Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 4

1. (**Bayesian Equilibrium**) Two workers can each choose to or not to make an effort for their joint project. The project generates one unit of utility to each worker if at least one worker chooses to make the effort. Making effort incurs a disutility c_i to worker i, where c_i is worker i's private information, and worker j believes that c_i is uniformly distributed over $[0, 2]$. Exante it is common knowledge that c_1 and c_2 are independent random variables. Find a symmetric pure-strategy Bayesian equilibrium.[1]

Solution. Suppose that such a pure-strategy BE exists, and that given worker j's equilibrium strategy, a type c_i chooses to make the effort in equilibrium and obtain the equilibrium payoff

$$1 - c_i \geq \pi_{i,d},$$

where $\pi_{i,d}$ stands for the deviation payoff that the type c_i would obtain if he chose to make no effort; note that $\pi_{i,d}$ is independent of worker i's type c_i. We claim that a type c_i' must also choose to make the effort in equilibrium, if $c_i' < c_i$. To see this, simply note that

$$1 - c_i' > 1 - c_i \geq \pi_{i,d}.$$

On the other hand, if a type c_i'' chooses to make no effort in equilibrium; i.e., if

$$1 - c_i'' < \pi_{i,d},$$

then a type c_i''' must also choose to make no effort if $c_i''' > c_i''$. It follows that given worker j's equilibrium strategy, there must exist some c_i^* such

[1]**Hint**: There should be a cut-off level of c_i, say c_i^*, such that a type-c_i chooses to make an effort if and only if $c_i \leq c_i^*$.

that a type c_i makes the effort if and only if $c_i \leq c_i^*$. The argument applies to worker j as well.

The above argument suggests that the type c_i^* must feel indifferent about making and not making the effort, given worker j's strategy. That is, we must have

$$1 - c_i^* = \int_0^{c_j^*} 1 \cdot \frac{1}{2} dc_j + \int_{c_j^*}^2 0 \cdot \frac{1}{2} dc_j$$

$$\Rightarrow 1 - c_i^* = \frac{c_j^*}{2}.$$

It follows that

$$1 - c_1^* = \frac{c_2^*}{2} = \frac{1 - \frac{c_1^*}{2}}{2} \Rightarrow c_1^* = \frac{2}{3} = c_2^*.$$

Thus this game has a symmetric pure-strategy BE in which, for $i = 1, 2$, worker i makes an effort if and only if $c_i \leq \frac{2}{3}$.

Note that to fulfill productive efficiency we should have exactly one worker making the effort in equilibrium. Thus the equilibrium inefficiency takes place in two manners. First, with probability

$$\int_0^{\frac{2}{3}} \frac{1}{2} dc_1 \times \int_0^{\frac{2}{3}} \frac{1}{2} dc_2 = \frac{1}{9}$$

both workers make the effort; one worker's effort is totally redundant in this event. Second, with probability

$$\int_{\frac{2}{3}}^2 \frac{1}{2} dc_1 \times \int_{\frac{2}{3}}^2 \frac{1}{2} dc_2 = \frac{4}{9}$$

none of the workers make the effort, when each worker i has $c_i < 1+1 = 2$.

2. In the following two signaling games, player 1 is equally likely to be of type t_1 and type t_2, and can send signal m_1 or m_2 or m_3, and player 2 can respond by taking action a_1 or a_2 or a_3. The three tables indicate their payoffs following each of the 3 signals sent by player 1.

- There is a separating PBE for the following game, where m_3 is not an equilibrium signal. Find this PBE. Is this PBE an intuitive equilibrium?

m_1	a_1	a_2	a_3
t_1	$(1,0)$	$(4,3)$	$(2,4)$
t_2	$(10,5)$	$(4,4)$	$(4,1)$

m_2	a_1	a_2	a_3
t_1	$(2,2)$	$(6,0)$	$(8,1)$
t_2	$(2,2)$	$(2,3)$	$(6,2)$

m_3	a_1	a_2	a_3
t_1	$(6,1)$	$(4,-2)$	$(1,2)$
t_2	$(6,2)$	$(2,3)$	$(0,-1)$

Solution. According to the hint, in the separating PBE either the type-t_j player 1 sends m_j, $j = 1, 2$, or the type-t_j player 1 sends m_{3-j}, $j = 1, 2$.

Suppose that the former is the case. Then upon seeing m_2, player 2 belives that player 1 is of type t_2, and hence player 2 must respond by choosing a_2, yielding a payoff of 2 for the type-t_2 player 1, who can however ensure himself a payoff of at least 4 by sending signal m_1 instead. This is a contradiction.

Thus suppose that the latter is the case. Then upon seeing m_2, player 2 belives that player 1 is of type t_1, and hence player 2 must respond by choosing a_1, yielding a payoff of 2 for the type-t_1 player 1. Upon seeing m_1, player 2 belives that player 1 is of type t_2, and hence player 2 must respond by choosing a_1, yielding a payoff of 10 for the type-t_2 player 1. It is clear that this type of player 1 would never deviate unilaterally by sending m_2 or m_3. What about the type-t_1 player 1? By sending m_1, he will be regarded as type t_2 for sure, and player 2 will choose a_1 accordingly, which yields only a payoff of 1 for him. If he sends m_3, player 2 must respond by choosing a_3 in order to sustain the separating PBE, and for a_3 to be player 2's best response, player 2 must believe that the signal-sender is of type t_1 with a probability exceeding $\frac{3}{4}$. Hence we have verified that these beliefs and strategies indeed constitute a separating PBE.

Finally, to see that the separating PBE is intuitive, note that only m_3 is an off-equilibrium signal, and hence seeing m_3 is the only zero-probability event relevant in Cho-Kreps refinement. Intuition suggests that the type-t_2 player 1 should not have sent m_3, after player 2 receives m_3, because no matter what action player 2 will take after seeing m_3, the type-t_2 player 1 would get a payoff strictly less than 10, where 10 is his equilibrium payoff. But if player 2 believes for sure that it was the type-t_1 player 1 that has sent m_3, then player 2's best response would be a_3, which yields a deviation payoff of 1 for the type-t_1 player 1, so that this type would never deviate in the first place. To sum up, the separating PBE is intuitive.

- There is a pooling PBE for the game below, where player 1's equilibrium signal is not m_1. Find this PBE. Is this PBE an intuitive equilibrium?

m_1	a_1	a_2	a_3
t_1	$(8,0)$	$(4,3)$	$(2,4)$
t_2	$(10,5)$	$(4,4)$	$(4,1)$

m_2	a_1	a_2	a_3
t_1	$(2,2)$	$(6,0)$	$(8,1)$
t_2	$(2,2)$	$(2,3)$	$(6,2)$

m_3	a_1	a_2	a_3
t_1	$(6,1)$	$(4,-2)$	$(2,2)$
t_2	$(6,2)$	$(2,3)$	$(0,-1)$

Solution. It is not possible that both types of player 1 choose m_2 in a pooling equilibrium. If that happened, then player 2's best response upon seeing m_2 would be a_1, so that player 1 would get 2 in equilibrium, regardless of player 1's type. However, the type-t_2 player 1 is definitely better off deviating and choosing m_1 instead!

Thus in the pooling PBE both types of player 1 must choose m_3. Apparently B's best response upon seeing m_3 is a_1, so that both types of player 1 get 6 in equilibrium. Immediately, we have $\mu_3 = \frac{1}{2}$.

Now, upon seeing the deviation signal m_1, player 2 can never choose a_1, for that would induce player 1 to deviate and send m_1.

This requires that $\mu_1 \geq \frac{1}{4}$.

Upon seeing the deviation signal m_2, player 2 can never choose a_3, but then any $\mu_2 \in [0, 1]$ would be consistent with this PBE.

This PBE is intuitive, because the Cho-Kreps criterion has no bite when the deviation signal is m_1, and when the deviation signal is m_2, the only intuitive belief is $\mu_2 = 1$, which does not induce the type-t_1 player 1 to deviate.

3. Recall the game of beer and quiche discussed in Lecture 4, and consider a modified version of that game as follows. Here we assume that everything is the same as described in section 22 of Lecture 4 except that the strong type of A prefers to fight B. More precisely, let F denote the event that there is a fight between A and B, and NF the event of no fight. Let b and q denote respectively the signals of ordering beer and quiche respectively. Let s and w denote A's two possible types. Let u_A and u_B denote A's and B's payoffs respectively. Then the payoff functions of A and B in this modified game of beer and quiche can be summarized as follows.

$$u_A(F, b, s) = 3,\ u_A(F, q, s) = 2,\ u_A(NF, b, s) = 1,\ u_A(NF, q, s) = 0,$$

$$u_A(F, b, w) = 0,\ u_A(F, q, w) = 1,\ u_A(NF, b, w) = 2,\ u_A(NF, q, w) = 3,$$

$$u_B(F, w) = 2,\ u_B(F, s) = 0,\ u_B(NF, w) = u_B(NF, s) = 1.$$

Let x be B's prior probability assigned to the event that A is of the strong type. In this exercise, we shall assume that $x > \frac{1}{2}$.

Find all PBEs of this game. Determine if these PBEs are intuitive or not.[2]

[2]**Hint**: This game has a continuum of PBEs where both types of A adopt mixed strategies. Besides, this game also has two PBEs in which at least one type of A uses a pure strategy, and the latter two PBEs have the following features: In one PBE, the weak type of A orders quiche with probability one, the strong type of A orders beer and quiche with respectively probability $2 - \frac{1}{x}$ and $\frac{1}{x} - 1$, and B chooses to not fight A after seeing the signal b, and B chooses to fight A with probability $\frac{1}{2}$ after seeing the signal q. In another PBE, both types of A order b, and B chooses to not fight A after seeing b, but B chooses to fight A with probability $\frac{1}{2}$ after seeing q.

Solution. This game has two PBEs in which at least one type of A uses a pure strategy,[3] and the game has a continuum of PBEs where both types of A adopt mixed strategies. These PBEs are all intuitive.

Now we give details. At first, we can rule out PBEs where B always adopts a pure strategy after seeing any equilibrium or off-the-equilibrium signal. It is clear that in such a PBE, each type of A must also adopt a pure strategy.

- If there existed a pure-strategy PBE where B chooses the same action with probability one after seeing b and after seeing q, then the two types of A must order differently, but then taking the same action after seeing b and after seeing q is not B's best response.
- If there existed a pure-strategy PBE where B chooses different actions with probability one after seeing b and after seeing q, then the strong type of A will send the signal that ensures that there will be a fight with B, and the weak type of A will send the signal that ensures that there will be no fight with B. Thus the two types of A must order differently in equilibrium. But then B would choose to fight after and only after seeing the weak type's signal. Then the strong type would have an incentive to deviate and send the same signal as the weak type does, which is a contradiction.

Next, observe that in a PBE where both types of A use pure strategies, B must adopt a pure strategy after seeing an equilibrium signal.[4] Thus a PBE where both types of A use pure strategies and where B sometimes randomizes over fighting and not fighting must be pooling,

[3]These two PBEs have the following features. In one PBE, the weak type of A orders quiche with probability one, the strong type of A orders beer and quiche with respectively probability $2 - \frac{1}{x}$ and $\frac{1}{x} - 1$, and B chooses to not fight A after seeing the signal b, and B chooses to fight A with probability $\frac{1}{2}$ after seeing the signal q. In another PBE, both types of A order b, and B chooses to not fight A after seeing b, but B chooses to fight A with probability $\frac{1}{2}$ after seeing q.

[4]Indeed, B will fight A with zero probability in any pooling PBE, and B will fight the weak type of A with probability one and the strong type of A with probability zero in any separating PBE.

with B's randomization over fighting and not fighting appearing after A sends the off-the-equilibrium signal. If a pooling PBE where both types of A order q existed, then the strong type's equilibrium payoff would be 0, but he would get at least 1 by ordering b instead. Thus we should only consider the pooling PBE where both types of A order b. For B to randomize over fighting and not fighting upon seeing q, B's posterior belief must be such that both types of A are equally likely, and the probability λ that B may fight A must satisfy

$$1 = u_A(NF, b, s) \geq \lambda u_A(F, q, s) + (1-\lambda) u_A(NF, q, s) = \lambda \cdot 2 + (1-\lambda) \cdot 0 = 2\lambda,$$

and

$$2 = u_A(NF, b, w) \geq \lambda u_A(F, q, w) + (1-\lambda) u_A(NF, q, w) = \lambda \cdot 1 + (1-\lambda) \cdot 3 = 3 - 2\lambda,$$

implying that $\lambda = \frac{1}{2}$. Hence we have a pooling PBE where both types of A order b, and B will not fight A after seeing b, but B will fight A with probability 0.5 after seeing q. Note that in the latter PBE, we have

$$u_A(F, q, s) = 2 > u_A(NF, b, s) = 1$$

and

$$u_A(NF, b, w) = 2 < u_A(NF, q, w) = 3,$$

and hence following the appearance of the off-the-equilibrium signal q, we cannot rule out intuitively the possibility that q was sent by either type of A. Thus the above PBE is an intuitive equilibrium.

The previous two steps have exhausted all the possible PBEs where both types of A use pure strategies. Now it remains to consider PBEs where at least one type of A uses a mixed strategy.

- There is no PBE where the strong type of A chooses q for sure and the weak type of A randomize between b and q: in such a PBE, B would fight A after seeing b, but then the strong type A can be made better off by ordering b in the first place.
- There is no PBE where the weak type of A chooses b for sure and the strong type of A randomize between b and q: in such a PBE, B would not fight A after seeing q, but then the strong type of A's equilibrium payoff must be zero (as ordering q is one of his

best responses in equilibrium), which is a contradiction because ordering b is also the strong type of A's equilibrium best response, which implis a positive equilibrium payoff for this type of A.

- There is no PBE where the strong type of A chooses b for sure and the weak type of A randomize between b and q: in such a PBE, upon seeing b B must randomize between F and NF also, and hence B must consider the strong and weak types equally likely after seeing b, which requires that

$$\frac{1}{2} = \frac{x}{(1-x)(1-\pi)+x},$$

where $\pi \in (0,1)$ denotes the weak type of A's probability of sending q in this supposed PBE. This last equation is impossible to hold, because $x > \frac{1}{2}$.

- Consider a PBE where the weak and strong types of A both randomize between b and q. We claim that in such a PBE, B must adopt a pure strategy either after seeing b or after seeing q. If instead B randomizes between F and NF both after seeing b and seeing q, then we must have

$$\frac{1}{2} = \frac{x\xi}{(1-x)(1-\pi)+x\xi} = \frac{x(1-\xi)}{(1-x)\pi + x(1-\xi)},$$

where ξ is strong A's probability of sending signal b. From these two equations, we can show that $x = \frac{1}{2}$, which is a contradiction.

Thus we have four cases to consider.

(a) B chooses F upon seeing q. In this PBE, the weak and strong types of A get respectively 1 and 2 by choosing q. For them to both feel indifferent about choosing b or q in the first place, B must fight with probability $\frac{1}{2}$ upon seeing b. Let ξ and π be respectively the probabilities that the strong type of A and the weak type of A may choose b. We must have

$$\frac{x\xi}{x\xi+(1-x)\pi} = \frac{1}{2} \geq \frac{x(1-\xi)}{x(1-\xi)+(1-x)(1-\pi)}$$

$$\Leftrightarrow \frac{1-\xi}{1-\pi} \leq \frac{1-x}{x} = \frac{\xi}{\pi} \Rightarrow \xi \geq \pi \Rightarrow x < \frac{1}{2},$$

which is a contradiction.

(b) B chooses NF upon seeing q. In this PBE, the weak and strong types of A get respectively 3 and 0 by choosing q. It is impossible for the weak type of A to feel indifferent about choosing b or q in the first place, a contradiction.

(c) B chooses F upon seeing b. In this PBE, the weak and strong types of A get respectively 0 and 3 by choosing b. It is impossible for the strong type of A to feel indifferent about choosing b or q in the first place, a contradiction.

(d) B chooses NF upon seeing b. In this PBE, the weak and strong types of A get respectively 2 and 1 by choosing b. For them to both feel indifferent about choosing b or q in the first place, B must fight with probability $\frac{1}{2}$ upon seeing q. Let ξ and π be respectively the probabilities that the strong type of A and the weak type of A may choose b. We must have

$$\frac{x\xi}{x\xi + (1-x)\pi} \geq \frac{1}{2} = \frac{x(1-\xi)}{x(1-\xi) + (1-x)(1-\pi)}$$

$$\Leftrightarrow \frac{1-\xi}{1-\pi} = \frac{1-x}{x} \leq \frac{\xi}{\pi},$$

which defines a continuum of PBEs, and each of them is intuitive.

- Finally, consider a PBE where the weak type of A orders quiche with probability one, where the strong type of A orders beer and quiche with respectively probability $2-\frac{1}{x}$ and $\frac{1}{x}-1$, and where B chooses to not fight A after seeing the signal b, and B chooses to fight A with probability $\frac{1}{2}$ after seeing the signal q. In this equilibrium, the strong type of A gets 1, the weak type of A gets 2, and B gets 1. Since both b and q are equilibrium signals, there is no off-the-equilibrium signals in this PBE, and hence this PBE is intuitive.

4. In the following dynamic game with incomplete information, player 1 has two equally likely types, denoted by t_1 and t_2, and given his type,

the informed player 1 must choose either strategic game A or strategic game B. After observing player 1's choice, the informed player 1 and the uninformed player 2 must simultaneously take actions in the chosen strategic game. In each strategic game, player 1 can choose either U or D, and player 2 can choose either L or R. The resulting payoff x for the type-t_1 player 1, y for the type-t_2 player 1, and z for player 2, are written as a row vector (x, y, z). The following two tables summarize the players' type-and-action-contingent payoffs. For example, if player 1 chooses game A and then action U, and if player 2 chooses action L in game A, then $x = 2$, $y = 1$, and $z = 3$.

Strategic Game A

	L	R
U	$(2, 1, 3)$	$(1, 2, 5)$
D	$(1, 2, 0)$	$(0, 12, 10)$

Strategic Game B

	L	R
U	$(\frac{3}{2}, 21, 3)$	$(\frac{3}{4}, 2, 1)$
D	$(0, 0, 0)$	$(0, 10, 4)$

We shall only consider PBEs in which players use pure strategies in each and every subgame. For supporting beliefs, let us define $\mu_A \equiv \text{prob}(t_1|A)$ and $\mu_B \equiv \text{prob}(t_1|B)$, where A and B stand for "strategic game A" and "strategic game B" respectively.

(i) Find all separating and pooling PBEs of this game.[5]

[5]**Hint**: For each PBE, you must write down explicitly player 1's and player 2's strategies, together with μ_A and μ_B. In particular, for player 1's strategy, you must state clearly

$$\begin{pmatrix} t_1 \to \text{(A,U) or (A,D) or (B,U) or (B,D)} \\ \\ t_2 \to \text{(A,U) or (A,D) or (B,U) or (B,D)} \end{pmatrix},$$

and for player 2's strategy, you must state clearly

(ii) For each PBE obtained in part (i), determine whether it is a Cho-Kreps intuitive equilibrium or not.[6]

Solution. For part (i), we can show that this game has two pooling but no separating equilibria. For part (ii), we can show that both pooling PBEs are intuitive. Now we give details.

At first, there is a pooling PBE where player 1's strategy is

$$\begin{pmatrix} t_1 \to \text{(A,U)} \\ \\ t_2 \to \text{(A,D)} \end{pmatrix},$$

and player 2's strategy is

$$\begin{pmatrix} A \to \text{R} \\ \\ B \to \text{R} \end{pmatrix},$$

and where

$$\mu_A = \frac{1}{2}, \ \mu_B \leq \frac{2}{3}.$$

This PBE is intuitive, because by sending the off-the-equilibrium signal B, (1) type-t_1 may get $\frac{3}{2}$ (if player 2 is willing to choose L), which is greater than t_1's equilibrium payoff, which is 1; and (2) type-t_2 may get 21 (if player 2 is willing to choose L), which is greater than t_1's equilibrium payoff, which is 12. Thus any $\mu_B \in [0, 1]$ is consistent with the intuitive criterion.

$$\begin{pmatrix} A \to \text{L or R} \\ \\ B \to \text{L or R} \end{pmatrix}.$$

[6]**Hint**: For part (i), show that this game has two pooling but no separating equilibria; and for part (ii), show that both pooling PBEs are intuitive.

There is another pooling PBE where player 1's strategy is

$$\begin{pmatrix} t_1 \to \text{(B,U)} \\ \\ t_2 \to \text{(B,U)} \end{pmatrix},$$

and player 2's strategy is[7]

$$\begin{pmatrix} A \to \text{R} \\ \\ B \to \text{L} \end{pmatrix},$$

and where

$$\mu_A \in [0,1],\ \mu_B = \frac{1}{2}.$$

This PBE is also intuitive, because (1) by sending the off-the-equilibrium signal A, type-t_2's maximal possible payoff is 12 (if player 2 is willing to choose R), which is still less than t_2's equilibrium payoff, which is 21; and (2) any μ_A that rules out t_2 must be such that $\mu_A = 1$, and given $\mu_A = 1$, player 2's best response is R, so that by sending the off-the-equilibrium signal A, type-t_1's payoff would become 1 (as long as player 2 adopts the intuitive belief $\mu_A = 1$ and chooses R), which is still less than t_1's equilibrium payoff, which is $\frac{3}{2}$, showing that t_1 would not want to deviate from his equilibrium strategy.

There is no separating PBE for this game. To see this, suppose first that there were a separating PBE where the type-t_2 player 1 sends signal A. In this PBE, player 2 would correctly expect the type-t_2 player 1 to play D after sending signal A, and hence player 2's best response would be R, yielding 12 for the type-t_2 player 1. However, the type-t_2 player 1 could have deviated and sent signal B, which would convince player 2 that U would then follow, and hence player 2 would choose L

[7] Following the equilibrium signal B, player 2's pure strategy must be L: if R were to be taken, then the type-t_2 A would rather send signal A and then play D, which would yield a payoff 12, higher than the maximal payoff 10 that he could get in the supposed equilibrium.

after seeing B, yielding a deviation payoff of 21 for the type-t_2 player 1, which is a contradiction.

Next, suppose that there were a separating PBE where the type-t_1 player 1 sends signal A. In this PBE, player 2 would correctly expect the type-t_1 player 1 to play U after sending signal A, and hence player 2's best response would be R, yielding 1 for the type-t_1 player 1. On the other hand, the type-t_2 player 1 is expected to send signal B. Thus seeing B, player 2 expects to have 2 possible pure-strategy NEs, (U,L) and (D,R). To sustain the current PBE, however, player 2 must believe in (U,L) only: if player 2 believed in (D,R), so that the type-t_2 player 1 must play D after sending signal B, then the type-t_2 player 1's equilibrium payoff would be 10, but he could have deviated and sent signal A and then played D to reach the outcome (A,(D,R)), which would yield $12 > 10$ for the type-t_2 player 1. Thus we conclude that in this supposed separating PBE, player 2 would expect player 1 to play U after seeing B, and hence player 2's best reponse upon seeing B is L. But then the type-t_1 player 1 could have sent B and then played U to reach the outcome (B,(U,L)), which would yield $\frac{3}{2} > 1$ for the type-t_1 player 1, which is also a contradiction. Hence we conclude that this game has no separating PBEs.

Recall that in the game of beer and qiche discussed in Lecture 4, the uninformed player's payoff depends on his own action and the informed player's type, but not on the informed player's action. In the signalling games discussed in Example 3 of Lecture 4, the uninformed player's payoff depends on everything—his own action, the informed's action and the informed's type all affect the uninformed's payoff. Those games are said to have *common values*, because the two players' payoffs both depend on the informed's type. Here, we have a game with *private values*, where the informed's type per se does not affect the uninformed's payoff. The uninformed player still cares about the informed's type, because different types of the informed player may take different actions, and those action choices affect the uninformed's payoff. This distinction is relevant in certain signaling games where the informed's signals are "contracts" that the informed designs and offers to the uninformed player; see for example Maskin and Tirole's two articles in *Econometrica*, The Principal-Agent Relationship with an Informed Principal: The Case of Private Values (1990) and The Principal-Agent Relationship with an Informed Principal, II: Common Values (1992).

5. Consider the following stock trading model with one traded common stock and three classes of traders: one insider (or informed speculator), one noise trader, and several Bertrand-competitive market makers. Everyone is risk-neutral without time preferences. Stock trading takes place at date 0, and the true value of the stock, denoted v, will become public information at date 1. The extensive game proceeds as follows.

   - At the beginning of date 0, the insider alone learns about the realization of v, when everyone else only knows that v is equally likely to be $-2, -1, 1$ or 2.
   - Then simultaneously, the insider and the noise trader must each submit one market order. The insider's market order is denoted by X, and the noise trader's market order is denoted by u, and we assume that u is equally likely to be 1 or -1; that is, the noise trader is equally likely to buy one share or sell one share. By submitting a market order a trader commits to accepting order execution at the market-clearing price subsequently announced by the stock-trading platform.
   - At the same time when the insider and the noise trader submit their market orders, the market makers must each submit one *pricing schedule*, denoted by $P(\cdot)$. By submitting a schedule $P_i(\cdot)$, a market maker i commits to absorbing any market order $z \in \Re$ at the share price $P_i(z)$ that he specifies via $P_i(\cdot)$.
   - Then, the stock-trading platform receives X, u and the market makers' pricing schedules. The platform insists on matching X and u first, and in case $z = X + u \neq 0$, then the platform will pick one market maker i whose $P_i(z)$ appears to be the lowest when $z > 0$ or whose $P_i(z)$ appears to be the highest in case $z < 0$. In case $z = 0$, then the platform will just pick $P(0) = E[v]$.
   - Then, the date-0 stock trading session ends, and the game moves on to date 1. Then the realization of v becomes publicly known,

and each stock-trading participant gets his realized gain or loss from the date-0 stock-trading.

The above is a signaling game, where v is the informed insider's type, and X is the signal he sends. This is referred to as a signaling game *with noise*, because market makers (i.e., the uninformed players) do not observe X directly; rather, what they learn from the stock-trading platform is $z = X + u$ only (not u and X separately), where we recall that u is a zero-mean random variable.

We shall look for pure strategy perfect Bayesian equilibria in which the market makers submit the same $P(\cdot)$. Let us call them symmetric PBEs. A symmetric PBE is formally a pair $\{P(z), X(v)\}$ such that (i) given $P(\cdot)$, $X(v) \in \arg\max_y E[y(v - P(y+u))|v]$; and (ii) given any $z = X + u$, either $P(z)$ would ensure that no trade would occur between the selected market maker and the traders submitting market orders, or in the opposite case, the selected market maker must break even by absorbing $z = X(v) + u$; that is, $P(z) = E[v|X(v) + u = z]$.

Show that for each $a \in (0, \frac{2}{3})$, $\{P_a(\cdot), X_a(\cdot)\}$ is one symmetric PBE, where $X_a(\cdot)$ is such that

$$X_a(2) = -X_a(-2) = 1 + a, \quad X_a(1) = -X_a(-1) = 1 - a,$$

and $P_a(\cdot)$ is such that

$$\forall z \in \{-2 - a, -2 + a, -a, a, 2 - a, 2 + a\}, \; P_a(z) = -P_a(-z),$$

$$P_a(a) = \frac{1}{2}, \; P_a(2 + a) = 2, \; P_a(2 - a) = 1,$$

$$\forall z > 0, \; z \neq 2 + a, 2 - a, a, \; P_a(z) = 2,$$

and

$$\forall z < 0, \; z \neq -2 + a, -2 - a, -a, \; P_a(z) = -2.$$

Solution. Given $X_a(\cdot)$, define the order imbalance observed by the market makers by $Z(u, v)$, with

$$Z(-1, 2) = Z(1, -1) = a, \quad Z(1, -2) = Z(-1, 1) = -a.$$

Thus, for example, when seeing an order imbalance a, the market makers think that $(u, v) = (-1, 2)$ and $(u, v) = (1, -1)$ are equally likely, and hence they set $P_a(a) = \frac{2+(-1)}{2} = \frac{1}{2}$. Similarly, they set $P_a(-a) = \frac{-2+1}{2} = -\frac{1}{2}$.

On the other hand, the order imbalance $2 + a$, if it appears, is fully revealing: it must be that $(u, v) = (1, 2)$, so that $P_a(2 + a) = 2$. The order imbalance $2 - a$, likewise, can appear only when $(u, v) = (1, 1)$, and hence $P_a(2 - a) = 1$. Finally it is easy to check that

$$\forall z \in \{-2 - a, -2 + a, -a, a, 2 - a, 2 + a\},\ P_a(z) = -P_a(-z).$$

Now consider the off-the-equilibrium order imbalances. Let us specify the following supporting posterior beliefs for the market makers:

$$\forall z > 0,\ z \neq 2 + a, 2 - a, a,\ \text{prob.}(v = 2|z) = 1,$$

and

$$\forall z < 0,\ z \neq -2 + a, -2 - a, -a,\ \text{prob.}(v = -2|z) = 1.$$

Apparently, with these posterior beliefs, $P_a(z)$ is as asserted when z is not contained in $\{-2 - a, -2 + a, -a, a, 2 - a, 2 + a\}$.

It remains to check that $X_a(\cdot)$ is the insider's best response given $P_a(\cdot)$. Given a, define the type-v insider's payoff in equilibrium a from submitting market order x as

$$B_a(v, x) \equiv x(v - E[P_a(x + u)]),\ \forall x \in \Re,\ v = -2, -1, 1, 2.$$

Given the above supporting beliefs, if x is such that $x + u$ is not contained in the set $\{-2 - a, -2 + a, -a, a, 2 - a, 2 + a\}$, then x can never be optimal. Finally, it is easy to show that when $a \in (0, \frac{2}{3})$,

$$B_a(2, 1 + a) \geq B_a(2, 1 - a),\quad B_a(1, 1 - a) \geq B_a(1, 1 + a).$$

Thus $X_a(\cdot)$ is indeed optimal. This finishes the proof.

6. Firm A has a single owner-manager Mr. A, who needs to raise \$100 for a positive-NPV investment project at date 0. There are two possible date-0 states, called G and B, and the date-0 state is Mr. A's private

information. In state G, the assets in place of firm A are worth \$150 and the new project's NPV equals \$20. In state B, the assets in place are worth only \$$x$ and the new project's NPV is accordingly \$$y$. The public investors (also referred to as the outsiders) believe that the state may be G with prob. a. Mr. A and public investors are all risk-neutral without time preferences.

The game proceeds as follows. Mr. A first decides to or not to issue new equity to raise \$100 (two feasible signals!). Then, upon seeing Mr. A's decision, the public investors form posterior beliefs about the date-0 state, and they engage in Bertrand competition to determine the fraction α of equity that Mr. A must sell in order to raise \$100.

(i) Suppose that $x = 50$ and $y = 10$. Find all the pure-strategy PBE's of this signaling game.

(ii) Suppose that $x = 60$ and $y = -25$. Assume that the firm, after raising \$100 from new investors, can either undertake the new investment project or put \$100 in a riskless money market account. The risk-free interest rate is zero. In this case, a pooling equilibrium where both types of the firm choose to issue new equity exists if and only if the prior probability a for the good state is such that $a \geq a^*$. Compute a^*.

(iii) Suppose that $x = 60$ and $y = -25$. Unlike in part (ii), assume instead that the firm, after raising \$100 from new investors, must spend it on the new investment project, regardless of the state. In this case, a pooling equilibrium where both types of the firm choose to issue new equity exists if and only if the prior probability a for the good state is such that $a \geq a^{**}$. Compute a^{**}.

(iv) Suppose that $x = 60$ and $y = -25$. Suppose that $a = a^{**}$. Then in the pooling equilibrium obtained in part (ii), Mr. A ends up possessing a fraction $1-\alpha$ of firm A's equity. Compute α.

Solution. Consider part (i). We first look for separating equilibria. We shall refer to Mr. A of type j as simply "type j."

In a separating PBE where only type G issues new equity, in exchange of the $100 raised, the outsiders must ask for a share $\alpha = \frac{100}{150+20+100}$ of the ownership. But then type B will deviate: by deviating and issuing, type B would get

$$(1-\frac{100}{150+20+100})(50+10+100) = 100.74,$$

which is greater than 50, the payoff of type B if abanoning the new project. Hence, there is no such separating equilibrium.

Now, consider the separating PBE where only type B issues new equity. Then, the public investors would ask for a share of ownership equal to $\alpha = \frac{100}{10+50+100}$. Type B would indeed want to issue new equity: by issuing, he would get

$$(1-\frac{100}{10+50+100})(100+50+10) = 60,$$

greater than 50. On the other hand, type G insider would not issue new equity if and only if

$$(1-\frac{100}{10+50+100})(100+150+20) = 101.25 < 150,$$

which indeed is true. Thus this separating equilibrium does exist. The supporting beliefs for this PBE are all equilibrium beliefs, and can be pinned down by the Bayes Law.

Next, we look for pooling equilibria. Suppose that in equilibrium neither type issues new equity. But then type B wants to deviate: by issuing, type B cannot do worse than being identified, but even in that case, issuing is preferred to not issuing. Therefore there is no such pooling equilibrium.

Finally, consider the PBE where both types issue new equity. The outsiders would ask for

$$\alpha[a(150 + 20 + 100) + (1 - a)(50 + 10 + 100)] = 100,$$

and hence

$$\alpha = \frac{100}{160 + 110a}.$$

Type G must be willing to issue new equity in equilibrium:

$$(1 - \frac{100}{160 + 110a})(100 + 150 + 20) > 150;$$

and so must type B insider:

$$(1 - \frac{100}{160 + 110a})(100 + 50 + 10) > 50.$$

Thus the pooling equilibrium exists if and only if $a > \frac{13}{22}$.

Note that in this pooling equilibrium the outsiders' beliefs following the off-equilibrium signal "not issuing" is irrelevant. Note also that there does not exist an off-equilibrium signal in a separating equilibrium. Thus in part (i), both pure-strategy PBE's are robust against Cho and Kreps' intuitive criterion.

Consider part (ii). In the assumed pooling equilibrium, we must have

$$\alpha[270a + (160 + 0)(1 - a)] = 100,$$

and hence

$$\alpha = \frac{100}{[270a + 160(1 - a)]}.$$

Implicitly we are assuming here how the firm uses the 100-dollar cash is not verifiable, and since the new project has a negative NPV in the bad state, it is in Mr. A's interest to put the cash in the riskless money market account. For the pooling equilibrium to be viable, we need

$$270(1-\alpha) \geq 150, \quad (160+0)(1-\alpha) \geq 60,$$

so that we must require that

$$a \geq a^* = \frac{13}{22},$$

as obtained in part (i).

Consider part (iii). In the assumed pooling equilibrium, we must have

$$\alpha[270a + (160-25)(1-a)] = 100,$$

and hence

$$\alpha = \frac{100}{[270a + (160-25)(1-a)]}.$$

Implicitly we are assuming here whether the firm undertakes the new project is verifiable, and even though the new project has a negative NPV in the bad state, Mr. A has to spend the 100-dollar cash on the new project if he wants to pool with his counterpart in the good state. For the pooling equilibrium to be viable, we need

$$270(1-\alpha) \geq 150, \quad (160-25)(1-\alpha) \geq 60,$$

so that we must require that

$$a \geq a^{**} = \frac{2}{3}.$$

Finally, for part (iv), we obtain

$$\alpha = \frac{100}{[270a^{**} + (160-25)(1-a^{**})]} = \frac{4}{9}.$$

Remark. Unlike in part (i), where a pooling equilibrium is always productively efficient, in part (iii) we have a pooling equilibrium with *over-investments*. Thus in this pooling equilibrium, neither informational efficiency nor productive efficiency is attained. In equilibrium the bad-type firm is willing to undertake a project with negative NPV because it can share with the new investors the proceeds of \$100 that it obtains from issuing the new equity. In this case, the new investors lose more than the negative NPV pertaining to the new project.

7. Let us modify the game of chain-store paradox in Lecture 4 by assuming 5 entrants instead of 3. Find as many PBE's as possible for this reputation game.

Solution. It is easy to verify that no entrants will ever enter if $x_1 \geq \frac{1}{2}$. There are two remaining cases.

- **Case 1**: $x_1 \in [\frac{1}{4}, \frac{1}{2})$.
  Consider the subgame where E_1 has just entered. It is easy to see that preying E_1 with probability zero is not the sane incumbent's equilibrium behavior: if it were, then the incumbent's action would fully reveal whether the incumbent is sane or not so that the sane incumbent would get zero by not preying, but by preying the sane incumbent would be recognized as the crazy incumbent, which would yield a strictly positive payoff for the sane incumbent. Preying E_1 with probability one, on the other hand, is consistent with a PBE: it will induce both E_2 and E_3 to stay out, but after that E_4 and E_5 will enter. More precisely, because of the sane and the crazy incumbents' pooling behavior after E_1 enters, upon seeing E_1 being preyed, the rest 4 potential entrants believe that $x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and according to our analysis for the 4-entrant case in Lecture 4, E_2 would rather stay out, which implies that $x_3 = x_2 \in [\frac{1}{4}, \frac{1}{2})$, so that E_3 would also stay out, according to our analysis for the 3-entrant and 2-entrant cases in Lecture 4. In this *pooling* equilibrium preying E_1 thus yields for the sane incumbent a payoff of $-1 + \frac{3}{4} + \frac{3}{4} > 0$, which is indeed higher than the deviation payoff generated by not preying E_1, which is zero.

  Rationally expecting that both types of the incumbent will prey after E_1 enters, E_1 will stay out for sure. It follows that $x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and hence E_2 will stay out also; according to our analysis for the 4-entrant case in Lecture 4. But then $x_3 = x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and hence E_3 will stay out also, according toour analysis for the 3-entrant and 2-entrant cases in Lecture 4. It follows that $x_4 = x_3 = x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and hence both E_4 and E_5 will enter, according to our analysis for the 3-entrant and 2-entrant cases in Lecture 4.

- **Case 2**: $x_1 \in (0, \frac{1}{4})$.
  It is easy to verify that in equilibrium of the subgame where E_1 has just entered, the sane incumbent must randomize between preying and not preying.[8]

  Thus after E_1 enters, the sane incumbent may prey or not prey both with positive probabilities. When E_1 is not preyed, the rest 4 potential entrants know immediately that the incumbent is sane, and hence $x_2 = 0$, which implies that all the rest 4 potential entrants will then enter and the sane incumbent gets zero by not preying E_1. In order for the sane incumbent to get zero continuation payoff by preying E_1, it is necessary that upon seeing E_1 being preyed, $x_2 = \frac{1}{4}$. This fact can be proved by contraposition as follows.

  (a) Suppose instead that $x_2 < \frac{1}{4}$ after E_1 is preyed, so that according to our analysis for the 4-entrant case in Lecture 4 E_2 will enter with probability one. If that did happen, then the sane incumbent would get $-1 + 0 < 0$ by preying E_1, so that the sane incumbent would be better off by not preying E_1, a contradiction.

  (b) Suppose instead that $x_2 > \frac{1}{4}$ after E_1 is preyed, so that according to our analysis for the 4-entrant case in Lecture 4 E_2 will stay out with probability one, which implies that $x_3 = x_2 > \frac{1}{4}$, and hence E_3 will also stay out. If that did happen, then the sane incumbent would get $-1 + \frac{3}{4} + \frac{3}{4} + 0 + 0 > 0$ by preying

[8] Unlike in the previous case, now there can be no pooling behavior on the part of the incumbent after E_1 enters: if instead such an equilibrium did exist, then $x_2 = x_1 < \frac{1}{4}$ and hence E_2 would enter after seeing E_1 being preyed, but according to our analysis for the 4-entrant case in Lecture 4, by preying E_1 the sane incumbent's payoff would be $-1 + 0 < 0$, which is a contradiction because the sane incumbent could become better off by not preying E_1.

E_1, implying that the sane incumbent would not feel indifferent about preying and not preying E_1, which is again a contradiction.

Now, if the sane incumbent preys E_1 with probability $y_1 \in (0,1)$ after E_1 gets in, y_1 must satisfy

$$\frac{x_1}{x_1 + (1 - x_1)y_1} = \frac{1}{4} \Rightarrow y_1 = \frac{3x_1}{1 - x_1}.$$

Note that for y_1 to lie strictly between 0 and 1, it is necessary that $x_1 < \frac{1}{4}$. (This proves that the sane incumbent cannot randomize between preying E_1 and not preying E_1 in the above **Case 1**, and hence the pooling behavior reported there is indeed the unique outcome when the incumbent is facing E_1.)

What about the equilibrium behavior of E_2, E_3, E_4 and E_5 upon seeing E_1 being preyed? Let E_2 stay out with probability a upon seeing E_1 being preyed, and E_3 stay out with probability b upon seeing E_2 stay out. Show that any $a, b \in (0,1)$ satisfying

$$3a(1 + b) = 4$$

are now consistent with equilibrium, for then the sane incumbent will indeed feel indifferent between preying E_1 and not preying E_1. For example, $a = 1$ and $b = \frac{1}{3}$ are consistent with a PBE. In this particular equilibrium, after seeing E_1 being preyed, E_2 stays out for sure because $x_2 = \frac{1}{4}$, and following that E_3 enters with probability $\frac{2}{3}$. In equilibrium, E_1 knows that preying will occur with probability $4x_1$ and hence E_1 will enter if and only if $x_1 < \frac{1}{8}$. Now you can summarize the equilibrium path as follows.

(a) If $x_1 \in (0, \frac{1}{8})$, then E_1 enters, and if E_1 is preyed, then in one equilibrium (where $a = 1, b = \frac{1}{3}$) E_2 stays out and E_3 enters with probability $\frac{2}{3}$. Then if E_3 is preyed, then E_4 may or may not enter, and if E_4 does not enter, then E_5 may or may not enter.

(b) If $x_1 \in [\frac{1}{8}, \frac{1}{4})$, then E_1 stays out, E_2 enters, and if and only if E_2 is preyed, then E_3 stays out and E_4 and E_5 may or may not enter.

8. We shall consider here a modified version of the *Chain-store Paradox* with 3 entrants E_1, E_2 and E_3.

   As in the original version considered in Lecture 4, here there are two types of incumbent, referred to as the sane and the crazy. Again, x_j is the entrants' common probability for the event that the incumbent may be crazy at the time E_j is about to enter. Note that the prior probability x_1 is an exogenous parameter, but the posterior probabilities x_2 and x_3 must be derived in equilibrium.

   The players' payoffs in this new version are different from those in the original version, as explained below.

   - By preying following entry, the sane gets an immediate payoff of -2 and the crazy gets an immediate payoff of $\frac{3}{2}$.
   - By accomodating following entry, the sane gets an immediate payoff of 0 and the crazy gets an immediate payoff of 1.[9]
   - As in the original version discussed in Lecture 4, the incumbent gets $\frac{3}{4}$ in a period without entry, and an entrant gets 0 from staying out, 1 from entering and then being accomodated, and -1 from entering and then being preyed.

   The following table summarizes the players' payff information in a one-entrant case.

[9] That the crazy gets $1 > \frac{3}{4}$ from accomodating a current entrant may seem odd at the first glance. This, however, may be explained by a network externality. In a telecommunication industry for example, entry by a new firm may raise consumers' valuations for the incumbents' products. Alternatively, this may be due to an advertising effect when the incumbent and entrants are operating in a market for an unconventional new product. In any case, this payoff assumption implies that the crazy may have an incentive to pool with the sane and accomodate a current entrant, as an attempt to lure future entrants. The bottom line here is that whether pooling occurs because the sane mimics the crazy (as in Lecture 4) or because the crazy mimics the sane (as in the current problem) depends crucially on the nature of the product.

	Entrant staying out	Entrant preyed	Entrant accomodated
Entrant	0	-1	1
The sane	$\frac{3}{4}$	-2	0
The crazy	$\frac{3}{4}$	$\frac{3}{2}$	1

Find *all* the PBEs of this modified *Chain-store Paradox* with 3 entrants.[10]

Solution.

Let us use backward induction. First consider the incumbent's reaction to entry by E_3. Because there is no other entrant following E_3, the sane will accomodate because $0 > -2$, and the crazy will prey because $\frac{3}{2} > 1$. It follows that E_3 will enter if $x_3 < \frac{1}{2}$, will stay out if $x_3 > \frac{1}{2}$, and may randomize between entering and staying out if $x_3 = \frac{1}{2}$.

Now consider the subgame where E_2 has just entered. Observe that

$$-2 + \frac{3}{4} + \frac{3}{4} < 0,$$

which implies that the sane will never prey following entry by E_1 or E_2. The question is then whether the crazy will always prey following entry by E_2.

- Suppose that there is a separating equilibrium for this subgame, where the crazy preys E_2 with probability one. Then following entry by E_2 the crazy must obtain a continuation payoff by preying

[10]**Hint**: The sane has a dominant strategy in this game. The PBE of this game depends on x_1. Summarize the PBEs for the cases of $x_1 > \frac{7}{8}$, $x_1 = \frac{7}{8}$, $\frac{7}{8} > x_1 > \frac{3}{4}$, and $x_1 \leq \frac{3}{4}$.

which is equal to or higher than the continuation payoff that the crazy would get by accomodating. Note that by preying E_2 the crazy immediately gets $\frac{3}{2}$, which then results in $x_3 = 1$, and from the preceding analysis we know that E_3 will stay out, which generates $\frac{3}{4}$ to the crazy. Thus by preying E_2 the crazy's continuation payoff is $\frac{3}{2} + \frac{3}{4} = \frac{9}{4}$. On the other hand, by accomodating E_2, the crazy gets 1 immediately, which results in $x_3 = 0$, and from the preceding analysis we know that E_3 will enter and will be preyed by the crazy with probability one, yielding for the crazy the payoff of $\frac{3}{2}$. Thus by deviating and accomodating E_2, the crazy's continuation payoff is $1 + \frac{3}{2} = \frac{10}{4}$. Since $\frac{10}{4} > \frac{9}{4}$, the crazy will deviate from the supposed separating outcome, proving that there cannot be a separating equilibrium for the subgame following entry by E_2.

- Suppose instead that there is a pooling equilibrium for the subgame following E_2's entry, in which the crazy accomodates E_2 with probability one.

  We first claim that, to sustain such a pooling equilibrium, upon seeing that E_2 was accomodated, E_3's belief must be $x_3 = x_2 \leq \frac{1}{2}$. Indeed, if instead $x_3 = x_2 > \frac{1}{2}$, then E_3 will stay out according to the preceding analysis, so that the crazy would get the continuation payoff $1+\frac{3}{4}$ by accomodating E_2. However, by preying E_2, the crazy would get $\frac{3}{2}$ immediately, which results in $x_3 = 1$, so that E_3 will stay out, implying that the crazy's continuation payoff from deviating and preying E_2 is $\frac{3}{2} + \frac{3}{4} > 1 + \frac{3}{4}$, a contradiction! Hence to sustain a pooling equilibrium for the subgame following E_2's entry, it is necessary that $x_2 \leq \frac{1}{2}$.

  We claim that $x_2 \leq \frac{1}{2}$ is also sufficient for the pooling equilibrium to exist. Recall that if $x_3 = x_2 < \frac{1}{2}$ then E_3 will enter and will be preyed by the crazy, so that by accomodating E_2 the crazy's continuation payoff is $1 + \frac{3}{2}$, which is greater than the crazy's continuation payoff from preying E_2 (where preying results in $x_3 = 1$ so that E_3 will stay out), $\frac{3}{2} + \frac{3}{4}$. Thus the crazy will not deviate from this pooling equilibrium. On the other hand, if $x_3 = x_2 = \frac{1}{2}$, and if E_3 enters with probability $a \in [0, 1]$ then the crazy's continuation payoff from accomodating E_2 is

$$1 + a \times \frac{3}{2} + (1 - a) \times \frac{3}{4},$$

which is greater than or equal to the crazy's continuation payoff from preying E_2, $\frac{3}{2} + \frac{3}{4}$, when $a \geq \frac{2}{3}$. Thus a pooling equilibrium does exist following E_2's entry if and only if $x_2 < \frac{1}{2}$, and in equilibrium E_3 enters with proability $a \geq \frac{2}{3}$ after seeing E_2 being preyed.

- Finally, suppose that there is a semi-pooling equilibrium for the subgame following E_2's entry, where the crazy feels indifferent about preying and accomodating E_2. This implies that preying E_2 is one best response for the crazy, and hence the crazy's continuation payoff in this semi-pooling equilibrium must be equal to $\frac{3}{2} + \frac{3}{4} = \frac{9}{4}$.

  Let E_3 enter with probability a upon seeing E_2 being accomodated. Then by accomodating E_2, the crazy gets the continuation payoff $1 + \frac{3(1-a)}{4} + \frac{3a}{2}$, which must equal $\frac{3}{2} + \frac{3}{4} = \frac{9}{4}$ in the supposed semi-pooling PBE. Hence we have $a = \frac{2}{3}$, which implies that E_3 must feel indifferent about getting in and staying out upon seeing E_2 being accomodated, and hence in the latter event E_3's belief must be $x_3 = \frac{1}{2}$. This requires that the crazy accomodate E_2 with probability $\frac{1-x_2}{x_2} < 1$, implying the necessary condition that $x_2 > \frac{1}{2}$.

Now, let us move backwards and consider E_2's decision about entry. E_2 will get 0 by staying out. If E_2 gets in, then E_2 will be accomodated with probability one (pooling outcome!) if $x_2 \leq \frac{1}{2}$ and with probability

$$x_2 \times \frac{1 - x_2}{x_2} + (1 - x_2) \times 1 = 2(1 - x_2)$$

(semi-pooling outcome!) if $x_2 > \frac{1}{2}$. Hence E_2 should always enter if $x_2 \leq \frac{1}{2}$. If $x_2 > \frac{1}{2}$, then E_2 should enter (respectively, stay out) if

$$x_2 < (\text{respectively}, >) \frac{3}{4}.$$

When $x_2 = \frac{3}{4}$, E_2 feels indifferent about entering and staying out.

Now we can summarize the perfect Bayesian equilibria for the subgame where E_2 is about to enter as follows.

- If $x_2 \leq \frac{1}{2}$, then E_2 enters with probability one, and following entry the incumbent accomodates E_2 with probability one regardless of his type. Following that, E_3 will enter with a probability $a \geq \frac{2}{3}$ if $x_2 = \frac{1}{2}$, and E_3 will enter with probability one if $x_2 < \frac{1}{2}$.
- If $\frac{3}{4} > x_2 > \frac{1}{2}$ (respectively, if $x_2 = \frac{3}{4}$), then E_2 enters with probability one (respectively, with probability $b \in [0, 1]$), and following entry the crazy preys with probability $\frac{2x_2-1}{x_2}$ and accomodates with probability $\frac{1-x_2}{x_2}$, and the sane accomodates with probability one. E_3 stays out if E_2 was preyed, and E_3 enters with probability $\frac{2}{3}$ if E_2 was accomodated.
- If $x_2 > \frac{3}{4}$, then both E_2 and E_3 stay out.

It is useful to summarize the crazy's continuation payoff when E_2 is about to enter.

x_2	The crazy's continuation payoff
$x_2 < \frac{1}{2}$	$1 + \frac{3}{2} = \frac{10}{4}$
$x_2 = \frac{1}{2}$	$\frac{7}{4} + \frac{3a}{4} \geq \frac{9}{4}$
$\frac{1}{2} < x_2 < \frac{3}{4}$	$\frac{3}{2} + \frac{3}{4} = \frac{9}{4}$
$x_2 = \frac{3}{4}$	$b(\frac{3}{2} + \frac{3}{4}) + (1-b)(\frac{3}{4} \times 2) = \frac{6+3b}{4}$
$x_2 > \frac{3}{4}$	$\frac{3}{4} \times 2 = \frac{6}{4}$

Now consider the subgame where E_1 has just entered.

- Suppose that there is a separating outcome at this subgame, where the crazy preys with probability one. Then the crazy's continuation payoff in this subgame is $\frac{3}{2} + \frac{3}{4} \times 2 = 3$. By deviating and accomodating, the crazy can get 1 immediately, and with $x_2 = 0$, the crazy will get a continuation payoff of $1 + \frac{3}{2} = \frac{10}{4}$ from E_2 and E_3, proving that there cannot be a separating outcome following entry by E_1.
- Suppose that there is a pooling outcome at this stage, in which the crazy accomodates E_1 with probability one. The crazy's deviation from this subgame equilibrium will generate $\frac{3}{2}$ immediately and it results in $x_2 = 1$, so that the crazy will get a continuation payoff $\frac{3}{4} \times 2$ from E_2 and E_3. Thus the crazy will get a deviation payoff of 3 if the crazy chooses to prey following E_1's entry. To sustain the pooling outcome, therefore, the crazy's equilibrium payoff in this subgame must be greater than or equal to 3. Since

by accomodating E_1 the crazy will get 1 immediately, to sustain this pooling outcome the crazy's continuation payoff from E_2 and E_3 given $x_2 = x_1$ must be greater than or equal to 2. According to the above table that summarizes the crazy's continuation payoff from E_2 and E_3, we conclude that such a pooling outcome exists if and only if $x_1 \leq \frac{3}{4}$.

- Finally, consider a semi-pooling outcome where the crazy feels indifferent about preying and accomodating after E_1 enters. This implies that the crazy's continuation payoff is equal to $\frac{3}{2} + \frac{3}{4} \times 2 = 3$, which the crazy would obtain by preying with probability one. According to the above table that summarizes the crazy's continuation payoff from E_2 and E_3, we conclude that such a semi-pooling outcome exists if and only if E_2 enters with a probability $b = \frac{2}{3}$, and this requires that $x_2 = \frac{3}{4}$, which in turn requires that the crazy accomodate E_1 with probability $\frac{3(1-x_1)}{x_1} < 1$, implying the necessary condition that $x_1 > \frac{3}{4}$.

Now we can summarize the perfect Bayesian equilibria as follows.

- If $x_1 \leq \frac{3}{4}$, then E_1 enters with probability one, and following entry the incumbent accomodates E_1 with probability one regardless of his type. Following that, $x_2 = x_1 \leq \frac{3}{4}$, and the subgame equilibrium is as summarized above.
- If $\frac{7}{8} > x_1 > \frac{3}{4}$ (respectively, if $x_1 = \frac{7}{8}$), then E_1 enters with probability one (respectively, with probability $c \in [0, 1]$), and following entry the crazy preys with probability $\frac{4x_1-3}{x_1}$ and accomodates with probability $\frac{3(1-x_1)}{x_1}$, and the sane accomodates with probability one. Following E_1 being preyed, $x_2 = 1$ and both E_2 and E_3 will stay out, and following E_1 being accomodated, $x_2 = \frac{3}{4}$, and E_2 may enter with probability $\frac{2}{3}$. The rest of the subgame equilibrium is as summarized above.
- If $x_1 > \frac{7}{8}$, then E_1, E_2, and E_3 will all stay out. The incumbent will then get $\frac{3}{4} \cdot 3 = \frac{9}{4}$ regardless of the incumbent's type.