Game Theory with Applications to Finance and Marketing, I

Solutions to the Final Exam

Name: __________ ID: __________

Write your solutions into the following table:

Question No.	No. 1	No. 2	No. 3	No. 4	No. 5
Your Solution	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{7}{4}$	$\frac{3}{4}$	$\frac{43}{16}$
Question No.	No. 6	No. 7	No. 8	No. 9	No. 10
Your Solution	$\frac{15}{16}$	$\frac{29}{32}$	$\frac{9}{4}$	$\frac{1}{4}$	$\frac{53}{16}$
Question No.	No. 11	No. 12	No. 13	No. 14	No. 15
Your Solution	$\frac{5}{16}$	$\frac{17}{32}$	$\frac{5}{4}$	$\frac{13}{8}$	$\frac{27}{32}$
Question No.	No. 16	No. 17	No. 18	No. 19	No. 20
Your Solution	m_1	a_2	m_2	5	38
Question No.	No. 21	No. 22	No. 23	No. 24	No. 25
Your Solution	80	40	30	drop	$=$
Question No.	No. 26	No. 27	No. 28	No. 29	No. 30
Your Solution	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{2x_2+1}$	$\frac{47}{12} - \frac{1}{2x_2+1}$
Question No.	No. 31	No. 32	No. 33	No. 34	No. 35
Your Solution	$+\infty$	0	$\frac{9}{5}$	$\frac{15}{11}$	$\frac{11}{9}$
Question No.	No. 36	No. 37	No. 38	No. 39	No. 40
Your Solution	$\frac{1718}{495}$	2	$\frac{15}{8}$	$\frac{16}{9}$	$\frac{241}{72}$

1. A large bank is facing a small borrowing firm, which may be of type H or type L, with probability α and $1-\alpha$ respectively. The firm has nothing but an illiquid asset at date 0, and it needs to borrow cash from the bank in order to operate at date 0. If the bank lends $q_j \geq 0$ dollars to a type-j firm at date 0, which costs the bank $\frac{(q_j)^2}{2}$ at date 0 (because of imperfections in the credit market), then the firm will generate a date-1 cash inflow $\theta_j q_j$. The bank wishes to design a menu of loan contracts $\{(q_H, T_H), (q_L, T_L)\}$ to maximize its own expected profits, where T_j is the date-1 repayment that the type-j borrowing firm must make to the bank for the loan q_j that it takes from the bank at date 0.[1] The firm and the bank are both risk-neutral without time preferences.

   The timing of relevant events prior to date 0 is as follows.

   - (**Stage 1.**) The bank announces $\mathcal{M} = \{(q_H, T_H), (q_L, T_L)\}$; that is, the bank will ask the firm to choose one element in the menu $\mathcal{M}$ in **Stage 3**, or else the firm should just leave. By choosing an element (q_j, T_j) in $\mathcal{M}$ in **Stage 3**, the firm gets q_j dollars from the bank and agrees to put up its illiquid asset as a collateral at date 0, which will be liquidated at date 1 in case the firm fails to repay T_j at date 1.
   - (**Stage 2.**) The firm can decide whether or not to spend a cost $c \geq 0$ to see its own type. If c is not spent, then there is no information asymmetry between the firm and the bank, and in this event, the firm simply thinks that its (average) type is $\theta_m = \alpha\theta_H + (1-\alpha)\theta_L$.
   - (**Stage 3.**) Given its type $j \in \{H, L, m\}$, the firm can choose one element in $\mathcal{M}$ (and the game moves on to date 0, with the bank lending q_j to the firm), or simply leave (and the game ends).

   From now on, we shall assume that

$$\alpha = \frac{1}{2},\ \theta_H = \frac{7}{4},\ \theta_L = \frac{5}{4}.$$

[1] We assume that the firm's date-1 cash inflow is not verifiable in the court of law, but the firm will *never* default on its date-1 debt obligations: its illiquid asset serves as a collateral for the bank loan, which would be sold by the bank in case default did take place at date 1. We assume that from the firm's perspective, fully repaying the debt is always better than losing the illiquid asset.

(i) Suppose that $c = +\infty$. Then it is optimal for the bank to set $q_H = q_L =$ ___No. 1___, which generates a payoff ___No. 2___ for the bank.

(ii) Suppose that $c = 0$. Then it is optimal for the bank to set $q_H =$ ___No. 3___, $q_L =$ ___No. 4___, $T_H =$ ___No. 5___, and $T_L =$ ___No. 6___, so that the bank's optimal payoff is equal to ___No. 7___.

(iii) Now, suppose that $c = \frac{1}{4}$, and suppose that the bank, when designing $\mathcal{M}$ in **Stage 1**, would like the firm to spend c in **Stage 2**.[2] Then the bank would optimally set $q_H =$ ___No. 8___, $q_L =$ ___No. 9___, $T_H =$ ___No. 10___, and $T_L =$ ___No. 11___, so that the bank's optimal payoff is equal to ___No. 12___.

(iv) Now, suppose that $c = \frac{1}{32}$, and suppose that the bank, when designing $\mathcal{M}$ in **Stage 1**, would prevent the firm from spending c in **Stage 2**.[3] Then the bank would optimally set $q_H = q_L =$ ___No. 13___,

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[2]Formally, the bank seeks to

$$\max_{q_H, T_H, q_L, T_L} \alpha[T_H - \frac{1}{2}q_H^2] + (1-\alpha)[T_L - \frac{1}{2}q_L^2]$$

subject to

$$\theta_H q_H - T_H \geq 0; \tag{1}$$

$$\theta_L q_L - T_L \geq 0; \tag{2}$$

$$\theta_H q_H - T_H \geq \theta_H q_L - T_L; \tag{3}$$

$$\theta_L q_L - T_L \geq \theta_L q_H - T_H; \tag{4}$$

$$-c + \alpha[\theta_H q_H - T_H] + (1-\alpha)[\theta_L q_L - T_L] \geq \max[\theta_m q_H - T_H, \theta_m q_L - T_L, 0]. \tag{5}$$

The first 4 constraints are the familiar IR and IC constraints. The last constraint, (5), ensures that the firm would first find out its own type and then choose between (q_H, T_H) and (q_L, T_L), rather than give up buying, or pick between (q_H, T_H) and (q_L, T_L) without first knowing its type.

[3]Formally, the bank seeks to

$$\max_{q,T} T - \frac{1}{2}q^2$$

subject to

$$\theta_m q - T \geq 0; \tag{6}$$

$$\theta_m q - T \geq -c + \alpha \max[\theta_H q - T, 0] + (1-\alpha)\max[\theta_L q - T, 0]. \tag{7}$$

Note that (7) ensures that the firm would make a borrowing directly rather than spend c and then decide whether to make a borrowing based on its type j, and that (6) ensures that the firm can break even by borrowing directly.

and $T_H = T_L =$ ___No. 14___, so that the bank's optimal payoff is equal to ___No. 15___.

Solution. Consider part (i). Given $c = +\infty$, it is impossible for the firm to learn about its type before dealing with the bank. Thus the bank seeks to

$$\max_{T,q\geq 0} T - \frac{1}{2}q^2,$$

subject to

$$\theta_m q - T \geq 0.$$

The constraint must be binding at optimum: or else we could raise T slightly and further raise the value of the objective function. Thus the optimal q must solve

$$\theta_m q - \frac{1}{2}q^2,$$

so that we have at optimum

$$q = \theta_m = \frac{3}{2},\ T = \frac{9}{4} \Rightarrow T - \frac{1}{2}q^2 = \frac{9}{8}.$$

The bank's optimal payoff is thus equal to $\frac{9}{8}$.

Now, consider part (ii). With $c = 0$, the firm would always learn about its type before dealing with the bank, so that the problem reduces to the screening problem examined in sections 5-6 in Lecture 4. Theorem AS-1 implies that IR_L and IC_H (i.e., (2) and (3) in footnote 2) must be binding at optimum, so that

$$T_L = \theta_L q_L = \frac{5q_L}{4},\ T_H = \theta_H q_H - (\theta_H - \theta_L)q_L = \frac{7q_H - 2q_L}{4}.$$

Thus the bank seeks to

$$\max_{q_H \geq q_L \geq 0} \frac{1}{2}\{\frac{7q_H - 2q_L - q_H^2}{4} + \frac{5q_L - 2q_L^2}{4}\},$$

so that at optimum we have

$$q_H = \frac{7}{4},\ q_L = \frac{3}{4} \Rightarrow T_L = \frac{15}{16},\ T_H = \frac{43}{16}.$$

The bank's optimal payoff is thus equal to $\frac{29}{32}$.

Now, consider part (iii). We can quickly make a few observations.

Step 1. The right-hand side of (5) can be re-written as $\max[\theta_m q_H - T_H, \theta_m q_L - T_L]$.

Step 1 is true because, by (1), we have $\theta_m q_L - T_L \geq \theta_L q_L - T_L \geq 0$.

Step 2. We can re-write (5) as

$$\theta_H(q_H - q_L) - \frac{c}{\alpha} \geq T_H - T_L, \tag{8}$$

and

$$T_H - T_L \geq \theta_L(q_H - q_L) + \frac{c}{1-\alpha}. \tag{9}$$

Step 3. We have (8) $\Rightarrow$ (3), (9) $\Rightarrow$ (4). Moreover, we have (8)+(2) $\Rightarrow$ (1). Hence we only need to impose (2), (8), and (9).

Step 4. At optimum, (2) and (8) must be binding. (This follows from the same reasoning that we used to prove that in Theorem AS-1, IR_1 and LDIC must be binding at optimum.) Thus we have

$$T_L = \theta_L q_L,\ T_H = \theta_L q_L + \theta_H(q_H - q_L) - \frac{c}{\alpha}. \tag{10}$$

Step 5. Thus the bank's problem becomes

$$(\text{P}) \quad \max_{q_H, q_L} \alpha[\theta_L q_L + \theta_H(q_H - q_L) - \frac{c}{\alpha} - \frac{1}{2}q_H^2] + (1-\alpha)[\theta_L q_L - \frac{1}{2}q_L^2]$$

subject to (9), or $q_H - q_L \geq \frac{c}{\alpha(1-\alpha)(\theta_H - \theta_L)}$.

Step 6. The unconstrained solution to (P) violates (9), so that (9) must be binding at optimum also. Thus the bank seeks to

$$(\text{P}') \quad \max_{q_L \geq 0} \alpha[\theta_L q_L + \theta_H(\frac{c}{\alpha(1-\alpha)(\theta_H - \theta_L)}) - \frac{c}{\alpha} - \frac{1}{2}(q_L + \frac{c}{\alpha(1-\alpha)(\theta_H - \theta_L)})^2]$$
$$+ (1-\alpha)[\theta_L q_L - \frac{1}{2}q_L^2].$$

Solving (P′), we obtain

$$q_L = \frac{1}{4} \Rightarrow q_H = \frac{9}{4} \Rightarrow T_H = \frac{53}{16},\ T_L = \frac{5}{16},$$

so that the bank's optimal payoff is equal to $\frac{17}{32}$.

Now, consider part (iv). First we examine the right-hand side of constraint (7). We claim that at optimum, (q, T) must be such that

$$\theta_H q \geq T > \theta_L q.$$

Indeed, if $T > \theta_H q$, then (7) is implied by (6), and hence the optimal $(q, T) = (\theta_m, \theta_m^2)$, which violates $T > \theta_H q$! If, on the other hand, $T \leq \theta_L q$, then (7) becomes redundant, so that once again we have $(q, T) = (\theta_m, \theta_m^2)$, which violates $T \leq \theta_L q$!

It follows that (7) can be re-written as

$$\theta_m q - T \geq -c + \alpha(\theta_H q - T). \tag{11}$$

We can combine (11) and (6) and impose one single constraint:

$$T \leq \min[\theta_m q, \theta_L q + \frac{c}{1-\alpha}],$$

where the minimum is equal to $\theta_L q + \frac{c}{1-\alpha}$ when c is very small (as in the current case, where $c = \frac{1}{32}$), and in the latter case we have $T = \theta_L q + \frac{c}{1-\alpha}$ at optimum. Thus the bank seeks to

$$\max_{q \geq 0} \theta_L q + \frac{c}{1-\alpha} - \frac{1}{2} q_L^2,$$

showing that the optimal solution is such that

$$q = \frac{5}{4},\ T = \frac{13}{8},$$

and the bank's optimal payoff is equal to $\frac{27}{32}$.

2. The following signaling game is taken from Example 3 (section 30) of Lecture 4 but with different payoffs for the players. In the unique intuitive pooling equilibrium, the informed's signal is __No. 16__, which induces the uninformed to choose action __No. 17__. This game also has a pooling PBE which is not intuitive, and in the latter PBE, the informed's signal is __No. 18__ and the uninformed's expected payoff is __No. 19__.

m_1	a_1	a_2	a_3
t_1	$(0,5)$	$(4,4)$	$(4,3)$
t_2	$(2,2)$	$(6,4)$	$(4,3)$

m_2	a_1	a_2	a_3
t_1	$(2,4)$	$(4,3)$	$(6,2)$
t_2	$(0,3)$	$(6,3)$	$(4,8)$

m_3	a_1	a_2	a_3
t_1	$(2,8)$	$(8,3)$	$(4,2)$
t_2	$(0,2)$	$(8,3)$	$(0,8)$

Solution. We shall use the same notation as in Example 3 (section 30) of Lecture 4. In the unique intuitive pooling equilibrium, the informed's strategy is

$$\begin{pmatrix} t_1 \to m_1 \\ \\ t_2 \to m_1 \end{pmatrix},$$

and the uninformed's strategy is

$$\begin{pmatrix} m_1 \to a_2 \\ \\ m_2 \to a_1 \\ \\ m_3 \to a_1 \text{ or } a_3 \end{pmatrix},$$

with the uninformed's beliefs being such that $\mu_1 = \frac{1}{2}$, $\mu_2 \in [\frac{5}{7}, 1]$, $\mu_3 \in [0,1]$. This PBE is intuitive, because intuition would suggest that the uninformed rule out t_2 when seeing m_2 or m_3, but given $\mu_2 = 1 = \mu_3$, the uninformed's best choice among a_1, a_2, and a_3 would still prevent t_1 from deviating from sending m_1.

In the un-intuitive pooling PBE, on the other hand, the informed's strategy is

$$\begin{pmatrix} t_1 \to m_2 \\ \\ t_2 \to m_2 \end{pmatrix},$$

and the uninformed's strategy is

$$\begin{pmatrix} m_1 \to a_1 \\ \\ m_2 \to a_3 \\ \\ m_3 \to a_1 \text{ or } a_3 \end{pmatrix},$$

with the uninformed's beliefs being such that $\mu_2 = \frac{1}{2}$, $\mu_1 \in [\frac{2}{3}, 1]$, $\mu_3 \in [0, 1]$. The uninformed's equilibrium payoff is $\frac{1}{2}(8 + 2) = 5$. This PBE is not intuitive because intuition would suggest that the uninformed rule out t_1 upon seeing m_1, but then his best choice is a_2, which would exactly induce t_2 to deviate from sending m_2.

3. Firm X is all-equity financed, and it has three shareholders (T, U, and V) and three shares of common stock outstanding. The three shareholders (and the public investors also) are risk neutral without time preference, and each shareholder is holding $\frac{1}{3}$ of the ownership at date 0.

   It is common knowledge at date 0 that the firm's date-1 assets will consist of \$40 in cash and an investment project that will generate $\$\tilde{x}$ at date 2, where $\tilde{x}$ may take on 80 or 50 with probability π and $1 - \pi$ respectively, and that shareholder T, who is also the CEO of the firm, will privately learn about the realization of $\tilde{x}$ at date 1.

   Thus a signaling game will be played at date 1, where the informed CEO, shareholder T, can decide to or not to buy back one share from the other two uninformed shareholders U and V. More precisely, the signaling game proceeds as follows.

   - At the beginning of date 1, T first decides *to* or *not to* repurchase 1 share from U and V (signaling with two possible signals!).
   - The game ends with T's payoff being $\frac{40+\tilde{x}}{3}$ if T gives up share repurchase.
   - In case T has announced a share repurchase, then U and V must compete in price to tender one share to T. In the latter case, T's payoff is equal to $\frac{40+\tilde{x}-P_1}{2}$.

(i) Suppose that $\pi = \frac{4}{5}$. Then the date-0 stock price for firm X is equal to ___No. 20___.

(ii) Let $j = r$ and $j = nr$ denote respectively the event that T announces a share repurchase at date 1 and the event that T does not. Let $P_1(j)$ denote the date-1 stock price for firm X in event $j \in \{r, nr\}$. Then in a date-1 separating equilibrium, T announces the event $j = r$ if and only if the realization of $\tilde{x}$ is equal to ___No. 21___. In this equilibrium, we have $P_1(r) =$ ___No. 22___ and $P_1(nr) =$ ___No. 23___. In this equilibrium, the stock price for firm X may (answer ‘rise’ or ‘drop’) ___No. 24___ from P_0 when absolutely nothing occurs at date 1.

(iii) The above signaling game also has a pooling PBE where at date 1, $P_1 - P_0$ is (answer ‘$>$’, or ‘$<$’, or ‘$=$’) ___No. 25___ zero.

Solution. Consider part (i). At date 0, when all three investors are equally uninformed, the stock price is

$$\frac{1}{3}[\pi(40 + 80) + (1 - \pi)(40 + 50)] = 30 + 10\pi = 38.$$

Consider part (ii). First conjecture that there is a separating PBE where T would announce a share repurchase when $\tilde{x} = 80$ but T would not when $\tilde{x} = 50$. In this PBE, T's announcing a share repurchase is taken as direct evidence that $\tilde{x} = 80$, and hence U and V would compete in price to make sure that $P_1(r) = 40$. Before making an announcement, T would expect $P_1(r) = 40$. Then, would T choose $j = r$ or $j = nr$? Upon seeing $\tilde{x} = 80$, T actually feels indifferent about $j = r$ or $j = nr$, and hence choosing $j = r$ is indeed one of T's best responses. Upon seeing $\tilde{x} = 50$, on the other hand, T would get $\frac{50+40}{3}$ by choosing $j = nr$, but T would only get $\frac{40-40+50}{2}$ by announcing $j = r$. Thus expecting $P_1(r) = 40$, T would announce $j = r$ if and only if $\tilde{x} = 80$, proving that our conjecture is correct. Thus there does exist a separating PBE where T would announce a share repurchase if and only if $\tilde{x} = 80$.

Note that when T chooses $j = nr$, nothing occurs at date 1, but $P_1(nr) < P_0$; that is, the stock price can drop at date 1 when absolutely there is no external bad news arriving at the market. This happens because no news released by firm X is taken as bad news at date 1. After all, a share repurchase is expected to happen if there is good news arriving at date 1!

It is easy to show that there is no other type of separating PBE, and that for part (iii), there exists a unique pooling PBE where T never announces a share repurchase, and in that equilibrium we have $P_1 = P_0$. This equilibrium can be sustained by the off-the-equilibrium belief that a share repurchase can only be announced when $\tilde{x} = 80$.

4. A firm can operate at date 1, date 2, and date 3, and it has two possible types, denoted by G and B. At the beginning of date t, the Bertrand competitive banks believe that the firm is of type G with probability x_t. The firm is endowed with $w \in [0, 1)$ dollars at the beginning of date 1, and it has no endowments at date 2 and date 3. The firm needs to spend one dollar at the beginning of each date $t \in \{1, 2, 3\}$ in order to operate at date t.

   If a type G firm can come up with one dollar at the beginning of each date $t \in \{1, 2, 3\}$, it will generate two dollars at the end of date t. A type G firm will always pay out all of its current earnings as cash dividends, after repaying its debt.

   If a type B firm can come up with one dollar at the beginning of each date $t \in \{1, 2, 3\}$, it will get a non-transferable private benefit $b > 0$ at the end of date t, and it must then decide whether or not to exert an unobservable effort. Let $e_t = 1$ (respectively, $e_t = 0$) denote the event that a type B firm chooses to (respectively, not to) exert the effort at date t. Exerting the effort will incur a dis-utility $\phi > 0$ to the type B firm, but with probability π the effort may generate two dollars (or nothing with probability $1 - \pi$) at the end of date t. Choosing $e_t = 0$, on the other hand, will generate nothing for sure. The cash earnings generated by $e_t = 1$ and the cash earnings generated by $e_s = 1$ are stochastically independent, for any $t, s \in \{1, 2, 3\}$, $t \neq s$.

   The firm can only borrow short-term debt, in the sense that if the firm borrows from a bank at the beginning of date t, then it is required to repay F_t at the end of date t, after it generates date-t earnings, where the face value of debt, F_t, is determined via Bertrand competition among the banks, based on their common date-t information. The firm is protected by limited liability, and even if it has defaulted on earlier debt obligations, it can still turn to the banks for a new loan later on. Similarly, if the firm's request for a loan at date $t \in \{1,2\}$ has been turned down by all the banks, it can still come back at date $t+1$ and request for a new loan.

We assume that the firm and the banks are risk neutral without time preferences. We shall examine the equilibrium of this reputation game, where the type B firm may try to mimic the type G firm's behavior in equilibrium. From now on, assume the following parameter values:

$$b = 3,\ \phi = \frac{3}{4},\ \pi = \frac{1}{3}.$$

Let $z_t \in \{2,0\}$ denote the (observable and verifiable) cash earnings generated by the firm at the end of date t. Let E_t denote the event that $z_s = 2$ for all $s \leq t$, and E_t^c denote the complement of E_t. Apparently, F_{t+1} depends on whether event E_t has occurred or not, so that we shall write $F_{t+1}(E_t)$ or $F_{t+1}(E_t^c)$. If at the beginning of date t the banks decide to turn down the firm's request for a loan, then we define $F_t = +\infty$. In the following, the type B firm's *payoff* is defined as the sum of the expected profits that it makes over date 1, date 2, and date 3.

(i) At the date-3 subgame where the type B firm has obtained a loan with face value F_3, we must have $e_3 =$ ___No. 26___ in equilibrium, so that at the beginning of date 3, the banks would approve for the firm's request for a loan if and only if $x_3 \geq x_3^*$, where $x_3^* =$ ___No. 27___.

(ii) At the date-2 subgame where the type B firm has obtained a loan with face value F_2, it would optimally choose $e_2 = 1$ if $x_2 \geq x_2^*$, where $x_2^* =$ ___No. 28___. Now, if $x_2 > x_2^*$, then as a function of x_2, we have $F_2 =$ ___No. 29___, so that the type B firm's continuation payoff in this date-2 subgame as a function of x_2 is equal to ___No. 30___. On the other hand, if $x_2 < x_2^*$, then as a function of x_2, we have $F_2 =$ ___No. 31___, so that the type B firm's continuation payoff in this date-2 subgame as a function of x_2 is equal to ___No. 32___.

(iii) Suppose that $x_1 = \frac{1}{3}$ and $w = 0$ at the beginning of date 1. In equilibrium, we have $F_1 =$ ___No. 33___, $F_2(E_1) =$ ___No. 34___,

$F_3(E_2) =$ ___No. 35___. The type B firm's equilibrium payoff is equal to ___No. 36___.

(iv) Suppose that $x_1 = \frac{1}{8}$ and $w = \frac{1}{6}$ dollars at the beginning of date 1. Thus at date 1 (and only at date 1), the firm needs to borrow $\frac{5}{6}$ dollars only. In equilibrium, we have $F_1 =$ ___No. 37___, $F_2(E_1) =$ ___No. 38___, $F_3(E_2) =$ ___No. 39___. The type B firm's equilibrium payoff is equal to ___No. 40___.

Solution. Consider part (i). At date 3, a type-B firm no longer has an incentive to build a reputation. Since

$$\phi = \frac{3}{4} > \frac{2}{3} = \pi \cdot 2,$$

a type-B firm would choose $e_3 = 0$ for sure. Thus given x_3, a bank lending a dollar to the firm at the beginning of date 3 would expect a cash inflow of $x_3 \cdot F_3 + (1 - x_3) \cdot 0$ at the end of date 3, and since limited liability requires that $F_3 \leq 2$, the lending bank can expect to break even if and only if $x_3 \geq \frac{1}{F_3} \geq \frac{1}{2} \equiv x_3^*$.

Consider part (ii). In an equilibrium where $e_2 = 1$ given x_2 and F_2, we must have

$$x_3 = \begin{cases} \frac{x_2 \cdot 1}{x_2 \cdot 1 + (1-x_2)\cdot \pi} = \frac{3x_2}{2x_2+1}, & \text{if } z_2 = 2; \\ 0, & \text{if } z_2 = 0, \end{cases}$$

and

$$F_2 = \frac{1}{x_2 \cdot 1 + (1 - x_2) \cdot \pi} = \frac{3}{2x_2 + 1}.$$

If $x_3 < \frac{1}{2}$ following $z_2 = 2$, then a type-B firm would rather choose $e_2 = 0$ instead, according to part (i). Thus for $e_2 = 1$ to be consistent with an equilibrium it is necessary that

$$\frac{3x_2}{2x_2 + 1} \geq \frac{1}{2} \Leftrightarrow x_2 \geq \frac{1}{4} \equiv x_2^*.$$

Now, we show that $x_2 \geq \frac{1}{4}$ is also sufficient for ensuring an equilibrium with $e_2 = 1$. In this equilibrium, a type-B firm's continuation payoff is equal to

$$b - \phi + \pi \cdot (\max[2 - F_2, 0] + b) = \frac{4b}{3} + \frac{2}{3} - \phi - \frac{F_2}{3} = \frac{47}{12} - \frac{F_2}{3},$$

which, given that $F_2 \geq 2$, is indeed greater than a type-B firm's continuation payoff from choosing $e_2 = 0$, which is b alone.

It follows from the preceding discussion that $e_2 = 0$ if $x_2 < \frac{1}{4}$. In the latter case, we have $x_2 < \frac{1}{2} \equiv x_3^*$, we conclude by part (i) that the banks would turn down the firm's date-2 request for a loan. Thus we have $F_2 = +\infty$, which leads to $x_3 = x_2 < \frac{1}{4}$, and by part (i), a type-B firm's continuation payoff is zero.

Consider part (iii). Mimicking the analysis for the date-2 subgame, we conclude that a bank lending one dollar to the firm at the beginning of date 1 can expect to break even only if

$$x_1 + \pi(1 - x_1) \geq \frac{1}{F_1} \geq \frac{1}{2} \Rightarrow x_1 \geq \frac{1}{4}.$$

When $x_1 \geq \frac{1}{4}$, then $x_2 \geq \frac{1}{2}$ following $z_1 = 2$, so that a type-B firm by choosing $e_1 = 1$ would obtain the continuation payoff

$$b - \phi + \pi \cdot \{\max(2 - F_1, 0) + b - \phi + \pi \cdot [\max(2 - F_2(E_1), 0) + b]\},$$

which is indeed greater than b alone. Thus, once again, $e_1 = 1$ if and only if $x_1 \geq \frac{1}{4}$.

With $x_1 = \frac{1}{3}$, we conclude by $e_1 = 1$ that

$$F_1 = \frac{3}{2x_1 + 1} = \frac{9}{5},$$

and in event E_1, we have

$$x_2 = F_1 x_1 = \frac{3}{5} \Rightarrow F_2(E_1) = \frac{3}{2x_2 + 1} = \frac{15}{11},$$

so that in event E_2, we have

$$x_3 = F_2(E_1)x_2 = \frac{9}{11}, \Rightarrow F_3(E_2) = \frac{1}{x_3} = \frac{11}{9}.$$

A type-B firm's equilibrium payoff is

$$b - \phi + \pi \cdot \{\max(2 - F_1, 0) + b - \phi + \pi \cdot [\max(2 - F_2(E_1), 0) + b]\} = \frac{1718}{495} > 3 = b.$$

Finally, consider part (iv). Since $x_1 < \frac{1}{4}$, the banks would turn down the firm's date-1 request for a loan if $w = 0$. Fortunately, here we have $w = \frac{1}{6}$. As can be conjectured, here we have $e_1 = 1$, so that

$$F_1 = \frac{\frac{5}{6}}{x_1 + \pi(1 - x_1)} = 2.$$

In event E_1, we have

$$x_2 = \frac{3x_1}{2x_1 + 1} = \frac{3}{10} \Rightarrow F_2(E_1) = \frac{3}{2x_2 + 1} = \frac{15}{8},$$

so that in event E_2, we have

$$x_3 = F_2(E_1)x_2 = \frac{9}{16}, \Rightarrow F_3(E_2) = \frac{1}{x_3} = \frac{16}{9}.$$

A type-B firm's equilibrium payoff is $\frac{241}{72} > 3 > \frac{1}{6}$.