Game Theory with Applications to Finance and Marketing, I

Solutions to the Final Exam

Name: __________ ID: __________

Questions	A	B	C	D	E
Solutions	$\frac{5}{2c}$	$\frac{25}{36c^2}$	$\frac{65}{6c}$	$\frac{25}{4c^2}$	$\frac{175}{72c}$
Questions	F	G	H	I	J
Solutions	$\frac{3}{4}$	$\frac{2}{3}$	-1	0	$\frac{5}{4}$
Questions	K	L	M	N	O
Solutions	$300 + 450a$	$\frac{13}{22}$	$\frac{10}{23}$	750	375
Questions	P	Q	R	S	T
Solutions	$\frac{75}{4}$	$\frac{23}{2}$	$\frac{75}{4}$	12	$\frac{1}{2}$
Questions	U	V	W	X	Y
Solutions	$\frac{9}{2}$	anything	5	anything	ABC

1. Recall the screening game reviewed in sections 3-6 in Lecture 4, where a seller must design the optimal *non-linear pricing scheme* to serve n segments of consumers. In this exercise, we assume that

$$n = 3,\ V(q) = \sqrt{q},\ \theta_1 = 1,\ \theta_2 = 3,\ \theta_3 = 5,\ \pi_1 = \frac{1}{6},\ \pi_2 = \frac{1}{2},\ \pi_3 = \frac{1}{3}.$$

By **Theoerm AS-1**, given $c > 0$, the second-best solution must be such that $T_2 =$ ___A___, $q_2 =$ ___B___, $T_3 =$ ___C___, $q_3 =$ ___D___, so that the seller's second-best payoff is equal to ___E___.

Solution. At optimum, IR_1 and LDICs must be binding, and hence we have

$$T_1 = \sqrt{q_1},\ T_2 = 3\sqrt{q_2} - 3\sqrt{q_1} + T_1,\ T_3 = 5\sqrt{q_3} - 5\sqrt{q_2} + T_2,$$

so that the seller seeks to

$$\pi_1(\sqrt{q_1} - cq_1) + \pi_2(3\sqrt{q_2} - 3\sqrt{q_1} + \sqrt{q_1} - cq_2)$$
$$+\pi_3(5\sqrt{q_3} - 5\sqrt{q_2} + 3\sqrt{q_2} - 3\sqrt{q_1} + \sqrt{q_1} - cq_3),$$

subject to

$$0 \le q_1 \le q_2 \le q_3.$$

Note that

$$\frac{1}{6} = \pi_1 < 2(\pi_2 + \pi_3) = \frac{10}{6},$$

so that the seller's objective function is decreasing in q_1, and hence we have

$$q_1 = T_1 = 0.$$

It is easy to obtain that

$$q_2 = \frac{25}{36c^2},\ q_3 = \frac{25}{4c^2} \Rightarrow T_2 = \frac{5}{2c},\ T_3 = \frac{65}{6c},$$

so that the seller's second-best payoff is equal to $\frac{175}{72c}$.

2. Consider the following static game with incomplete information, where before the two players simultaneously choose between C and D, player 1 privately learns about the realization of $\tilde{x}$, and player 2 privately learns about the realization of $\tilde{y}$, where it is ex-ante common knowledge that $\tilde{x}$ and $\tilde{y}$ are independently and identically uniformly distributed over the interval $(-a, a)$, where $a > 0$.

Player 1/Player 2	D	C
D	$0, 0$	$3 + \tilde{x}, 1$
C	$1, 4 + \tilde{y}$	$1, 1$

(i) First assume that $\tilde{x} = \tilde{y} = 0$. This complete-information game has a (non-pure) mixed strategy equilibrium, where player 1 may play D with probability ___F____ and player 2 may play D with probability ___G___.
(ii) Next, return to the original static game with incomplete information and assume that $a = 2$. This game has a unique pure-strategy Bayesian equilibrium where player 1 would play D if and only if the realization of $\tilde{x}$ is no less than ___H____ and where player 2 would play D if and only if the realization of $\tilde{y}$ is no less than ___I____. In equilibrium, when player 2 learns that the realization of $\tilde{y}$ is 1, player 2's equilibrium payoff is equal to ___J____.

Solution. Part (i) is easy. Suppose players 1 and 2 play D with respectively probabilities π_1 and π_2. Then we require that

$$(-1)\pi_2 + 2(1 - \pi_2) = 0,$$

$$(-1)\pi_1 + 3(1 - \pi_1) = 0,$$

so that

$$\pi_1 = \frac{3}{4}, \pi_2 = \frac{2}{3}.$$

Now, consider part (ii). Since player 1's payoff from playing D increases with $\tilde{x}$ and player 2's payoff from playing D increases with $\tilde{y}$, we conjectur that there exist x^* and y^* such that player 1 would rather play

D than C if and only if $\tilde{x} > x^*$ and player 2 would rather play D than C if and only if $\tilde{y} > y^*$. This implies that player 1 would feel indifferent about D and C when $\tilde{x} = x^*$ and player 2 would feel indifferent about D and C when $\tilde{y} = y^*$. This gives rise to the following system of equations:

$$1 = \frac{y^* - (-a)}{2a}(3 + x^*),\ 1 = \frac{x^* - (-a)}{2a}(4 + y^*),$$

and upon solving this system of equations using $a = 2$, we obtain

$$x^* = -1,\ y^* = 0.$$

When player 2 learns that the realization of $\tilde{y}$ is 1, player 2's equilibrium payoff is equal to

$$\frac{x^* - (-a)}{2a}(4 + y) = \frac{x^* - (-a)}{2a}(4 + 1) = \frac{5}{4}.$$

3. Firm Y has a single owner-manager Mr. Y, who needs to raise \$500 for a positive-NPV investment project at date 0. (Firm Y is Mr. Y's sole asset.) There are two possible date-0 states, called G and B, and the date-0 state is Mr. Y's private information. In state G, the assets in place of firm A are worth \$750 and the new project's NPV equals \$100. In state B, the assets in place are worth only $\$x$ and the new project's NPV is accordingly $\$y$. The public investors (also referred to as the outsiders) believe that the state may be G with prob. a. Mr. Y and public investors are all risk-neutral without time preferences.

   The game proceeds as follows. Mr. Y first decides to or not to issue new equity to raise \$500 (two feasible signals!). We assume that after the firm succeeds in raising the \$500, Mr. Y *must* spend it on the new investment project, regardless of the state. Then, upon seeing Mr. Y's decision, the public investors form posterior beliefs about the date-0 state, and they engage in Bertrand competition to determine the fraction α of equity that Mr. Y must sell in order to raise \$500. For $j \in \{G, B\}$, we shall refer to the type-j Mr. Y as simply *type* j.

Moreover, by *type j's payoff* we mean Mr. Y's expected *wealth* in state $j \in \{G, B\}$.

(i) Suppose that $x = 250$ and $y = 50$. It can be shown that this game has one separating PBE and may also have one pooling PBE. Before learning his own type, given a, Mr. Y's payoff in the unique separating PBE is equal to __K__. This game has a pooling PBE if and only if $a \geq \hat{a}$, where $\hat{a} =$ __L__. In particular, when $a = \frac{7}{11}$, to raise \$500 in a pooling PBE Mr. Y must sell a fraction α^p of equity, where $\alpha^p =$ __M__.

(ii) Suppose that $x = 300$ and $y = -125$. In this case, a pooling PBE where both types of the firm choose to issue new equity exists if and only if the prior probability a for the good state is such that $a \geq a'$, and if the public investors' prior probability for state G is exactly $a = a'$, then in this pooling PBE type G's payoff is __N__ while type B's payoff is __O__.

Solution. This is exactly the last exercise of Homework 5; the current payoffs are five times of the payoffs in that exercise. Thus you can look up the solution in the file ngt2019sh5.pdf. In particular, in part (ii), we have $a' = \frac{2}{3}$, implying that when $a = \frac{2}{3}$, the firm must sell $\frac{4}{9}$ of the equity to the public investors in the pooling PBE, so that type G's equilibrium payoff is $(1 - \frac{4}{9})(750 + 100 + 500) = 750$ and type B's equilibrium payoff is $(1 - \frac{4}{9})(300 - 125 + 500) = 375$.

4. Recall the reputation game that we solved in Quiz 10, where there is a Cournot-competitive industry that extends for 3 periods with one incumbent firm and three potential entrants, E_j, $j = 1, 2, \cdots, 3$. There is no discounting, and each firm seeks to maximize (the sum of) expected profits. Thus the incumbent's payoff is defined as the sum of expected profits that it makes over the three operating periods.

   (i) Suppose that $x_1 = \frac{1}{2}$, $k = \frac{7}{3}$, and $A = 5$. Then the type-0 incumbent's equilibrium payoff is equal to __P__ and the type-1 incumbent's equilibrium payoff is equal to ___Q___.

(ii) Suppose instead that $x_1 = \frac{2}{3}$, $k = \frac{8}{3}$, and $A = 6$. Then the type-0 incumbent's equilibrium payoff is equal to ___R___ and the type-1 incumbent's equilibrium payoff is equal to ___S___. The sum of the three entrants' equilibrium payoffs is equal to ___T___.

Solution. Consider part (i). Since $x_1 < 1 + 2k - A$, there is pooling at date 1 and date 2, and all three entrants would stay out. Thus both types of the incumbent would produce $\frac{A}{2}$ at date 1 and date 2, leading to a price of $\frac{A}{2}$. At date 3, there is a separating outcome, where the type-0 incumbent would again produce $\frac{A}{2}$, but the type-1 incumbent would produce $\frac{A-1}{2}$. Thus the type-0 incumbent's equilibrium payoff is

$$3 \times \frac{A^2}{4} = \frac{75}{4},$$

and the type-1 incumbent's equilibrium payoff is

$$2 \times \frac{A}{2}(A - \frac{A}{2} - 1) + \frac{(A-1)^2}{4} = \frac{23}{2}.$$

Consider part (ii). Since $x_1 \geq 1+2k-A$, there is separating at date 1, and only E_1 enters in equilibrium. Thus E_2 and E_3 have zero payoffs in equilibrium. At each date, the type-0 incumbent would produce $\frac{A-1}{2}$, so that its equilibrium payoff is $3 \times \frac{(A-1)^2}{4} = \frac{75}{4}$; and the type-1 incumbent would produce $\frac{A-2}{2}$, so that its equilibrium payoff is $3 \times \frac{(A-2)^2}{4} = 12$. Finally, we compute E_1's equilibrium payoff. Following its entry, with probability x_1 the incumbent is of type 1, so that the product price would be $A - 1 - \frac{A-2}{2} = \frac{A}{2} = 3$; and with probability $1 - x_1$ the incumbent is of type 0, so that the product price would be $A-1-\frac{A-1}{2} = \frac{A-1}{2} = \frac{5}{2}$. Thus E_1's equilibrium payoff is

$$-3k + 3 \times [x_1 \cdot 3 + (1 - x_1) \cdot \frac{5}{2}] = \frac{1}{2}.$$

5. Consider the following signaling game taken from Example 3 (pages 32-36) of Lecture 4 but with different payoff functions. (Again, player 1 is equally likely to be of type t_1 or type t_2.)

m_1	a_1	a_2	a_3
t_1	$(6, 1)$	$(2, 4)$	$(2, 5)$
t_2	$(5, 5)$	$(5, 4)$	$(3, 1)$

m_2	a_1	a_2	a_3
t_1	$(4, 1)$	$(2, 4)$	$(5, 3)$
t_2	$(2, 4)$	$(2, 1)$	$(4, 3)$

m_3	a_1	a_2	a_3
t_1	$(4, 4)$	$(3, 1)$	$(3, 3)$
t_2	$(6, 4)$	$(2, 5)$	$(5, 1)$

(i) This game has a pooling PBE where after player 1 sends the equilibrium signal, player 2's best response is a_3. In this PBE, before learning his own type, player 1's expected payoff is equal to __U__. To sustain this PBE, we must require that $b \leq \mu_2 \leq a$, where $a+b =$ __V__.

(ii) This game also has another PBE where after player 1 sends the equilibrium signal, player 2's best response is a_1. In this PBE, before learning his own type, player 1's expected payoff is equal to __W__. To sustain this PBE, either $\mu_2 \leq c$ or $\mu_2 \geq d$ must hold, where $d - c =$ __X__.

(iii) Which statements below are correct? __Y__.
(A) The PBE described in part (i) is an intuitive equilibrium.
(B) The PBE described in part (ii) is an intuitive equilibrium.
(C) In the PBE described in part (ii), player 2's equilibrium payoff is equal to 4.
(D) None of the above is true.

Solution. It is a routine to verify that this game has two pooling PBEs: (1) $x = y = m_2,\ z = a_3,\ \mu_1 \in [\frac{3}{4}, 1],\ \mu_2 = \frac{1}{2},\ \mu_3 \in [0, \frac{1}{4}]$; (2) $x = y = m_3,\ z = a_1,\ \mu_1 \in [\frac{1}{4}, 1],\ \mu_2 \in [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1],\ \mu_3 = \frac{1}{2}$.

For part (i), it is easy to see that if m_1 or m_2 is the signal sent by player 1 in a pooling PBE, then a_3 is *not* player 2's equilibrium best response. Thus player 1 must send a_2 in equilibrium, and hence $\mu_2 = \frac{1}{2}$, so that if a, b are any real numbers satisfying $a \geq \frac{1}{2} \geq b$, then $a + b$ can be any real number. Player 1's equilibrium payoff is $\frac{5+4}{2} = \frac{9}{2}$.

For part (ii), note that if $\mu_2 \in [0, \frac{1}{3}]$ then $c \in [\frac{1}{3}, +\infty)$ and if $\mu_2 \in [\frac{2}{3}, 1]$, then $d \in (-\infty, \frac{2}{3}]$. Thus, once again, $d - c$ can be any real number.

We claim that both PBEs (1) and (2) satisfy the intuitive criterion. Take (1) for example. The deviation with m_1 fails the first supposition.The deviation with m_3 satisfies the first supposition as t_1 could do strictly better by sticking to its equilibrium signal. However, given that the deviator in this case must be t_2, the uninformed player would choose a_2 so that t_2 would rather stick to its equilibrium signal also. Thus PBE (i) is intuitive. The reasoning that (2) is intuitive is similar.