Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 2

1. Consider the following strategic game:

player 1/player 2	L	R
U	1,1	0,0
D	0,0	3,2

Any NE can be represented by (p, q), where p is the probability that player 1 adopts U and q the probability that player 2 adopts L.

(i) Show that this game has 3 NE's: (1,1), (0,0), and $(\frac{2}{3}, \frac{3}{4})$.

(ii) Now, consider the following new version of the above strategic game. At the first stage, player 1 can invite either A or B to become player 2 for the above strategic game. At the second stage, player 1 and the selected player 2 then play the above strategic game. A (or B) gets the player 2's payoffs described in the above strategic game, if he accepts the invitation to play the game. Without playing the game, A can get a payoff of $\frac{1}{200}$ on his own, and B can get a payoff of $\frac{3}{2}$ on his own.

The game proceeds as follows. First, player 1 can invite either A or B, and if the invitation is accepted, then the game moves on to the second stage; and if the invitation gets turned down, then player 1 can invite the other candidate. If both A and B turn down player 1's invitations, then the game ends with A getting $\frac{1}{200}$, B getting $\frac{3}{2}$, and player 1 getting 0.

Which one between A and B should player 1 invite first? Compute player 1's equilibrium payoff.

Solution. Part (i) is straightforward. Player 2's payoff is 1, 2, and $\frac{2}{3}$ in respectively the equilibria $(p, q) = (1, 1)$, $(0, 0)$, and $(\frac{2}{3}, \frac{3}{4})$.

Consider part (ii). If player 1 invites A first, then A will get $\frac{1}{200}$ if A turns down the invitation, and A will get at least $\frac{2}{3}$ if A accepts the invitation. Thus A will always accept player 1's invitation. Player 1 will not get the chance to invite B again. Thus player 1's payoff from inviting A first may equal 1, or 3, or $\frac{3}{4}$.

On the other hand, if player 1 invites B first, then B will get $\frac{3}{2}$ if B turns down the invitation, and B will get more than $\frac{3}{2}$ if and only if B expects to attain the equilibrium $(0,0)$ subsequently. Thus B will accept player 1's invitation if and only if B is prepared to play L with probability one in the strategic game subsequently. Thus when B turns down player 1's invitation player 1 will get the same payoff as he would when he invited A first, and when B accepts player 1's invitation player 1 would get the payoff of 3 for sure. To sum up, forward induction implies that player 1 should invite B first.

Remark. When a firm recruits new employees, it typically gives offers first to those job applicants that other firms would also like to recruit, even if all job applicants are expected to deliver similar job performances once recruited. This exercise gives an explanation to this phenomenon. A newly recruited job applicant that gives up a high salary that he or she could otherwise have by accepting another job opportunity signals that he or she intends to work hard, and that he or she expects to earn more by working hard given that his or her intention is correctly understood (via forward induction) by the employer (so that the employer is also expected to work hard accordingly).

We have assumed that A and B do not know their co-existence, as in the case of a firm recruiting new employees. In this case, A and player 1 must interact without knowing the presence of B, and similarly, B must interact with player 1 without knowing the presence of A. We show that it is a better choice for player 1 to contact B first, which would allow player 1 to use forward induction and to ensure (D,R) as the unique equilibrium outcome after B accepts the job offer (and B will because B knows that player 1 would interpret B's accepting the offer as a clear indication that B is planning to play R).

If instead it is common knowledge that A and B both exist and have the assumed reservation payoffs, then forward induction can be used by all three players. In the latter case, player 1 can ensure that the (D,R) equilibrium will prevail no matter which job applicant he is to contact first. Essentially, player 1 can ensure the (D,R) equilibrium by first contacting B, and hence when player 1 actually chooses to contact A first, A must interpret that player 1 is planning to play D in the subsequent normal form game, and in response A would then play R with probability one.

2. Consider the following strategic game:

player 1/player 2	L	M	R
U	2,0	2,2	4,4
M	6,8	8,4	5,0
D	10,6	4,4	6,5

(i) Assume that players are restricted to using only pure strategies. Find the strategy profiles that survive the procedure of *iterative deletion of strictly dominated strategies.*

(ii) Assume that players are restricted to using only pure strategies. Find the strategy profiles that survive the procedure of *iterative deletion of non-best-response strategies.*

(iii) How would your solutions for parts (i) and (ii) change if players are allowed to use also mixed strategies?[1]

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[1]**Hint**: Define for part (i)

$$S_1^0 = S_1 = \{U, M, D\}, \quad S_2^0 = S_2 = \{L, M, R\},$$

and let S_j^n be the subset of S_j^{n-1} such that S_j^n contains player j's pure strategies that are not strictly dominated when player i is restricted to using only pure strategies contained in S_i^{n-1}. Then define

$$S_1^\infty \equiv \bigcap_{n=1}^{\infty} S_1^n, \quad S_2^\infty \equiv \bigcap_{n=1}^{\infty} S_2^n.$$

Solution. Consider part (i). Define

$$S_1^0 = S_1 = \{U, M, D\}, \quad S_2^0 = S_2 = \{L, M, R\}.$$

Let S_j^n be the subset of S_j^{n-1} such that S_j^n contains player j's pure strategies that are not strictly dominated when player i is restricted to using only pure strategies contained in S_i^{n-1}. Then we have

$$S_1^1 = \{M, D\}, \quad S_2^1 = \{L, M, R\},$$

$$S_1^2 = \{M, D\}, \quad S_2^2 = \{L\},$$

$$S_1^3 = \{D\}, \quad S_2^3 = \{L\},$$

$$S_1^n = \{D\}, \quad S_2^n = \{L\}, \ \forall n \geq 3,$$

and hence

$$S_1^\infty \equiv \bigcap_{n=1}^{\infty} S_1^n = \{D\}, \quad S_2^\infty \equiv \bigcap_{n=1}^{\infty} S_2^n = \{L\}.$$

That is, in this game $\{D, L\}$ is the unique strategy profile that survives the procedure of *iterative deletion of strictly dominated strategies.*

Consider part (ii). Define

$$H_1^0 = S_1 = \{U, M, D\}, \quad H_2^0 = S_2 = \{L, M, R\}.$$

The strategy profiles that survive the procedure of *iterative deletion of strictly dominated strategies* are the elements of the Cartesian product $S_1^\infty \times S_2^\infty$.

Define for part (ii)

$$H_1^0 = S_1 = \{U, M, D\}, \quad H_2^0 = S_2 = \{L, M, R\},$$

and let H_j^n be the subset of H_j^{n-1} such that H_j^n contains all player j's pure-strategy best responses when player i is restricted to using only pure strategies contained in H_i^{n-1}. Then define

$$H_1^\infty \equiv \bigcap_{n=1}^{\infty} H_1^n \quad H_2^\infty \equiv \bigcap_{n=1}^{\infty} H_2^n.$$

The strategy profiles that survive the procedure of *iterative deletion of non-best-response strategies* are the elements of the Cartesian product $H_1^\infty \times H_2^\infty$.

Let H_j^n be the subset of H_j^{n-1} such that H_j^n contains all player j's pure-strategy best responses when player i is restricted to using only pure strategies contained in H_i^{n-1}. Then we have

$$H_1^1 = \{M, D\}, \quad H_2^1 = \{L, R\},$$

$$H_1^2 = \{D\}, \quad H_2^2 = \{L\},$$

$$H_1^n = \{D\}, \quad H_2^n = \{L\}, \; \forall n \geq 2,$$

and hence

$$H_1^\infty \equiv \bigcap_{n=1}^{\infty} H_1^n = \{D\}, \quad H_2^\infty \equiv \bigcap_{n=1}^{\infty} H_2^n = \{L\}.$$

That is, in this game $\{D, L\}$ is the unique strategy profile that survives the procedure of *iterative deletion of non-best-response strategies*.

Now, consider part (iii). First consider allowing mixed strategies in part (i). Let Σ_j be the set of mixed strategies available for player j. Let $\Sigma_j^0 = \Sigma_j$, and Σ_j^n be the set of elements in Σ_j^{n-1} that are not strictly dominated mixed strategies from player j's perspective, given that player i is restricted to using mixed strategies in Σ_i^{n-1}. Now, observe that Σ_1^1 does not contain any mixed strategies that assign a positive probability to U. That is, Σ_1^1 contains only (some) mixed strategies that randomize over M or D. Observe also that against any element in Σ_1^1, L strictly dominates any other mixed strategy for player 2. Thus $\Sigma_2^n = \{L\}, \; \forall n \geq 2$, so that $\Sigma_1^n = \{D\}, \; \forall n \geq 3$. It follows that

$$\Sigma_1^\infty \equiv \bigcap_{n=1}^{\infty} \Sigma_1^n = \{D\}, \quad \Sigma_2^\infty \equiv \bigcap_{n=1}^{\infty} \Sigma_2^n = \{L\},$$

and hence $\{D, L\}$ must be the unique strategy profile that survives the procedure of *iterative deletion of strictly dominated strategies*, even if the players are allowed to use mixed strategies.

Finally, consider allowing mixed strategies in part (ii). It is clear that each mixed strategy of player 1 that can become a best response against player 2 using any mixed strategies must assign zero probability to U.

Player 2's best response (in mixed strategy) against any mixed strategy of player 1 that assigns zero probability to U is L. Player 1's best response (in mixed strategy) against L is D. It follows that, in this game, $\{D, L\}$ must be the unique strategy profile that survives the procedure of *iterative deletion of non-best-response strategies*, even if the players are allowed to use mixed strategies.

3. Players 1 and 2 are living in a city where on each day the weather is equally likely to be sunny (S), cloudy (C), or rainy (R). Players 1 and 2 are supposed to play the following strategic game at date 1.

player 1/player 2	L	R
U	15,3	0,0
D	12,12	3,15

(i) Suppose that the above strategic game must be played before players 1 and 2 know anything about the date-1 weather. Verify that the game has two pure-strategy NE's and one mixed-strategy NE. Suppose that before playing the strategic game, players 1 and 2 both believe that they may attain each pure-strategy NE with probability $a < \frac{1}{2}$ and they may attain the mixed-strategy NE with probability $1 - 2a$. Compute the expected Nash-equilibrium payoff for player 1 given a.

(ii) Now, suppose that for $i = 1, 2$, player i receives a weather report s_i right before playing the above strategic game at date 1. The weather report s_1 tells player 1 whether the weather will or will not be sunny. The weather report s_2 tells player 2 whether the weather will or will not be rainy. That the two players will receive these two weather reports is their common knowledge at the beginning of date 1. Consider the following strategy profile:

- Player 1 uses U if the weather will be sunny, and he uses D if the weather will not be sunny.
- Player 2 uses R if the weather will be rainy, and he uses L if the weather will not be rainy.

Does this strategy profile constitute a Nash equilibrium?[2] If it does, compute player 1's equilibrium payoff. Compare this payoff to player 1's expected Nash-equilibrium payoff that you obtained in part (i). Explain.[3]

Solution. Consider part (i). Let p be the probability that player 1 may use U, and q the probability that player 2 may use L. We have 3 NE's for this game, in which (p, q) equals respectively $(1, 1)$, $(0, 0)$, and $(\frac{1}{2}, \frac{1}{2})$. Given a, player 1's expected Nash-equilibrium payoff is equal to

$$a \cdot 15 + a \cdot 3 + (1 - 2a) \cdot \frac{1}{4}(15 + 0 + 12 + 3)$$

$$= 18a + \frac{15}{2} - 15a = 3a + \frac{15}{2}.$$

Consider part (ii).

- First suppose that the true weather state is sunny.

  In this event, player 1 knows that the state is sunny, and he knows that player 2 knows that the state is not rainy, and according to player 2's strategy described above, player 1 expects player 2 to

[2]This strategy profile is not an NE of the original strategic game without weather reports, which has been analyzed in part (i). In part (ii), with weather reports, we have a new game where players' strategies are functions that map weather information into actions.

[3]**Hint**: Show that

- when the state is sunny, given player 2's strategy described above it is optimal for player 1 to use U, and given player 1's strategy described above it is optimal for player 2 to use L;
- when the state is cloudy, given player 2's strategy described above it is optimal for player 1 to use D, and given player 1's strategy described above it is optimal for player 2 to use L; and
- when the state is rainy, given player 2's strategy described above it is optimal for player 1 to use D, and given player 1's strategy described above it is optimal for player 2 to use R.

use L with probability one. Player 1's best response against player 2 using L is indeed U, according to our analysis in part (i).

On the other hand, player 2 knows that the weather state is not rainy, and hence is equally likely to be sunny or cloudy, and according to player 1's strategy described above, player 2 expects player 1 to use U or D with equal probability. It is clear from our analysis in part (i) that player 2 indeed feels indifferent about using L or R, and in equilibrium player 2 uses L with probability one.

- Next, suppose that the true weather state is cloudy.

  In this event, player 1 knows that the state is not sunny, and hence is equally likely to be cloudy or rainy, and according to player 2's strategy described above, player 1 expects player 2 to use L and R with equal probability. Player 1 feels indifferent about U and D, according to our analysis in part (i), and in equilibrium player 1 uses D with probability one.

  On the other hand, player 2 knows that the weather state is not rainy, and hence is equally likely to be sunny or cloudy, and according to player 1's strategy described above, player 2 expects player 1 to use U and D with equal probability. It is clear from our analysis in part (i) that player 2 indeed feels indifferent about using L or R, and in equilibrium player 2 uses L with probability one.

- Finally, suppose that the true weather state is rainy.

  In this event, player 1 knows that the state is not sunny, and hence is equally likely to be cloudy or rainy, and according to player 2's strategy described above, player 1 expects player 2 to use L and R with equal probability. Player 1 feels indifferent about U and D, according to our analysis in part (i), and in equilibrium player 1 uses D with probability one.

  On the other hand, player 2 knows that the weather state is rainy, and according to player 1's strategy described above, player 2 expects player 1 to use D with probability one. It is clear from our analysis in part (i) that player 2's best response against player 1 using D is indeed R.

To sum up, the aforementioned strategy profile does constitute an equilibrium. In this equilibrium, player 1's payoff is

$$15 \cdot \text{prob.(sunny)} + 12 \cdot \text{prob.(cloudy)} + 3 \cdot \text{prob.(rainy)}$$

$$= 10 > 3a + \frac{15}{2}, \quad \forall a \in [0, \frac{1}{2}].$$

Remark. To see why this "correlated equilibrium" in part (ii) generates for each player an expected payoff higher than the expected Nash equilibrium payoff in part (i), note that by making their date-1 actions contingent on the date-1 (imperfect) weather reports, the two players can make sure that the undesirable outcome (U,R) never arises in equilibrium, and the pleasant outcome (D,L), which is not an NE of the original normal-form game, can now arise when the weather is cloudy. Indeed, player 1 would adopt U only when the weather state is sunny, but player 2 would adopt R only when the weather state is rainy, and hence (U,R) never arises in any weather state. On the other hand, (D,L) is now implemented when the weather is cloudy. This cannot be done in a mixed strategy Nash equilibrium without a correlated device (i.e., the two weather reports): in the mixed-strategy NE obtained in part (i), the two players must randomize over their pure strategies in a stochastically independent manner, which implies that (U,R) may arise with probability $\frac{1}{4}$!

That the weather reports do not always deliver precise information is also important in leading to the above result. To see this, suppose instead that both players' weather reports tell them the exact weather state at date 1. In this case, given a realized weather state, the two players can only attain one Nash equilibrium payoff profile in part (i), which implies, in particular, that (D,L) can never arise as an equilibrium profile when the weather is cloudy. With imprecise weather information when the weather state is cloudy, however, player 1 thinks that player 2 may adopt L or R with equal probability, and player 2 thinks that player 1 may adopt U or D with equal probability, and that is why player 1 feels indifferent about U and D and player 2 feels indifferent about L and R, and in equilibrium player 1 adopts D with probability one and player 2 adopts L with probability one. The outcome (D,L) generates 12 for each player, which, together with the fact that (U,R) never arises in equilibrium, explains why the two players expect a payoff from this correlated equilibrium which is higher than the expected Nash equilibrium payoff of the original game without any correlated device.[4]

4. (**A Strategic Role of Futures Contracts**) Consider example 1 in Lecture 1, part I, where firms 1 and 2 can costlessly produce a product and engage in Cournot competition with the inverse demand being, in the relevant range,

$$P(q_1 + q_2) = 1 - q_1 - q_2.$$

This problem is a modification of the above Cournot game.
(i) Assume that there are two dates. The two firms will compete at

[4]When the weather reports always deliver precise information, an attainable expected payoff profile is simply a weighted average of the 3 Nash equilibrium payoff profiles in the original normal-form game. Indeed, the following are the attainable payoff profiles:

$$(15,3),\ (3,15),\ (\frac{15}{2},\frac{15}{2}),$$

$$\frac{2}{3}(15,3)+\frac{1}{3}(3,15)=(11,7),$$

$$\frac{2}{3}(15,3)+\frac{1}{3}(\frac{15}{2},\frac{15}{2})=(\frac{9}{2},\frac{9}{2}),$$

$$\frac{1}{3}(15,3)+\frac{2}{3}(3,15)=(7,11),$$

$$\frac{1}{3}(\frac{15}{2},\frac{15}{2})+\frac{2}{3}(3,15)=(\frac{9}{2},\frac{9}{2}),$$

$$\frac{1}{3}(15,3)+\frac{2}{3}(\frac{15}{2},\frac{15}{2})=(10,6),$$

$$\frac{1}{3}(3,15)+\frac{2}{3}(\frac{15}{2},\frac{15}{2})=(6,10),$$

$$\frac{1}{3}(15,3)+\frac{1}{3}(3,15)+\frac{1}{3}(\frac{15}{2},\frac{15}{2})=(\frac{17}{2},\frac{17}{2}).$$

In the above, if payoff profiles (x,y) and (y,x) are equally likely to arise, then the expected payoff profile always falls short of 10, where recall 10 is the expected payoff that each player obtains in the correlated equilibrium of part (ii).

date 1, but at date 0, both firms can correctly expect the date-1 inverse demand function, which is the $P(\cdot)$ defined above. At date 0, the futures market opens for the product produced by the two firms. There are price-competitive investors in the futures market, who, just like the two firms, are risk neutral without time preferences (that is, there will be no discounting for anyone). The extensive game is as follows.

- At date 0, (only) firm 1 can sign a futures contract with the competitive investors. In the futures contract, firm 1 promises to deliver f_1 units of the product at date 1 to one of the investors (say, Mr. A), and Mr. A promises to pay the price F (referred to as the date-0 futures price of the product). We assume that firm 1 announces f_1, and the competitive investors then determine the futures price F. Assume that investors have ratinal expectations; that is, upon seeing f_1, they can use backward induction to anticipate the date-1 price of the product (called the date-1 spot price of the product), and to rule out arbitrage opportunities, in the date-0 equilibrium, F must equal the anticipated date-1 price $P(q_1, q_2)$ so that Mr. A would get zero profits from futures trading.
- At date 1, upon seeing firm 1's date-0 futures contract (f_1, F), the two firms choose q_1 and q_2 simultaneously.
- Then, after firms set q_1 and q_2, firm 1 must deliver f_1 units of the product to Mr. A, and Mr. A must pay firm 1 Ff_1 dollars.
- Then, consumers arrive, and they purchase f_1 units of the product from Mr. A, $q_1 - f_1$ units from firm 1, and q_2 units from firm 2. Since consumers purchase $q_1 + q_2$ units in total, the date-1 spot transaction price is $P(q_1, q_2)$. Mr. A's profit is then $[P(q_1, q_2) - F]f_1$. Firm 1's profit as a function of q_1, q_2 is

  $$\Pi_1(q_1, q_2; f_1) = [1 - q_1 - q_2][q_1 - f_1] + Ff_1.$$

  Firm 2's profit function is still

  $$\Pi_2(q_1, q_2) = [1 - q_1 - q_2]q_2.$$

Find the SPNE of this extensive game. Explain why firm 1 may benefit from futures trading.[5]

(ii) Now, suppose that both firms can engage in futures trading at date 0, with f_1 and f_2 units sold respectively at the futures price F determined at date 0. Again, assume that all investors in the futures market have rational expectations when they compete in price to determine F. Re-derive the SPNE. Explain why the two firms might be hurt by the availability of futures trading.[6]

Solution. Consider part (i). It is straightforward to show that the two firms' date-1 reaction functions are

$$r_1^1(q_2; f_1) = \frac{1 + f_1 - q_2}{2}, \quad r_2^1(q_1) = \frac{1 - q_2}{2}.$$

Hence we have the subgame equilibrium

$$q_1^*(f_1) = \frac{1}{3} + \frac{2}{3}f_1, \quad q_2^*(f_1) = \frac{1}{3} - \frac{1}{3}f_1.$$

Now consider firm 1's date-0 choice of f_1. Since $F = P((q_1^*(f_1), q_2^*(f_1))$

[5]**Hint**: Use backward induction. First consider the date-1 subgame with f_1 given. This is just a Cournot game with the two firms' profit functions being Π_1 and Π_2 specified above. Let the subgame equilibrium be $(q_1^*(f_1), q_2^*(f_1))$, which depends on f_1. Now move backwards to consider firm 1's date-0 choice of f_1. Remember that the investors in the futures market can rationally expect the date-1 spot price of the product, which is $P((q_1^*(f_1), q_2^*(f_1))$, and given f_1, they will compete in price so that in the date-0 futures market equilibrium, $F = P((q_1^*(f_1), q_2^*(f_1))$. Given that $F = P((q_1^*(f_1), q_2^*(f_1))$, find firm 1's optimal f_1.

[6]**Hint**: Again, consider the date-1 subgame with f_1, f_2 given. Now for $i = 1, 2,,$ firm i's profit function becomes

$$\Pi_i(q_i, q_j; f_i) = [1 - q_i - q_j][q_i - f_i] + F f_i.$$

Find the Nash equilibrium $(q_1^*(f_1, f_2), q_2^*(f_1, f_2))$for this subgame. Now return to the date-0 futures market, where the two firms must simultaneously choose f_1 and f_2. For each pair (f_1, f_2) announced, the investors can correctly expect the date-1 spot price, which must be $P((q_1^*(f_1, f_2), q_2^*(f_1, f_2))$. Knowing that the futures price will be such that $F = P((q_1^*(f_1, f_2), q_2^*(f_1, f_2))$, the two firms' choices (f_1, f_2) must form a Nash equilibrium at date 0.

(a no-arbitrage condition!), at date 0 firm 1 seeks to

$$\max_{f_1} P((q_1^*(f_1), q_2^*(f_1))q_1^*(f_1) = \frac{1}{3}(1-f_1)(\frac{1}{3}+\frac{2}{3}f_1),$$

for which the necessary and sufficient first-order condition gives

$$f_1 = \frac{1}{4},$$

implying that, in equilibrium,

$$F^* = P^* = \frac{1}{4},\ q_1^* = \frac{1}{2},\ q_2^* = \frac{1}{4},\ \Pi_1^* = \frac{1}{8},\ \Pi_2^* = \frac{1}{16}.$$

Next consider part (ii). Given (f_1, f_2), now the subgame equilibrium becomes

$$q_1^*(f_1, f_2) = \frac{1}{3} + \frac{2}{3}f_1 - \frac{1}{3}f_2,\quad q_2^*(f_1, f_2) = \frac{1}{3} + \frac{2}{3}f_2 - \frac{1}{3}f_1,$$

$$P^*(f_1, f_2) \equiv P(q_1^*(f_1, f_2), q_2^*(f_1, f_2)) = \frac{1}{3}(1 - f_1 - f_2).$$

Now consider the date-0 futures market equilibrium. Firm i's problem is to, given the conjectured f_j,

$$\max_{f_i} P(q_i^*(f_i, f_j), q_j^*(f_i, f_j))q_i^*(f_i, f_j) = \frac{1}{3}(1 - f_i - f_j)(\frac{1}{3} + \frac{2}{3}f_i - \frac{1}{3}f_j).$$

The necessary and sufficient first-order condition gives firm i's date-0 reaction function

$$r_i^0(f_j) = \frac{1-f_j}{4},\ i, j = 1, 2,\ i \neq j.$$

Thus the date-0 equilibrium is

$$f_1^* = f_2^* = \frac{1}{5},$$

implying that

$$q_1^* = q_2^* = \frac{2}{5},\ F^* = P^* = \frac{1}{5},\quad \Pi_1^* = \Pi_2^* = \frac{2}{25}.$$

Remark. In part (i), firm 1 is better off with futures trading. The reason is that after commtting to sell f_1 units at a fixed price F, which will not fall when firm 1 expands output at date 1, firm 1 has an incentive to choose a higher total output at date 1. This fact results in firm 2 lowering output accordingly (because output choices are strategic substitutes). In essense, firm 1's selling futures contracts serves as a commitment that tells its rival that its reaction function is now shifted upwards. Consequently, firm 1 benefits from futures trading, which hurts firm 2 at the same time.

Compared to the Cournot equilibrium profit, however, both firms are worse off in part (ii). The reason is that, as in the game of prisoner's dilemma, here each firm intends to hold a short position in the futures contract as an attempt to force its rival to produce less. With the short positions in the futures contract, both firms are faced with a residual inverse demand with lower elasticity to their output expansion. Consequently, both firms choose to produce more in the subgame where futures contracts have been signed, leading to a lower spot and futures price for the product, and lower profit for each firm.[7]

5. (**A Strategic Role of Option Contracts**) This exercise can be applied to joint ventures, but we shall consider a simpler interpretation. There are two players in this sequential game, a landlord (L) and a tenant (T). The landlord can first spend $a \in [0, 1]$ to build a house, and then after the tenant moves in, the tenant can spend $b \in [0, 1]$ to make improvements on the house. The resale value of the house is $v(a, b) = a^f + b^h$, where the constants $f, h \in (0, 1)$. (Of course the landlord charges a rent from the tenant, say r, for renting the house for a given period, say a year, but this rental transaction has nothing to do with our main analysis and so we shall forget about it at this moment.) Let us call

$$S(a, b) \equiv v(a, b) - a - b$$

the *social benefit*, and the solution

[7] This exercise is adapted from Biaise Allaz and Jean-Luc Vila, 1993, Cournot Competition, Forward Markets and Efficiency, *Journal of Economic Theory*, 59, 1-16.

$$(a^*, b^*) = \arg\max_{a,b\in[0,1]} S(a, b)$$

will be called the *first-best* investments. We shall assume that a, b can only be observed by the landlord and the tenant but not by the court of law (i.e., they are *non-verifiable* variables), and hence cannot be put into a legally binding contract. Moreover, $S(a, b)$ is not verifiable either.[8] What L and T can do is to sign a contract to decide who owns the house. The timing of the game is as follows. The two first sign an ownership contract, and then given the contract L first chooses a, and upon seeing a, T must choose b. Then the house is sold after the rental period, and the two people share the proceeds according to the ownership contract.

(i) Compute a^*, b^*. Suppose first that a, b are contractible. Show that if L and T are both rational, they will put $a = a^*, b = b^*$ in the contract.

From now on, return to our initial assumption that a, b cannot be verified in the court of law, and hence L and T can only try to "implement" efficient a, b by choosing a smart "ownership contract."

[8]We claim that if instead $S(a, b)$ is verifiable, then there exists a simple sharing rule that gives L and T respectively the payoffs $\alpha S(a^*, b^*)$ and $(1-\alpha)S(a^*, b^*)$ for some $\alpha \in [0, 1]$, and that contract induces L and T to choose respectively a^* and b^*.

To see this, recall that $v(a, b)$ is verifiable, and if $S(a, b)$ is verifiable also, then $a + b$ must also be verifiable. Consider the following contract: If $S(a, b) = S(a^*, b^*)$, and if $a + b = a^* + b^*$ also, then T would get a fraction $(1 - \lambda)$ of the proceeds $v(a^*, b^*)$ from selling the house, where λ satisfies both L's and T's individual rationality conditions; and in any other event regarding (a, b), both L and T would get nothing from the proceeds of selling the house (the entire $v(a, b)$ would be donated to charity). Now, given λ, define α associated with this λ as such that $(1 - \alpha)S(a^*, b^*) + b^* = (1 - \lambda)v(a^*, b^*)$.

Now, if L chooses any $a \neq a^*$, then by the uniqueness of (a^*, b^*), it is in T's interest to choose $b = 0$ rather than any $b' > 0$ such that $S(a, b') = S(a^*, b^*)$, as there is not other pair (a, b') satisfying both $a + b' = a^* + b^*$ and $S(a, b') = S(a^*, b^*)$. Thus by choosing some $a \neq a^*$, S would get the payoff $-a$. On the other hand, if L chooses $a = a^*$, then it is obviously in T's interest to choose $b = b^*$. Thus with the above contract, in equilibrium L and T get respectively $\alpha S(a^*, b^*)$ and $(1 - \alpha)S(a^*, b^*)$.

(ii) Suppose that L owns the house exclusively (so that T cannot share a cent when the house is sold), determine the a, b and $v(a, b)$ by backward induction.

(iii) Suppose that before building the house, L sells the house to T by making a take-it-or-leave-it offering price q (so that L cannot share a cent when the house is sold). Determine the a, b and $v(a, b)$ by backward induction. Find q.

(iv) Suppose that before building the house, T agrees to pay L some money z to jointly own the house with L, and L and T will subsequently receive respectively $\lambda v(a, b)$ and $(1-\lambda)v(a, b)$ when selling the house (where λ is exogenously given). Determine the a, b and $v(a, b)$ by backward induction. Find z, assuming that L has all the bargaining power in determining z.

(v) Finally, consider the following contingent ownership contract: L owns the house initially, and he gives an option for free (why for free?) to T, and the option allows T to buy the house at the exercise price $p = v(a^*, b^*) - b^*$ after L chooses a but before T chooses b. Find the SPNE by backward induction. Determine the equilibrium a, b and $v(a, b)$.

(vi) Explain why the contingent ownership contract attains the first-best efficiency, while the other ownership contracts do not.

(vii) Now suppose instead that after L chooses a but before T decides to or not to exercise the option, L can offer a new contract to T. (We call this re-contracting event a "renegotiation.") This new contract will replace the existing option contract if and only if both L and T agree to do so. The new contract states a (probably) different exercise price p' that allows T to pay p' to L and get the house before T chooses b. Find the equilibrium a and b chosen by L and T respectively.

Solution. Consider part (i). The first-best investment levels (a^*, b^*) must solve the following maximization problem

$$\max_{a,b} S(a,b) = v(a,b) - a - b = a^f + b^h - a - b.$$

The necessary and sufficient first-order conditions yield $a^* = f^{1/(1-f)}$ and $b^* = h^{1/(1-h)}$. Since rational people must sign a Pareto efficient contract, these will be L and T's choices if they can sign complete contracts.

Consider part (ii). Obviously, T will choose $b = 0$ since he cannot share the proceeds from selling the house. Thus L seeks to

$$\max_a v(a,0) - a = a^f - a.$$

The solution is $a = a^*$. Hence when L owns the house exclusively, $v(a,b) = v(a^*,0)$ and L's payoff is $S(a^*,0) < S(a^*,b^*)$.

Consider part (iii). Suppose that T has already paid q to L before L chooses a. Then L will choose $a = 0$. Thus T seeks to

$$\max_b v(0,b) - b = b^h - b,$$

yielding $b = b^*$. Thus, the proceeds from selling the house will be $v(0,b^*)$. For T to be willing to pay q for the house in the first place, it must be that $q \leq v(0,b^*) - b^*$. Thus L optimaly chooses $q = v(0,b^*) - b^*$. It follows that L's payoff is $S(0,b^*) < S(a^*,b^*)$.

Consider part (iv). Consider the subgame where T has already paid z to L for the right of jointly owning the house. Given that L has chosen a, T seeks to

$$\max_b \ (1-\lambda)v(a,b) - b = (1-\lambda)(a^f + b^h) - b.$$

Thus T optimally chooses $b = [(1-\lambda)h]^{1/(1-h)} \equiv b(\lambda)$. Rationally expecting T's behavior, in choosing a, L seeks to

$$\max_a \ \lambda v(a, b(\lambda)) - a = \lambda\{a^f + [b(\lambda)]^h\} - a.$$

The solution is $a = (\lambda f)^{1/(1-f)} \equiv a(\lambda)$. The proceeds from selling the house will thus be $v(a(\lambda), b(\lambda))$. Thus T will accept z if and only if $z \leq (1-\lambda)v(a(\lambda), b(\lambda)) - b(\lambda)$. Consequently, L will choose $z = (1-\lambda)v(a(\lambda), b(\lambda)) - b(\lambda)$, which yields for L the payoff $S(a(\lambda), b(\lambda))$. It is easy to see that $S(a(\lambda), b(\lambda)) < S(a^*, b^*)$.

Consider part (v). If T does not exercise the option, then he must choose $b = 0$ because he does not get to share the proceeds from selling the house. If T exercises the option, then given any a he will choose b to

$$\max_b \; v(a, b) - b = a^f + b^h - b,$$

since he exclusively owns the house. Thus T will choose $b = b^*$ after he exercises the option.

Should T exercise the option? T knows that he will choose $b = b^*$ if he exercises the option, and hence he chooses to exercise the option if and only if

$$v(a, b^*) - b^* - p = v(a, b^*) - b^* - [v(a^*, b^*) - b^*] \geq 0 \Leftrightarrow a \geq a^*.$$

The result is not surprising. The house value depends not only on b but also on a. From T's perspective, given the strike price, the house is worth buying only if a is large enough. Indeed, the higher the strike price chosen by L, the higher a must be in order to induce T to exercise the option. By wisely setting $p = v(a^*, b^*) - b^*$, L knows that T will exercise the option if and only if L chooses some $a \geq a^*$.

Now, what is L's optimal choice about a? If L chooses some $a < a^*$, then T will not exercise the option, and T will subsequently choose $b = 0$, leading to the payoff $S(a, 0)$ for L. If L chooses some $a \geq a^*$, then T will exercise the option and L's payoff would become $S(a, b^*)$. Thus L's optimal choice is $a = a^*$, which generates for L the first-best payoff $S(a^*, b^*)$.

An interesting question here is why L offers the option for free? In fact, regardless of the strike price chosen by L, T will refuse to pay anything for the option. Why? Note that after T obtains the option, L will choose some a that makes T feel indifferent between to and not to excercise the option. In other words, L will choose some a that ensures that T makes zero profits by exercising the option. Therefore, for any strike price chosen by L, T will attach zero value to the option.

Consider part (vi). The above discussion shows that the first-best efficiency is attained in part (v) but not in parts (ii), (iii), or (iv). There is a free-rider problem in parts (ii), (iii) and (iv), which prevents the first-best efficiency from prevailing. On the other hand, in part (v), T's incentive to choose b^* can be ensured by making T the sole owner at the time the house is sold (or equivalently, making T the sole *residual claimant*). For L, on the other hand, by wisely choosing the strike price $v(a^*, b^*) - b^*$ for the option, L can be induced to choose $a = a^*$. This explains how the first best efficiency is attained in part (v).

Finally, consider part (vii). Note that in part (v), given the existing option contract T will not exercise the option if $a < a^*$, which is not efficient because $b = 0$ rather than b^* will then be chosen by T. We have assumed in part (v) that the existing option contract cannot be renegotiated, even though such inefficiency may exist. What if L and T can renegotiate the existing option contract? Does the opportunity of renegotiating an inefficient old contract undermine our result that option contracts can help attain the first-best efficiency?

Recall from part (v) that T will exercise the option if and only if

$$v(a, b^*) - b^* - p = v(a, b^*) - b^* - [v(a^*, b^*) - b^*] \geq 0 \Leftrightarrow a \geq a^*.$$

Now, if a new contract specifies a strike price $p' > p$, T will never agree to replace the old contract p by this new contract p'. Thus if L wants to offer a new contract to T, he must choose some $p' \leq p$. Suppose that L has already spent some $a \geq a^*$. Since T is willing to exercise the option under old contract, L will optimally choose $p' = p$ in this case, so that contract renegotiation does not arise in this case. What if L has spent some $a < a^*$? To induce T to agree to replace the old contract p by this new contract p', it is necessary and sufficient that the new strike price p' satisfies

$$v(a, b^*) - b^* - p' \geq 0.$$

Hence from L's perspective the optimal $p' = v(a, b^*) - b^*$. Therefore, if L has chosen some $a < a^*$, he will offer a new contract that yields for L the payoff $v(a, b^*) - b^* - a = S(a, b^*)$. It follows that L should optimally choose $a = a^*$! Our conclusion is that, allowing renegotiation does not change our main result that option contracts can help resolve the free-rider problem and attain the first-best efficiency.

Remark. As we explained in part (vi), the free-rider problem in parts (ii), (iii) and (iv) that prevents the first-best efficiency from prevailing is removed in part (v), where T has the correct incentive to choose b^* because T is the *residual claimant*) when choosing b and the wisely chosen strike price $v(a^*, b^*) - b^*$ induces L to optimally choose $a = a^*$. The problem with this "wisely designed" contract is that it leads to $b = 0$ even if a is only slightly lower than a^*, an outcome which is not productive efficient. Thus subgame perfection implies that L and T may wish to replace this contract by a new one as a remedy, if it did happen that somehow L has chosen $a < a^*$.

By giving L full bargaining power in contract renegotiation, we show that (a^*, b^*) are still the two players' equilibrium choices, even if they are allowed to replace an old contract by a new one after a is chosen. The idea here is that L knows that he will get all the surplus (and T will get zero surplus) generated from the replacement of p by p', given that he has all the bargaining power against T in the regenotiation subgame, and for this reason L should choose a^* to maximize the social benefit (given that, by backward induction, T will always choose b^* after T agrees to exercise the new option under the price p').

Although we have assumed in this exercise a special functional form for $v(a, b)$, the above results stand valid rather generally even if v is not additively separable in a and b.[9]

[9] This exercise is adapted from George Nöldeke and Klaus M. Schmidt, 1998, Sequential Investments and Options to Own, *Rand Journal of Economics*, 29, 633-653.