Game Theory, Solutions to Quiz 2

Name: __________ ID: __________

Questions	1	2	3	4	5
Solutions	BD	false	ABCD	BC	ABC

1. Consider the following voting game. There are three voters ($i = 1, 2, 3$) and three candidates ($j = X, Y, Z$). The candidate receiving the most votes would win; in case each and every candidate receives one vote, then each of them would win with probability $\frac{1}{3}$. A winning candidate's payoff is one, and a losing candidate's payoff is zero. The following table summarizes the three voters' payoffs after one of the candidates is announced the winner.

voter/candidate	X	Y	Z
1	2	1	0
2	0	2	1
3	1	0	2

A pure-strategy NE for this game will be denoted by (a_1, a_2, a_3), where $a_i \in \{X, Y, Z\}$ denotes the candidate chosen by voter i.

(i) Which statements below are correct? __No. 1__.
(A) (X,X,Y) is a pure-strategy NE.
(B) (X,Y,X) is a pure-strategy NE.
(C) (Y,X,X) is a pure-strategy NE.
(D) (X,Y,Z) is a pure-strategy NE.
(E) The above 4 statements are all false.

(ii) Determine whether the following statement is **true** or **false**: __No. 2__.
Given that the voters' payoffs are now given in the following table, (X,Y,Z) becomes a pure-strategy NE.

voter/candidate	X	Y	Z
1	2	1	0
2	0	1.98	1.03
3	1	0	2

(iii) Now, suppose instead that there are 9 voters, whose payoffs are summarized in the table below.

voters/candidate	X	Y	Z
1-3	2	1	0
4-5	0	1	0
6-9	0	0	1

Assume that voters would never use (weakly) dominated strategies. We shall consider only *symmetric* pure-strategy NEs; that is, in equilibrium players having the same payoff functions must vote for the same candidate. Such an equilibrium will be denoted by (a, b, c), where a is the candidate chosen by voters 1-3, b the candidate chosen by voters 4-5, and c the candidate chosen by voters 6-9.

(iii-1) Which statements below are correct? No. 3 .
(A) (X,Y,Z) is a pure-strategy NE.
(B) Voters 1-3 may vote for X or Y in a pure-strategy NE.
(C) Voters 4 and 5 would only vote for Y in a pure-strategy NE.
(D) Voters 6-9 would only vote for Z in a pure-strategy NE.
(E) The above 4 statements are all false.

(iii-2) Which statements below are correct? No. 4 .
(A) There is a pure-strategy NE where X is the winner.
(B) There is a pure-strategy NE where Y is the winner.
(C) There is a pure-strategy NE where Z is the winner.
(D) The above 4 statements are all false.

(iii-3) Which statements below are correct? No. 5 .
(A) The absence of X would benefit Y.
(B) The presence of X has benefited Z.
(C) The absence of Z would benefit X.

(D) The above 4 statements are all false.

Solution. Consider part (i). (A) is false, because voter 2 can deviate unilaterally and choose Y, which allows him to obtain a payoff of 2.

(B) is true: voter 3 gets 0 if he deviates and votes for Y, and he gets 1 again if he votes for Z instead; and what voter 2 does is immaterial given the other two voters choose to stick to their equilibrium behavior.

(C) is also false: voter 2 would deviate and vote for Y.

(D) is correct: each voter's equilibrium payoff is 1, and he cannot obtain a higher payoff by making a unilateral deviation. To sum up, the answer for part (i) is BD.

Consider part (ii). Now the statement is false! While (X,Y,Z) is a pure-strategy NE in part (i), this is no longer true in part (ii). Voter 2 can deviate and vote for Z instead and obtain a payoff of 1.03, which is strictly greater than his equilibrium payoff $\frac{3.01}{3}$![1]

Consider part (iii). First observe that voting for X or Z is a dominated strategy for voters 4-5, and voting for X or Y is a dominated strategy for voters 6-9. Thus in a pure-strategy NE (a, b, c), we must have $b = Y$ and $c = Z$. Also, voting for Z is a dominated strategy for voters 1-3, and hence we are left with two possible equilibria: (X, Y, Z) and (Y, Y, Z). It is easy to verify that both of them are symmetric pure-strategy NEs. Thus the answer for (iii-1) is ABCD.

The answer to part (iii-2) is now obvious: Z would be the winner in equilibrium (X, Y, Z) and Y would be the winner in equilibrium (Y, Y, Z). Thus the answer for part (iii-2) is BC.

Note that more than half of the voters (voters 1-5, to be specific) hate candidate Z, and yet Z still wins the campaign in equilibrium (X, Y, Z). Why?

[1] The new payoffs in part (ii) indicate that voter 2 does not like Y and hate Z as much as in part (i), and voter 2 would indeed rather give up his favorite candidate Y (who may win with probability $\frac{1}{3}$ only) and go for the sure payoff of 1.03 by voting for his second-favorite candidate Z.

This happens because in equilibrium (X, Y, Z) voters 1-5 fail to vote for the same candidate, namely Y. (By contrast, in the other equilibrium, (Y, Y, Z), voters 1-5 all vote for Y, and hence Y wins and Z loses.)

Note that (X, Y, Z) is not a strong equilibrium, as voters 1-3 can jointly deviate and vote for Y, which would generate a payoff of 1 for voters 1-3, better than the zero payoff that voters 1-3 obtain in equilibrium (X, Y, Z). The equilibrium (Y, Y, Z), on the other hand, is coalition-proof: a self-enforcing coalition consisting of no more than 8 voters can never strictly improve the well-being of the coalition members given that the voters outside of the coalition would still behave as in equilibrium (Y, Y, Z).[2]

Note also that, although voters 1-3 attach 2 utils to the event that X wins and yet voters 4-5 attach only 1 util to the event that Y wins, X can never win this campaign, unlike Y. Why not?

The difference lies in the fact that voters 1-3 are still willing to consider Y, but voters 4 and 5 would never consider X. Thus voters 1-3 are potential *switchers*, exhibiting less loyalty to their favorite candiate X. Although the group consisting of voters 1-3 is larger in size than the group consisting of voters 4-5, and voters 1-3 feel more strongly (2 utils) about their favorite candidate than voters 4 and 5 do, the former group's favorite candidate can never win. In this sense, *loyalty is more important than the the size* in an election.

Finally, consider part (iii-3). Note that if X chooses to withdraw from the election, then Y will definitely win the campaign, which hurts Z. In other words, X's running the campaign implies that Z may win with a positive probability, and hence it benefits Z. Also, if Z decides to withdraw, then X can become a winner, although Y can be a winner too. Thus the answer for (iii-3) is ABC.

[2] To see this, note that voters 4-9 are voting for their favorite candidates already. Given that voters 6-9 would still vote for Z, voters 1-3 cannot get more than 1 by jointly voting for X.