Game Theory, Solutions to Quiz 10

Name: __________ ID: __________

A	B	C	D	E
$\frac{A-m_3-1}{2}$	$\frac{A-m_3+x_3}{2}$	1	$x_3 \geq 1+2k-A$	$\frac{2A-3}{2}$
F	G	H	I	J
$\frac{A(A-2)}{4}$	separating	$3(\frac{A-2}{2})^2$	$x_1 < 1+2k-A$	$\frac{2A(A-2)+(A-1)^2}{4}$

1. Consider the following reputation game about a Cournot-competitive industry that extends for n periods. In each period t, the price of the homogeneous product (referred to as product X) supplied by all the firms is $P_t = A - Q_t$, where $A > n+1$ is a positive constant, and Q_t is the sum of supply quantities chosen by the firms.

   This industry has an incumbent firm, I, and n potential entrants, E_j, $j = 1, 2, \cdots, n$. There is no discounting, and each firm seeks to maximize (the sum of) expected profits.

   At $t = i \in \{1, 2, \cdots, n\}$, E_i can decide whether to spend a one-time cost $k_i \equiv k(n+1-i)$ to enter the industry, where $k > 0$ is a constant. Once it enters, it can costless supply 1 unit of product X at each period $t = i, i+1, \cdots, n$. Let m_i denote the number of entrants among E_1, E_2, $\cdots$, E_i which are operating at $t = i$. The incumbent firm's unit cost is $\tilde{c}$, and at $t = i$, *all* entrants believe that $\tilde{c}$ may take on 1 with probability x_i or 0 with probability $1 - x_i$. (Bayesian updating is applied whenever possible.) The timing of the relevant events is as follows.

   - At $t = i$, before E_i enters, E_i can observe whether entry has occurred at an earlier point in time, and the supply quantities chosen by all the firms operating at that point in time. However, $\tilde{c}$ and the incumbent firm's past profits remain unobservable to E_i.
   - Then, E_i must decide whether to spend k_i and enter the industry or stay out and get zero payoffs.

- Then, given m_i all the firms operating at $t = i$ must make output decisions simultaneously, where $m_i = m_{i-1}$ if E_i stays out and $m_i = m_{i-1} + 1$ if E_i enters.
- Then P_t is realized at $t = i$ and the date-t profits accrue to the firms. Then the game ends if $i = n$; or else the game moves on to $t = i + 1$.

We shall assume that $n = 3$, $A > 4$, and $A - 1 < 2k < A$. Notice that $k_1 = 3k$, $k_2 = 2k$, and $k_3 = k$. The outcome of $\tilde{c}$ will be referred to as the incumbent's *type*.

(i) First consider $t = 3$. Given $m_3 \in \{0, 1, 2, 3\}$, the date-t supply quantity chosen by the type-1 incumbent is ___A___, and given x_3 and m_3, the expected date-t product price is ___B___. Thus E_3 enters if and only if $m_3 =$ ___C___ and x_3 satisfies the *weak* inequality (write it down!) ___D___.[1]

(ii) Now, consider $t = 2$. Suppose first that E_1 has entered at $t = 1$. In this case we can get m_2, so that the type-0 incumbent's date-2 output quantity plus the type-1 incumbent's date-2 output quantity must be equal to ___E___.

(iii) Continue with $t = 2$. Now, suppose that E_1 did not enter at $t = 1$. In this latter case, we can get m_2 also, and show that E_2 would stay out if and only if x_2 satisfies a *strict* inequality, and when E_2 does stay out, the type-1 incumbent's profit at $t = 2$ is equal to ___F___.

(iv) Now, consider $t = 1$. If E_1 has entered, then there is a (answer 'pooling' or 'separating') ___G___ PBE, where the type-1 incumbent's sum of expected profits over the date-1-date-3 period is equal to ___H___. Thus E_1 would stay out if and only if x_1 satisfies the

[1] Thus we are making the tie-breaking assumption that E_i would enter when feeling indifferent about entering or staying out.

strict inquality (write it down!) __I___, and following that, the type-1 incumbent's sum of expected profits over the date-1-date-3 period is equal to __J___.

Solution. We shall solve the PBE using backward induction, and we shall record our findings as a series of lemmas along the way.

Since once entering the industry, an entrant can supply 1 unit without incurring any costs, and since $A > n+1$ (which implies that the product price is never negative), the optimal choice of output quantity in any operating period for such an entrant is exactly 1 unit.

Lemma 0. *The sum of output quantities supplied by the entrants operating at date t is m_t.*

Now observe that the type-0 incumbent has no concerns for reputation.

Lemma 1. Given m_t, the type-0 incumbent's date-t output choice is $\frac{A-m_t}{2}$.

Now we solve the PBE of the above reputation game using backward induction.

First consider the date-3 subgame where E_3 has just made its entry decision. Since this is the last period of the game, the incumbent has no reputation concern any more. By **Lemma 0**, given m_3 and $\tilde{c}$, the incumbent would seek to

$$\max_q q(A - m_3 - q - \tilde{c})$$

so that the type-$\tilde{c}$ incumbent's date-3 output choice is

$$q(\tilde{c}) = \frac{A - m_3 - \tilde{c}}{2}.$$

It follows that in state $(m_3, \tilde{c})$, the realized date-3 product price is

$$P_3(\tilde{c}) = A - m_3 - q(\tilde{c}) = \frac{A - m_3 + \tilde{c}}{2},$$

and hence given (m_3, x_3), E$_3$ expects its post-entry expected profit to be

$$1 \cdot [x_3 \times P_3(1) + (1 - x_3) \times P_1(0)] = \frac{A - m_3 + x_3}{2}.$$

Thus E$_3$ will enter in equilibrium if and only if, by our tie-breaking assumption,

$$\frac{A - m_3 + x_3}{2} \geq k \Leftrightarrow x_3 \geq 2k + m_3 - A,$$

where note that with E$_3$'s entry we have $m_3 \geq 1$. Note that if E$_3$ enters and yet $m_3 \geq 2$, then

$$1 \geq x_3 \geq 2 + 2k - A > 1,$$

which is a contradiction. Thus we conclude that E$_3$ would enter in equilibrium if and only if $m_3 = 1$ and x_3 satisfies

$$x_3 \geq 1 + 2k - A.$$

Lemma 2. *If $m_2 \geq 1$ so that either E_1 or E_2 has already entered prior to date 3, then we have $m_3 = m_2$; and in the opposite case, we have $1 \geq m_3 \geq m_2 = 0$, so that $m_3 = 1$ if and only if $x_3 \geq 1 + 2k - A$.*

Now, consider the date-2 subgame where E$_2$ has just entered the industry.

Since $1 \geq m_3 \geq m_2 \geq 1$, the incumbent knows that by **Lemma 2** $m_3 = m_2$ and E$_3$ would never enter, so that the type-$\tilde{c}$ incumbent's date-3 profit, according to part (i), will be

$$(\frac{A - m_2 - \tilde{c}}{2})^2,$$

which is independent of the incumbent's choice of date-2 output quantity. Thus there is a separating date-2 equilibrium, with the type-$\tilde{c}$ incumbent's date-2 output choice being

$$\frac{A - m_2 - \tilde{c}}{2}.$$

It follows that in state $(m_2, \tilde{c})$, the realized date-2 product price is

$$P_2(\tilde{c}) = \frac{A - m_2 + \tilde{c}}{2},$$

so that before making its entry decision, E_2 would expect its post-entry profit at date 2 (and at date 3 also, why?) to be

$$1 \cdot [x_2 \times P(1) + (1 - x_2) \times P(0)] = \frac{A - m_2 + x_2}{2}.$$

Note that E_2 would not deviate and stay out if and only if

$$2 \times \frac{A - m_2 + x_2}{2} \geq 2k \Leftrightarrow x_2 \geq 2k + m_2 - A,$$

implying that $m_2 = 1$. Thus we conclude that there exists a PBE at the date-2 subgame where E_2 enters for sure if and only if E_1 did not enter at date 1 *and* if $x_2 \geq 1 + 2k - A$.

Next, consider the date-2 subgame where E_2 has just decided to stay out. Then $m_2 = m_1 = 1$ if E_1 entered at date 1 and $m_2 = m_1 = 0$ if E_1 did not.

In the former case, by **Lemma 2** E_3 would never enter, so that the type-$\tilde{c}$ incumbent's date-3 profit, according to part (i), will be independent of the incumbent's choice of date-2 output quantity. Thus there is again a separating date-2 equilibrium where the expected date-2 product price is

$$\frac{A - 1 + x_2}{2},$$

and we must check that E_2 indeed would not deviate and make entry: following a deviation the expected date-2 product price would become

$$\frac{A-2+x_2}{2},$$

and we must require that

$$2 \times \frac{A-2+x_2}{2} < 2k \Leftrightarrow x_2 < 2+2k-A,$$

but the last inequality holds always! Thus if E_1 has entered at date 1, there is a separating PBE at date 2 where E_2 does not enter, and following that the type-$\tilde{c}$ incumbent would choose the output quantity

$$\frac{A-1-\tilde{c}}{2}$$

at both date 2 and date 3.

Now, consider the lattter case, where E_1 and E_2 have both chosen to stay out. Can there be a separating PBE at this point, where the two types of the incumbent choose different output quantities? In such an equilibrium, the type-1 incumbent would expect E_3 to enter at date 3 after seeing its date-2 output choice, which differs from the type-0 incumbent's output choice $\frac{A}{2}$; recall **Lemma 2**. Thus in this supposed separating PBE, the type-1 incumbent would choose the output quantity $\frac{A-1}{2}$, yielding for the type-1 incumbent the continuation payoff $\frac{(A-1)^2+(A-2)^2}{4}$. If the type-1 incumbent deviates and chooses $\frac{A}{2}$ instead, then it would get the date-2 payoff

$$\frac{A}{2} \times (A - \frac{A}{2} - 1) = \frac{A(A-2)}{4},$$

but this would lead to $x_3 = 0$ and $m_3 = 0$, so that the type-1 incumbent would get

$$\frac{A-1}{2} \times (A - \frac{A-1}{2} - 1) = \frac{(A-1)^2}{4}.$$

Thus the type-1 incumbent would surely want to deviate! This proves that there cannot be a separating PBE.

Can there be a pooling PBE for the latter case, where both types of the incumbent produce $\frac{A}{2}$ units at date 2? Note that a deviation will be taken as evidence that the deviator is the type-1 incumbent, so that the optimal deviating output choice for the type-1 incumbent is $\frac{A-1}{2}$. In this PBE, we have $x_3 = x_2$, and if $x_2 \geq 1+2k-A$, then by **Lemma 2** E_3 would enter even though no deviation at date 2 is detected, which would then induce the type-1 incumbent to strictly prefer producing $\frac{A-1}{2}$ units instead of $\frac{A}{2}$ units. Thus for such a PBE to prevail at date 2, it is necessary that $x_2 < 1 - 2k + A$. When this inequality does hold, the type-1 incumbent would get $\frac{A(A-2)}{4}$ at date 2 and $\frac{(A-1)^2}{4}$ at date 3 in equilibrium, and he would get $\frac{(A-1)^2}{4}$ at date 2 and $\frac{(A-2)^2}{4}$ at date 3 after a deviation. Thus this pooling PBE does exist given that $x_2 < 1 - 2k + A$.

Lemma 4. The date-2 equilibrium given (m_1, x_2) is as follows.

- If $m_1 = 0$ and $x_2 \geq 1 + 2k - A$, then E_2 would enter for sure, leading to $m_2 = 1$, and following that there is a *separating* date-2 equilibrium, with the type-$\tilde{c}$ incumbent's date-2 and date-3 common output choice being
$$\frac{A-1-\tilde{c}}{2}.$$
- If $m_1 = 0$ and $x_2 < 1 - 2k + A$, then E_2 would stay out for sure, leading to $m_2 = 0$, and following that there is a *pooling* date-2 equilibrium, with $\frac{A}{2}$ being the equilibrium date-2 output choice for both types of the incumbent, and upon seeing this date-2 output choice E_3 would stay out for sure. The type-$\tilde{c}$ incumbent would then produce $\frac{A-\tilde{c}}{2}$ units at date 3.
- If $m_1 = 1$, then regardless of x_2, E_2 would stay out for sure, leading to $m_2 = 1$, and following that there is a *separating* date-2 equilibrium, with the type-$\tilde{c}$ incumbent's date-2 and date-3 common output choice being
$$\frac{A-1-\tilde{c}}{2}.$$

Now, consider the date-1 subgame where E_1 has just entered, so that $m_1 = 1$, and by **Lemma 4** and **Lemma 2**, E_2 and E_3 would both stay out for sure. We claim that following entry by E_1, there is a separating PBE, where the type-1 incumbent gets

$$3 \times \frac{(A-2)^2}{4}.$$

To see this, note that if the type-1 incumbent deviates and produces $\frac{A-1}{2}$ at date 1, then it would choose exactly the same output quantity at dates 2 and 3, just like deviation never occurs; recall the last statement in **Lemma 4**. Thus following E_1's entry, this separating PBE exists always! It follows that there is a date-1 equilibrium where E_1 would enter for sure if and only if $x_1 \geq 1 + 2k - A$.

Finally, consider the date-1 subgame where E_1 has just chosen to stay out, so that $m_1 = 0$. We claim that there is no separating equilibrium at date 1. If there were, then $x_2 = 1$ after the type-1 incumbent makes the equilibriu date-1 output choice, and by **Lemma 4** E_2 and E_3 would enter at date 2 and stay out at date 3 respectively. The type-1 incumbent's payoff in this supposed equilibrium would be

$$\frac{A(A-2)}{2} + 2 \times \frac{(A-2)^2}{4}.$$

By deviating and choosing the output $\frac{A}{2}$ at date 1, the type-1 incumbent can ensure that $x_2 = 0$, so that by **Lemma 4** E_2 would stay out for sure, and following that the type-1 incumbent can again choose $\frac{A}{2}$ as its date-2 output to ensure that E_3 would stay out for sure; the type-1 incumbent would then produce $\frac{A-1}{2}$ units at date 3. Thus with a series of deviations, the type-1 incumbent can get the payoff

$$2 \times \frac{A(A-2)}{2} + \frac{(A-1)^2}{4},$$

showing that the deviation payoff is higher!

Now, can there be a pooling equilibrium following E_1's staying out? Note that if $x_1 \geq 1 + 2k - A$, then upon seeing the incumbent's date-1 outpout choice $\frac{A}{2}$ in the pooling equilibrium, by **Lemma 4** E_1 would enter, and following that there would be a date-2 separating outcome. It is clear that the type-1 incumbent had better deviate at date 1 in this case!

Thus we focus on the case where $x_1 < 1 + 2k - A$. By **Lemma 4**, following the date-1 pooling choice of output, E_2 would stay out, and following that there is again a date-2 pooling equilibrium that induces E_3 to also stay out. Thus in this pooling PBE the type-1 incumbent gets the equilibrium payoff

$$2 \times \frac{A(A-2)}{4} + \frac{(A-1)^2}{4},$$

whereas after choosing the date-1 output $\frac{A-1}{2}$ during a deviation, by **Lemma 4**, the type-1 incumbent would expect both E_2 and E_3 to enter, so that its deviation payoff is

$$\frac{(A-1)^2}{4} + 2 \times \frac{(A-2)^2}{4}.$$

Clearly, no deviation would occur.

Lemma 5. The date-1 equilibrium depends on x_1.

- If $x_1 < 1 + 2k - A$, then there is pooling at date 1 and date 2, and all three entrants would stay out.
- If $x_1 \geq 1 + 2k - A$, then there is separating at date 1, and only E_1 enters in equilibrium.

The type-1 incumbent would always pool with the type-0 incumbent as long as no entrants have ever entered before. The type-1 incumbent would instead distinguish itself from the type-0 incumbent following the first occurrence of entry. Note that the condition

$$0 < 1 + 2k - A < 1$$

says that under full information E_1's decision is to enter if and only if the incumbent is of type 1. Given x_1, the incumbent's expected output quantity following entry of E_1 is at least x_1, and it is exactly equal to x_1 if following entry of E_1 the type-1 incumbent would rather distinguish itself from the type-0 incumbent, which is exactly what would happen given $x_1 \geq 1 + 2k - A$. Thus E_1 would stay out if and only if $x_1 < 1 + 2k - A$.