# Game Theory with Applications to Finance and Marketing

## Lecture 2: Multistage Games with Observable Actions and Repeated Games

1. A multistage game is an extensive game where the game tree can be naturally divided into stages $t = 0, 1, 2, \cdots$. A multistage game with observable actions is a multistage game where at the beginning of any stage $t$, all the actions taken by the players in earlier stages are common knowledge. A special class of multistage games with observable actions are *repeated games*, in which a stage game is repeated for a finite or infinite number of times. Our agenda here is as follows. First, we shall analyze two special multistage games with observable actions with an infinite time horizon, one being Rubinstein's (1982) infinite-horizon bargaining game and the other the war of attrition. There does not exist a "last period" in these games, and hence the backward induction technique that we learned in Lecture 1 cannot be used to solve for the SPNE of such games. We shall demonstrate a procedure to solve such games. Second, we shall consider infinitely repeated games, and introduce the folk theorems that have been used to solve such games, which require understanding of such concepts as minmax strategy, trigger strategy, and so on. We shall then apply the theory to collusive pricing of imperfectly competitive firms and to strategic cooperation between upstream and downstream firms in a distribution channel. We shall also apply the theory to study the role of forward transactions in changing imperfectly competitive firms' profits. Third, we consider finitely repeated games, and introduce a new equilibrium concept called *renegotiation-proof equilibrium.* We shall demonstrate the ideas using a series of examples. Finally, we consider a series of multistage games with observable actions in corporate finance. Issues to be examined include bank runs, corporate agency problems, strategic default of debt, and so on.

2. (**Rubinstein's Bargaining Game**) Consider Rubinstein's (1982) bargaining model with alternating offers. Players 1 and 2 are bargaining over 1 dollar. In period $i$, where $i$ is odd, player 1 can make an offer $(x_i, 1 - x_i)$ to player 2, where $x_i$ is player 1's share, and player 2 can

either accept or reject that offer. If the offer is accepted, then the dollar is so divided; or else, the game moves on to the $i+1$st period. In period $j$, where $j$ is even, player 2 can make an offer $(x_j, 1-x_j)$ to player 1, where $x_j$ is again player 1's share, and player 1 can either accept or reject that offer. The game ends here if acceptance is player 1's decision (with the dollar divided as player 2 proposed); or else, the game moves on to the $j+1$st period. If an offer $(x, 1-x)$ is accepted in period $i$, then player 1's payoff is $\delta_1^{i-1}x$ and player 2's payoff is $\delta_2^{i-1}(1-x)$, where the discount factors $\delta_1, \delta_2 \in (0,1)$.

Shaked and Sutton (1984) prove that there is a unique SPNE of this infinite-horizon game. At the beginning of period $i$ where $i$ is odd and no consensus has been reached, the subgame looks exactly the same as at period 1. This is referred to as the *stationarity* property. Because of this property, we know that the set of SPNE's in the subgame starting at the beginning of period $i$ where $i$ is odd and no consensus has been reached is the same as the set of SPNE's of the entire extensive game. Let $\underline{v}_i$ and $\overline{v}_i$ be the infimum and supremum of player $i$'s continuation payoffs in any SPNE of a subgame starting at a period where it is player $i$'s turn to make an offer. Let $\underline{w}_i$ and $\overline{w}_i$ be the infimum and supremum of player $i$'s continuation payoffs in any SPNE of a subgame starting at a period where it is player $j$'s turn to make an offer.[1] Then in any SPNE, player $j$ cannot reject an immediate offer from $i$ that gives player $j$ a share greater than $\delta_j\overline{v}_j$, proving that[2]

$$\underline{v}_i \geq 1 - \delta_j\overline{v}_j.$$

[1]The set of SPNE payoffs for player $i$ in the subgame where concensus has never been reached and player $i$ is about to make an offer is bounded above by 1 and below by zero. This set is nonempty so long as this game has at least one SPNE. A non-empty subset of $\Re$ that is bounded above must have a lowest upper bound (the supremum), and a non-empty subset of $\Re$ that is bounded below must have a greatest lower bound (the infimum).

[2]Note that by promising player $j$ the payoff $\delta_j\overline{v}_j$ player $i$ can get the payoff $1-\delta_j\overline{v}_j$ immediately. In any SPNE of the subgame where it is player $i$'s turn to make an offer, player $i$'s equilibrium payoff must be greater than or equal to this "feasible" payoff; that is, $1-\delta_j\overline{v}_j$ is a lower bound for the set of SPNE payoffs that player $i$ can get in a subgame where it is player $i$'s turn to make an offer. Since $\underline{v}_i$ is the maximum of all such lower bounds, the following inequality must hold.

This also implies that player $i$ when making offers to player $j$ will not promise player $j$ a share greater than $\delta_j \overline{v}_j$, and hence

$$\overline{w}_j \leq \delta_j \overline{v}_j.$$

On the other hand, player $j$ will definitely reject an offer from player $i$ that promises a share to player $j$ less than $\delta_j \underline{v}_j$. That is, when player $i$ is the one making an offer, he cannot get more than $1 - \delta_j \underline{v}_j$ when his offer gets accepted. Recall that if player $j$ rejects player $i$'s offer in this period and the game moves on to the next period, then player $i$'s payoff in the subgame starting from the next period cannot exceed $\overline{w}_i$. Now, in a subgame where no concensus has been reached and player $i$ is about to make an offer to player $j$, player $i$ either gets nothing in equilibrium (if the two players will never reach a concensus in this subgame), or player $i$'s equilibrium payoff is positive because player $i$ will ultimately make an offer that is accepted by player $j$ (and in this event player $i$'s payoff is maximized when that offer is made in the first period of the current subgame), or player $i$'s equilibrium payoff is positive because player $i$ will ultimately accept an offer made by player $j$ (and in that event player $i$'s payoff is maximized when that offer is made in the second period of the current subgame). Thus we conclude that

$$\begin{aligned} \overline{v}_i &\leq \max(1 - \delta_j \underline{v}_j, \delta_i \overline{w}_i) \\ &\leq \max(1 - \delta_j \underline{v}_j, \delta_i^2 \overline{v}_i), \end{aligned}$$

implying that the last maximum equals $1 - \delta_j \underline{v}_j$. (Why?) It follows that

$$\begin{aligned} \overline{v}_i \leq 1 - \delta_j \underline{v}_j &\leq 1 - \delta_j (1 - \delta_i \overline{v}_i) \\ \Rightarrow \overline{v}_i &\leq \frac{1 - \delta_j}{1 - \delta_i \delta_j}. \end{aligned}$$

It also follows that

$$\underline{v}_i \geq 1 - \delta_j \overline{v}_j \geq 1 - \delta_j (1 - \delta_i \underline{v}_i)$$

$$\Rightarrow \underline{v}_i \geq \frac{1-\delta_j}{1-\delta_i\delta_j}.$$

This implies that

$$v_i \equiv \overline{v}_i = \underline{v}_i = \frac{1-\delta_j}{1-\delta_i\delta_j},$$

so that all SPNE's must generate the same payoff profile. We have $1 - v_j = \delta_i v_i$, and hence $1 - \delta_j \underline{v}_j \geq \delta_i^2 \overline{v}_i$. Now the two players must reach concensus at period 1 in the unique SPNE: player 1 must offer $x_1 = \underline{v}_1$, which player 2 feels indifferent about accepting or rejecting but chooses to accept with probability one. (When $\delta_i = \delta_j$, we have $v_i = v_j$; compare this result with section 53 of Lecture 1, Part I.)

3. Here is another proof for the uniqueness of SPNE's in Rubinstein's bargainging game. Note that when making offers to player $j$, player $i$ knows that player $j$ will never accept a negative share, and player $j$ will accept any share greater than $\delta_j$. It follows that player $i$ will not offer a share greater than $\delta_j$, and player $j$ will reject any share less than $\delta_j(1-\delta_i)$. Repeating this argument, we now claim that if for some $k \in \mathbf{Z}_+$, player 1 accepts any share $x > x^k$ and player 2 accepts any share $x < y^k$, with $y^k < x^k$, (so that player 1 will never offer $x < y^k$ and will reject any $x < \delta_1 y^k$, and player 2 will never offer $x > x^k$ and will reject any $x > 1 - \delta_2(1-x^k)$), then player 1 must accept any $x > x^{k+1} = \delta_1(1-\delta_2) + \delta_1\delta_2 x^k$, and player 2 must accept any $x < y^{k+1} = 1 - \delta_2 + \delta_1\delta_2 y^k$.[3]

[3] Note that $x^1 = \delta_1$ and $y^1 = 1 - \delta_2$. Note also that $x^{k+1} - x^k = \delta_1\delta_2(x^k - x^{k-1})$, and hence $\{x^k\}$ is a monotone sequence. To figure out whether $\{x^k\}$ is monotone increasing or monotone decreasing, we simply check

$$x^{k+1} - x^1 = -\delta_1\delta_2(1-x^k) \leq 0.$$

Similarly, note that $y^{k+1} - y^k = \delta_1\delta_2(y^k - y^{k-1})$ and $y^{k+1} - y^1 = \delta_1\delta_2 y^k \geq 0$. Thus $\{x^l\}$ and $\{y^k\}$ are respectively decreasing and increasing sequences contained in the compact interval $[0,1]$, and hence they both have limits, which we denote by $x^\infty$ and $y^\infty$ respectively. Since $x^\infty < y^\infty$, it is incorrect to assert that $x^{k+1} > y^{k+1}$ for all $k$ even if it is true that $x^1 > y^1$ (or if, equivalently, $1 < \delta_1 + \delta_2$). The latter error appears on page 130 of Fudenberg and Tirole's book, *Game Theory*, MIT Press.

To see this, note that if player 1 rejects player 2's offer in some subgame, then one of three things may happen. Either (i) no agreement will ever be reached, and player 1 gets 0; or (ii) player 2 accepts one of player 1's offers, which allows player 1 to get a payoff no greater than $x^{k+1} = \delta_1[1-\delta_2(1-x^k)]$; or (iii) player 1 accepts one of player 2's offers, which yileds a payoff for player that is at most $\delta_1^2 x^k$. The payoff in (ii) is the highest among the 3 possibilities.[4] Thus player 1 should accept any share $x > x^{k+1}$. The reasoning for player 2's behavior is similar.

Now $x^{k+1} - x^k = (1-\delta_1\delta_2)(1-x^k) - (1-\delta_1) < 0$, and similarly $y^{k+1} - y^k > 0$, by the axiom of continuity in $\Re$, the two monotone sequences $\{x^k; k \in \mathbf{Z}_+\}$ and $\{y^k; k \in \mathbf{Z}_+\}$ are contained in $[0,1]$ and have limits $x^\infty = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ and $y^\infty = \frac{1-\delta_2}{1-\delta_1\delta_2}$ respectively. Since player 2 rejects any $x > y^\infty$ and accepts any $x < y^\infty$, it is the unique SPNE outcome that player 1 offers $y^\infty$ to player 2 in period 1, which player 2 accepts with probability one.

Here the unique SPNE result depends crucially on the facts that there are only two players engaging in bargaining and that the players can offer any number contained in the unit interval. When there are more than two players, or when the set of feasible offers is finite, there tend to be multiple equilibria for this infinite-horizon bargaining game.

4. (**War of Attrition**) Consider the following timing game, called the *war of attrition* (Maynard Smith, 1974). Two animals are fighting for a prey, which they both attach value $v > 1$. The current time is $t = 0$, and time is discrete. Fighting costs 1 per date. If one animal gives up fighting at date $t$, the other animal gets the prey without incurring the fighting cost at that date. If both stop fighting at date $t$, both get zero. The two animals have common discount factor $\delta \in (0, 1)$.

   Thus this is a timing game, where players must decide when to make a move.[5] (Here, the move is to give up fighting and leave.) Denote the

[4] Note that, because $\delta_1 < 1$ and $x^k \leq 1$,

$$\delta_1[1-\delta_2(1-x^k)] - \delta_1^2 x^k = \delta_1[(1-\delta_2) - (\delta_1 - \delta_2)x^k] > 0.$$

[5] Another important class of timing games is the *preemption games*, in which the first

payoff of the animal that stops first by $L(t)$, where

$$L(t) = -(1 + \delta + \delta^2 + \cdots + \delta^{t-1}) = -\frac{1-\delta^t}{1-\delta},$$

and the payoff of the animal that does not stop first by $F(t)$, where[6]

$$F(t) = L(t) + \delta^t v.$$

Note that both animals have payoff $L(t)$ if they both stop fighting at date $t$. Among the SPNE's of this game, there is a unique symmetric *stationary* one, where each player chooses to stop right away with a probability $p$ if the rival is still present.[7] For this to be a symmetric equilibrium, $p$ must satisfy

$$pF(t) + (1-p)L(t+1) = L(t),$$

where the right-hand side is the payoff generated by "stop immediately," and the left-hand side the payoff generated by "stop at the next date unless the rival is no longer present the next date." Solving, we have $p = \frac{1}{1+v}$. Note that in this equilibrium, both players have zero payoff.

The continuous-time counterpart of this game (where $\delta^t$ is replaced by $e^{-rt}$, the distribution function of the exit time $t$ is denoted by $G(t)$,

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player that makes a move wins, in contrast with the current *war of attrition*, in which the first player that makes the move loses. For example, there are two firms considering entering a new market. Suppose that neither entered before $t$ and one of them enters exactly at $t$. In this case the entering firm gets a time-$t$ payoff $g > 0$ and the other gets zero. If both enter at $t$, then both get $l < 0$ at $t$.

[6]Waiting for one more period does not automatically raise the waiting cost by one; it does if and only if the rival is also waiting at the same time.

[7]Usually, the players' temporal equilibrium behavior in an SPNE may depend both on $t$ and on the history at time $t$. We call such an SPNE a closed-loop or feedback SPNE. (Here, in the war of attrition, if we let $\phi_t$ represent the set of players who did not exit before time $t$, and the history at time $t$ can be represented by $\{\phi_s; s \in [0,t]\}$.) If the players' temporal equilibrium behavior depends only on $t$, then the SPNE is an open-loop equilibrium. If the players' temporal equilibrium behavior depends only on $\phi_t$, then the SPNE is stationary.

and fighting costs $\Delta t$ for any time interval with length $\Delta t$) also has symmetric stationary SPNE where both players use the same behavior strategy at all dates. For $G(\cdot)$ to define a symmetric mixed strategy equilibrium, it must be that given both firms stay up to time $t$, each firm feels indifferent about quitting right away or staying for $dt$ longer. That is, by incurring a cost $-dt$, one gets $v$ in the next instant with probability $\frac{G(t+dt)-G(t)}{1-G(t)}$, and this must yield a zero incremental payoff for the firm:

$$vdG(t) - [1 - G(t)]dt = 0.$$

It follows that

$$G'(t) = \frac{dG(t)}{dt} = \frac{1 - G(t)}{v}.$$

It follows that

$$G(t) = 1 - ce^{-\frac{t}{v}},$$

for some constant $c$. Note that $G(0) > 1$ if $c < 0$, and $G(0) < 0$ if $c > 1$. Thus $c \in [0, 1]$. If $c < 1$, then $G(0) > 0$. Hence a firm can get essentially $G(0)v > 0$ by exiting right after time 0, implying that it should not exit at time 0 with a strictly positive probability. To be consistent with a symmetric SPNE, we must have $c = 1$. Thus in the unique symmetric stationary SPNE, we have

$$G(t) = 1 - e^{-\frac{t}{v}}.$$

5. (**Repeated Games**) A repeated game is a supergame of some stage (or constituent) game, and it takes the simple form of repeating the stage game for either a finite or an infinite number of times. Let the stage game in normal form be

$$G(1) = \{(A_i)_{i=1}^I,\ (u_i)_{i=1}^I\},$$

where $I$ is the number of players, $A_i$ the set of feasible actions available to player $i$ in $G(1)$, and $u_i : \Pi_{i=1}^I A_i \to \Re$ player $i$'s (current) payoff in

$G(1)$. The set $A = \Pi_{i=1}^{I} A_i$ is the set of *action profiles* at each stage $t$. We shall enlarge players' action spaces by allowing them to use any correlated device on $A$. If a device $\Omega$ is adopted, and if at stage $\tau$, $\omega^\tau$ is the outcome of the device, we call the sequence

$$h^t \equiv \{\omega^0, \omega^1, \cdots, \omega^{t-1}; a^0, a^1, \cdots, a^{t-1}\}, \ \forall t \in \mathbf{Z}_+ \bigcup \{0\}$$

the *history* of the game at the beginning of stage $t$. (The notation $\mathbf{Z}_+$ stands for the set of strictly positive integers.) A pure strategy $s_i$ for player $i$ is a sequence of mappings $\{s_i^t : h^t \to A, \forall t\}$. Let $S_i$ be the set of all pure strategies and $\Sigma_i$ the set of all mixed strategies for player $i$.

Now, denote $G(T)$ the supergame of $G(1)$ that would result if we repeat $G(1)$ for $T$ times, with player $i$'s payoff being given by

$$\frac{1-\delta}{1-\delta^{T+1}} \sum_{t=0}^{T} \delta^t u_i(a^t),$$

where $a^t$ is the action profile chosen by the players at stage $t$ and $\delta \in (0, 1]$ the players' common discount factor (in general players can have different discount factors $\delta_i$). In this case, $s_i$ is a sequence containing $T$ terms of $s_i^t$.

Let $G(\infty)$ be the supergame of $G(1)$ that will result if we repeat $G(1)$ for a countably infinite number of times. Correspondingly, the payoff function of player $i$ in $G(\infty)$ is[8]

$$(1-\delta) \sum_{t=0}^{\infty} \delta^t u_i(a^t).$$

Note that this payoff function has already been normalized so that it is readily comparable with the payoff function in $G(1)$: observe simply that $(1-\delta)\sum_{t=0}^{\infty} \delta^t = 1$. In this case, $s_i$ is an infinite sequence of $s_i^t$. Given $G(\infty)$, the following is called the *continuation payoff* at stage $t$:

[8]Game theorists also consider other payoff functions. For example, the *time-average* payoff function takes the form of $\frac{1}{T}\sum_{t=1}^{T} u_i(s^t)$ in $G(T)$ and $\underline{\lim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(s^t)$ in $G(\infty)$. Recall that if $\{x_t; t \in \mathbf{Z}_+\}$ is a real sequence, then $\underline{\lim}_{t\to\infty} x_t = \sup_{t \in \mathbf{Z}_+} \inf_{s \geq t} x_s$ always exists in the extended real line $\overline{\Re}$.

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u_i(a^\tau).$$

Note that when players choose actions at stage $t$, all the current-stage payoffs in the earlier stages are sunk, and only the *continuation payoffs* will affect their choices of actions at stage $t$.

6. Define $V$ as the set of feasible payoff profiles that the players may attain by taking some action profiles in $G(1)$. Depending on our assumption about what the players can do in $G(1)$, $V$ may or may not be a convex set. ( Recall that $A \subset \Re^n$ is convex if the line segment connecting any two points in $A$ is also contained in $A$.) For example, if the players are confined to using mixed strategies only, then this set is in general non-convex.[9] At the other extreme where the players can collectively choose *any* correlated device (not necessarily a correlated equilibrium), which randomly selects elements in $A$, then it is easy to see that $V$ is convex. For example, consider the following $G(1)$:

| Player 1/Player 2 | L | R |
|---|---|---|
| U | $5,1$ | $0,0$ |
| D | $4,4$ | $1,5$ |

where a correlated device assigning (U,L), (U,R), (D,L), and (D,R) with respectively probability $a$, $b$, $c$, and $d$ will generate a payoff profile

$$\begin{pmatrix} 5\cdot a+0\cdot b+4\cdot c+1\cdot d \\ 1\cdot a+0\cdot b+4\cdot c+5\cdot d \end{pmatrix}$$

$$= a\begin{pmatrix} 5 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 4 \\ 4 \end{pmatrix} + d\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Thus if we allow the two players to adopt <u>any</u> correlated device from

$$\mathcal{C} = \{(a,b,c,d) : a+b+c+d = 1, a,b,c,d \geq 0\},$$

[9] Exceptions exist. For example, in a two player finite game where player 1 has only one feasible action, the set $V$ is convex.

then the set of payoff profiles attainable by the two players is a convex set, which is exactly the convex hull generated by the four vectors

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

In the following, we shall mostly assume that $V$ is convex.[10] Any payoff profile in $V$ will be referred to as *feasible*.

7. (**Minmax Value and Individually Rational Payoff**) Given $G(1)$, define player $i$'s *minmax value* or *reservation utility* as

$$\underline{v}_i \equiv \min_{\sigma_{-i}\in\Sigma_{-i}} [\max_{\sigma_i\in\Sigma_i} u_i(\sigma)],$$

where in this section only, $\sigma$ denotes a mixed strategy profile in the game $G(1)$. Define *minmax strategy profiles against player* $i$ by the following

$$m^i_{-i} \in \arg \min_{\sigma_{-i}\in\Sigma_{-i}} [\max_{\sigma_i\in\Sigma_i} u_i(\sigma)].$$

Define

$$m^i_i \in \arg \max_{\sigma_i\in\Sigma_i} u_i(\sigma_i, m^i_{-i}).$$

With these notations, of course, we have

$$\underline{v}_i = u_i(m^i_i, m^i_{-i}).$$

As an example, consider the following stage game $G(1)$:

[10] Even if the players cannot freely commit to any correlated device that they like, *intertemporal switching* may lead to approximately the same result if $\delta$ is close to 1. For example, imagine that in $G(\infty)$, for all $n \in \mathbf{Z}_+$ and at all stage $10(n-1)+t$, the two players will play (U,L) if $t = 1, 2, 3$, (U,R) if $t = 4, 5, 6, 7$, (D,L) if $t = 8, 9$, and (D,R) if $t = 10$. This will generate an average (per-period) payoff profile close to that generated by a correlated device with $a = 0.3$, $b = 0.4$, $c = 0.2$, and $d = 0.1$.

| player 1/player 2 | L | R |
|---|---|---|
| U | $-2, 2$ | $1, -2$ |
| M | $1, -2$ | $-2, 2$ |
| D | $0, 1$ | $0, 1$ |

Let us find $\underline{v}_1$. If $q$ is the probability that player 2 may play L in $G(1)$, then $m_2^1$ is the $q$ that minimizes

$$\max(u_1(U, q), u_1(M, q), u_1(D, q)) = \max(1 - 3q, 3q - 2, 0),$$

so that we have $m_2^1 = [\frac{1}{3}, \frac{2}{3}]$, and

$$\underline{v}_1 = u_1(D, m_2^1) = 0.$$

Similarly, to determine $\underline{v}_2$, let $p_U$ and $p_M$ be the probabilities that player 1 may play U and M respectively, and we have

$$\underline{v}_2 = \min_{p_U, p_M} [\max(2(p_U - p_M) + (1 - p_U - p_M), -2(p_U - p_M) + (1 - p_U - p_M))].$$

8. Note that in any SPNE of $G(\infty)$, player $i$'s equilibrium payoff can not be lower than $\underline{v}_i$, since player $i$ can always take a best response against the other players' actions in each and every period! Thus, define accordingly

$$V^* = \{v \in V : \ v_i > \underline{v}_i, \ \forall i = 1, 2, \cdots, I\},$$

and payoff profiles in $V^*$ will be referred to as *feasible and individually rational.*[11] [12]

[11] The minmax strategy is typically a mixed strategy. Since a player's minmax value is what she can get in $G(1)$ when she uses her best response against her rivals' joint minmax strategies, and since a player's payoff in a Nash equilibrium of $G(1)$ is what she can get in $G(1)$ when she uses her best response against her rivals' equilibrium strategies, we conclude that a player's equilibrium payoff in $G(1)$ is always greater than or equal to her minmax value.

[12] What if players are restricted to using only pure strategies? Note that if a penalized player's best response against the other players' joint minmax strategy is a mixed strategy, then there exists an equally good pure-strategy best response for the ponalized player. Hence restricting the players to use only pure strategies only restricts the set of feasible penalizing strategies, and hence it weakly raises the minmax value for each player.

9. (**One-stage Deviation Principle**) To check if a conjectured profile $\sigma$ is an SPNE of a repeated game, one would presumably need to check if, from each player' perspective, any (complex) unilateral deviations can improve upon $\sigma$. The following theorem, however, asserts that verifying the simplest form of unilateral deviations will suffice.[13]

**Theorem 1**. (One-stage Deviation Principle)

(a) A pure strategy profile $s$ is a subgame perfect NE for $G(T)$ if and only if no player $i$ can at any stage $t$ benefit from a unilateral deviation $s_i'$ that differs from his equilibrium strategy $s_i$ only in the stage-$t$ mapping $s_i^t$.

(b) If $G(\infty)$ is *continuous at infinity* in the sense that for all $i$,

$$\lim_{t\to\infty} \sup_{h,h' \text{s.t. } h^t=[h']^t} |u_i(h) - u_i(h')| = 0,$$

then a pure strategy profile $s$ is a subgame perfect NE for $G(\infty)$ if and only if no player $i$ can at any stage $t$ benefit from a unilateral deviation $s_i'$ that differs from his equilibrium strategy $s_i$ only in the stage-$t$ mapping $s_i^t$.

**Proof**. Consider assertion (a). Necessity is definitional. To check sufficiency, suppose instead that for some player $i$, for some $t$ and $h^t$, there exists $\hat{s}_i$ better than $s_i$ in the subgame starting at $h^t$. Let $\hat{t}$ be the largest $t'$ such that for some $h^{t'}$, $\hat{s}_i(h^{t'}) \neq s_i(h^{t'})$. Apparently, $\hat{t} > t$: otherwise, at $t$ and $h^t$, $\hat{s}_i$ relative to $s_i$ is simply a one-stage deviation, and we have assumed that no one-stage deviation can improve upon $s_i$!

Define $\tilde{s}_i$ as such that it coincides with $\hat{s}_i$ at all $\tau < \hat{t}$, and it coincides with $s_i$ from stage $\hat{t}$ on. At $\hat{t}$, $\hat{s}_i$ is a one-stage deviation relative to $s_i$, and since we have assumed that no one-stage deviation can improve upon $s_i$, we conclude that $\tilde{s}_i$ is a weakly better response than $\hat{s}_i$ in every subgame starting at stage $\hat{t}$. Since $\tilde{s}_i$ coincides with $\hat{s}_i$ at all $t < \hat{t}$, we conclude that $\tilde{s}_i$ is also a weakly better response than $\hat{s}_i$ starting at

[13]Theorem 1 actually applies to all multi-stage games with observable actions.

stage $t$ with history $h^t$. Now if $\hat{t} = t + 1$, then at $t$ given history $h^t$, $\tilde{s}_i$ becomes a one-stage deviation relative to $s_i$, but we have assumed that such a deviation cannot improve upon $s_i$. This contradicts the assumption that $\hat{s}_i$, which is a weakly worse response than $\tilde{s}_i$, can improve upon $s_i$ at $t$ given history $h^t$.

It remains to discuss the case where $\hat{t} > t + 1$. In this case, we can move the spot light from $\hat{s}_i$ to $\tilde{s}_i$, and find a strategy $s_i^{(1)}$ that coincides with $\tilde{s}_i$ at all $\tau < \hat{t} - 1$ and coincides with $s_i$ from stage $\hat{t} - 1$ on. The same reasoning as above shows that $s_i^{(1)}$ is a weakly better response than $\tilde{s}_i$, and hence a weakly better response than $\hat{s}_i$ at $t$ given history $h^t$. Either $\hat{t} - 1 = t + 1$, in that case we shall obtain a contradiction, or we can move the spot light from $\tilde{s}_i$ to $s_i^{(1)}$, and find a strategy $s_i^{(2)}$ that coincides with $s_i^{(1)}$ at all $\tau < \hat{t} - 2$ and coincides with $s_i$ from stage $\hat{t} - 2$ on. Repeating this argument, we can utimately show that $\hat{s}_i$ is a weakly worse response than some $s_i^{(k)}$, and yet $s_i^{(k)}$ is a one-stage deviation at $t$ given history $h^t$, which by assumption cannot improve upon $s_i$ at $t$ given history $h^t$, therey establishing a contradiction. This completes the proof for sufficiency in assertion (a).

Next, consider assertion (b). From the proof to assertion (a), if $s$ satisfies the one-stage deviation condition, then it cannot be improved by any finite sequence of deviations in any subgame. Suppose instead that for some player $i$, for some $t$ and $h^t$, there exists $\hat{s}_i$ better than $s_i$ in the subgame starting at $h^t$. Let $e > 0$ be the amount of extra payoff brought about by the improved strategy $\hat{s}_i$. Continuity at infinity implies that there exists $T$ such that the strategy $\tilde{s}_i$ that coincides with $\hat{s}_i$ before $T$ and coincides with $s_i$ from stage $T$ on must improve on $s_i$ by at least $\frac{e}{2} > 0$, which contradicts the fact that no finite sequence of deviations can make any improvement on $s_i$ at all.

10. (**Infinitely Repeated Games.**) We shall first focus on infinitely repeated games. How is Theorem 1 helpful? Note that if for all $i = 1, 2, \cdots, I$, $u_i(\cdot)$ is bounded, which will be true if $A$ is a finite set or if $u_i(\cdot)$ is continuous and $A$ is a compact subset of some finite-dimensional Euclidean space, then with the payoff function $(1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(a^t)$, $G(\infty)$ is continuous at infinity. In this case, Theorem 1 is always applicable.

11. **Theorem 2**. (Nash Folk Theorem) Every feasible and individually rational payoff profile (i.e. every element of $V^*$) can be sustained as an NE payoff profile in $G(\infty)$ as long as $\delta$ is close to 1.

    The idea is that when some deviation occurs, the deviator will keep receiving his minmax payoff, starting from the very stage after the deviation is detected. In other words, the deviator is penalized forever if he once deviated from an implicitly agreed upon profile in $V^*$. The problem with this penalizing scheme is that it is generally not subgame perfect, since minmaxing the opponent and expecting the opponent to make a best response does not constitute an SPNE in general. In other words, the threat to penalize the deviator may not be credible. This motivates Friedman's perfect Folk Theorem.[14]

12. **Theorem 3**. (Friedman's (Nash-threats) Perfect Folk Theorem) Let $u(s) = e$ be an NE payoff profile in $G(1)$. For all $v \in V$ with $v_i > e_i$ for all $i$, $v$ is a payoff profile of some subgame perfect NE in $G(\infty)$ as long as $\delta$ is close to 1.[15]

    The idea is that when some deviation occurs, the players will play the Pareto dominated NE in $G(1)$ forever from the next stage on. This penalizing scheme is referred to as the *trigger strategy*. Again, we must check if this scheme is itself an SPNE in $G(\infty)$. First observe that playing the NE in $G(1)$ in each and every stage is certainly an NE for $G(\infty)$. Next, observe the stationarity of $G(\infty)$: at any stage $t$, the subgame is again $G(\infty)$. This implies that playing the NE in $G(1)$ in each and every stage is indeed an SPNE, for it specifies NE strategy

[14] We did not claim that there exist feasible and individually rational payoff profiles that cannot be sustained as an SPNE payoff profile in $G(\infty)$ even when $\delta$ is close to 1. We only pointed out that such a payoff profile may not be sustained by the naive minmax threat. In fact, when $I = 2$, the assertion in Theorem 2 is *correct*, but we need to design another penalizing scheme, and make sure that the scheme is itself an SPNE.

[15] As we remarked earlier that $e_i \geq \underline{v}_i$ for all $i = 1, 2, \cdots, I$, Theorem 3 may give us an (incorrect) impression that requiring subgame perfection reduces the set of payoff profiles in $V$ that can be sustained as a reasonable equilibrium outcome of $G(\infty)$. The problem is, again, that Theorem 3 assumes that the players can use only a particular type of penalizing schemes, called the *trigger strategy*, and if we consider (and characterize) all the possible SPNE penalizing schemes, we can actually implement a lot of payoff profiles not stated in Theorem 3; see below.

profile to the players in each and every subgame (which is again $G(\infty)$). We thus have proven that the trigger strategy is itself an SPNE, and hence after a unilateral deviation occurs, playing the trigger strategy from the next stage on is credible![16]

13. As an application of the perfect folk theorem, consider an infinitely repeated version of the prisoners' dilemma discussed in Lecture 1.

| player 1/player 2 | C | D |
|---|---|---|
| C | 1,1 | -3,2 |
| D | 2,-3 | 0,0 |

What is the lowest common discount factor $\rho$ that sustains cooperation in each period as an SPNE?

Consider the following pure strategy: play C (meaning "Cooperate") as long as in the history no one has ever played D (meaning "Defect" or "Confess"), and play D forever if otherwise. Then both players' playing this pure strategy constitutes an SPNE as long as $\rho$ is high enough.

To see this, consider first the subgame where in the history there has been some player playing D before. In this case both will play D forever, and since (D,D) is the unique NE in the stage game, of course it remains to be an SPNE in the current infinitely repeated subgame. Next, consider the subgame where in the history nobody has played D before. Let the equilibrium continuation payoff for a player be denoted by $\pi$. Given this history, we show that no one has incentives to deviate unilaterally if $\rho$ is large enough, and we shall derive the lowest $\rho$ as required.
Expecting one's rival to play C, if a player plays C also, he gets

$$\pi = 1 + \rho\pi,$$

implying that $\pi = \frac{1}{1-\rho}$; and if the player plays D instead, he gets 2 immediately, but this implies that at the beginning of the next round

[16] The term "Folk Theorem" originates from the fact that its contents had long been recognized by people before a formal proof was created.

the history will be such that some player has played D before, implying that he will get nothing from the next round on. Thus for both players to play C, it is necessary and sufficient that

$$2+\frac{\rho\cdot 0}{1-\rho}\leq \pi=\frac{1}{1-\rho},$$

or equivalently $\rho\geq\frac{1}{2}$.[17]

14. A manufacturer and a retailer are playing the following infinitely repeated supergame of $G(1)$, where the stage game $G(1)$ is described as follows. At each date $t$, the manufacturer can first decide whether to spend $c_m>0$ and if only if it does, there will be consumers of population one that visit the retailer. The manufacturer also determines a wholesale price $w(t)$ for its product, which for simplicity can be produced without costs. If there are consumers visiting the retailer, they will buy certain goods from the retailer and the retailer will get $x>0$, and moreover, if the retailer is willing to spend $c_r>0$ on in-store promotion for the manufacturer's product, then with probability $\pi\in(0,1)$ these consumers will also have a valuation $v>0$ for the manufacturer's product. After the retailer decides whether to spend $c_r$, a retail price $p(t)$ must be chosen for the manufacturer's product. Note that without the retailer's promotion effort, no consumers will buy the manufacturer's product. (Thus essentially we have a *moral hazard in team* problem.) Assume that whether the retailer has spent $c_r$ cannot be detected by the manufacturer.[18] At date $t+1$, however, the manufacturer can observe its own sales at date $t$, and it can try to infer from the sales data whether the retailer has promoted its product at date $t$. On the other hand, if the retailer has spent $c_r$ at date $t$, then the retailer would observe whether consumers choose to purchase the manufacturer's product. Assume that $x+\pi v>c_m+c_r$, but $c_r\geq\pi v$.

(i) Show that in $G(1)$ neither the manufacturer nor the retailer promotes.

[17] The discount rate $r$ and its corresponding discount factor $\rho$ satisfy the familiar relationship: $\rho=\frac{1}{(1+r)}$. With continuous compounding, we have $\lim_{n\to\infty}(1+\frac{r}{n})^{-n}=e^{-r}$.

[18] Thus this multi-stage game involves unobservable actions!

(ii) Consider $G(\infty)$. Assume that $\rho \in (0,1)$ is the common discount factor for the two firms. Show that with $T$ large enough, the following strategies constitute an SPNE for $G(\infty)$: Both firms spend on promotions, and if at date $t+1$ the manufacturer knows that its profit is positive at date $t$, then both firms promote at date $t+1$; or else, both firms play the stage game NE for a number of $T$ periods, and then they return to cooperation at date $T+1$. Find the smallest $T^*$ in this class of SPNE's. Find conditions on $c_r, c_m, \pi, \rho$, and $x$ so that such an SPNE with the least penalty can be sustained.

**Solution**. Consider part (i). Note that $c_r \geq \pi v$ and hence given any $w(t)$ promoting the manufacturer's product is a dominated strategy for the retailer in the stage game $G(1)$. Thus the retailer does not spend $c_r$ in any NE of $G(1)$. It follows that the manufacturer does not promote either. The unique NE for the stage game involves no channel promotion at all.

Next consider part (ii). We shall look for SPNE's in Markov strategy where both firms promote at date $t$ if $v_{t-1} = v$, where $v_{t-1}$ is consumers' valuation for the manufacturer's product at date $t-1$. A Markov strategy makes the actions of punishment contingent only on the current state variable. If $v_{t-1} = 0$, then the Markov strategies trigger a $T$-period punishment. In the current context, an action of punishment consists of the firms' stopping promoting the product. We restrict attention to pure strategies. Let $V_i^+$ be the sum of discounted future profits of firm $i \in \{r, m\}$ at the beginning of a period $t$ where $v_{t-1} = v$, and let $V_i^-$ be the corresponding sum of discounted future profits at the beginning of period $t$ where $v_{t-1} = 0$ and either $t = 2$ or $v_{t-2} = v$. Observe that along the equilibrium path, $w_t = p_t = v$. By definition, we thus have

$$V_r^+ = -c_r + x + \pi\rho V_r^+ + (1-\pi)\rho V_r^-,$$

$$V_r^- = \rho^T V_r^+,$$

$$V_m^+ = -c_m + \pi(v + \rho V_m^+) + (1-\pi)\rho V_m^-,$$

$$V_m^- = \rho^T V_m^+.$$

Solving, we have

$$V_m^+ = \frac{\pi v - c_m}{1-(1-\pi)\rho^{T+1} - \pi\rho}, \ V_m^- = \frac{\rho^T(\pi v - c_m)}{1-(1-\pi)\rho^{T+1} - \pi\rho},$$

$$V_r^+ = \frac{x - c_r}{1-(1-\pi)\rho^{T+1} - \pi\rho}, \ V_r^- = \frac{\rho^T(x - c_r)}{1-(1-\pi)\rho^{T+1} - \pi\rho}.$$

The retailer does not want to deviate if and only if

$$c_r \leq \pi\rho(V_r^+ - V_r^-),$$

or equivalently,

$$(\pi x - c_r)\rho^{T+1} \leq \pi\rho x - c_r.$$

This IC is always violated unless $c_r < \pi\rho x < \pi x$, which we assume hereafter. In this case, the IC implies that

$$(\Gamma) \ \ T \geq \frac{\log(\frac{\pi\rho x - c_r}{\pi x - c_r})}{\log(\rho)} - 1.$$

Next, the manufacturer does not want to deviate if and only if

$$c_m - \pi v \leq \pi\rho(V_m^+ - V_m^-).$$

This inequality holds true always as long as $V_m^+ > 0$, which is true if and only if

$$\pi v > c_m.$$

Since $V_m^+$ and $V_r^+$ are both decreasing in $T$, the optimal SPNE in the class we described has $T^*$, which is the smallest integer exceeding

$$\frac{\log(\frac{\pi\rho x - c_r}{\pi x - c_r})}{\log(\rho)} - 1.$$

The conditions required to sustain this SPNE are hence

$$c_r < \pi\rho x < \pi x, \ \ \pi v > c_m.$$

15. Two identical firms facing a random, unobservable market demand try to sustain a tacit collusion in an infinitely repeated game of price competition. In each period, demand is either 0 with probability $\alpha$ or $D(p)$ with prob. $1-\alpha$. When $D(p)$ is there, if two firms collude, they set prices at the monopoly level, i.e. $p_1 = p_2 = p_m$, and each gets a profit $\frac{\Pi_m}{2}$, where $\Pi_m > 0$ is the monopoly profit. If one firm deviates by reducing price to $p_m - \epsilon$, then it gets $\Pi_m$ itself leaving nothing to its opponent. Firms cannot observe opponents' (current or past) actions. We now derive conditions under which the following trigger strategy sustains firms' collusive pricing behavior: If one firm finds out that it had no sales volume in the last period, then they begin to play Bertrand game for T periods, but after that, they go back to the collusive pricing again. (Wait! How can the other firm know that this firm's sales volume in the last period was zero and so it is prepared to play the Bertrand outcome in the current period?) Assume that the common discount factor is $\delta \in (0,1)$. Define $V^+$ and $V^-$ as the continuation payoffs of a firm at the beginning of, respectively, a period where both firms had sales volumes last period, and a period where at least one firm had no sales volume in the last period but its sales volume in the period before the last period was still positive.[19]

(i) Show that

$$V^+ = (1-\alpha)(\frac{\Pi_m}{2} + \delta V^+) + \alpha\delta V^-,$$

$$V^- = \delta^T V^+.$$

(ii) Show that the collusion can be sustained iff

$$V^+ \geq (1-\alpha)(\Pi_m + \delta V^-) + \alpha\delta V^-.$$

(iii) Suppose that $\alpha = \frac{1}{4}$ and $\delta = \frac{7}{12}$. Find the smallest positive integer $T$ such that the collusion can be sustained by the above described trigger-strategy penalizing scheme.

[19] Recall that the continuation payoff starting at date $t$ is the (discounted) sum of the temporal payoff at date $s$, for all $s = t, t+1, \cdots$. That is, the continuation payoff disregards the payoffs generated at date $1, 2, \cdots, t-1$, because the latter payoffs are sunk at date $t$ and are irrelevant regarding the subgame starting at date $t$.

16. Consider example 1 in Lecture 1, part I, where firms 1 and 2 can costlessly produce a product and engage in Cournot competition with the inverse demand being, in the relevant range,

$$P(q_1 + q_2) = 1 - q_1 - q_2.$$

This problem is a modification of the above Cournot game.

(i) Assume that there are two dates. The two firms will compete at date 1, but at date 0, both firms can correctly expect the date-1 inverse demand function, which is the $P(\cdot)$ defined above. At date 0, the futures market opens for the product produced by the two firms. There are price-competitive investors in the futures market, who, just like the two firms, are risk neutral without time preferences (that is, there will be no discounting for anyone). The extensive game is as follows.

- At date 0, (only) firm 1 can sign a futures contract with the competitive investors. In the futures contract, firm 1 promises to deliver $f_1$ units of the product at date 1 to one of the investors (say, Mr. A), and Mr. A promises to pay the price $F$ (referred to as the date-0 futures price of the product). We assume that firm 1 announces $f_1$, and the competitive investors then determine the futures price $F$. Assume that investors have rational expectations; that is, upon seeing $f_1$, they can use backward induction to anticipate the date-1 price of the product (called the date-1 spot price of the product), and to rule out arbitrage opportunities, in the date-0 equilibrium, $F$ must equal the anticipated date-1 price.
- At date 1, upon seeing firm 1's date-0 futures contract $(f_1, F)$, the two firms choose $q_1$ and $q_2$ simultaneously. Note that firm 1's profit as a function of $q_1, q_2$ is

  $$\Pi_1(q_1, q_2; f_1) = [1 - q_1 - q_2][q_1 - f_1] + Ff_1.$$

  Firm 2's profit function is still

  $$\Pi_2(q_1, q_2) = [1 - q_1 - q_2]q_2.$$

- Then, after firms set $q_1$ and $q_2$, the date-1 price $P(q_1, q_2)$ is realized, and firm 1 must deliver $f_1$ units of the product to Mr. A, and Mr. A must pay firm 1 $Ff_1$ dollars.

Find the SPNE of this extensive game. Explain why firm 1 may benefit from futures trading.[20]

(ii) Now, suppose that both firms can engage in futures trading at date 0, with $f_1$ and $f_2$ units sold respectively at the futures price $F$ determined at date 0. Again, assume that all investors in the futures market have rational expectations when they compete in price to determine $F$. Re-derive the SPNE. Explain why the two firms might be hurt by the availability of futures trading.[21]

(iii) Now, call the extensive game described in part (ii) $G(1)$. Consider an infinitely repeated version $G(\infty)$ of $G(1)$, where both firm seek to maximize the sum of discounted profits with common discount factor $\rho \in (0,1)$.[22] We shall consider both the case where the futures market is always open and the case where the futures market is always closed.

[20]**Hint**: Use backward induction. First consider the date-1 subgame with $f_1$ given. This is just a Cournot game with the two firms' profit functions being $\Pi_1$ and $\Pi_2$ specified above. Let the subgame equilibrium be $(q_1^*(f_1), q_2^*(f_1))$, which depends on $f_1$. Now move backwards to consider firm 1's date-0 choice of $f_1$. Remember that the investors in the futures market can rationally expect the date-1 spot price of the product, which is $P((q_1^*(f_1), q_2^*(f_1))$, and given $f_1$, they will compete in price so that in the date-0 futures market equilibrium, $F = P((q_1^*(f_1), q_2^*(f_1))$. Given that $F = P((q_1^*(f_1), q_2^*(f_1))$, find firm 1's optimal $f_1$.

[21]**Hint**: Again, consider the date-1 subgame with $f_1, f_2$ given. Now for $i = 1, 2,$, firm $i$'s profit function becomes

$$\Pi_i(q_i, q_j; f_i) = [1 - q_i - q_j][q_i - f_i] + Ff_i.$$

Find the Nash equilibrium $(q_1^*(f_1, f_2), q_2^*(f_1, f_2))$for this subgame. Now return to the date-0 futures market, where the two firms must simultaneously choose $f_1$ and $f_2$. For each pair $(f_1, f_2)$ announced, the investors can correctly expect the date-1 spot price, which must be $P((q_1^*(f_1, f_2), q_2^*(f_1, f_2))$. Knowing that the futures price will be such that $F = P((q_1^*(f_1, f_2), q_2^*(f_1, f_2))$, the two firms' choices $(f_1, f_2)$ must form a Nash equilibrium at date 0.

[22]More precisely, each date contains two subperiods, where in period 1 firms can trade

Show that if the futures market is always open, then as long as $\rho$ is large enough (i.e., it is sufficiently close to 1), there exists an SPNE supported by the trigger strategy, in which both firms produce $\frac{11}{30}$ at each and every date. Can this SPNE be supported by the trigger strategy if the futures market is always closed?

**Solution**. Consider part (i). It is straightforward to show that the two firms' date-1 reaction functions are

$$r_1^1(q_2; f_1) = \frac{1 + f_1 - q_2}{2}, \quad r_2^1(q_1) = \frac{1 - q_2}{2}.$$

Hence we have the subgame equilibrium

$$q_1^*(f_1) = \frac{1}{3} + \frac{2}{3}f_1, \quad q_2^*(f_1) = \frac{1}{3} - \frac{1}{3}f_1.$$

Now consider firm 1's date-0 choice of $f_1$. Since $F = P((q_1^*(f_1), q_2^*(f_1)),$ at date 0 firm 1 seeks to

$$\max_{f_1} P((q_1^*(f_1), q_2^*(f_1))q_1^*(f_1) = \frac{1}{3}(1 - f_1)(\frac{1}{3} + \frac{2}{3}f_1),$$

for which the necessary and sufficient first-order condition gives

$$f_1 = \frac{1}{4},$$

implying that, in equilibrium,

$$F^* = P^* = \frac{1}{4}, \; q_1^* = \frac{1}{2}, \; q_2^* = \frac{1}{4}, \; \Pi_1^* = \frac{1}{8}, \; \Pi_2^* = \frac{1}{16}.$$

Compared to the Cournot equilibrium profit $\frac{1}{9}$, firm 1 is better off with futures trading. The reason is that after commtting to sell $f_1$ units at a fixed price $F$, which will not fall when firm 1 expands output at date

futures, and then in period 2, firms select total outputs to engage in the Cournot competition. There is no discounting over the 2 periods in one date. There is discounting (with discount factor $\rho$) over any two consecutive dates.

1, firm 1 has an incentive to choose a higher total output at date 1. This fact results in firm 2 lowering output accordingly (because output choices are strategic substitutes). Consequently, firm 1 benefits from futures trading, which hurts firm 2 at the same time.

Next consider part (ii). Given $(f_1, f_2)$, now the subgame equilibrium becomes

$$q_1^*(f_1, f_2) = \frac{1}{3}+\frac{2}{3}f_1-\frac{1}{3}f_2,\ \ q_2^*(f_1, f_2) = \frac{1}{3}+\frac{2}{3}f_2-\frac{1}{3}f_1,\ \ P^*(f_1, f_2) = \frac{1}{3}(1-f_1-f_2).$$

Now consider the date-0 futures market equilibrium. Firm $i$'s problem is to, given the conjectured $f_j$,

$$\max_{f_i} P(q_i^*(f_i, f_j), q_j^*(f_i, f_j))q_i^*(f_i, f_j) = \frac{1}{3}(1 - f_i - f_j)(\frac{1}{3} + \frac{2}{3}f_i - \frac{1}{3}f_j).$$

The necessary and sufficient first-order condition gives firm $i$'s date-0 reaction function

$$r_i^0(f_j) = \frac{1 - f_j}{4},\ \ i, j = 1, 2,\ \ i \neq j.$$

Thus the date-0 equilibrium is

$$f_1^* = f_2^* = \frac{1}{5},$$

implying that

$$q_1^* = q_2^* = \frac{2}{5},\ \ F^* = P^* = \frac{1}{5},\ \ \ \Pi_1^* = \Pi_2^* = \frac{2}{25}.$$

Compared to the Cournot equilibrium profit, each firm is worse off. The reason is that, as in the game of prisoners' dilemma, here each firm intends to hold a short position in the futures contract as an attempt to force its rival to produce less. With the short positions in the futures contract, both firms are faced with a residual inverse demand with lower elasticity to their output expansion. Consequently, both firms choose to produce more in the subgame where futures contracts have been signed, leading to a lower spot and futures price for the product, and lower profit for each firm.

Finally, consider part (iii). Suppose that the futures market stays open forever. If the two firms choose $q_1 = q_2 = \frac{11}{30}$, their profits are $\Pi_1 = \Pi_2 = \frac{22}{225}$ in each period, so that they are better off than in the single-period equilibrium, where each earns $\frac{2}{25}$ by (ii). However, it is easy to verify that $q_1 = q_2 = \frac{11}{30}$ does not form a Nash equilibrium in the stage game. We claim that if $\rho$ is sufficiently large, these outputs can indeed be sustained as an SPNE in $G(\infty)$ using a trigger strategy. Note that if firm 1 wishes to deviate, its optimal output choice would be $\frac{1-q_2}{2} = \frac{19}{60}$. Thus by making the optimal deviation firm 1's current profit would rise immediately to

$$\Pi_1(q_1 = \frac{19}{60}, q_2 = \frac{11}{30}) = \frac{19}{60}\left(1 - \frac{19}{60} - \frac{11}{30}\right) = \frac{361}{3600} > \frac{22}{225}.$$

Following the current period, firm 1 would earn $\frac{2}{25}$ in each of the remaining periods; recall part (ii). Thus making the deviation is not worthwhile if and only if

$$\begin{aligned}\frac{22}{225} + \frac{22}{225}\rho + \frac{22}{225}\rho^2 + \cdots &\geq \frac{361}{3600} + \frac{2}{25}\rho + \frac{2}{25}\rho^2 + \cdots \\ \Leftrightarrow \frac{22}{225} \times \frac{1}{1-\rho} &\geq \frac{361}{3600} + \frac{2}{25} \times \frac{\rho}{1-\rho} \\ \Leftrightarrow \rho &\geq \frac{9}{73}.\end{aligned}$$

Thus as long as $\rho \geq \frac{9}{73}$ there does exist an SPNE supported by the trigger strategy, where both firms produce $\frac{11}{30}$ in every period. Now, note the role of futures trading in leading to this conclusion. If the futures market stays closed forever, there does not exist such an SPNE sustained by the trigger strategy; simply note that both firms earn $\frac{1}{9}$ in the single-period equilibrium, where $\frac{1}{9} > \frac{22}{225}$.

17. Consider the following repeated game $G(\infty)$ with a seller and a buyer engaged in repeat purchase decisions.[23] At each date $t \in \mathbf{N}$, the seller

[23] This model has a name; it is called a *relational model* in contract theory.

first chooses to offer either a high-quality product or a low-quality product (together with a price $p$), which the buyer must decide to or not to accept. The buyer cannot tell the quality of the product on the selling spot (hence the product is an *experience good*). The buyer needs exactly one unit of the product at each date, and he attaches value $v > c$ to the high-quality good, where $c > 0$ is the seller's cost of producing one unit of the high-quality good. The low-quality good is considered worthless to the buyer and it incurs no production cost to the seller.

(i) First consider the stage game $G(1)$. Show that in the uniqe SPNE the seller produces only the low-quality product and prices it at $p = 0$.

(ii) Next consider $G(\infty)$ and let $\delta_s$ and $\delta_b$ be the discount factors of respectively the seller and the buyer. Show that for $\delta_s$ close to one, there exists an SPNE sustained by the trigger strategy, where at every date, the seller offers the high-quality good and prices it at $p = v$.

**Solution**. Consider part (i). Suppose that there exists an equilibrium where the seller produces the high-quality product and prices it at $p$, with $c \leq p \leq v$. In the equilibrium, the buyer believes that the seller will provide the high-quality product. Given the buyer's belief, if the seller produces high-quality product, she earns $p - c$; if the seller deviates to produce low-quality product, she earns $p$ (recall that the buyer can not tell the quality of the product on the selling spot). Thus, the seller has an incentive to deviate and produce a low-quality product. This proves that there does not exist an equilibrium where the seller produces the high-quality product. By similar reasoning, one can show that there does not exist an equilibrium where the seller randomizes between producing a high-quality product and a low-quality product either.

Now, does there exist an equilibrium where the seller produces a low-quality product? If such an equilibrium exists, the equilibrium product price is of course 0. In this equilibrium, the buyer believes that the seller will provide the low-quality product. Given the buyer's belief, even if the seller provides a high-quality product, the buyer would not know that and would still want to pay zero for the product. Thus, the seller has no incentive to deviate. Therefore, the uniqe SPNE is that the seller produces only the low-quality product and prices it at $p = 0$.

Consider part (ii). We shall show that for $\delta_s$ close to one, the following strategies constitute an SPNE for $G(\infty)$: The seller offers the high-quality good with price $p = v$ at every date, and at the beginning of date t, the buyer purchases the good as long as the seller has never offered the low-quality product before date t. Otherwise, the buyer purchases the good only if $p = 0$. Given the buyer's strategy, the seller would like to offer the high-quality product at every date if and only if

$$\frac{v - c}{1 - \delta_s} \geq (v - 0) + 0,$$

or equivalently, $\delta_s \geq \frac{c}{v}$. Therefore, for $\delta_s$ close to one, there exists an SPNE where the seller offers the high-quality good at every date, and prices it at $p = v$. This game can be modified by assuming an infinite sequence of short-term buyers with a perfect word-of-mouth effect, and the conclusion will be identical. The date-$n$ buyer would be ready to pay $v$ for the date-$n$ product offered by the seller if all preceding buyers were served with the high-quality product; or else, all buyers from date $n$ on would pay zero.

18. Consider a seller and a buyer interacting with each other at dates $n = 0, 1, \cdots$. At each date $n$, the seller can first decide to or not to spend $F > 0$ on advertising its price $p_n$. If the seller spends $F$ on date $n$, then the buyer knows $p_n$, before he chooses to or not to spend $t > 0$ to visit the seller's store. The buyer has unit demand for the seller's product, and his valuation is $v > 0$ at each date $n$. The common discount factor is $\rho \in (0, 1)$ for the seller and the buyer. Note that if the seller chooses not to spend $F$ at date $n$, then the buyer must choose to or not to spend $t$ before learning $p_n$. We assume that at the beginning of each date $n$, with probability $1 - q \in (0, 1)$ another seller with a much better product may appear, and when she does, the buyer will not deal with the existing seller any more; that is, the old seller would face with no demand from date $n$ on, if that new seller emerges.

(We shall let $q$ represent the market status of the old seller's brand; a lower $q$ indicates that the old seller has a weak product, which may be easily replaced by a newly introduced product.)

(i) Derive a condition on $t, q, \rho, v$, and $F$ which ensures the existence of an SPNE sustained by the trigger strategy, where in equilibrium the seller never spends $F$ at any date, but the buyer keeps buying from the seller at each date till the superior new selller shows up. Find one such SPNE which is most favorable to the seller.

(ii) Derive a condition on $t, q, \rho, v$, and $F$ which ensures the existence of an SPNE in which the seller spends $F$ at each date till the superior new seller appears. Find one such SPNE which is most favorable to the seller.

**Solution**. For part (i), if such an SPNE exists, then in equilibrium the seller must price at $p_n \leq v - t$ at each date $n$, and in the SPNE most favorable to the seller (we focus on this one), the seller will price exactly at $v - t$ at each date $n$. The buyer's equilibrium strategy is to visit the seller at date $n$ if and only if the seller has spent $F$ at date $n$ or if for all $m \leq n - 1$, $p_m \leq v - t$.

The seller's equilibrium behavior is to price at $v - t$ at each and every date, so that the seller's equilibrium continuation payoff at date $n$ is

$$(v-t) + \rho q(v-t) + \rho^2 q^2 (v-t) + \cdots = \frac{(v-t)}{1-\rho q}.$$

By making a unilateral deviation and pricing at $v$ at date $n$, the seller's continuation payoff would become

$$v + \frac{\rho q \max(0, v-t-F)}{1-\rho q}.$$

In equilibrium, the seller should have no incentives to make unilateral deviations. Thus we require that, if $F \geq v - t$,

$$t \leq \frac{\rho q(v-t)}{1-\rho q} \Leftrightarrow t \leq \rho q v.$$

If instead $F < v - t$, then we require that

$$t \leq \frac{\rho q F}{1-\rho q}.$$

Written compactly, the condition that we are looking for in part (i) is

$$t \leq \frac{\rho q \min(F, v-t)}{1-\rho q}.$$

This finishes part (i).

Now, the SPNE described in part (ii) can arise only if $F + t < v$ and only if the *buyer would stop visiting the seller had the seller not spent $F$ at least once in the past*. The latter is impossible, for the seller can always spend $F$ and price slightly below $v - t$ in any period following a deviation. Thus no SPNE described in part (ii) can arise.

19. As we remarked in a preceding footnote, one might get the impression from the two preceding folk theorems that requiring subgame perfection reduces the set of payoff profiles that can be sustained in the SPNE's of $G(\infty)$. Fudenberg and Maskin (1986) show that this need not be the case.
**Theorem 4**. (Fudenberg and Maskin, 1986) Every $v \in V^*$ can be sustained in some SPNE of $G(\infty)$ if $I = 2$ or if $I \geq 3$ but $V$ is convex with dimension $I$.[24]

To prove the assertion for two-player games (i.e. $I = 2$), Fudenberg and Maskin employ Abreu's (1988) result (see Fudenberg and Tirole's *Game Theory*, Theorem 5.6) that SPNE penalizing schemes can without loss of generality be confined to simple punishments: a punishment

[24]The dimension of a convex set in $\mathcal{R}^N$ is the dimension of the smallest affine set containing the convex set.

is *simple* if it depends only on the identity of the latest unilateral deviator. Assuming that $\underline{v} = 0$,[25] Fudenberg and Maskin show that given $v \in V^* = V \bigcap \Re^I_+$, the following strategies constitute an SPNE: play the profile $s$ with $u(s) = v$ until unilateral deviations occur, and in the latter case minmax the deviator for a long enough period of time ("enough" means to wipe out the deviator's gain from deviation) and then return to the profile $s$. If a further deviation occurs, start a new punishment phase against the new deviator (so that all old deviators are exonerated at this point).[26]

20. To demonstrate Fudenberg and Maskin's idea for the case $I = 2$, consider the infinitely repeated version $G(\infty)$ of the Cournot game in sec-

[25]This is a harmless assumption, given that von Neumann-Morgenstern utility function is determined only up to a positive affine transformation.

[26]Note that Theorem 4 only shows that minmax strategies can be used in the penalty phase so that a cooperative SPNE can be sustained when $\delta$ is sufficiently high. It does not assert that this class of penalizing strategies will make that SPNE most likely to be sustained. For example, consider the repeated Cournot duopoly where the two firms face the temporal inverse demand $P = 1 - q_1 - q_2$ at each period $t$, and consider how the collusive outcome where each produces $\frac{1}{4}$ units in each period can be sustained as an SPNE in $G(\infty)$. If the trigger strategy is used to penalize the deviator in the penalty phase, then this collusive outcome is sustained as an SPNE in $G(\infty)$ as long as $\delta > \frac{9}{17}$. (To see this, recall that each firm's temporal profit is $\frac{1}{9}$ and the temporal collusive profit is $\frac{1}{8}$. Deviation unilaterally by producing the output $\frac{3}{8}$ implies an immediate gain of $\frac{1}{64}$.) Now, if instead the minmax strategy is used to penalize the deviator(s), then to wipe out the immediate gain from unilateral deviation from the collusive outcome it requires that the other firm minmax the deviator for just one period, as long as $\delta > \frac{1}{8}$. But this also requires the penalizing firm to produce 1 unit in that penalizing period, and the penalizing firm may wish to deviate and produce $\frac{1}{2}$ units and get an immediate gain of $\frac{1}{4}$ instead, which is much larger than $\frac{1}{64}$! To wipe out the gain from this penalizing firm, when it does deviate, the original deviator (who is now exonerated!) must minmax it for, say, 4 periods if $\delta = \frac{3}{4}$ for example:

$$\frac{1}{8}[\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256}] > \frac{1}{4}.$$

This explains why $\delta$ has to be much higher in order to make the minmax penalizing strategy work. In fact, for the minmax strategy to successfully sustain the collusive outcome in each and every period, we require $\delta + \delta^2 + \cdots > \frac{\frac{1}{4}}{\frac{1}{8}} = 2$, or $\delta > \frac{2}{3}$. That is, for $\delta$ lying between $\frac{9}{17}$ and $\frac{2}{3}$, the trigger penalizing strategy can be used to sustain the collusive outcome as an SPNE in $G(\infty)$, but the minmax strategy fails to do so.

tion 11 of Lecture 1, part I, where firms seek to maximize the sum of discounted profits with $\delta \in (0,1)$ being the two firms' common discount factor. First verify that for firm $i$, $m^i_{-i} = 1$, and $m^i_i = \underline{v}_i = 0$. Is there an SPNE where each firm earns a profit $\frac{5}{72}$ in each and every period for $\delta$ large enough? (**Hint**: Consider $Q_0 = \{(q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), (q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), (q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), \cdots\}$, supported by $Q_{-i}$ and $Q_i$, where $Q_i = \{(q_{(t+1)i} = 0, q_{(t+1)j} = 1), (q_{(t+2)i} = 0, q_{(t+2)j} = 1), (q_{((t+3)i} = 0, q_{(t+3)j} = 1), \cdots, (q_{(t+T)i} = 0, q_{(t+T)j} = 1), (q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), (q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), \cdots\}$, and $Q_{-i} = \{(q_{(t+1)i} = 1, q_{(t+1)j} = 0), (q_{(t+2)i} = 1, q_{(t+2)j} = 0), (q_{((t+3)i} = 1, q_{(t+3)j} = 0), \cdots, (q_{(t+T')i} = 1, q_{(t+T')j} = 0), (q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), (q_1 = \frac{5}{12}, q_2 = \frac{5}{12}), \cdots\}$. The following is how these strategies work: The two firms play $Q_0$ until at some date $s$ firm $i$ makes a unilateral deviation, and from the next date on, firm $i$ is minmaxed for $T$ periods, and if further deviations occur, all old deviators are exonerated, and the new deviator will be minmaxed for $T'$ periods. After minmax stage is over, the two firms start playing $Q_0$ until further deviations take place.)

**Solution**. If firm $j$ wants to minmax firm $i$, then firm $j$ should choose $q_j$ to minimize $r_i(q_j) = \frac{1-q_j}{2}$, and hence firm $j$ should choose $q_j = 1$, which implies that, when minmaxed, firm $i$ does not produce anything, and it gets zero profit. Now, by symmetry, the same conclusion applies to firm $j$. It follows that the output pair $(q_1 = \frac{5}{12}, q_2 = \frac{5}{12})$ is both *feasible* and *individually rational* (meaning that the pair generates for each firm a profit higher than a firm's minmax profit level, which is zero). Theorem 4 tells us that, yes, this output pair can be sustained as an SPNE outcome in each period of the above infinitely repeated game. Call the path where both firms produce $\frac{5}{12}$ at each date the equilibrium path $Q_0$. Can we sustain $Q_0$ by the trigger strategy? No, because the trigger strategy SPNE is one where both firms produce $\frac{1}{3}$, yielding for each firm a profit higher than the profit in equilibrium $Q_0$ at each date.

So, how can we sustain this SPNE $Q_0$? Theorem 4 tells us that, in the current case, we can distinguish between two types of deviators, the one who should be minmaxed, and the one who should minmax its rival. For this reason, we must require different lengths in the penalty stage, depending on which type of deviator the punishment is intended to be imposed on.

Imagine that firm $i$ deviates from $Q_0$ at date $n$ by setting $q_i = \frac{1-\frac{5}{12}}{2} = \frac{7}{24}$, leading to a one-time gain

$$\frac{7}{24}[1 - \frac{5}{12} - \frac{7}{24}] - \frac{5}{12}[1 - \frac{5}{12} - \frac{5}{12}] = \frac{9}{576}.$$

From date $n+1$ on, firm $j$ must minmax firm $i$ for $T$ periods, such that

$$T = \min\{\tau : \frac{5}{12}[1 - \frac{5}{12} - \frac{5}{12}](\delta + \delta^2 + \cdots + \delta^\tau) \geq \frac{9}{576}\}.$$

After minmaxing firm $i$ for $T$ periods, firm $j$, together with firm $i$, start playing $Q_0$ again. This path, starting with date $n+1$, is referred to as $Q_i$.

If firm $j$ deviates from $Q_i$ at any date $m \in \{n+1, n+2, \cdots, n+T\}$ by setting $q_j = \frac{1}{2}$ at date $m$, then the one-time gain that firm $j$ obtains from the deviation at date $m$ is

$$\frac{1}{2}[1 - 0 - \frac{1}{2}] - 1[1 - 0 - 1] = \frac{1}{4}.$$

Starting from date $m+1$ on, firm $i$ has to minmax firm $j$ for $T'$ periods, where

$$T' = \min\{m - n + \tau : \frac{5}{12}[1 - \frac{5}{12} - \frac{5}{12}]\delta^{n-m}(\delta + \delta^2 + \cdots + \delta^\tau) \geq \frac{1}{4}\}.$$

After minmaxing firm $j$ for $T'$ periods, firm $i$, together with firm $j$, start playing $Q_0$ from the next date on. This path, starting with date $m+1$, is referred to as $Q_{-i}$.

Once gain, if firm $i$ deviates from $Q_{-i}$ at some date $m'$, then from date $m'+1$ on, the two firms play $Q_{-j}$, and so on and so forth. It is easy to verify that the above strategies do form an SPNE, where the equilibrium path $Q_0$ is supported by $Q_1, Q_2, Q_{-1}$, and $Q_{-2}$.

21. How about Fudenberg and Maskin's proof for the case $I \geq 3$? It turns out the above penalizing schemes used in the proof for the case $I = 2$ cannot work in general, since threatening to minmax all $I \geq 3$ players in different punishment phases is rather difficult to be compatible with an SPNE. The following is an example.

    **Example 1**. There are three players. Each player can choose between $a$ and $b$ in each stage. Payoffs are zero for everyone unless their choices are the same, and in the latter case each gets 1. Show that, although the common minmax value for the three players is zero (where for all distinct $i, j, k \in \{1, 2, 3\}$, $m_{-i}^i$ consists of player $j$ playing $a$ and player $k$ playing $b$), for $\delta < 1$ no SPNE of $G(\infty)$ gives an average payoff that lies in the interval $[0, \frac{1}{4}]$.[27]

    **Proof**. Define $\alpha$ as the infimum of a player's average payoff which she obtains in any SPNE of $G(\infty)$. (By symmetry, $\alpha$ does not depend on the identity of the player.) We must show $\alpha \geq \frac{1}{4}$. At any stage, the equilibrium strategy profile must be such that there exists player $i$ such that the other two players both assign to $a$ or to $b$ a probability that is less than or equal to $\frac{1}{2}$. Either way, player $i$ can ensure herself an immediate payoff of $\frac{1}{4}$ by betting on the choice between $a$ and $b$ which her rivals both assign a probability of at least $\frac{1}{2}$. This is one feasible deviation from the supposed equilibrium profile, and it generates a payoff for player $i$ of at least

$$\frac{1}{4} + \frac{\delta\alpha}{1-\delta},$$

    which, by the assumption that initially we are in an SPNE, should be less than any perfect equilibrium payoff for player $i$. Let $\{\alpha_n\}$ be a sequence of SPNE payoffs for player $i$ converging to $\alpha$. (Recall that the infimum of a set $A \subset \mathcal{R}$ must be a limit point of $A$.) Then, for each $n$, the $n$-th SPNE must be robust against the feasible deviation we just designed for player $i$, which implies that

[27] Note that in minmaxing some player $i$, each of the two penalizers gets zero payoff in each penalizing stage, but one of the penalizers can get an immediate payoff of at least $\frac{1}{2}$ by deviation.

$$\frac{1}{4} + \frac{\delta\alpha}{1-\delta} \leq \frac{\alpha_n}{1-\delta}, \ \forall n.$$

Passing $n$ to $\infty$, we have

$$\frac{1}{4} + \frac{\delta\alpha}{1-\delta} \leq \frac{\alpha}{1-\delta},$$

implying that $\alpha \geq \frac{1}{4}$. This example shows that, unlike in the two-player case, with 3 or more than 3 players, not all feasible and enforceable payoff vectors in $G(1)$ are sustainable as the average payoff of some SPNE in $G(\infty)$.[28]

22. Because of Example 1, Fudenberg and Maskin turn to reward schemes in proving their assertion for the case $I \geq 3$. The idea is that after some unilateral deviation by deviator $i$ from $v$, all other players should minmax the deviator for a long enough period of time and then all players except the punished deviator get to play some profile which gives them payoffs which are higher than their individual minmax values (but still lower than the payoffs they would have received in the original SPNE if player $i$ had not deviated). Again, if a further deviation occurs, then the players follow the same procedure to implement a new punishment phase against the new deviator, and all the previous deviators are exonerated. This penalizing scheme works because a player who is supposed to join force with other players to minmax an earlier deviator would lose the chance to obtain a higher payoff after the minmaxing

[28]For example, imagine that we want to implement a payoff profile where all 3 players get $\frac{1}{8}$. Assume that mixed strategies are detectable for the moment, and think about the strategy profile to be implemented consists of players 1 and 2 playing $a$ and player 3 choosing $a$ with probability $\frac{1}{8}$. Now if player 3 has deviated and chosen $a$ for sure, and in coordinating their actions to minmax player 3 player 1 should play $a$ and player 2 should play $b$, in anticipation of player 3's response of, say, choosing $a$ or $b$ with probability $\frac{1}{2}$, then player 1 can deviate and choose $b$ for sure instead, which, as we can verify, also imply that the other two players can get the immediate payoff $\frac{1}{2}$. (This payoff becomes 1 if player 3's $m_3^3$ is a pure strategy!) Thus in anticipation of the high payoff during the period a player is to be minmaxed, that player will not tolerate an equilibrium payoff of $\frac{1}{8}$, even though $\frac{1}{8}$ is higher than that player's minmax value in $G(1)$.

phase if she chose to deviate and maximize her one-period payoff. For such a scheme to work, it must be required that offering rewards to the penalizing players without simultaneously benefiting the original deviator be always possible; i.e., $V^*$ must possess some "interior" property. (Example 1 fails to possess this property.) Being convex and having full dimension in $\mathcal{R}^I$, $V^*$ has a non-empty interior.[29]

23. Here we give an example for the SPNE's in some $G(\infty)$.

**Example 2**. Consider the following 2-person stage game $G(1)$ in normal form:

| 1/2 | L | M | R |
|---|---|---|---|
| L | 10,10 | 3,15 | 0,7 |
| M | 15,3 | 7,7 | -4,5 |
| R | 7,0 | 5,-4 | -15,-15 |

Players can use only pure strategies. Suppose that $\delta = \frac{4}{7}$. A perfect equilibrium in simple strategy profile is a triple of SPNE's $(Q_0, Q_1, Q_2)$ such that in equilibrium the outcome path is $Q_0$ and in case player $i$ deviates unilaterally the outcome path becomes $Q_i$. Show that the following is an SPNE:

$$Q_0 = (L, L), \ \forall t;$$

$$Q_1 = (M, R), (L, M), (L, M), \cdots.$$

[29]We mentioned earlier that the joint minmax strategy against player $i$ is typically a profile of mixed strategies in $G(1)$. For example, suppose that player 1 and player 2 must jointly minmax player 3, with player 1 choosing action L and M with probability 0.4 and 0.6 respectively, leaving another action R un-used. A deviation by player 1 can be detected if player 1 is found to have used action R. Thus player 1 will become the latest deviator and will be minmaxed subsequently after the players observe player 1's action R in the previous stage. For the penalizing scheme to work, there is more that needs to be satisfied: player 1, without wanting to take action R, should also feel indifferent between action L and action M! Thus the penalizing scheme must give player 1 different payoffs following player 1's action L than action M. The bottom line here is that, the penalizing scheme can be very complicated, exactly because *a mixed strategy cannot be observed, only actions that the mixed strategy randomly chooses may be observed*, and hence the penalizing scheme must trigger different actions to be taken by the players following different observed actions taken in the preceding stage by the penalizing players.

$$Q_2 = (R, M), (M, L), (M, L), \cdots.$$

Show that $Q_i$ gives the lowest payoff to player $i$ among all the SPNEs for $G(\infty)$.[30]

**Proof**. First verify that there is a unique (pure strategy) NE in $G(1)$, which is $(M, M)$. In this game, we interpret $(L, L)$ as the collusive outcome that the two players (two firms, say) would like to sustain. Suppose that $\sigma$ is an SPNE for $G(\infty)$ following a deviation from $Q_0$, with $v_i(\sigma)$ being player $i$'s average payoff in the SPNE $\sigma$. Then for $Q_0$ to be sustained by $\sigma$, it must be that

$$5 \leq \frac{\delta}{1-\delta}(10 - v_i(\sigma)), \ \forall i.$$

If we are restricted to using only the trigger strategy in punishing deviations, then $v_i(\sigma) = 7$, and hence $Q_0$ cannot be sustained as an SPNE: we have assumed that $\delta = \frac{4}{7}$, but $Q_0$ can be sustained by the trigger strategy if and only if $\delta \geq \frac{5}{8}$. This exercise shows that more severe punishments than the trigger strategy such as $Q_1$ and $Q_2$ can still be designed to sustain $Q_0$ as an SPNE for $G(\infty)$.

The triple $(Q_0, Q_1, Q_2)$ defined above actually contains three SPNE profiles. We shall show that both players playing $L$ all the time is an SPNE which is supported by $Q_1$ and $Q_2$: If player $i$ deviates from

[30] Fudenberg and Levine (1983) show that if the stage game is a finite game, then for each player $i$ the worst possible SPNE $Q_i$ in $G(\infty)$ always exists. Abreu (1988) show that the same is true if $I$ is finite, and for all $i = 1, 2, \cdots, I$, $A_i$ is a compact subset of some finite-dimensional Euclidean space, with the one-stage payoff function $u_i(\cdot)$ being continuous. Moreover, for a finite game $G(1)$, any SPNE outcome of $G(\infty)$ can be sustained as an SPNE outcome when unilateral deviation by player $i$ is always followed by an immediate switch from the original SPNE to $Q_i$. Abreu (1988) also shows that, if $I$ is finite, and for all $i = 1, 2, \cdots, I$, $A_i$ is a compact subset of some finite-dimensional Euclidean space, with the one-stage payoff function $u_i(\cdot)$ being continuous, and if there is a pure-strategy NE in $G(1)$, then the same conclusion holds; that is, any SPNE outcome of $G(\infty)$ can be sustained as an SPNE outcome when unilateral deviation by player $i$ is always followed by an immediate switch from the original SPNE to $Q_i$. Note that these penalizing schemes are *simple*, in the sense that they depend only on the identity of the latest deviator: if player $i$ is the latest deviator, then the implemented penalizing scheme is $Q_i$, regardless of what happened before player $i$'s latest deviation.

playing $L$ unilaterally at stage $t$, then start $Q_i$ at stage $t+1$, with player $i$ playing $M$ and player $j$ playing $R$ at stage $t+1$, and with player $i$ playing $L$ and player $j$ playing $M$ in each and every subsequent stage. We shall also show that $Q_1$ and $Q_2$ are themselves SPNEs, and are self-supported: If player $j$ deviates from $Q_k$ at stage $\tau$, where $k \in \{1, 2\}$, then restart $Q_j$ at stage $\tau + 1$. Let us now verify that these claims are all true.

First we verify that each $Q_i$ is an SPNE supported by the set of punishments $\{Q_1, Q_2\}$. Suppose that player 1 just deviated unilaterally at stage $t-1$ so that $Q_1$ is now in force at stage $t$. Then player 1's optimal deviation (recall that we only need to check "one-stage" deviations) at this stage is to play $L$. This will make player 2 play $R$ in the subsequent stages until player 1 finally plays $M$. But, by conforming, player 1 gets

$$-4 + \frac{\delta \cdot 3}{1-\delta} = 0,$$

so that player 1 had better conform. (Note that player 1 attains her minmax payoff, which is zero, in the SPNE $Q_1$.) On the other hand, player 2 will play $R$ at stage $t$ because by conforming she gets

$$5 + \frac{\delta \cdot 15}{1-\delta} > 7 + \delta[-4 + \frac{\delta \cdot 3}{1-\delta}] = 7,$$

where the right-hand side is what she gets by deviating and playing $M$, which is followed by $Q_2$ from stage $t+1$ on. Thus $\{Q_1, Q_2\}$ are self-supported SPNE in $G(\infty)$, with the equilibrium payoff of player $i$ in $Q_i$ attaining player $i$'s minmax payoff level, proving that these profiles do attain the minimum payoffs within the class of SPNE's of $G(\infty)$.

Finally, we need to verify that players do not want to unilaterally deviate from $Q_0$, knowing that $Q_i$ will be in force after player $i$ unilaterally deviates, but this is straightforward.

24. Fudenberg and Levine (1983) give a connection between finite- and infinite-horizon games. They show that under certain conditions, $\sigma^*$ is an SPNE for $G(\infty)$ if and only if it is the limit in product topology of a sequence $\sigma^T$ of $\epsilon_T$-perfect equilibrium (cf. Lecture 1, Part II) of a

sequence of truncated games $G(T)$ with $\epsilon_T \to 0$. In this case the set of SPNE of $G(\infty)$ is non-empty and compact.

25. (**Finitely Repeated Games.**) Now we review one important result for $G(T)$. As we remarked earlier, if $G(1)$ is the prisoners' dilemma introduced in Lecture 1, then $G(T)$ has a unique SPNE where the players simply play the unique NE in $G(1)$ in each and every stage. However, stage games with a unique NE are rare. In the remaining case, we have

    **Theorem 5**. (Benoit and Krishna, 1985)
    If for all $i$, there are two NE's in $G(1)$ yielding different payoffs for player $i$, and if the dimension of $V$ is $I$, then for all feasible and enforceable payoff profile $v$, for all $\epsilon > 0$, there exists some $T(\epsilon, v)$ such that some SPNE in $G(T)$, $T \geq T(\epsilon, v)$, generates a payoff profile which is within $\epsilon$ of $v$.

    Instead of proving the Theorem, we shall consider a series of examples (Examples 3-6 below) that demonstrate the intuition behind this result. Since there is a unique SPNE in $G(T)$ whenever there is a unique NE in $G(1)$, these examples assume that $G(1)$ has at least 2 NE's.

26. **Example 3**. Consider the following stage game $G(1)$:

| 1/2 | L | R |
|---|---|---|
| U | 0,0 | 1, 2 |
| D | 2,1 | 0,0 |

Consider $G(T)$ where players receive "sum of stage payoffs." Suppose that players can only use pure strategies. Let $P(T)$ be the set of SPNE payoff profiles for $G(T)$. Let $Q(1) = P(1)$ and eff$Q(1)$ be the efficient frontier of $Q(1)$. Let $R(1) =$eff$Q(1)$, and for $t \in \{2, 3, \cdots, T\}$, define $Q(t) \equiv \{u(\sigma) \in P(t)$ : all continuation payoffs prescribed by $\sigma$ on $G(t-1)$ lie in $R(t-1)\}$ and $R(t) =$eff$Q(t)$. We say that an SPNE $\sigma$ of $G(T)$ is renegotiation-proof if $u(\sigma) \in R(T)$. Thus the players cannot commit to play Pareto dominated SPNE's in later stages. The idea is that any Pareto dominated SPNE may cause renegotiations (and hence not *renegotiation-proof*), which will Pareto improve the players' continuation payoffs by directing the players' attention to some Pareto superior SPNE. We shall show that

$$R \equiv \lim_{T\to\infty} \frac{1}{T} R(T) = \text{co}\{(1,2),(2,1)\}.$$

Note that $P(1) = \{(2,1),(1,2)\} = Q(1) = \text{eff}Q(1)$. Note that

$$P(2) = \{(4,2),(2,4),(3,3)\},\ Q(2) = \text{eff}Q(2) = P(2).$$

Continuing this way, one can show that for all $t \in \{1,2,\cdots,T\}$, $\frac{1}{T}[t(2,1)+(T-t)(1,2)]$ is a per-period average payoff profile attained by some renegotiation-proof SPNE in $G(T)$. Now for each $\alpha \in [0,1]$, and for each $n \in \mathbf{Z}_+$, there exists $m(n) \in \{0,1,\cdots,n-1\}$ such that $\frac{m(n)}{n} \leq \alpha \leq \frac{m(n)+1}{n}$ so that

$$\lim_{n\to\infty} |\alpha - \frac{m(n)}{n}| \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Of course, this only shows that the set of attainable payoff profiles in renegotation-proof SPNE's of $G(T)$ for some $T$ is *dense* in $\text{co}\{(2,1),(1,2)\}$; it does not mean that each point in the latter set is actually attained by some renegotiation-proof SPNE in $G(T)$ for some $T$.

27. Example 3 shows that only the efficient payoff profiles in $G(1)$ are retained in the set of reasonable equilibrium payoff profiles in $G(T)$, as long as $T$ is large enough. This example, however, is misleading, as we shall demonstrate below.

28. **Example 4**. Consider the following stage game $G(1)$:

| 1/2 | L | M | R |
|---|---|---|---|
| L | 5,3 | 0,0 | 2,0 |
| M | 0,0 | 2,2 | 0,0 |
| R | 0,0 | 0,0 | 0,0 |

Consider $G(T)$ where players receive "sum of stage payoffs." Suppose that players can only use pure strategies.
(i) Show that there are two pure NE's in $G(1)$.

(ii) Suppose players can use only pure strategies in $G(2)$, where players' payoffs are averages of temporal payoffs. Show that {(R,R), (L,L)} is an SPNE in $G(2)$. Show by means of this fact that it is not possible to sustain all SPNE's in $G(7)$ using Friedman's trigger strategies.

**Solution**. For part (i), $(L, L)$ and $(M, M)$ are the two NE's. In part (ii), it is easy to see that the claimed profiles form an SPNE in $G(2)$, which is supported by the threat of playing $(M, M)$ in stage 2 if unilateral deviations occur in stage 1. The average payoff for player 2 in this SPNE is $1.5 < 2$, where 2 is the worst NE payoff in $G(1)$ for both players. Obviously, 1.5 is the worst average SPNE payoff for player 2 in $G(2)$ (correspondingly, 2 is the worst average SPNE payoff for player 1). Consider the following profiles in $G(7)$:

$$Q = \{(L, R), (M, M), (M, M), (M, M), (M, M), (M, M), (M, M)\},$$

which cannot be sustained as an SPNE by trigger strategies. But, it is still an SPNE, and the supporting strategies can be

$$\{(R, R), (L, L), (R, R), (L, L), (R, R), (L, L)\}$$

in $G(6)$ in case unilateral deviations occur in stage 1.

29. **Example 5**. Consider the following stage game $G(1)$:

| 1/2 | L | M | R |
|---|---|---|---|
| L | 0,0 | 1,2 | 0,0 |
| M | 2,1 | 0,0 | 4,0 |
| R | 0,0 | 0,4 | 3,3 |

Consider $G(T)$ where players receive "sum of stage payoffs." Suppose that players can only use pure strategies. Find $R \equiv \lim_{T\to\infty} \frac{1}{T} R(T)$. Is it in eff$V^*$? (**Hint**: There are two pure strategy NE's in $G(1)$, yielding payoff profiles $(1, 2)$ and $(2, 1)$ respectively. Note that

$$Q(2) = \{(4, 2), (3, 3), (2, 4), (5, 2), (2, 5)\},$$

where $(5,2)$ is associated with the SPNE $\{(M,R),(L,M)\}$ and $(2,5)$ is associated with the SPNE $\{(R,M),(M,L)\}$. Hence

$$R(2)=\{(3,3),(5,2),(2,5)\}.$$

It follows that

$$Q(3)=\{(5,4),(7,3),(4,6),(4,5),(3,7),(6,4),(6,5),(5,6),(6,6)\},$$

where $(6,5)$ is associated with the SPNE $\{(M,R),(R,M),(M,L)\}$, $(5,6)$ is associated with the SPNE $\{((R,M),(M,R),(L,M)\}$, and $(6,6)$ is associated with the SPNE $\{(R,R),(L,M),(M,L)\}$. Hence

$$R(3)=\{(7,3),(3,7),(6,6)\}.$$

Note that, for $T\geq 3$, there exists an SPNE of $G(T)$ where $(R,R)$ can be played in the first stage.[31] Continuing this way, one can show that

$$Q(4)=\{(9,4),(5,8),(8,7),(8,5),(4,9),(7,8),(10,6),(7,7),(6,10),(9,9)\}$$

and

$$R(4)=\{(10,6),(6,10),(9,9)\}.$$

For $T$ large, we have

$$R(T)=\{(3T-3,3T-3),(3T-2,3T-6),(3T-6,3T-2)\},$$

so that

$$\lim_{T\to\infty}\frac{1}{T}R(T)=\{(3,3)\}\subset \text{eff}(V^*).$$

This example shows that the set $R$ may have a dimensionality smaller than the dimensionality of the set of Nash equilibrium payoff profiles of the stage game!)

30. **Example 6**. Consider the following stage game $G(1)$:

[31] The action pair (R,R) cannot be sustained as the first-stage outcome in $G(2)$ because *each of the two players* can unilaterally deviate and obtain an additional first-stage payoff of 1. If (R,R) is followed by the second-stage action pair (M,L), then player 2 strictly wants to deviate from (R,R) in the first stage; and if (R,R) is followed by the second-stage action pair (L,M), then player 1 strictly wants to deviate from (R,R) in the first stage.

| 1/2 | L | M | R | A |
|---|---|---|---|---|
| L | 0,0 | 2,4 | 0,0 | 6,0 |
| M | 4,2 | 0,0 | 0,0 | 0,0 |
| R | 0,0 | 0,0 | 3,3 | 0,0 |
| B | 0,6 | 0,0 | 0,0 | 5,5 |

Consider $G(T)$ where players receive "sum of stage payoffs." Suppose that players can only use pure strategies. We shall show that $R = \{(4,4)\}$, and verify that $(4,4)$ is not contained in $\text{eff}(V^*)$.

Note that

$$P(1) = \{(4,2),(2,4),(3,3)\} = Q(1) = \text{eff}Q(1),$$

and that

$$P(2) = \{(8,4),(4,8),(6,6),(5,7),(7,5),(8,8)\} \Rightarrow R(2) = \text{eff}Q(2) = \{(8,8)\},$$

where $(8,8)$ is sustained by the strategy profile where the two players play (B,A) and then (R,R), and if (B,A) was not observed, then play either (M,L) or (L,M) depending on who the deviator is. Continuing this way, one can show that for all $T = 2n$, $n \in \mathbf{Z}_+$, $\frac{1}{T}R(T) = \{(4,4)\}$, and for $T = 2n-1$, on the other hand, we have

$$\frac{1}{T}R(T) = \{(4, 4-\frac{2}{T}), (4-\frac{2}{T}, 4), (4-\frac{1}{T}, 4-\frac{1}{T})\}.$$

Thus we conclude that $R = \lim_{T\to\infty}\frac{1}{T}R(T) = \{(4,4)\}$. This example shows that elements of $R$ may not even be weakly Pareto efficient relative to payoffs in $G(1)$.

31. Example 6 shows that, unlike what it seems at first glance, the *number* of renegotiation-proof SPNE's of $G(T)$ can be smaller than the *number* of NE's of $G(1)$! Moreover, the attainable payoff profiles may not be contained in the efficient frontier of the set of feasible payoff profiles of $G(1)$. However, Krishna and Benoit show that $\lim_{T\to\infty}\frac{1}{T}R(T)$ is either a singleton or it must be contained in $\text{weff}(V^*)$, the *weakly efficient frontier* of the set $V^*$.[32] (Why does renegotiation-proofness not ensure higher equilibrium payoffs for the players?)

[32] For a two-player game where $(u_1,u_2) \in V^*$ denote a typical payoff profile of $G(1)$, $(u_1,u_2)$ is contained in $\text{weff}(V^*)$ if and only if there does not exist $(u_1',u_2') \in V^*$ such that $u_1' > u_1$ and $u_2' > u_2$.

32. (**Applications to Corporate Finance**.) We now give a series applications of multistage games with observable actions in finance.

**Example 7**. (Maksimovic, 1988, *Rand JE*) There are $n$ firms playing the Cournot game $G(1)$, and we shall consider the corresponding $G(\infty)$. Suppose that in the stage game $G(1)$, each firm gets $\pi^c$ in a symmetric collusive outcome, and each gets $\pi^{nc}$ in the Cournot equilibrium. Let $\pi^d$ be the optimal profit of a firm in $G(1)$ when all the remaining $n-1$ firms are producing the tacitly agreed collusive quantity $q^c$. Suppose that $r \in (0,1)$ is the common discount rate. Suppose that each firm has a sole equityholder. In this case, the Friedman's folk theorem says that if $r < \frac{\pi^c - \pi^{nc}}{\pi^d - \pi^c}$,[33] then the collusive outcome can be sustained as an SPNE outcome supported by the trigger strategy.

Suppose now that at date 0 all the $n$ firms can issue perpetual bonds. A bond is thus a constant $b > 0$ that specifies the interest payments the borrowing firm must make at each subsequent date. Show that in a symmetric equilibrium where all firms choose the same amount $b$ at date 0, there is an upper bound on $b$ so that the collusive SPNE can still be sustained from date 1 on, if $r < \frac{\pi^c - \pi^{nc}}{\pi^d - \pi^c}$. In fact, show that the

[33] Note that the equilibrium incentive compatibility (or the no-unilateral-deviation) condition requires

$$\frac{\delta(\pi^c - \pi^{nc})}{1-\delta} > \pi^d - \pi^c,$$

where $\delta = \frac{1}{1+r}$, and hence it is required that

$$r < \frac{\pi^c - \pi^{nc}}{\pi^d - \pi^c}.$$

upper bound on $b$ in this case is[34]

$$\max(\pi^{nc}, \pi^c - r(\pi^d - \pi^c)).$$

On the other hand, if $r \geq \frac{\pi^c - \pi^{nc}}{\pi^d - \pi^c}$, show that in a symmetric equilibrium, $b$ cannot exceed $\pi^{nc}$.

**Remark**. The above has mainly assumed that the firm can only consider vanilla debt contracts. The result crucially depends on this assumption. If the contracting parties can optimally design the covenants, then the upper bound on $b$ can be lifted. For example, the debt contract may impose restrictions on the amount of cash earnings that can be distributed as dividends, or it may put restrictions on the firm's production activities. In the above model, if the debt covenants restrict the amount of cash dividend per period to not exceed $\pi^c - b$, then with any debt level $b$, the firm will optimally conform to the collusive arrangement! Similarly, if the debt covenants forbid the firm to expand output beyond $q^c$, then any debt level $b$ is consistent with value

[34]If $r < \frac{\pi^c - \pi^{nc}}{\pi^d - \pi^c}$, then as verified above the collusive outcome can be sustained in an SPNE when $b = 0$ is chosen by firm $i$, for all $i$. In this case, if $b < \pi^{nc}$, then $b < \pi^c < \pi^d$, and hence the IC condition for producing the collusive output is the same as before:

$$\frac{\delta((\pi^c - b) - (\pi^{nc} - b))}{1 - \delta} > [(\pi^d - b) - (\pi^c - b)].$$

Since $b < \pi^c$ apparently, if $b \geq \pi^{nc}$ instead, then $b \in [\pi^{nc}, \pi^c)$. In this latter case, if in a symmetric equilibrium all firms choose such a $b$ at date 0, and if for the first time some firm deviates from the collusive behavior at date $t$, then from date $t + 1$ on it is an SPNE where all firms produce the static Nash equilibrium output and each of them obtains $\pi^{nc}$. Indeed, at date $t + 1$ no firm can fully repay its debt given that its rivals will all produce the static Nash equilibrium output, and although these firms are feeling indifferent about any output strategy, Maksimovic assumes that all firms will choose to produce the static Nash equilibrium output at date $t + 1$. Following default on debt at date $t + 1$, all firms will then be run by their creditors (as if they are all-equity firms) from date $t + 2$ on, and these reorganized firms will then produce the static Nash equilibrium output from date $t+2$ on. The bottom line is that, for the original equityholder(s) at the deviating firm, the IC condition for conforming to the collusive behavior is $\pi^d - b \leq (\pi^c - b)(1 + \delta + \delta^2 + \cdots)$, which reduces to $b < \pi^c - r(\pi^d - \pi^c)$. Clearly, when $b \geq \pi^{nc}$, a deviating firm no longer has to worry about its debt obligations due after its deviation, and this alters the incentive-compatibility condition for the collusive outcome.

maximization. Hence, I do not find Maksimovic's result about a debt upper bound very convincing. See Smith and Warner (1979, *JFE*) for an in-depth discussion of debt covenants. See also Rajan and Winton (1995, *JF*) for the strategic roles of debt covenants and collateral. Maksimovic (1988) does mention that the use of warrants and convertible debt may help lift the debt upper bound: by allowing the holders of the option-like securities to turn their securities into equity after the firm deviates from the collusive outcome, a firm convinces its rivals that it will not deviate at all. Thus issuing these option-like securities helps sustain collusion.

33. **Example 8**. (Jensen and Meckling, 1976, *JFE*) At date 0, Mr. B is the owner-manager of a firm protected by limited liability. The firm is endowed with \$50 in cash. There are two mutually exclusive investment projects available to B at date 0. Alternative 1 is a riskless project which incurs an immediate \$100 cash outflow and generates \$105 at date 1. Alternative 2 is a risky project which incurs an immediate cash outflow of \$100 and generates a random cash inflow $\$\tilde{X}$, where $\tilde{X}$ has two equally likely outcomes, 0 and 180. Note that alternative 1 generates a positive NPV of \$5, but alternative 2 leads to an expected loss of \$10. Since taking alternative 1 is productively efficient, we assume that Mr. B will take alternative 1 whenever he feels indifferent about the two investment alternatives.

    Mr. B decides to come to Mr. C for a loan of \$50. The game proceeds as follows. B offers a debt contract with face value $F$ to C, which C can either accept or reject. If C rejects the contract, no investment is made and both B and C get zero payoffs. If C accepts the contract, then B must choose between alternative 1 and alternative 2. After the investment decision is made, the state of nature is realized, and B and C get paid according to the debt contract. Find the subgame perfect Nash equilibrium of the game.

    **Solution**. First consider the subgame where $F$ is given and the loan is made (or else the game has ended). If B chooses alternative 1, she is sure that she will get $\max(0, 105 - F)$. If B chooses alternative 2, then her payoff is random: with prob. $\frac{1}{2}$, she gets 0; and with prob. $\frac{1}{2}$, she gets $\max(180 - F, 0)$. Thus, B chooses alternative 2 over alternative 1 if and only if

$$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \max(180 - F, 0) \geq \max(105 - F, 0). \tag{1}$$

The following table summarizes the investment behavior of B given different values of $F$:

| $F$ | alternative chosen |
|---|---|
| $\in [0, 30]$ | 1 |
| $\in (30, 105]$ | 2 |
| $\in (105, 180)$ | 2 |
| $\in [180, +\infty)$ | 1 |

Now we consider the subgame where C must decide whether to accept B's debt contract. According to the above table, B would subsequently invest in alternative 1 if and only if $F \leq 30$ or $F \geq 180$, but C is sure to lose money if she accepts any offer with $F \leq 30$ or with B choosing alternative 2. Thus, $C$ accepts B's offer if and only if $F \geq 180$. Now consider B's problem of making an offer to C. Given the above analysis, B can expect her offer to be accepted by C only if $F \geq 180$, but B would be better off giving up the new investment and keeping her 50 dollars at hand.

Our conclusion is that, in the unique subgame perfect Nash equilibrium of this game, B does not make any offers to C in the first place, and the game ends at the very beginning with the firm passing on the good investment opportunity (alternative 1).

This kind of shareholders' incentive problems is referred to as *risk shifting* or *asset substitution* in the finance literature. There are other kinds of incentive problems involving shareholders or creditors which we shall review later on. These incentive problems lead to investment inefficiencies and hence reductions in firm value.

**Remarks**. Implicitly assumed in the above extensive game is that B's investment decision cannot be *observed* by C, or it can be observed by C but cannot be *verified* by the contract enforer (usually the court of law). For if the choice of the investment alternative is both *observable and verifiable* (which will be referred to as *contractible*), then B can sign a contract with C saying that B will choose the riskless project, or else C can, say, break B's arms. This is called a *forcing contract*, which apprently removes the agency problem, as long as B cares enough about his arms. The problem is then, "Why can't C observe B's investment decision?" One may argue that, B, as the CEO, makes the decision in his office, and may not be observed by C. The problem is more delicate than that! Note that if ex-post cash flows are contractible, then by observing the cash flows C can prove whether B has invested the riskless project or not, and hence a forcing contract seems possible. (Of course, breaking somebody's arms may not be legal, and hence itself unenforceable; this could create a new problem: penalizing B in a monetary manner may not work as well as breaking arms, for B may not have enough money to implement a monetary penalty on him!) Therefore, it seems necessary to assume that the ex-post cash flows are not contractible. Alas! This is again not the end of the problem. One must then ask, "Why can't cash flows be observable?" Is it a reasonable assumption? As we shall see, a large body of literature in optimal design of financial contracts has assumed that cash flows can be costlessly observed only by the insiders of the firm (here, B). However, it has also been assumed that by spending some money, C may be able to verify the true cash flows. Of course this money, paid to an accountant for example, is a deadweight loss, and should be by all means avoided or minimized in an optimal contract, but allowing C an opportunity to verify is indeed a more reasonable assumption. The bottom line here is that, the above conclusion that external financing leads to the asset substitution problem actually stems from the somewhat arbitrary assumption that Mr. B can only use a standard debt contract when raising funds from outside investors. If B and C are rational, they should be able to use Pareto optimal contracts, and one of them is clearly equity contract.

34. **Example 9**. (Jensen and Meckling, 1976, *JFE*) Suppose A is the owner-manager of a firm whose value is

$$V = 1 - L,$$

where $L \in [0,1]$ is A's on the job leisure. A has utility function $U(V,L) = V^{\frac{4}{5}} L^{\frac{1}{5}}$.

(i) Compute the optimal leisure for A. What is the corresponding value of the firm?
(ii) Now suppose A wants to sell $\frac{1}{3}$ of his ownership to outsiders. The game proceeds as follows. A first sells his partial ownership to outsiders in exchange for money $M$. Then, after the transaction, A chooses his leisure $L$. Assume that outside investors are competitive and have perfect foresight, so that $M$ is exactly the worth of the partial ownership they obtain in equilibrium. What is the equilibrium value of the firm? Suppose there is no portfolio effect between ownership and money for A, determine if A should make this ownership transaction in the first place. What if there is a portfolio effect?

**Solution**. First, part (i). Recall the following consumption problem: with constants $a, b > 0$, $a + b = 1$, and $p_x, p_y, I > 0$ given, the solution to

$$\max_{x,y} U(x,y) = x^a y^b,$$
$$s.t.\ p_x x + p_y y \leq I$$

is simply

$$x^*(p_x, p_y, I) = \frac{aI}{p_x}, \quad y^*(p_x, p_y, I) = \frac{bI}{p_y}.$$

(The above utility function is called a Cobb-Douglas utility function.) Using this fact, we have for part (i),

$$V^* = \frac{4}{5}, \quad L^* = \frac{1}{5}.$$

That is, the firm value is $\frac{4}{5}$. In the following, we continue to denote the manager's monetary wealth by $V$. Consider part (ii). In the subgame where $M$ has been given, the manager's problem is to choose $L$ to maximize her utility. Let $V$ be the manager's monetary wealth including the cash $M$. Then, the value of the firm will be

$$\frac{V - M}{\frac{2}{3}}.$$

Thus, the manager seeks to

$$\max_{V,L} V^{\frac{4}{5}} L^{\frac{1}{5}}$$

$$s.t. \ \frac{V - M}{\frac{2}{3}} = 1 - L.$$

Using the above result for the Cobb-Douglas utility function, we have

$$L^* = \frac{1}{5}(1 + \frac{3}{2}M).$$

Observe that two things happen here. First, the price of ownership ($V$) relative to leisure ($L$) has increased from 1 to $\frac{3}{2}$. Second, before choosing the optimal $L$, the manager has received $M$ (as part of his $V$), which implies by the concavity of $U$ in $V$ that $L$ has become more desirable than in part (i). Thus it is not surprising that $L^* > \frac{1}{5}$, where $\frac{1}{5}$ is the optimal leisure in part (i), and moreover, the difference $L^* - \frac{1}{5}$ increases with $M$ and the fraction of ownership held by the outside investors.

Now, using backward induction, we can infer what $M$ must be in equilibrium: With rational expectations, the $M$ outsiders are willing to pay to the manager is exactly $\frac{1}{3}$ the value of the firm:

$$M = \frac{1}{3}(1 - L^*) = \frac{1}{3}(1 - \frac{1}{5}(1 + \frac{3}{2}M)).$$

Solving, we have

$$M^* = \frac{8}{33},$$

which implies that $L^* = \frac{3}{11}$, and the value of the firm becomes $\frac{8}{11}$ (which was originally $\frac{8}{10}$). How about the manager's utility in equilibrium? It is

$$(\frac{2}{3} \times \frac{8}{11} + \frac{8}{33})^{\frac{4}{5}} (\frac{3}{11})^{\frac{1}{5}} < (\frac{4}{5})^{\frac{4}{5}} (\frac{1}{5})^{\frac{1}{5}}.$$

This has assumed that there is no portfolio effect in the manager's utility function, and we conclude that in this case the manager will not sell the partial ownership willingly in the first place. On the other hand, if the manager considers cash different from ownership of the firm, then selling the ownership may still enhance her satisfaction. In this case, however, it is not clear if the manager will increase her on the job consumption after selling the shares.

A question comes to mind at this point: why are there so many professional managers in the real world? One possibility is that running business takes managerial expertise, and it is costly to acquire such expertise; in particular, the initial founder of the firm may not be able to manage the firm as well as a professional manager. It is natural then for the initial founder to sell the firm to the professional manager to avoid the agency problem discussed above, but the latter may not have enough money to buy out the firm! Another possibility is that an owner-manager may be hit by liquidity shock and must sell a fraction of equity for cash. This however raises the following question, "why can't the owner-manager borrow some money, if after all getting some cash will resolve his problem?" A likely answer is that borrowing also creates some agency costs.

These two possibilities both suggest that professional managers exist only when the equityholders are *forced* to hire them (either because of their exclusive expertise, or because the equityholders must liquidate their positions in the stock to get cash). An ingenius point is made by Hirshleifer and Thakor (1992, *Review of Financial Studies*), who suggest that professional managers may exhibit too much conservatism, which shareholders dislike, but by hiring conservative professional managers, shareholders can reduce or even remove the agency costs of debt that arises from the asset substitution problem. (Like the Chinese old saying suggests, one poison may be used as an antidote for another poison.) Thus shareholders may be *pleased* to hire professional managers, using the latter as a commitment to ensure the debtholders that the firm will not undertake projects that involve excessive risks.[35]

[35]A converse story (Hart and Moore, 1995, *AER*; Stulz, 1990, *JF*) is as follows. Suppose that managers derive private benefits from *empire-building* (taking too many negative NPV projects just to keep a large corporation under their control). Debt, although it creates agency costs on its own, can be used to force the manager to disgorge the free cash flows (Jensen, 1986, *AER*). Thus debt (a poison) can be used as the antidote to the managerial incentive problem (another poison).

35. **Example 10**. (Diamond and Dybvig, 1983, *JPE*)[36] Consider economy $E$ that consists of a continuum of consumers. Let the population of consumers be (normalized to) 1. At date 0, each consumer is endowed with one unit of consumption, which can be invested in a real investment project that generates $R > 1$ units of consumption at date 2. A consumer normally consumes at date 2, but with probability $x \in (0, 1)$, he may be hit by a liquidity shock at date 1, and in that event the consumer is forced to consume at date 1 (hence he must liquidate his investment at date 1 to get cash). Assume that if the project is terminated at date 1, it returns 1 unit of consumption. Assume that at date 1, the population of consumers who are hit by a liquidity shock is exactly $x$; this is called a *no-aggregate uncertainty* assumption.[37]

(A) At first, suppose there are no financial institutions. Then, the date-0 expected utility of a consumer is simply $xu(1) + (1 - x)u(R)$, where $u(\cdot)$ is the consumer's temporal utility function of consumption, with $u' > 0 > u''$.

(B) Now assume that the consumers can get together to form a financial institution (called a *commercial bank*), and each consumer gets to sign a deposit contract with the bank at date 0. A deposit contract is a pair of promised consumptions $\{c_1, c_2\}$ such that (1) in the event that the consumer is hit by a liquidity shock at date 1, then he receives $c_1$ at date 1 and nothing at date 2 from the bank (equivalently, the consumer withdraws $c_1$ from the bank at date 1); and (2) in the event that the consumer is not hit by a liquidity shock, then he receives nothing at date 1 but $c_2$ at date 2 from the bank. Such a deposit contract is called *feasible* if

$$[1 - xc_1]R = (1 - x)c_2; \tag{2}$$

that is, with $x$ consumers withdrawing at date 1, and each taking away

[36]This exercise examines why we need a commercial bank, and why a bank's functioning may fail because of inefficient bank runs, and inquires about the possibility of replacing the commercial bank by other financial institutions like insurance companies. It also discusses the role of deposit insurance.

[37]Although each individual is facing the uncertainty that he might need to consume early (at date 1), and that event may occur with probability $x$, the entire population is sure to have $x$ consumers who need to consume early, thus facing no uncertainty.

$c_1$ dollars, the bank gets to invest the rest $[1 - xc_1]$ dollars at date 1, which yields $[1-xc_1]R$ dollars at date 2 for the rest $(1-x)$ consumers.

A deposit contract $\{c_1^*, c_2^*\}$ is called *Pareto optimal* if it is *feasible* and it maximizes each individual's expected utility:

$$\max_{c_1, c_2} xu(c_1) + (1-x)u(c_2).$$

(i) Show that under the optimal contract, the following first-order condition must hold:

$$u'(c_1^*) = Ru'(c_2^*). \tag{3}$$

(ii) Show that if $u(c) = \log(c)$ for all consumers, then there is no need of forming the commercial bank.
(iii) Show that if for all consumers and for all $c$, $-\frac{cu''(c)}{u'(c)} > 1$, then the Pareto optimal deposit contract is such that $1 < c_1^* < c_2^*$. (**Hint**: Write $R$ in terms of $c_1^*$ and $c_2^*$ from the *feasibility* condition, and then substitute $R$ into the first-order condition, and then recall the fundamental theorem of calculus: if $f(\cdot)$ is continuously differentiable, then $f(x) = f(y) + \int_y^x f'(t)dt$. Now look at the first-order condition with $R$ substituted, and let $f(x) = xu'(x)$.)

From now on, we always assume that $-\frac{cu''(c)}{u'(c)} > 1$ for all $c$. Suppose that at date 1 the bank cannot tell if a consumer who wants to withdraw money was really hit by a liquidity shock, but it chooses to pay $c_1^*$ to everyone who wants to withdraw at date 1. Let $f$ be the population of consumers who want to withdraw at date 1. Note that if $fc_1^* \leq 1$, then each withdrawer gets $c_1^*$; or else, each gets $\frac{1}{f}$. (This simplifies the analysis. But in general, the bank holds a "first-come-first-serve" policy and hence the money a withdrawer can get is random.) Consumers who do not withdraw at date 1 receive $\max(0, \frac{R(1-c_1^* f)}{1-f})$ at date 2.
(iv) Show that there is a Nash equilibrium where consumers withdraw at date 1 (i.e. $f = x$) if and only if they were hit by liquidity shocks.
(v) Show that there is another Nash equilibrium where consumers all withdraw at date 1 (i.e. $f = 1$) regardless of whether or not a liquidity shock has occurred.
(vi) Now consider a *sunspot* $y$ which takes the values 0 and 1 with

probability $\pi$ and $1-\pi$. Show that it is an equilibrium that $f=x$ if $y=0$ and $f=1$ if $y=1$. Call this the *sunspot banking equilibrium*. For the rest of this problem, always assume the presence of the sunspot $y$.

(vii) Suppose that instead of signing the deposit contract, the consumers sign an *insurance contract* with the institution (with this new contract, the institution is defined as an *insurance company*): if a consumer intends to withdraw money at date 1, then a cost $k>0$ is spent (to hire a private investigator for example) by the institution to see if the withdrawer was hit by a liquidity shock, and at date 1 this consumer receives $c_1'$ if he was hit by a liquidity shock and nothing if he was not. Let $f$ stand for the population of consumers who receive $c_1'$ at date 1, and $h$ the population of consumers who intend but fail to withdraw money at date 1. For each consumer who did not receive $c_1'$ at date 1, his date-2 payoff is $c_2' = \frac{\max(0, 1-fc_1'-(f+h)k)R}{1-f}$. Show that in equilibrium $f=x$ and $h=0$. Show that under the Pareto optimal insurance contract $c_1'$ and $c_2'$ satisfy

$$[1-xk-xc_1']R=(1-x)c_2'; \tag{4}$$

and

$$u'(c_1')=Ru'(c_2'). \tag{5}$$

Compare $(c_1', c_2')$ to $(c_1^*, c_2^*)$.

(viii) Show that when $k$ is sufficiently small, an insurance company dominates a commercial bank, and when $\pi$ is sufficiently large, a commercial bank dominates an insurance company.

(ix) Suppose now that there is another economy $E'$, identical to $E$. However, we shall make several modifications. First, the investment opportunity that transforms one dollar into $R$ dollars is available to just one economy, where $1<R<2$, and ex-ante it is equally likely that $E$ and $E'$ may have this investment opportunity. Second, the consumers in the two economies can first sign a *deposit insurance contract* before learning to which economy nature assigns the investment opportunity. After signing the contract and after knowing that their economy is endowed with the investment opportunity, the consumers in the lucky economy can then form a commercial bank and sign deposit contracts with the bank, as in the above sections (i)-(vi). The deposit

insurance contract states that if $E$ is endowed with the investment opportunity while $E'$ is not (the same is true if $E$ and $E'$ are switched), and if at date 1 the bank in economy $E$ has $f > \frac{1}{c_1^*}$ withdrawers, then each consumer in economy $E'$ is obliged to pay $fc_1^* - 1$ to the bank of economy $E$. Show that under deposit insurance, withdrawing money from the bank is a weakly dominated strategy for a consumer in economy $E$ who was not hit by a liquidity shock. Show that in equilibrium consumers in economy $E'$ never have to pay to the bank of economy $E$.
(x) Can you think of any negative side of deposit insurance?

**Solution**. Consider questions (i)-(iii). The maximization problem in part (i) can be stated as follow:[38]

$$\max_{c_1, c_2} xu(c_1) + (1-x)u(c_2)$$

subject to

$$(1 - xc_1)R = (1-x)c_2$$

Equivalently, we can maximize the unconstrained function

$$f(c_1) = xu(c_1) + (1-x)u(\frac{(1-xc_1)R}{1-x}),$$

which is a strictly concave function of $c_1$ (because $x < 1$ and $f'' < 0$), so that its maximum appears at

$$f'(c_1^*) = 0 \Leftrightarrow u'(c_1^*) = Ru'(c_2^*).$$

In part (iii), since $R > 1$ and $u'(\cdot)$ is a strictly decreasing function ( because $u'' < 0$!), we conclude that $c_2^* > c_1^*$. Now, if we replace $R$ by $\frac{(1-x)c_2^*}{1-xc_1^*}$ in $u'(c_1^*) = Ru'(c_2^*)$ (note that $1 > xc_1^*$ because of the condition $u'(0) > \frac{u'(\frac{1}{x})}{R}$), we have the following equation:

$$u'(c_1^*) = \frac{(1-x)c_2^*}{1-xc_1^*}u'(c_2^*)$$

[38]The condition $u'(0) > \max(\frac{u'(\frac{1}{x})}{R}, Ru'(\frac{R}{1-x}))$ implies that the constraint $0 \leq xc_1 \leq 1$ will be automatically satisfied at optimum, and hence we can ignore this constraint.

$$\Rightarrow u'(c_1^*) - xc_1^*u'(c_1^*) = (1-x)\{c_1^*u'(c_1^*) + \int_{c_1^*}^{c_2^*}[u'(t) + tu''(t)]dt\}$$
$$< (1-x)c_1^*u'(c_1^*),$$

because by assumption for all $c$, $-\frac{cu''(c)}{u'(c)} > 1$. It follows that, if for all $c$, $-\frac{cu''(c)}{u'(c)} > 1$, then we have

$$0 < u'(c_1^*)(c_1^* - 1) \Rightarrow c_1^* > 1,$$

and hence $c_2^* > c_1^* > 1$ in this case. Note that if $u(c) = log(c)$, as in part (ii), then $u'(c) = \frac{1}{c}$. In this case we have $c_1^* = 1$ and $c_2^* = R$. Thus there is no room for welfare-improving commercial banks in this case.

Consider part (iv). Because $xc_1^* < 1$, if consumers believe that only those hit by a liquidity shock will withdraw at date 1, a consumer not hit by a liquidity shock knows that she can get $c_2^* = R(1 - xc_1^*)/(1 - x)$ if she decides to not withdraw her money from the bank at date 1, and she would otherwise get $c_1^*$ if she chooses to withdraw at date 1. Since $c_2^* > c_1^*$, a consumer not hit by a liquidity shock at date 1 indeed will not withdraw at date 1. Thus there exists an equilibrium where consumers withdraw at date 1 if and only if they are hit by a liquidity shock at date 1.

Consider part (v). When a consumer believes that all other consumers will withdraw at date 1, she believes that she will get nothing unless she also goes withdraw her money from the bank at date 1, in the latter case she would get 1 dollar back. Thus it is an equilibrium where all consumers withdraw their money from the bank at date 1 whether or not they are hit by a liquidity shock.

Consider part (vi). Suppose that all consumers believe that the equilibrium in part (iv) will prevail when everyone sees $y = 0$, and the equilibrium in part (v) will prevail when everyone sees $y = 1$. Then, it is easy to verify that these beliefs are self-fulfilling so that the above beliefs are realized in a *sunspot equilibrium*, where the equilibrium outcome depends on the realization of $y$.

Consider part (vii). With the insurance contract, a withdrawer gets money only if the institution proves that she was indeed hit by a liquidity shock. Thus there is no point for those not hit by a liquidity shock to try to withdraw. This means that in equilibrium $f = x$ and $h = 0$, and it rules out inefficient bank runs that appear in the equilibrium described in part (v). However, the insurance company has committed to verify withdrawers in equilibrium, even though it knows that all withdrawers are hit by a liquidity shock. Thus insurance company incurs a deadweight verification cost, which reduces the non-withdrawers' date-2 wealth. An insurance contract is feasible if and only if $[1 - xk - xc'_1]R = (1 - x)c'_2$. The Pareto optimal insurance contract $c'_1$and $c'_2$ must solve the following program:

$$\max_{c_1, c_2} xu(c_1) + (1 - x)u(c_2)$$

subject to

$$(1 - xk - xc_1)R = (1 - x)c_2.$$

It can be shown that $c'_1$and $c'_2$ satisfy $[1 - xk - xc'_1]R = (1 - x)c'_2$ and $u'(c'_1) = Ru'(c'_2)$.

Recall that $c^*_1$and $c^*_2$ satisfy $[1-xc^*_1]R = (1-x)c^*_2$ and $u'(c^*_1) = Ru'(c^*_2)$. Using $u'(c'_1) = Ru'(c'_2)$ and $u'(c^*_1) = Ru'(c^*_2)$, we have

$$\frac{u'(c'_1)}{u'(c^*_1)} = \frac{u'(c'_2)}{u'(c^*_2)}.$$

From $[1-xk-xc'_1]R = (1-x)c'_2$ and $[1-xc^*_1]R = (1-x)c^*_2$, we obtain

$$\frac{1 - x(k + c'_1)}{1 - xc^*_1} = \frac{c'_2}{c^*_2}.$$

It can be verified that $c'_1 < c^*_1 < c'_1+k$ and $c^*_2 > c'_2$. Thus the verification cost is shared by all consumers, regardless of whether or not a consumer withdraws money at date 1.

Consider part (viii). Under the optimal deposit contract, a consumer's expected payoff is

$$\pi[xu(c^*_1) + (1 - x)u(c^*_2)] + (1 - \pi)u(1)$$

Under the optimal insurance contract, a consumer's expected payoff is

$$xu(c_1') + (1-x)u(c_2'),$$

When $k$ tends to zero, $c_1'$ and $c_2'$ approach to $c_1^*$ and $c_2^*$ respectively. Therefore, when $k$ is sufficiently small, an insurance company dominates a commercial bank, and when $\pi$ is sufficiently large, a commercial bank dominates an insurance company.

Consider part (ix). Under the deposit insurance, if there are $y$ withdrawers in economy $E$ at date 1, then the bank in economy $E$ is left with the following amount of money for investment,[39]

$$1 + G(y) - yc_1^*.$$

Each non-withdrawer in economy $E$ will then get

$$\frac{[1 + G(y) - yc_1^*]R}{1-y} = c_2^*$$

at date 2. If such a person deviated and withdrew at date 1, then he or she received $c_1^*$ only! Thus only those who are forced to consume at date 1 would choose to withdraw at date 1 in the presence of deposit insurance provided by the other economy. We conclude that in equilibrium, the population of withdrawers at date 1 is exactly $x$ in economy $E$.

Consider part (x). Sometimes depositors withdraw money from the bank because they believe that the bank has a bad performance and is not operating in a way that is consistent with their best interest. Thus, withdrawing money is one costly way (sometimes the *only* way) to make sure that a bad bank quits the market. Under a deposit insurance contract with full coverage (such as the one in part (ix)), depositors have no incentives to withdraw money even if they believe that the

[39]We should make sure that people in economy $E'$ do not default on this deposit insurance contract; that is, $G(y) \leq 1$ for all $y \in [x, 1]$. It can be readily verified that $G(x) = 0$ and $G' > 0$. Thus it suffices to check that $G(1) \leq 1$. Note that $G(1) = c_1^* - 1 < c_2^* - 1 < R - 1 < 1$, because we have assumed that $R < 2$.

bank's performance is poor. The bank managers can continue to have their own way and to reduce the value of the bank without having to fear being terminated by the runs created by monitoring depositors. These issues did not appear in the above exercise, but their importance in reality cannot be overstated.

**Remarks**.

(i) It is important that to get $R$ at date 2, the investment must be made *at date 0*. For if it is feasible to invest the dollar at date 1 and get $R$ at date 2, then the only NE involves everyone withdrawing at date 1: a consumer who is not hit by the liquidity shock can realize an arbitrage profit by withdrawing at date 1. Certainly this cannot happen in equilibrium, and hence no arbitrage will rule out the existence of banks; that is, rule out the possibility of an insurance contract that Pareto improves the no trade outcome.

(ii) This model leaves out the banker's moral hazard problems. In the presence of these problems, bank runs may have a positive function: being afraid that the monitoring depositors may withdraw their money from the bank, the banker's incentives of making bad loans can be effectively removed. Deposit insurance (discussed in class) that aims at eliminating the inefficient runs may simultaneously eliminate the possibility of efficient runs, thereby aggravating the banker's moral hazard problem.

(iii) Observe that in the above model, banks essentially play the role of insurance companies: consumers are ensured a gain in the bad state (consumption equals 1) in exchange for a fee that must be paid in the good state (consumption equals $R$). The difference between an insurance policy and the demand deposit is that with an insurance policy, an investigation (or in more formal terminology, *state verification*) may take place to determine if the insuree is really in the bad state, and the insuree gets paid only if the bad state does occur. It is then not surprising why runs do not occur to the insurance companies. Now, if both demand deposit and insurance policy provide insurance to the consumers, and if inefficient runs are not likely to occur with insurance policy, what is the relative advantage of demand deposit to justify its existence? One distinct feature of demand deposit offered by commercial banks is the immediacy that it allows. Nowadays one can withdraw money by visiting an ATM or do transactions on the net. Investigation by the insurance company can be costly and time-consuming and hence may not be desirable if consumption is needed immediately. In terms of this observation, the above analysis can be further generalized by including immediacy and investigation (state verification), and the optimal choice of these features helps to explain why a commercial bank instead of an insurance company is more desirable in certain economic environments. The same line of reasoning now suggests that one also consider security innovation. What may happen if, for example, instead of forming a bank, the economy chooses to form an exchange, which then optimally designs a traded security for the public? A number of authors have published new research in this regard.

36. **Example 11**. (Myers, 1977, *JFE*) A growth firm may be more vulnerable to an agency problem (known as *debt overhang*) than a firm with no growth opportunities. The following is an example. A firm finances the date-0 cost $g > 0$ for its search for a valuable investment opportunity by borrowing, and it promises to repay the debtholder $F > 0$ at date 2. (The cost $g$ can be viewed as an R&D expense.) It is known at date 0 that , some public information will arrive at date 1, which will reveal how much cash inflow the project will generate at date 2. Suppose that it is investors' common knowledge that the date-2 cash inflow is equally likely to be either 20 or 10. To generate that cash flow, an additional $I$ dollars must be spent at date 1. However, The firm has no cash at date 1, and must issue junior debt at date 1 to raise the $I$ dollars. Now, assume that competitive investors are all risk neutral without time preferences (recall that this implies that asset prices are all marginales). At date 1, if the state is that the date-2 cash inflow is $C$, then the new investor (debtholder) will get $\min(C - F, F')$ at date 2, where $F'$ is the face value of junior debt. Thus the new investor will lend $I$ to the firm if and only if $I \leq \min(C - F, F')$, and since $F' \geq I$,

the latter condition is equivalent to $C \geq F + I$. In case

$$20 > F + I > 10 > I,$$

the new investor will refuse to lend to the firm, if $C = 10$ at date 1. Since $I < 10$, this creates a deadweight loss, and is referred to as an *agency cost* pertaining to debt.

Thus, solving the SPNE of this game, we conclude that when $20 > 2g + I > 10 > I$, then in equilibrium, $F = 2g$, so that the date-0 firm value is $\frac{1}{2}(20 - I) > g$, which justifies the firm's inital R&D effort. Note that if the firm were to have enough cash earnings at date 1, the date-0 firm value would be $\frac{1}{2}(20 - I) + \frac{1}{2}(10 - I)$.

It is not surprising that the standard debt contract is Pareto suboptimal in this example. Let us derive a Pareto optimal financial contract for the initially raised $g$ dollars, assuming more generally that $C = 20$ and 10 with probability $\pi$ and $1 - \pi$ (in the above we have assumed that $\pi = \frac{1}{2}$). Assume correspondingly

$$20 > I + \frac{g}{\pi} > I + g > 10 > I > 0. \quad (\Theta)$$

Before solving the optimal financial contract at date 0, let us first consider the equilibrium $F$ associated with the (asserted suboptimal) standard debt contract written at date 0. Quickly deduce that $F \leq 20 - I$. (Why?) Similarly, we claim that $F \geq 10 - I$. Thus given $F$, the firm can raise $I$ at date 1 if and only if $C = 20$. Rationally expecting this, the $F$ can be obtained by solving the zero expected profit condition of the senior debtholder:

$$\pi F = g \Rightarrow F = \frac{g}{\pi}.$$

This result is consistent with assumption $(\Theta)$. Thus at date 0, the value of debt is exactly $g$, showing that trading financial assets yields zero NPV at date 0. The date-0 value of equity is then $\pi(20 - I)$. What happens at date 1? It depends on $C$. In case $C = 20$, then the date-1 equity value is $20 - F - I$, and the date-1 value of the senior (old) debt is $F$ (the junior debt is fairly priced, and hence is worth $I$); and in case $C = 10$, then all securities are worthless.

Now we consider the Pareto optimal financial contracts at date 0. Such a contract must allow the firm to maximize its date-0 value (allowing the firm to adopt as many positive-NPV projects as possible) while allowing all investors to at least break even of the time financing is granted. Let $f(C)$ be the initial investor's payoff at date 2 when $C$ is the date-2 cash inflow. We must look for $f(10)$ and $f(20)$ such that

$$10 - f(10) \geq I;\ 20 - f(20) \geq I;$$

$$0 \leq f(20) \leq 20 - I;\ 0 \leq f(10) \leq 10 - I.$$

We shall maintain the assumption that

$$0 < g \leq 20\pi + 10(1-\pi) - I,$$

so that establishing the firm by spending $g$ in the first place makes sense to the entrepreneur. Note that this assumption implies that

$$g - 10\pi \leq 10 - I.$$

It is easy to show that (i) if $g < 10\pi$, then

$$f(10) = 0,\ f(20) = \frac{g}{\pi};$$

and (ii) if $g \geq 10\pi$, then

$$f(10) = g - 10\pi,\ f(20) = 10 + g - 10\pi,$$

are optimal contracts. (There are other optimal contracts, all leading to the same date-0 value.) Compared to the long-term debt maturing at date 2, these contracts allow the date-2 repayments to the date-0 investor to be indexed by the net present value of the date-1 project. Recall that the above *debt overhang* problem occurs because the firm promises to repay the senior debtholder too much in the poor state $C = 10$. Thus, by indexing the face value of debt to the realization of $C$, the problem is solved; see a profound analysis based on this idea in Froot, K., D. Scharfstein, and J. Stein, 1989, LDC Debt: Forgiveness, Indexation, and Investment Incentives, Journal of Finance, 44, 1335-1350.

There is also a second resolution to the above debt overhang problem: at date 0, issuing a properly designed convertible bond instead of the straight bond. This will give the initial bondholder an option to convert the senior debt into a fraction $\alpha$ of equity right before the firm tries to raise $I$ at date 1.

How does this work? Note that the firm fails to raise $I$ at date 1 if and only if the senior debt was not converted and the face value of the senior debt is $F > C - I$. In this event the senior debt will also be worthless, while by converting and holding a fraction $\alpha$ of the equity, the senior debtholder's payoff will be strictly positive: the new investor will be happy to lend $I$, as he will be the sole debtholder at date 2, and will hence be sure to get back the $I$ dollars he invests at date 1. Thus investment efficiency is attained at date 1.[40]

It remains to compute the pair $(F, \alpha)$, which completely describes the CB issued at date 0. The zero expected profit condition requires only that

$$\pi F + (1 - \pi)\alpha(10 - I) = g,$$

and hence there is more than one solution. For the bondholder to optimally convert in the date-1 subgame where $C = 10$, we need $\alpha > 0$, so that $F < g$. If we do not want the initial investor to convert the bond in the state $C = 20$, then we should choose $F > \alpha(20 - I)$.

Finally, we must re-consider Myers' reasoning that leads to the debt overhang problem. At date 1, when $C = 10$, what prevents the equityholder (assuming there is only one) and the senior debtholder from renegotiating the inefficient debt contract? This is a legitimate question, for both of them will get zero if they choose to do nothing. On the other hand, imagine that the equityholder says to the senior debtholder

[40]The point here is to make sure that the new investor holds the senior claim at date 2, if it is known that $C = 10$ at date 1. Thus, one even simpler solution is to issue outside equity at date 0. That is, in exchange of $g$ dollars raised at date 0, the firm gives the initial investor a fraction $\frac{g}{20\pi+10(1-\pi)-I+g}$ of equity. Can you give a story that justifies the seniority of the financial claim issued at date 0?

that, "if you can just reduce the face value to $x < 10 - I$, then you know that you will receive $x > 0$ for sure at date 2 instead of getting nothing." Of course, any $x \leq 10 - I$ will do, and which $x \in [0, 10 - I]$ will actually prevail at date 1 must depend on the relative bargaining power between the equityholder and the senior debtholder, but as you can see, renegotiation *should* occur, as long as renegotiation is costless (Coase, 1937, *Economica*).

Can renegotiation be costly anyway? Imagine that the senior debt is a corporate bond diffusely held by a large number of small investors. Renegotiation can be costly, although the equityholder may have more bargaining power, in this case. On the other hand, if the initial investor is a commercial bank, then renegotiation may not be very costly, although the equityholder may not enjoy as much bargaining power as when he is faced with a large number of small creditors. Thus the type of the debt instrument (bond or bank loan) and the ownership structure of the debt (diffuse or concentrated) may both affect the possibility of ex-post renegotiation.

Notice that unlike issuing CB or outside equity, the outcome of ex-post renegotiation is not guaranteed. Both the equityholder and the initial investor must form expectations about how much they may respectively get in the stage of renegotiation, and based on these expectations, the terms of the initial debt can be determined at date 0 (via backward induction). Although we are assuming risk neutrality for everyone in this model, it is important to notice the risk involved in the ex-post renegotiation.

37. **Example 12**. (Maksimovic, 1990, *JF*) Consider two firms playing a Cournot game, given the inverse demand

$$p = A - b(q_1 + q_2), \quad A, b > 0.$$

Suppose that the two firms have unit cost of 1 dollar (a normalization). Define $a = A - 1$. Suppose that the two firms have no funds and must borrow from competitive banks, who require a cost of capital $r \in (\frac{a}{6}, a)$. Assume that all players are risk neutral. In this case, firm $i$'s financing cost is

$$r \times \text{total cost} = r \times [1 \times q_i] = rq_i.$$

It follows that firm $i$'s profit as a function of the two firms' quantity profile is

$$\Pi_i(q_i, q_j) = [A - b(q_i + q_j) - 1]q_i - rq_i = [a - r - b(q_i + q_j)]q_i.$$

This game has a unique Nash equilibrium where firm $i$'s reaction function is

$$r_i(q_j) = -\frac{q_j}{2} + \frac{a - r}{2b},$$

and the equilibrium outputs are

$$q_1^* = q_2^* = \frac{a - r}{3b}.$$

Observe from here that firm $i$ will be better off if $r$ can be reduced: expanding output will appear to be credible to firm $j$, and firm $j$ must reduce output to prevent the price from dropping too much in this case.

Refer to the bank that lends to firm $i$ as bank $i$. Imagine that firm $i$ makes the following take-it-or-leave-it offer to bank $i$: I suggest that you charge me a fixed payment $f = \frac{r(a+r)}{3b}$ and lend me the amount $\frac{a+r}{3b}$ at zero interest rate. Bank $i$ should accept this offer, because it earns a rate of return $r$ on each dollar it lends to firm $i$. With the offer being accepted by bank $i$, firm $i$'s reaction function becomes

$$r_i(q_j) = -\frac{q_j}{2} + \frac{a}{2b},$$

so that in equilibrium $q_i = \frac{a+r}{3b}$, which is exactly the amount that firm $i$ borrows from bank $i$ given the above offer. Note that what happens is that firm $i$ successfully expands its output:

$$\frac{a + r}{3b} > \frac{a - r}{3b},$$

which raises $\Pi_i$ at firm $j$'s expense.

Thus it is a good idea for firm $i$ to get the above *loan commitment* from bank $i$. (The fixed payment $f$ is the fee that firm $i$ pays the bank in order to get a committed amount $\frac{a+r}{3b}$ of loan at the very low interest rate (zero interest rate).) The problem here is that firm $j$ would then want to do the same, resulting in a situation like the Pareto dominated equilibrium outcome for the prisoners' dilemma. To formally model the interactions, suppose that the two firms simultaneously propose $(f_i, r_i)$ and $(f_j, r_j)$ to bank $i$ and bank $j$, where $r_i, r_j \in [0, r]$. Either an offer is accepted by the bank, or $(0, r)$ will prevail. Then upon seeing both firms' financial arrangements, the two firms play the above Cournot game. Using backward induction, we first solve for the Cournot game, taking $(r_i, r_j)$ as given. It is easy to derive

$$q_i^*(r_i, r_j) = \frac{a - 2r_i + r_j}{3b},\ i, j \in \{1, 2\},\ i \neq j.$$

Substituting these optimal $q_i^*$'s into $\Pi_i$ and $\Pi_j$, we then consider the first stage of the game where the two firms simultaneously choose $r_i, r_j$. The reaction functions are

$$r_i = \frac{6r - a - r_j}{4},\ i, j \in \{1, 2\},\ i \neq j.$$

Thus we have

$$r_i = r_j = \frac{6r - a}{5} < r.$$

The corresponding

$$f_i = f_j = \frac{r(a - \frac{6r-a}{5})}{3b}.$$

As we expected, both firms expand outputs in equilibrium, resulting in lower profits, since with loan commitments the two firms are operating at lower unit costs (at the expense of higher fixed costs), which prompts them to both choose a higher quantity.

38. (**Example 13**). (Harris and Raviv, 1995, *Review of Financial Studies*) Suppose that at date 0 an enterpreneur (E) wants to raise no less than 75 dollars from competitive investors to implement some investment project. All people are risk neutral without time preferences. There are two equally likely states, and the realization of the true state becomes common knowledge at date 1. In state $s = 1$, the project generates 100 at date 1 and 200 at date 2; and in state $s = 2$, the project generates 40 at date 1 and 200 at date 2. The project can be partially or fully liquidated at date 1, and if fully liquidated, it is worth 100 in state $s = 1$ and 50 in state $s = 2$. We shall assume that at each date, after the cash flow at that date is generated, E can appropriate all of it if he likes. Thus the investors may refuse to commit money at date 0 unless their claims are due at date 1 and they are given the right to liquidate the firm's asset in case they are not fully repaid. Since rational agents will select Pareto optimal outcomes, we would like to know what Pareo optimal financial contracts look like in this game.

We shall distinguish two cases. First, the two parties can sign state-dependent financial contracts. In particular, they can sign debt contracts with face value indexed on the state $s$. Second, the two parties can only sign state-independent contracts. In particular, if they must sign debt contracts, then the face value of debt cannot be made contingent upon the state $s$. Apparently, whether the first case approximates the reality better depends on whether the state $s$ is verifiable in the court of law. But, in either case, recall that cash flows are *not* verifiable: E can take them away!

Consider the first case now. A state-dependent debt contract specifies $(T, F_1, F_2)$, where the chosen investor at date 0 pays E $T + 75$, with $T \geq 0$, and is entitled to be repaid $F_s$ in state $s$. With this class of contracts, the Pareto optimal one is $(0, 110, 40)$.[41] Note that E prefers

[41] Can there be a feasible contract that induces no liquidation in either state? If there is one such $(T, F_1, F_2)$, then the contract must satisfy

$$T + 100 \geq F_1,\ T + 40 \geq F_2,\ T + 100 - F_1 + 200 \geq T + 100,\ T + 40 - F_2 + 200 \geq T + 40,$$

implying that

$$\frac{1}{2}(F_1 + F_2) \leq T + 70,$$

paying cash to getting liquidated, and liquidation is less costly in state $s = 1$. In state $s = 1$, E will then liquidate assets with l0-dollar's worth and pay 100, and in state $s = 2$, E pays $F_2$ in full and avoids liquidation completely. As you can verify, this contract maximizes E's welfare (or minimizes E's expected cost) while keeping the investor's

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which violates the investor's individual rationality (IR) condition,

$$\frac{1}{2}(F_1 + F_2) \geq T + 75.$$

Thus a feasible contract must induce liquidation in either $s = 1$ or $s = 2$. Apparently, liquidation is less costly in state $s = 1$. Let us look for the optimal contract that induces liquidation only in state $s = 1$. Any such contract must satisfy:

$$T+100 < F_1,\ T+40 \geq F_2,\ 2\{100-[F_1-(T+100)]\} \geq T+100,\ T+40-F_2+200 \geq T+40,$$

and moreover,

$$\frac{1}{2}(F_1 + F_2) \geq T + 75.$$

It can be easily verified that the last constraint must be binding (equality must hold) at optimum. We thus search for contracts that satisfy the above constraints that maximize E's expected payoff at date 0, which is

$$\frac{1}{2}\left\{2\{100 - [F_1 - (T + 100)]\} + (T + 40 - F_2 + 200)\right\}.$$

The solution to the above constrained maximization problem is $(T^*, T^* + 110, T^* + 40)$, where $T^*$ is non-negative. Hence $(0, 110, 40)$ is optimal.

One can also derive the (dominated) contract that induces liquidation only in state $s = 2$. The corresponding maximization program is as follows.

$$\max_{(T,F_1,F_2)} \frac{1}{2}[(T + 100 - F_1 + 200) + 4(50 - [F_2 - (T + 40)])]$$

subject to

$$F_2 > T + 40, \quad F_1 \leq T + 100,$$
$$F_1 \leq 200, \quad 3T - 4F_2 + 320 \geq 0,$$

and

$$\frac{1}{2}(F_1 + F_2) = T + 75.$$

One can show that the optimal contract takes the form of $(T^*, T^* + 100, T^* + 50)$, which incurs a 10-dollar liquidation in state $s = 2$. Apparently, this contract is dominated by the optimal contract that induces liquidation only in state $s = 1$.

ex-ante individual rationality condition binding.

What happens if state-dependent contracts are infeasible? In this case, one face value $F$ of debt will be chosen to apply to both date-1 states, and it is clear that in this case at least in one state $s$, the date-0 chosen $F$ will not be Pareto optimal at date 1. That is, under the old contract signed at date 0, at least in one state $s$ the firm will liquidate too much, and it is feasible to find a Pareto improving new contract for E and the creditor such that under the new contract everyone can be made better off than under the old contract. Since renegotiation will generally occur at date 1, it matters how renegotiation proceeds. In the following we refine our attention to two cases, one where in renegotiation E can make a take-it-or-leave-it offer to the creditor, and the other where the creditor can make a take-it-or-leave-it offer to E. We assume that the choice between the two renegotiation processes can be stated in the date-0 old contract. Hence a debt contract in the current case is a triple $(i, T, F)$, where $i$, which is either the creditor or the debtor, represents the contracting party who has the right to make the take-it-or-leave-it renegotiation offer to the other party at date 1.

Let us call the debt contract $(T, F)$ with $i$ being the (initial) investor the *creditor-favored* debt contract. The creditor, in making a renegotiation offer, can propose to reduce the face value of debt from $F$ to a new face value $F' \leq F$, and then E must repay accordingly by either cash or the proceeds from asset liquidation; and if E does not repay accordingly, then the creditor can liquidate as much of the asset as needed to recoup up to the face value $F'$. It can be verified that the optimal creditor-favored debt contract consists of $(T, F) = (104, 200)$.[42]

[42]If liquidation occurs in state $s = 1$, then it has to occur in state $s = 2$ also. It is easy to show that no contracts can avoid liquidation in both states. Hence we may first look for a contract that implies liquidation only in state $s = 2$. If such a contract exists with $F'$ being the new face value in state $s = 2$ following renegotiation (where at this point, it remains a possibility that $F = F'$), then $F'$ must be the largest one satisfying $F \geq F' \geq 50$, and $4(50 - [F' - (T + 40)]) \geq T + 40$. The first set of inequalities say that in renegotiation face value of debt cannot move up, and the investor (creditor) can ensure at least a payoff of 50 by liquidating the whole firm. The second inequality makes sure that E is willing to accept $F'$ instead of taking $T + 40$ and running away. This gives the subgame equilibrium $F' = 80 + \frac{3}{4}T$ in state $s = 2$. Move backward to consider E's date-0

Given this contract, in state $s = 1$, E has cash $T + 100 = 204$, and hence the investor will propose $F' = F = 200$ and no liquidation will take place; but in state $s = 2$, E has cash $T + 40 = 144$, lower than $F = 200$. If the investor does not offer some $F' < F$, then E is better off taking the money at hand and run: if $F' = 158$, then by honoring the debt, E pays in total $144 + (158 - 144) \times \frac{200}{50} = 200$, which is equal to what E may lose if he chooses to simply take money at hand and forget about the income 200 at date 2. For the investor, he prefers that E honors the debt: taking 158 is better than taking 50. Thus in state $s = 2$, the investor proposes $F' = 158$, and E, besides paying cash, liquidates some assets to generate the remaining 14.

Note that the creditor-favored debt contract is rather inefficient: liquidation occurs in the wrong state (which arises because the same face

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contract choice. E seeks to maximize $\frac{1}{2}[100 + T - F + 200] + \frac{1}{2}[T + 40]$, where in state $s = 2$, note that E's payoff consists of the date-1 cash only, for the creditor has all the bargaining power in renegotiation. The constraints facing E include:

- $T + 75 \leq \frac{F+F'}{2}$, which is the creditor's break-even condition, with everyone knowing that renegotiation will occur in state $s = 2$ and a new face value of debt $F'$ will replace $F$, where $F' = 80 + \frac{3}{4}T$;
- $F \leq 100 + T$, so that the firm is solvent in state $s = 1$;
- $F \geq 40 + T$, so that the firm is not solvent in state $s = 2$;
- $100 + T - F + 200 \geq 100 + T$, so that in the solvent state E will not run away with cash; and
- $T + 40 \geq 4 \max(0, 50 - [F - (T + 40)])$, so that with the old contract, E would rather take the cash and run away, and that is why renegotiation will occur.

It can be shown that the creditor's break-even condition will be binding at optimum, implying that $F = 70 + \frac{5}{4}T$. Replacing $F', F$ by respectively, $80 + \frac{3}{4}T$ and $70 + \frac{5}{4}T$, and maximizing E's date-0 payoff with respect to $T$ subject to the remaining constraints, we have the optimal $T = 104$. It follows that $F = 200$ and $F' = 158$.

Why is $T > 0$ at optimum? Note that in renegotiation in state $s = 2$ the creditor will make sure that E gets the payoff $4(50 - [F' - (T + 40)]) = T + 40$, where the left-hand side tells us that an increase in $T$ by one dollar implies a reduction in the amount of liquidated assets by $\frac{1}{4}$ dollars. This efficiency enhancement improves E's welfare because the creditor will simply break even in equilibrium. But then, how come we did not raise $T$ beyond 104? Note that in order for E to not default strategically in state $s = 1$, it is necessary that $F \leq 200$, but if $T$ is greater than 104, the creditor would not be able to break even given that $F \leq 200$; recall that given $T$, $F' = 80 + \frac{3}{4}T$.

value $F$ must be applied to both date-1 states), and the assets have been liquidated *too much*! Why? This is because at the renegotiation stage, the creditor has full bargaining power.[43] Thus the proceeds from asset liquidation must be very high, because the cash that E has in state $s = 2$ is very little. From here, it is now understandable why $T > 0$ may help: in equilibrium, a fraction $\frac{\frac{200-(T+40)}{4}}{50}$ of date-1 assets must be liquidated in state $s = 2$, which is decreasing in $T$. Note that, however, $T$ cannot be too high either: intuitively, if $T$ is too high, then E will choose to steal the money and run away in state $s = 1$ at date 1.[44]

Finally, consider the debtor-favored debt contract. At the renegotiation stage, E simply offers some cash and some liquidation proceeds to replace the existing debt contract, and if the creditor rejects the offer,

[43] At first, renegotiation must maximize ex-post efficiency, and it is efficient in state $s = 2$ that the creditor gets the cash at date 1 and E gets the (remaining) cash at date 2. With full bargaining power, the creditor only leaves $T + 40 = 144$ to E at date 2, so that (200-144)/4=14 out of 50 of the date-1 assets will be liquidated. Note that the role of $T = 104$ in reducing the amount of liquidated assets at date 1.

[44] In state $s = 2$, E can get $T + 20$ by running away with the cash earnings, and to induce E to disgorge the cash earnings plus $T$ (which is necessary in order that the amount of liquidated assets can be reduced) the creditor will always choose $F'$ to make E feel indifferent about running away with cash earnings and repaying the debt. Note that increasing $T$ by one dollar raises E's payoff from running away with cash earnings by one dollar. Thus increasing $T$ by one dollar can reduce the amount of liquidated assets by $\frac{1}{4}$ dollars in state $s = 2$! Now, this should induce E to keep raising $T$ when designing the date-0 contract. However, in raising $T$, E should make sure that E will not run away with cash earnings in state $s = 1$ also! In other words, E must make sure that $F \leq 200$, where $F$, the old face value of debt determined at date 0, is the cost that E must incur if E chooses to repay the debt in state $s = 1$ at date 1, and 200 is the cost that E must incur if E chooses to run away with cash earnings plus $T$ in state $s = 1$ at date 1. Now, by the fact that $F'$ must make E indifferent about repaying and not repaying the debt in state $s = 2$, we have

$$4[50 - (F' - (T + 40))] = T + 40 \Rightarrow F' = 80 + \frac{3}{4}T.$$

Since the creditor must break even at date 0, we have

$$F + F' = (T + 75) \times 2 \Rightarrow F = 70 + \frac{5}{4}T.$$

Thus we require $70 + \frac{5}{4}T \leq 200$, implying that the optimal choice of $T$ is 104.

then E simply takes the cash, leaving the assets in place to the creditor. It can be verified that $(T, F) = (0, 100)$ is an optimal contract in this class.[45] With this contract, in state $s = 1$, E offers 100 cash to the investor, since if E does not, the investor can still get the full face value by selling all the assets in place; and in state $s = 2$, E offers 40 in cash and 10 from liquidation proceeds to replace the old debt contract, which the investor cannot reject (why not?).

Thus the allocation of bargaining power at the renegotiation stage does matter. In the creditor-favored contract, E gets no rent at date 1: his expected cost is 200 in the state $s = 2$ even if E has decided to repay the debt at date 1; and in debtor-favored contract, the creditor gets no rent at date 1: the creditor's accepting E's offer does prevent the assets in place from being liquidated, but E gets all the efficiency gain! It can be shown that creditor-favored debt contract is never optimal, but neither is the debtor-favored debt contract. We shall discuss the optimal contracts for this game when we review the literature of optimal financial contracts.

[45] At first, observe that at date 1, under the old contract, the payoff for the investor (creditor) never exceeds the minimum of $F$ and the liquidation value of the firm: if $F$ is equal to that minimum, then E will choose to repay the debt (probably by partially liquidating the assets in place) instead of leaving the whole firm to the creditor, and if $F$ is greater than that minimum, then E can force the creditor to accept a a repayment equal to the liquidation value of the firm, by threatening to leave the whole firm to the creditor if the creditor dares to turn down E's renegotiation offer. With this observation in mind, we see that E will force the investor to receive only 50 in state $s = 2$, which is what the investor would get if E simply took the cash 40 and run; and similarly the investor can get no more than the liquidation value 100 of the assets in place in state $s = 1$. Since the investor must provide at least 75, there is only one solution to the date-0 contract, which is $(T = 0, F = 100)$.