Game Theory, Solutions to Quiz 3

Name: __________ ID: __________

Questions	A	B	C	D	E
Solutions	$\frac{8}{3}$	$\frac{8}{9}$	$\frac{8}{3}$	$\frac{1}{4}$	42
Questions	F	G	H	I	J
Solutions	22	23	23	S	24

1. Consider a seller (S) endowed with an indivisible product X, and two potential buyers B and C. B and C would obtain payoffs v and u respectively when consuming X. Although B knows about product X, C does not. C would also know about product X if S is willing to spend F on advertising. The game proceeds as follows.

   - At the beginning of $t = 1$, S can decide whether to spend F, and this decision remains S's private information till the end of $t = 2$.
   - At $t = 1$, after S decides to or not to spend F, B must make a price offer p_1 to S, which S can either accept or reject. The game will end at once with B paying S p_1 and getting X if S accepts p_1, and the game will move on to $t = 2$ in case that S rejects p_1.
   - At $t = 2$, B must make a price offer p_2 to S, which S can either accept or reject. The game will end at once with B paying S p_2 and getting X if S accepts p_2, but if S rejects p_2, then S must either sell X to C at the price of u (in case S did spend F at $t = 1$) or keep X (in case S did not spend F at $t = 1$). Keeping X generates zero payoff to S at $t = 2$.

   The payoffs for S and B are as follows. If S sells X to B at price p_t at date $t \in \{1, 2\}$, then S would get either $\delta_S^{t-1} p_t$ or $\delta_S^{t-1} p_t - F$, depending on whether F has been spent at $t = 1$; and B would get $\delta_B^{t-1}(v - p_t)$. In case S sells X to C at price u at $t = 2$, then S would get $\delta_S u - F$. If B fails to obtain X by the end of $t = 2$, then B's payoff is zero.

Now, suppose that

$$u = 4,\ v = 3,\ \delta_S = \delta_B = \frac{2}{3}.$$

(i) This game has a pure-strategy NE where S does not spend F at $t = 1$, and this NE exists if and only if $F \geq F^*$, where $F^* = \underline{\quad A \quad}$.

(ii) Now, suppose that $F = 2$. This game has a unique NE, where at $t = 1$, S may spend F with probability π and B may offer two prices $p^H > p_L$ with probabilty α and $1 - \alpha$ respectively. One can show that $\pi = \underline{\quad B \quad}$, $p^H = \underline{\quad C \quad}$, and $\alpha = \underline{\quad D \quad}$.

Solution. Consider part (i). Ancipating (correctly) that F was not spent, B will offer zero price in both $t = 1$ and $t = 2$, so that S must have zero equilibrium payoff. If S deviates and spends F at $t = 1$, S can obtain a payoff of

$$-F + \delta_S u = \frac{8}{3} - F,$$

and hence this NE can be sustained if and only if $F \geq F^* = \frac{8}{3}$.

Consider part (ii). Since $F = 2 < F^*$, the pure-strategy NE depicted in part (i) does not exist. Moreover, spending F with probability one is not consistent with an NE either: in this supposed NE, B would optimally offer the price $\frac{8}{3}$ at $t = 1$, but then S can spare the expenditure F and accept B's offer, which would make S better off than following S's equilibrium move! To see this, note that if B expects F to have been spent with probability one, then because $v < u$, B's only chance to obtain X is to offer a price no less than $\delta_S u = \frac{8}{3}$ at $t = 1$, but S can then spare the cost F and simply take this offer.

Thus we must look for a mixed-strategy NE, where S may randomize between spending and not spending F.

Note that an offer p_1 made by B will be accepted by S when F was not spent *if* S is willing to accept p_1 when F was spent.

Thus we classify B's price offers into two categories: those that S will always accept, and those that S will accept when and only when F

was not spent. The optimal choice in the former category, from B's perspective, is $p^H = \frac{8}{3}$; and the optimal choice in the latter category (again, from B's perspective), is $p^L = 0$.

Note that if B would offer some p_1 for sure, then S would not randomize; sparing F is always better if B would offer p^H for sure, and spending F is always better if B would offer p^L for sure. Thus B must also randomize over p_1's.

Now, assume that B may offer p_H with probability α and p_L with probability $1-\alpha$. From S's perspective, by spending F and then selling to C (or accepting p^H when B does offer p^H), S can obtain a payoff of

$$-F + \delta_S u = \frac{8}{3} - 2 = \frac{2}{3};$$

and by not spending F S would obtain an expected payoff of

$$\alpha \cdot \frac{8}{3} + (1-\alpha) \cdot 0,$$

so that for S to randomize between spending and not spending F, we must have

$$\alpha = \frac{1}{4}.$$

On the other hand, by offering p^L, B would obtain the payoff of

$$(1-\pi) \cdot (v - p^L) = 3 - 3\pi,$$

and by offering p^H instead, which S will accept for sure, B would obtain the payoff of

$$v - p^H = \frac{1}{3},$$

and hence for B to randomize between offering p^H and offering p^L, we must have

$$\pi = \frac{8}{9}.$$

Remark. We can interpret S as the owner of a private firm, and B an acquirer in a takeover attempt. S can spend a cost F to search for a while knight C, but whether S has spent F is not known to B. This game analyzes how the possibility that S may look for a white knight may affect the takeover offer made by B. For a formal analysis on this subject, see for example Shleifer, A, and R. Vishny, 1986, Greenmail, White Knights, and Shareholders' Interest, *Rand Journal of Economics*, 17, 293-309.

2. Re-consider Problem 1. This time, assume that $\delta_S = \delta_B = 1,\ F = 0$, and hence C is sure to be present at $t = 2$, and there is no discounting for the payoffs of B and S.

   We also make the following changes:

   - S's valuation for X is 12, rather than zero.
   - B's valuation for X, v, is a random variable, which may equal $42, 18$, or 9 with equal probability. C's valuation for X is 9 for sure. The players' valuations for X are their common knowledge at $t = 1$.
   - B would privately learn about the realization of v at $t = 2$, when C shows up to try to purchase X. (Thus C is a *late buyer*, and B an *early buyer*.)
   - Three trading formats will be compared:

     (a) (**Date-2 spot selling**.) S can announce a price p_2, and upon seeing p_2, B and C can simultaneously express willingness to buy. All willing buyers get the same chance to get X by paying p_2 to S. In case no willing buyer exists, S will keep X.

     (b) (**Date-1 advance selling**.) S can make a price offer p_1 to the early buyer B, and B can either accept or reject the offer. Whether trade takes place at $t = 1$ or not, there will be *no* transaction for product X at $t = 2$.

     (c) (**Date-1 advance selling with date-2 resale.**) The same as in trading format (b), but whoever keeps X at the beginning of $t = 2$, can either keep it till the end of $t = 2$ or announce a price p_2' at $t = 2$. Again, in the latter event, all willing buyers have equal chance to get X at $t = 2$.

(i) Under trading format (a), the equilibrium price $p_2 =$ ___E___, and S has equilibrium payoff ___F___.[1]

(ii) Under trading format (b), the equilibrium price $p_1 =$ ___G___, and S has equilibrium payoff ___H___.

(iii) Under trading format (c), the buyer during the date-2 resale must be (answer S, B, or C) ___I___, and the equilibrium payoff for S is equal to ___J___.

Solution. Consider part (i). As in Problem 1 of Homework 1, one can easily verify that only the prices 42, 18 and 9 are un-dominated choices for S. Since S would not offer a price over 42 or below 9, and since S's own valuation is 12, we only need to compare the price 42 to the price 18.

By setting $p_2 = 42$, S's payoff is

$$\frac{1}{3} \cdot 42 + \frac{2}{3} \cdot 12 = 22.$$

By setting $p_2 = 18$, S's payoff is

$$\frac{2}{3} \cdot 18 + \frac{1}{3} \cdot 12 = 16.$$

Thus the equilibrium price choice for S is $p_2 = 42$ and S obtains a payoff of 22.

Consider part (ii). B's expected valuation for X is

$$\frac{42 + 18 + 9}{3} = 23$$

at $t = 1$, and hence S will offer the price $p_1 = 23$, and obtain a payoff of 23.

Consider part (iii). Now, with resale B can expect to sell X back to S at $t = 2$ after B obtains X at $t = 1$ and after B learns at $t = 2$ that his

[1]Note that S's payoff is 12 if S chooses to keep X.

valuation for X is 9. Thus with resale B is willing to accept any price less than or equal to 24 at $t = 1$. Recognizing this fact, S will then offer the price 24 at $t = 1$.

Remark. Why does trade format (b) (i.e., advance selling at $t = 1$) benefit S more than trade format (a) (i.e., spot selling at $t = 2$)?[2]

This happens because when S offers p_2 in spot selling S knows that B would have already known his valuation for X, and S would rather offer a high price to bet on the event that B's valuation is 42 than offer a low price to bet on the event that B's valuation is 18. In other words, S's intention to extract more rent from B when B's valuation is 42 destroys the opportunity of completing an efficient trade with B when B's valuation is actually 18!

With advance selling at $t = 1$, on the other hand, S knows that when p_1 is offered to B, B, just like S, does not know the exact realization of v. Thus S can offer a "bundle," saying that X will be sold to B no matter which realization of v will come out. This way, S ensures that he can sell X to B even if B's valuation is 18.

The cautious reader must have discovered that, selling X to B for sure also has a problem: selling X to B is inefficient when B's valuation is actually 9, less than S's valuation!

However, the efficiency loss from completing an inefficient trade when B's valuation is 9, which is $12 - 9 = 3$, is less than the efficiency gain from completing an otherwise lost trade when B's valuation is 18, which is $18 - 12 = 6$! This explains why S benefits more from trade format (b) than trade format (a), even if trade format (b) is less than perfect.

[2]Finance students would know that other things equal, advance selling has the advantage of getting the cash inflow early. The assumption that $\delta_S = \delta_B = 1$ removes this advantage, and allows us to focus on the strategic aspect of advance selling.

Now it becomes clear why trade format (c) is better than trade format (b). The resale provides a remedy for the problem pertaining to (b)! When B learns that his valuation for X is 9 at $t = 2$, he would be happy to sell X back to S at the price of 12, which removes the efficiency loss that (b) incurs at $t = 1$. With rational expectations (or with backward induction), both S and B can see how the resale opportunity would change B's valuation for X at $t = 1$, and hence S raises p_1 from 23 to 24 at $t = 1$.

Finally, note that the resale opportunity in (c) can be replaced by a returns policy, and the latter, like advance selling, is an important topic in marketing theory.[3] More precisely, (b) can attain the same trading efficiency as (c) if we can append to (b) the following refund policy: B can get a refund of 12 if B returns X (in a good condition) to S at $t = 2$.

Note also that we have assumed in (c) that B has all bargaining power against S during resale at $t = 2$. It is your homework to show that, as long as S and B have rational expectations, assuming any other allocation of bargaining power for the date-2 resale does not alter our conclusion. (Why not?)

[3] For a formal theory of advance selling, see Shugan, S., and J. Xie, 2005, Advance-selling as a Competitive Marketing Tool, *International Journal of Research in Marketing*, 22, 351-373. For returns policy, see Anderson, E., K. Hansen, and D. Simester, 2009, The Option Value of Returns: Theory and Empirical Evidence, *Marketing Science*, 28, 405-423.