \frac{1}{2}\max(0 - D, 0) - k$$

$$= \frac{1}{2}\max(12 - D, 0) + \frac{1}{2}\max(0 - D, 0) - 5$$

$$= \begin{cases} 1 - \frac{D}{2}, & D \leq 12; \\ \\ -5, & D > 12. \end{cases}$$

Thus we can summarize F's investment decision as follows.

- If $D > 12$, then $\max(4 - D, 0) = 0 > -5$, and so F will choose project S.
- If $12 \geq D > 4$, then $\max(4 - D, 0) = 0 > 1 - \frac{D}{2}$, and so F will choose project S.
- If $4 \geq D > 2$, then $\max(4 - D, 0) \geq 0 > 1 - \frac{D}{2}$, and so F will choose project S.
- If $2 \geq D \geq 0$, then $\max(4 - D, 0) \geq 2 > 1 - \frac{D}{2}$, and so F will again choose project S.

We conclude that F will always choose project S.

Now, we can move backward to consider B's choice of D at the beginning of the game. Given that F accepts only $D \leq 4$, and given that F will choose project S after accepting a loan contract with $D \leq 4$, B's payoff as a function of D is $D - 2$, and hence B's optimal choice is $D^* = 4$. It follows that B's equilibrium payoff is 2, F's equilibrium payoff is $\max(4 - 4, 0) = 0$, and project S will be implemented in equilibrium. This finishes part (i).

Now, consider part (ii), The equilibrium for $G(1)$ obtained in part (i) is inefficient because B would get $\frac{7}{2}$ if B could commit to lending F another two dollars when the date-1 earnings generated by project R is $\tilde{z} = 0$. To see this, note that with this commitment, F would choose project R over project S, even if $D = X = 12$ and $D' = r = 1$. Indeed, by choosing project R over project S in this case, F's payoff would be

$$-k + (1 - \pi)u = -5 + \frac{1}{2} \times 12 = 1 > 0.$$

Now, with this commitment, and with F choosing project R, B can optimally choose $D \geq X = 12$ and $D' \geq r = 1$ to obtain

$$-2 + \pi \times \min(D, X) + (1 - \pi)[-2 + \min(D', r)]$$
$$= -2 + \frac{1}{2} \times 12 + \frac{1}{2} \times [-2 + 1] = \frac{7}{2} > 2,$$

where 2 is B's equilibrium payoff in part (i). Now we show that in $G(\infty)$, B possesses such a commitment power if ρ is sufficiently large. Indeed, assuming that F and B will play the SPNE obtained in part (i) forever from stage n on whenever B has refused to lend another 2 dollars upon seeing $\tilde{z} = 0$ at stage $n - 1$, we claim that B will lend another 2 dollars at all stages n in equilibrium if $\rho \geq \rho^* = \frac{2}{5}$. To find ρ^*, note that for B to not deviate at stage $n - 1$, it is necessary and sufficient that

$$(-2 + D') + \frac{7}{2}[\rho + \rho^2 + \cdots] \geq 2[\rho + \rho^2 + \cdots]$$

$$\Leftrightarrow (-2 + r) + \frac{7}{2}[\rho + \rho^2 + \cdots] \geq 2[\rho + \rho^2 + \cdots]$$

$$\Leftrightarrow (\frac{7}{2} - 2)[\rho + \rho^2 + \cdots] \geq 1$$

$$\Leftrightarrow \frac{\rho}{1-\rho} \geq \frac{2}{3}$$

$$\Leftrightarrow \rho \geq \rho^* = \frac{2}{5},$$

where, note that with full bargaining power, B will propose $D' = r = 1$ when lending another 2 dollars. This finishes part (ii).

Remark. This exercise is about a bank's willingness to rescue a troubled borrowing firm. If the bank can commit to practice this kind of *relationship banking*, then the borrowing firm is encouraged to undertake a more efficient long-term, risky project, which will benefit both the lending bank and the borrowing firm. In a static setting, where by definition relationships are out of the question, the self-interested lending bank never rescues a troubled borrowing firm.[8]

In an infinitely repeated version of the stage game, rescuing becomes possible if the bank is sufficiently patient. Recall that ρ can also stand for the conditional probability that the borrowing firm may remain active in the next period given that it is active in the current period. Thus our result shows that efficiency can prevail only if the bank is sufficiently optimistic about the borrowing firm's future viability.

4. Consider a Hotelling main street denoted by the unit interval $[0, 1]$. The population of consumers is 1, and consumers are uniformly distributed along the Hotelling main street. We shall refer to the consumer located at $x \in [0, 1]$ by "consumer x."

   In the following stage game $G(1)$, there are two firms (firm 0 and firm 1) producing a homogeneous product (called product Y) and trying to sell it to the consumers living along the Hotelling main street. Consumers

[8] The bank has a hard-budget-constraint problem in this case, as opposed to the more familiar soft-budget-constraint problem emphasized in the banking literature; for the latter see for example Dewatripont, M., and E. Maskin, 1995, Credit and Efficiency in Centralized and Decentralized Economies, *Review of Economic Studies*, 62, 541-555.

have unit demand. For $i = 0, 1$, firm i is located at the point $i \in [0, 1]$. For all $x \in [0, 1]$, consumer x must incur a transportation cost x if he wants to visit firm 0, and he must incur a transportation cost $1 - x$ if he wants to visit firm 1. Consumers' common valuation for product Y is v. Let p_i be the unit price for product Y charged by firm i. Then consumer x will buy from firm 0 if and only if

$$\text{(IR)} \quad v - p_0 - x \geq 0 \text{ and (IC)} \quad v - p_0 - x \geq v - p_1 - (1 - x).$$

Similarly, consumer x will buy from firm 1 if and only if

$$\text{(IR)} \quad v - p_1 - (1 - x) \geq 0 \text{ and (IC)} \quad v - p_0 - x \leq v - p_1 - (1 - x).$$

Given (p_0, p_1), if there exists $x^* \in [0, 1]$ such that

$$v - p_0 - x^* = v - p_1 - (1 - x^*) \geq 0,$$

then x^* is firm 0's sales volume, and $1 - x^*$ is firm 1's sales volume. Assume that firm i's unit production cost is c_i, where $c_1 = 1$ and $c_0 = 0$.

The above stage game $G(1)$ proceeds as follows.

- Firm 0 and firm 1 must simultaneously announce p_0 and p_1.
- Then, given (p_0, p_1), consumers must simultaneously decide whether to buy 1 unit of product Y from firm 0, or to buy 1 unit of product Y from firm 1, or not to buy anything.
- Then profits are realized for the two firms, and the game ends.

(i) Suppose that $v = \frac{5}{4}$. Find the Nash equilibrium (p_0^*, p_1^*) for the stage game $G(1)$.[9] What is firm 0's equilibrium sales volume? What is firm 0's equilibrium profit Π_0? What is firm 1's equilibrium profit Π_1?

(ii) Now, suppose that $v = 4$,[10] and consider the infinitely repeated version $G(\infty)$ of $G(1)$, where firm j's discount factor is $\rho_j \in (0, 1)$.

[9] Since v is very small, we conjecture that in equilibrium (i) the two firms are local monopolists; and (ii) some consumers are left unserved.

[10] Since v is rather large, we conjecture that in equilibrium of $G(1)$ no consumers are left unserved.

Show that there exist ρ_0^* and ρ_1^* such that whenever $\rho_1 \geq \rho_1^*$ and $\rho_0 \geq \rho_0^*$, there exists an SPNE for $G(\infty)$ sustained by the trigger strategy, where in each and every period the two firms' equilibrium prices are $(p_0, p_1) = (\frac{13}{4}, \frac{15}{4})$.[11] Compute ρ_1^* and ρ_0^*.

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Consider part (i). Since v is very small, we conjecture that the two firms are local monopolists in equilibrium. That is, given p_0^*, firm 0's sales volume is x with $v - p_0^* - x = 0$, so that firm 0's equilibrium profit is

$$f(p_0^*) \equiv (v - p_0^*)(p_0^* - c_0),$$

which must satisfy

$$f'(p_0^*) = 0 \Rightarrow p_0^* = \frac{v + c_0}{2} = \frac{5}{8},$$

implying that in equilibrium firm 0's sales volume is

$$v - p_0^* = \frac{5}{4} - \frac{5}{8} = \frac{5}{8},$$

and firm 0's profit is

$$f(p_0^*) = \frac{5}{8} \cdot (\frac{5}{8} - 0) = \frac{25}{64}.$$

Similarly, we can derive

$$p_1^* = \frac{v + c_1}{2} = \frac{9}{8},$$

implying that in equilibrium firm 1's sales volume is

$$v - p_1^* = \frac{5}{4} - \frac{9}{8} = \frac{1}{8},$$

[11] Conjecture that in each and every period all consumers are served in the SPNE of $G(\infty)$.

and firm 1's profit is

$$(p_1^* - c_1) \cdot \frac{1}{8} = \frac{1}{64}.$$

As conjectured, in equilibrium consumers located in the interval $[0, \frac{5}{8}]$ purchase from firm 0, and consumers located in the interval $[\frac{7}{8}, 1]$ purchase from firm 1. Consumers located in $(\frac{5}{8}, \frac{7}{8})$ are left unserved.

Consider part (ii). Since v is large relative to other parameters, we conjecture that in equilibrium of $G(1)$ all consumers are served. There must exist some $x \in [0, 1]$ such that consumers located at x feel indifferent about purchasing from firm 0 or purchasing from firm 1. That is, given the equilibrium prices (p_0, p_1) for $G(1)$,

$$v - p_0 - x = v - p_1 - (1 - x) \Rightarrow x = \frac{p_1 - p_0 + 1}{2}.$$

It follows that given p_1, p_0 is the solution to

$$\max_p \frac{p_1 - p + 1}{2} \cdot (p - c_0),$$

and that given p_0, p_1 is the solution to

$$\max_p \frac{p_0 - p + 1}{2} \cdot (p - c_1),$$

so that we have

$$p_0 = \frac{p_1 + c_0 + 1}{2}, \ p_1 = \frac{p_0 + c_1 + 1}{2},$$

implying that

$$p_0 = \frac{4}{3}, \ p_1 = \frac{5}{3}.$$

It follows that firm 0's equilibrium sales volume is

$$\frac{p_1 - p_0 + 1}{2} = \frac{2}{3},$$

and firm 1's equilibrium sales volume is

$$\frac{p_0 - p_1 + 1}{2} = \frac{1}{3}.$$

In equilibrium, consumers located in the interval $[0, \frac{2}{3}]$ purchase from firm 0, and consumers located in the interval $(\frac{2}{3}, 1]$ purchase from firm 1.[12] Firm 0's equilibrium profit is

$$\frac{2}{3} \cdot (\frac{4}{3} - 0) = \frac{8}{9},$$

and firm 1's equilibrium profit is

$$\frac{1}{3} \cdot (\frac{5}{3} - 1) = \frac{2}{9}.$$

Now, we claim that in the above-mentioned SPNE for $G(\infty)$, given the two firms' equilibrium prices $(p_0, p_1) = (\frac{13}{4}, \frac{15}{4})$ all consumers are served in each and every period. Indeed, consider a consumer located at $\frac{3}{4}$. This consumer would obtain a surplus of

$$4 - \frac{13}{4} - \frac{3}{4} = 0$$

if he chooses to purchase from firm 0, and he would obtain

$$4 - \frac{15}{4} - \frac{1}{4} = 0$$

if he chooses to purchase from firm 1. All consumers in the interval $[0, \frac{3}{4})$ are then better off purchasing from firm 0 and obtaining a positive surplus than purchasing from firm 1. Similarly, all consumers in the interval $(\frac{3}{4}, 1]$ are better off purchasing from firm 1 than from firm 0. In this SPNE, firm 0's per-period profit is

$$(\frac{13}{4} - 0) \cdot \frac{3}{4} = \frac{39}{16},$$

and firm 1's per-period profit is

$$(\frac{15}{4} - 1) \cdot \frac{1}{4} = \frac{11}{16}.$$

Now, let us determine ρ_0^*. We claim that if firm 0 wishes to deviate from the above SPNE in period n, then its optimal deviation is to price

[12] Consumers located at $\frac{2}{3}$ may purchase from firm 1 instead, but that is immaterial.

at $\frac{11}{4}$, which generates a sales volume of 1 and a period-n profit of $\frac{11}{4}$. To see this, note that the optimal deviating price q for firm 0 must solve the following maximization problem:

$$\max_q \ (q - c_0)\min[1, v - q, \frac{\frac{15}{4} + q + 1}{2}]$$

$$= \begin{cases} q, & \text{if } q \in [0, \frac{11}{4}]; \\ q(\frac{19}{8} - \frac{q}{2}), & \text{if } q \in [\frac{11}{4}, \frac{13}{4}]; \\ q(4 - q), & \text{if } q \geq \frac{13}{4}. \end{cases}$$

The solution is $q = \frac{11}{4}$.

Now, by the trigger strategy, following its deviation in period n, firm 0 will lose $(\frac{39}{16} - \frac{8}{9})$ in each and every period $m \geq n+1$. Thus firm 0 will not deviate in period n if and only if

$$\frac{11}{4} - \frac{39}{16} = \frac{5}{16} \leq \frac{\rho_0(\frac{39}{16} - \frac{8}{9})}{1 - \rho_0}$$

$$\Leftrightarrow \rho_0 \geq \rho_0^* \equiv \frac{45}{268}.$$

Similarly, from firm 1's perspective, if if firm 1 wishes to deviate from the above SPNE in period n, then its optimal deviation is to price at $\frac{21}{8}$, which generates a sales volume of $\frac{13}{16}$ and a period-n profit of $\frac{13}{16}(\frac{21}{8} - 1) = \frac{169}{128}$. To see this, note that the optimal deviating price q for firm 1 must solve the following maximization problem:

$$\max_q \ (q - c_1)\min[1, v - q, \frac{\frac{13}{4} - q + 1}{2}]$$

$$= \begin{cases} (q - 1), & \text{if } q \in [0, \frac{9}{4}]; \\ (q - 1)(\frac{17}{8} - \frac{q}{2}), & \text{if } q \in [\frac{9}{4}, \frac{15}{4}]; \\ (q - 1)(4 - q), & \text{if } q \geq \frac{15}{4}. \end{cases}$$

The solution is $q = \frac{21}{8}$.

Now, by the trigger strategy, following its deviation in period n, firm 1 will lose $(\frac{11}{16} - \frac{2}{9})$ in each and every period $m \geq n+1$. Thus firm 1 will not deviate in period n if and only if

$$\frac{169}{128} - \frac{11}{16} = \frac{81}{128} \leq \frac{\rho_1(\frac{11}{16} - \frac{2}{9})}{1 - \rho_1}$$

$$\Leftrightarrow \rho_1 \geq \rho_1^* \equiv \frac{729}{1265}.$$

This finishes part (ii).

Remark. The theory of infinitely repeated games allows us to formally define brand image and reputation for competitive firms. This exercise shows that these concepts are related to firm-specific characteristics such as the unit cost of production (c_i) and product quality (as reflected by the magnitude of v).

5. Consider the following strategic game, where y is a real number.

player 1/player 2	L	R
U	$(2, 1)$	$(1, 5)$
D	$(y, 10)$	$(0, 10)$

(i) Find all the (pure- and mixed-strategy) Nash equilibria of the above strategic game.

(ii) Suppose that $y = 3$. Call the above strategic game $G(1)$. Consider the finitely repeated version of $G(1)$, denoted by $G(T)$, where a player j's payoff in $G(T)$ is simply $\sum_{t=1}^{T} u_j(s_1^t, s_2^t)$, where $u_j(s_1^t, s_2^t)$ is player j's payoff in the t-th stage game $G(1)$ that he plays, given that the two players' actions taken at that time are respectively $s_1^t \in \{U, D\}$ and $s_2^t \in \{L, R\}$.

- What is the minimum T such that (D,R) appears on the equilibrium path of $G(T)$?
- What is the minimum T such that (U,L) appears on the equilibrium path of $G(T)$?

(iii) Suppose that $y = \frac{3}{2}$. Consider the infinitely repeated version $G(\infty)$ of the above stage game $G(1)$, where the two players' common discount factor in $G(\infty)$ is δ. It can be shown that for the trigger strategy to sustain an SPNE in $G(\infty)$ in which the two players play (D,L) in each and every stage, it is necessary and sufficient that $\delta \geq \delta^*$. Compute δ^*.[13]

Solution. Consider part (i). A Nash equilibrium can be denoted by (p, q), where p is the probability that player 1 adopts U and q is the probability that player 2 adopts L.

- Can there be an NE with $p = 1$? In such an equilibrium, $p = 1$ must be one of player 1's equilibrium best responses against player 2's equilibrium strategy q; that is,

  $$2q + (1 - q) = 1 + q \geq yq \Rightarrow (y - 1)q \leq 1.$$

  The last inequality always holds if $q = 0$, and given $p = 1$, $q = 0$ is indeed player 2's *unique* equilibrium best response. Hence there exists a unique NE with $p = 1$, which is $(p, q) = (1, 0)$.
- Can there be an NE with $p = 0$? In such an equilibrium, $p = 0$ must be one of player 1's equilibrium best responses against player 2's equilibrium strategy q; that is,

  $$2q + (1 - q) = 1 + q \leq yq \Rightarrow (y - 1)q \geq 1,$$

  which requires that

  $$y \geq 2, \quad q \geq \frac{1}{y - 1}.$$

[13]**Hint**: Show that with $y = \frac{3}{2}$, there is a unique NE in G(1), which gives the minmax values in G(1) to both players. Thus in G(∞) the trigger strategy coincides with the worst possible SPNE from each player's perspective.

When $y = 2$, then in such an equilibrium we must have $q = 1$, and it is easy to verify that $(p, q) = (0, 1)$ is indeed a pure-strategy NE. This NE $(p, q) = (0, 1)$ remains an NE if $y > 2$, but when $y > 2$, there also exists a continuum of mixed-strategy NE's with $p = 0$ and $1 > q \geq \frac{1}{y-1}$.

- Can there be an NE with $1 > p > 0$? In such an equilibrium, U and D must both be player 1's equilibrium best responses against player 2's equilibrium strategy q; that is,

$$2q + (1 - q) = 1 + q = yq \Rightarrow (y - 1)q = 1,$$

which requires that

$$y \geq 2, \quad q = \frac{1}{y - 1}.$$

When $y = 2$, we must have $q = 1$, but against player 1's mixed strategy $1 > p > 0$, $q = 1$ is dominated by $q = 0$ from player 2's perspective! In fact, we must have $q = 0$ in any NE with $1 > p > 0$, but given $q = 0$, D is dominated by U from player 1's perspective, and hence any strategy with $1 > p > 0$ is dominated by $p = 1$, showing again that such an equilibrium cannot exist.

The NE's for part (i) can now be summarized as follows.

- If $y < 2$, then this game has a unique NE in pure strategy, which is $(p, q) = (1, 0)$.
- If $y = 2$, then this game has two pure-strategy NEs, which are $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$.
- If $y > 2$, then this game has two pure-strategy NEs, $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$, together with a continuum of mixed-strategy NEs with $p = 0$ and $1 > q \geq \frac{1}{y-1}$.

Consider part (ii). Apparently, by part (i), $G(1)$ has 2 pure-strategy NEs, (D,L) and (U,R). Can (D,R) appears in the SPNE of G(2)? If it does, then it must appear in the first stage of G(2). If this is the case, then player 2 would wish to deviate and choose U in the first stage but he cannot: the two playes will play (D,L) in the second stage if they play (D,R) in the first stage, but they will play (U,R) in the second stage if (D,R) did not appear in the first stage. Indeed, this profile of intertemporal strategies removes player 1's incentive to deviate from playing (D,R) in the first stage. To sum up, it is indeed an SPNE in G(2) that

$$\begin{pmatrix} (D,R) \to (D,L) \\ \\ \text{Not } (D,R) \to (U,R) \end{pmatrix}.$$

Similarly, one can verify that it is an SPNE in G(2) that

$$\begin{pmatrix} (U,L) \to (D,L) \\ \\ \text{Not } (U,L) \to (U,R) \end{pmatrix}.$$

Thus we conclude that to both questions in part (ii), the minimum T required is 2.

Now, consider part (iii). It is clear that with $y = \frac{3}{2}$, there is a unique NE in G(1), which is (U,R). Thus there is a unique trigger strategy that can be used to sustain any SPNE. Note that (U,R) gives the minmax values in G(1) to both players. Thus no other SPNE can provide more severe penalty than the trigger strategy in sustaining (D,L) as a perpetual SPNE outcome.

To sustain (D,L) in each and every stage of G(∞), we must take care of player 1's incentive to deviate and play U in any stage. The one-time gain from this deviation is $2 - \frac{3}{2} = \frac{1}{2}$. The present value of the lost payoffs in the future due to the penalizing trigger strategy is

$$\frac{\delta(\frac{3}{2}-1)}{1-\delta},$$

so that we need

$$\frac{1}{2} \leq \frac{\delta(\frac{3}{2}-1)}{1-\delta} \Rightarrow \delta \geq \delta^* = \frac{1}{2}.$$

6. Consider a single-product manufacturer M that wishes to sell its product to two segments of consumers, referred to as H (the highs) and L

(the lows), whose populations are respectively α and $1-\alpha$. Consumers have unit demand at each date n, for all positive integers n. H-buyers and L-buyers are respectively willing to pay V and v for 1 unit of the product. For simplicity, M has no production costs.

At each date n, there is a (different) physical retailer R_n that can sell the product to the consumers on behalf of M. Except for the wholesale price charged by M, R can operate costlessly. M, R, and the consumers are all risk-neutral.

The above physical market is modelled as an infinitely repeated game as follows.

- At date 0, M can decide whether to build and operate its own online channel by spending a cost F.[14]

[14] The marketing literature has shown that a dual-channel strategy may be beneficial for a manufacturer for several reasons.

- First, buyers may be endowed with heterogeneous costs/benefits of visiting a physical or online outlet, and it may be efficient to direct different buyers to purchase at different outlets. For example, a monopolistic manufacturer M is trying to serve two buyers A and B, both willing to spend 10 for M's product. Suppose that trading online is costless for A but prohibitively costly for B (because B is unfamiliar with the internet), and trading at the physical outlet is costless for B but prohibitively costly for A (because A has a high transportation cost). If building online and physical outlets is costless, then M should serve A at its online outlet and B at its physical outlet.
- Second, a dual-channel strategy may allow M to better discriminate buyers, even if doing so may reduce efficiency. Take again the above example, but assume that A is willing to pay 6 for the product instead of 10. Both A and B can trade costlessly in the physical market, but A and B must incur respectively 1 and 10 if they wish to trade online. Efficiency would require that M serve both A and B at the physical outlet. However, if M can build an online outlet costlessly, then M can set an online price 5 and an offline price 10 and direct A and B to trade respectively at the online and physical outlets. Letting A to trade online is inefficient, but it allows M to identify B at the physical outlet and extract B's surplus.
- Third, a dual-channel strategy can be valuable for imperfectly competitive manufacturers who wish to reduce competition. When a firm builds an online channel, it induces some buyers in the physical market to migrate to the online market, and this may alleviate competition in the physical market.

- At each date $n = 1, 2, \cdots$, after learning the interactions between M and $\{R_1, R_2, \cdots, R_{n-1}\}$, M and R_n must play the following date-n stage game.

  At the beginning of date n, M and R_n will first learn about whether there is a demand for M's product at date n. Given that there is a demand at date $n-1$, with probability $1 - q_M$ the demand may vanish from date n on.[15] The game ends at the first date the demand for M's product vanishes in the physical market. On the other hand, if there is a demand at date n, then the date-n stage game proceeds as follows.

  - M must first announce its online price P_n and then must offer a wholesale price w_n to the physical retailer R_n, which R_n can either accept or reject. If R_n accepts w_n, then R_n must announce its retail price p_n.
  - Then consumers learn about both p_n and P_n (where $p_n = +\infty$ if R_n has rejected w_n and where $P_n = +\infty$ if M did not build its online channel at date 0) and they must simultaneously decide whether to buy from R_n at the price p_n or to buy online from M at the price P_n, or to make no purchase at all. M must incur a unit cost $c > 0$ when selling through its online channel, and all buyers must incur a cost $\lambda > 0$ to buy online.[16]
  - In case some consumers have chosen to buy from M at the price P_n, M can decide whether to avoid trade by telling those

In addition to these reasons, this exercise provides yet another rationale for the dual-channel strategy: if a manufacturer can build an online channel to compete with its physical retailer, then it can prevent the physical retailer from pricing too high and dropping too many low-valuation buyers (which is against the manufacturer's interest).

[15] An interpretation is that with probability $1 - q_M$ a new brand may emerge at date n and take all the existing customers from M. With this interpretation, q_M measures the strength of M's brand image.

[16] The assumption that $c, \lambda > 0$ implies that selling the product via the physical channel is more efficient than selling it via the online channel. The assumption that the highs and the lows share the same parameter λ implies that the manufacturer cannot adopt a dual channel strategy to better screen consumers. However, we shall show that a dual channel strategy can still be beneficial to M.

consumers that the product has been sold out (i.e., to renege), or to sell to those consumers at the price P_n as promised. We assume that renege has no direct costs for M.

- Consumers that have tried to purchase online at the price P_n without success (in case M chose to renege P_n) can then decide whether to return to the physical market and purchase from R_n at the price p_n.
- Then the the date-n profits are realized for M and R_n respectively. Then the game moves on to date $n+1$.

We assume that each R_n seeks to maximize expected profits, M seeks to maximized the sum of expected profits accrued at all transaction dates, and consumers seek to maximize expected consumer surplus.[17] Assume that

$$V > v > \lambda > 0 > v - c - \lambda, \; 1 > \alpha, q_M > 0. \tag{1}$$

(i) Suppose that $F = +\infty$ and that M and R_n are the same firm, for all n. (This implies that $w_n = 0$ for all n.) Find the equilibrium p_n and M's equilibrium payoff, assuming that

$$v > \alpha(2-\alpha)V.$$

(ii) Suppose that $F = +\infty$ and that M and R_n are different firms, for all n. Find the equilibrium w_n and p_n, assuming that

$$v > \alpha(2-\alpha)V.$$

(iii) Suppose that $F = +\infty$ and that R_n are the same physical retailer R at each date n. Like M, R also seeks to maximize the sum of expected

[17] A distribution channel consists of an upstream manufacturer M and a downstream retailer R. Channel efficiency is attained by the distribution channel, if the sum of M's and R's profits are maximized in equilibrium. Apparently, when M and R are the same firm, there is no conflict of interests between the two firms, and the integrated firm's marketing strategy always fulfills channel efficiency.

profits accrued at all transaction dates. Find the equilibrium w_n and p_n, assuming that

$$\alpha(2-\alpha)V > v > \alpha V[1+(1-\alpha)(1-q_M)] > \alpha V.$$

Show that this game has an SPNE supported by the trigger strategy, where in equilibrium $w_n = \frac{v-\alpha(1-q_M)V}{1-\alpha(1-q_M)} \in (0,v)$ and $p_n = v$ for all n. Show that if instead R_n are different firms, then we would have $w_n = V = p_n$ in equilibrium.

(iv) Suppose that $v > \alpha(2-\alpha)V$ and that R_n are different firms. Define

$$\Pi_M^0 \equiv \frac{v-\alpha V}{1-\alpha}[1+q_M+q_M^2+\cdots] = \frac{v-\alpha V}{(1-q_M)(1-\alpha)}, \tag{2}$$

and

$$q_M^* \equiv \frac{\max[F, c+\lambda-v]}{v+\max[F, c+\lambda-v]-\frac{v-\alpha V}{1-\alpha}}. \tag{3}$$

Show that this game has an SPNE supported by the trigger strategy such that in equilibrium $w_n = p_n = v$ and $P_n = v-\lambda$ for all n if and only if $q_M \geq q_M^*$.[18]

Solution. Consider part (i). By assumption, there is a vertically integrated M at each and every date n. If M sets $p_n = v$, then M's

[18]**Hint**: Show that pricing at $p_n = w_n = v$ is indeed R_n's equilibrium best response as long as R_n believes that M will never renege P_n. Show that consumers facing $p_n = v$ and $P_n = v-\lambda$ will choose to trade with R_n. Show that when $p_n > v$ consumers all wish to trade online at date n, and in the latter event, by reneging at date n, M's payoff from date n on is

$$q_M \Pi_M^0,$$

and by selling to the consumers at P_n as promised, M's payoff from date n on becomes

$$(v-c-\lambda)+\frac{q_M v}{1-q_M}.$$

date-n payoff is v; if M sets $p_n = V$, then M's date-n payoff is αV. Since

$$v > \alpha(2-\alpha)V \Rightarrow v > \alpha V,$$

M's optimal choice is $p_n = v$ and M's date-n equilibrium payoff is v also. In equilibrium the date-0 present value of M is then $\frac{q_M v}{1-q_M}$.

Consider part (ii). In this case, in the subgame where R_n chooses p_n, R_n would choose $p_n = v$ if and only if

$$(v - w_n) \geq \alpha(V - w_n) \Rightarrow w_n \leq \frac{v - \alpha V}{1-\alpha}.$$

Among those w_n that satisfy the above inequality, M's favorite is $w_n = \frac{v-\alpha V}{1-\alpha}$, which yields for M the date-n payoff $\frac{v-\alpha V}{1-\alpha}$. Among those w_n that violate the above inequality, M's favorite is $w_n = V$, which implies that $p_n = V$ also, yielding for M the date-n payoff αV. Thus M would choose $w_n = \frac{v-\alpha V}{1-\alpha}$ over $w_n = V$ if and only if

$$\frac{v - \alpha V}{1-\alpha} \geq \alpha V \Leftrightarrow v \geq \alpha(2-\alpha)V.$$

We conclude that in equilibrium $w_n = \frac{v-\alpha V}{1-\alpha}$ and $p_n = v$.

Consider part (iii). Call the stage game where M and R can interact for only once (at date 1) $G(1)$. According to part (ii), in equilibrium of $G(1)$ we have $w_1 = V = p_1$. (The same is true then if R_n's are different retailers.) Now, in $G(\infty)$, if M and R were the same firm, then the vertically integrated firm would choose $p_n = v$ at each date n, according to part (i), which would fulfill the *channel efficiency*.[19] This outcome, by part (ii), is not a Nash equilibrium outcome in $G(1)$, given that

[19] A distribution channel consists of an upstream manufacturer M and a downstream retailer R. Channel efficiency is attained by the distribution channel, if the sum of M's and R's profits are maximized in equilibrium. Apparently, when M and R are the same firm, there is no conflict of interests between the two firms, and the integrated firm's marketing strategy always fulfills channel efficiency.

R and M are not the same firm and given that $v < \alpha(2-\alpha)V$. However, thanks to the fact that R, like M, is also a long-term player in part (iii), we shall show that $p_n = v$ can arise as an SPNE outcome in $G(\infty)$.

Indeed, if in an SPNE M offers w_n and R chooses $p_n = v$ at each date n, then R's equilibrium date-n payoff would be $v - w_n$; and when the trigger strategy is at work, R's date-n payoff would become 0 (because $w_n = p_n = V$). Hence for R to conform to the equilibrium pricing strategy $p_n = v$, we must have

$$\alpha(V - w_n) - (v - w_n) \leq \frac{q_M(v - w_n)}{1 - q_M}$$

$$\Leftrightarrow \alpha(1 - q_M)(V - w_n) \leq v - w_n$$

$$\Leftrightarrow [1 - \alpha(1 - q_M)]w_n \leq v - \alpha(1 - q_M)V,$$

so that, as asserted, M would offer

$$w_n = \frac{v - \alpha(1 - q_M)V}{1 - \alpha(1 - q_M)}.$$

Now, consider part (iv). Now R_n's are short-term players, unlike in part (iii), and hence we cannot fulfill channel efficiency by simply letting M interact with physical retailers. However, we have removed the assumption that $F = +\infty$ in part (iv), so that M can use a dual-channel strategy to enhance channel efficiency.

We are asked to show that there is an SPNE where M spends F at date 0 and prices online at $P_n = v - \lambda$ at each date $n \geq 1$ as long as q_M is sufficiently large. In this equilibrium, R_n cannot price higher than v if R_n wishes to win the patronage of any consumer: otherwise, a consumer would rather buy online from M. Thus M can set $w_n = v$ accordingly to extract all the surplus from R. However, there is this problem of

$$(v - \lambda) - c = P_n - c < 0,$$

and hence M would like to renege P_n when consumers really come and try to make a purchase online. The one-time gain from reneging is $c - (v - \lambda)$. Following reneging the trigger strategy would be at work, and by our analysis in part (ii), M's date-n payoff with the trigger strategy is

$$\frac{v - \alpha V}{1 - \alpha}.$$

Thus M will not renege at date n when consumers really come and try to buy online if and only if

$$(v - c - \lambda) + \frac{q_M w_n}{1 - q_M} = (v - c - \lambda) + \frac{q_M v}{1 - q_M} \geq 0 + q_M \Pi_M^0$$
$$\Leftrightarrow q_M \geq \frac{c + \lambda - v}{c + \lambda - \frac{v - \alpha V}{1 - \alpha}}.$$

When the above inequality holds, recognizing that M will never renege its online prices, R_n will choose $p_n = v$ at each and every date n, so that no consumers would really come and try to buy online. Finally, M should be willing to spend F at date 0. Thus we require that

$$F \leq \frac{q_M v}{1 - q_M} - q_M \Pi_M^0$$
$$\Leftrightarrow q_M \geq \frac{F}{v + F - \frac{v - \alpha V}{1 - \alpha}}.$$

The above two inequalities can be compactly written as

$$q_M \geq q_M^* = \frac{\max[F, c + \lambda - v]}{v + \max[F, c + \lambda - v] - \frac{v - \alpha V}{1 - \alpha}},$$

since the function

$$h(z) \equiv \frac{z}{v + z - \frac{v - \alpha V}{1 - \alpha}}$$

is strictly increasing in z. This finishes part (iv).

Remark. The marketing literature has shown that a dual-channel strategy may be beneficial for a manufacturer for several reasons.

- First, buyers may be endowed with heterogeneous costs/benefits of visiting a physical or online outlet, and it may be efficient to direct different buyers to purchase at different outlets. For example, a monopolistic manufacturer M is trying to serve two buyers A and B, both willing to spend 10 for M's product. Suppose that trading online is costless for A but prohibitively costly for B (because B is unfamiliar with the internet), and trading at the physical outlet is costless for B but prohibitively costly for A (because A has a high transportation cost). If building online and physical outlets is costless, then M should serve A at its online outlet and B at its physical outlet.
- Second, a dual-channel strategy may allow M to better discriminate buyers, even if doing so may reduce efficiency. Take again the above example, but assume that A is willing to pay 6 for the product instead of 10. Both A and B can trade costlessly in the physical market, but A and B must incur respectively 1 and 10 if they wish to trade online. Efficiency would require that M serve both A and B at the physical outlet. However, if M can build an online outlet costlessly, then M can set an online price 5 and an offline price 10 and direct A and B to trade respectively at the online and physical outlets. Letting A to trade online is inefficient, but it allows M to identify B at the physical outlet and extract B's surplus.
- Third, a dual-channel strategy can be valuable for imperfectly competitive manufacturers who wish to reduce competition. When a firm builds an online channel, it induces some buyers in the physical market to migrate to the online market, and this may alleviate competition in the physical market.

In addition to these reasons, this exercise provides yet another rationale for the dual-channel strategy: a manufacturer usually must use an independent retailer's service in the physical market, and there is always a conflict of interests between the manufacturer and the physical retailer. If a manufacturer can build an online channel to compete with its physical retailer, then it can prevent the physical retailer from pricing too high and dropping too many low-valuation buyers (which is against the manufacturer's interest). Such an online pricing strategy by the manufacturer is referred to as a "flank-attack" strategy against the physical retailer.

The problem with this "flank-attack" strategy is whether or not it is credible for the manufacturer to set a low online price. An overly low online price may not be able to cover the unit cost. However, if the physical retailer does not believe that the manufacturer can really trade at that low price at its online outlet, then such a pricing strategy is not credible, and the physical retailer would simply ignore it.

This exercise shows that, building on the perfect folk theorem, if the manufacturer has a strong image (i.e., q_M is sufficiently large), then it is credible for M to stand by its promised online price, and this makes the flank-attack strategy work. In equilibrium, M is able to force the physical retailer to cooperate and price low, serving both high- and low-valuation buyers and fulfilling channel efficiency.

We have assumed in part (iv) that R_n's are different firms. We claim that the manufacturer does not gain if the physical retailers are the same firm R, and hence M will choose a dual-channel strategy (together with the above flank-attack pricing strategy) over a single-channel strategy (together with the SPNE pricing strategy supported by the trigger strategy as in part (iii)) if and only if $q_M \geq q_M^*$.

With the single-channel strategy, M can spare F at date 0. When the trigger strategy is at work, R will get

$$v - \frac{v - \alpha V}{1 - \alpha}$$

at each date, so that facing w_n R would not deviate and price above v if and only if

$$\alpha(V - w_n) + \frac{q_M(v - \frac{v-\alpha V}{1-\alpha})}{1 - q_M} \leq \frac{v - w_n}{1 - q_M},$$

implying that M should choose

$$w_n^* = \frac{q_M(\frac{v-\alpha V}{1-\alpha}) + (1-q_M)(v-\alpha V)}{1-\alpha(1-q_M)}$$

$$= \frac{q_M(\frac{v-\alpha V}{1-\alpha}) + (1-q_M)(1-\alpha)(\frac{v-\alpha V}{1-\alpha})}{1-\alpha(1-q_M)}$$

$$= \frac{[q_M + (1-q_M)(1-\alpha)](\frac{v-\alpha V}{1-\alpha})}{1-\alpha(1-q_M)}$$

$$= \frac{[1-\alpha(1-q_M)](\frac{v-\alpha V}{1-\alpha})}{1-\alpha(1-q_M)}$$

$$= \frac{v-\alpha V}{1-\alpha} < v,$$

which yields for M a date-0 present value of

$$\frac{q_M w_n^*}{1-q_M}.$$

The intuition is that, the long-lived physical retailer R realizes that it would receive $v - w_n^*$ in each and every period when the trigger strategy is at work, and hence to induce R to cooperate in an SPNE that attains channel efficiency M must offer some w_n that represents a weakly deeper trade promotion than w_n^*, and the optimal w_n that meets this requirement from M's perspective is w_n^* itself. Thus M will adopt a dual-channel strategy in the presence of a forever-lived R if and only if, again,

$$q_M \geq q_M^*.$$

This exercise is adapted from Shan-Yu Chou (2014, The Optimal Product-line Extension, Pricing, Targeting, and Online-Channel Strategies for a Manufacturer Facing a Physical Independent Retailer, NTU working paper).