Game Theory with Applications to Finance and Marketing, I

Iterative Deletion of Strictly Dominated Strategies

Consider two players, labeled A and B, in a simultaneous game with their *finite* strategy spaces being respectively

$$S^A = \{a_1, a_2, \cdots, a_J\},$$

and

$$S^B = \{b_1, b_2, \cdots, b_K\}.$$

Suppose that iterative deletion of strictly dominated strategies[1] in two different sequential orders V and T leads to two maximal reductions (V_N^A, V_N^B) and (T_M^A, T_M^B) for (S^A, S^B), where N (respectively, M) is the number of steps taken to reach the maximal reduction (V_N^A, V_N^B) (respectively, (T_M^A, T_M^B)). We shall prove by contraposition that $V_N^A = T_M^A$ and $V_N^B = T_M^B$. To this end, assume without loss of generality that $a_j \in V_N^A \bigcap [T_M^A]^c$, and we shall demonstrate a contradiction.

Since T_M^A does not contain a_j, a_j must be strictly dominated by some mixed strategy σ_A and deleted in some step $m \leq M-1$ under T. Can we have $V_N^B \subset T_m^B$? We shall show that the answer is negative.

Suppose instead that $V_N^B \subset T_m^B$. If σ_A remains feasible when A is restricted to using pure strategies in V_N^A, then since a_j is strictly dominated by σ_A, a_j cannot be contained in V_N^A, which is a contradiction. What if σ_A becomes infeasible when A is restricted to using pure strategies in V_N^A? In the latter case, there must exist $a_i \in T_m^A \bigcap [V_N^A]^c$ with $\sigma_A(a_i) > 0$, but since V_N^A does not contain a_i, a_i must be strictly dominated by some σ_A^i and deleted at some step $n_i < N$ under V. For all a_i as such, we can replace a_i by σ_A^i in σ_A, in the sense that we assign probability $\sigma_A(a_i)$ to σ_A^i rather than a_i, and call the resulting A's mixed strategy σ_A'. Clearly, σ_A (and hence a_j) is

[1] Let $S_0^i \equiv S^i$ and $\Sigma_0^i \equiv \Sigma^i$. Define correspondingly S_0^{-i} and Σ_0^{-i}. Define for all $n \geq 1$, $\Sigma_n^i \equiv \{\sigma^i \in \Sigma^i : \sigma^i(s^i) > 0 \Rightarrow s^i \in S_{n-1}^i\}$ and $S_n^i = S_{n-1}^i \setminus \{s^i \in S_{n-1}^i : \exists \sigma^i \in \Sigma_n^i \ni u_i(\sigma^i, \sigma^{-i}) > u_i(s^i, \sigma^{-i}), \forall \sigma^{-i} \in \Sigma_n^{-i}\}$. Define $S_\infty^i \equiv \bigcap_{n=1}^\infty S_n^i$. Because S^A and S^B are finite sets, it takes a finite number of steps to reach S_∞^i, for $i = A, B$. We must show that S_∞^i is independent of the order of deletion of strictly dominated strategies.

dominated strictly by σ'_A and hence a_j cannot be contained in V_N^A, as long as σ'_A is feasible when A is restricted to using pure strategies in V_N^A. Note that if σ'_A is again infeasible when A is restricted to using pure strategies in V_N^A, then by repeating the above argument we can again create some σ''_A that dominates strictly σ'_A. In the end, since S^A is a finite set, there must be some σ^*_A that remains feasible when A is restricted to using pure strategies in V_N^A,[2] and σ^*_A dominates strictly a_j, implying that V_N^A cannot contain a_j, which is a contradiction.

Thus we conclude that there must exist some $b_k \in V_N^B \bigcap [T_m^B]^c$ such that when player B uses b_k, player A feels that using a_j is no worse than using σ_A. As T_m^B does not contain b_k, there exists σ_B that dominates strictly b_k and makes b_k deleted at some step $l \leq m-1$ under T. If $V_N^A \subset T_l^A$, then by mimicking the reasoning in the preceding paragraph, we can establish that V_N^B does not contain b_k, which would be a contradiction. Thus there must exist some $a_h \in V_N^A \bigcap [T_l^A]^c$, such that when player A uses a_h, player B feels that using b_k is no worse than using σ_B.

Note that a_j and a_h are distinct: a_j has not been deleted at step $l \leq m-1$ under T, and if $a_j = a_h$, then b_k cannot be deleted at step l under T!

Now we can turn the spotlight away from $a_j \in V_N^A \bigcap [T_M^A]^c$ but on to $a_h \in V_N^A \bigcap [T_l^A]^c$ instead. Define $a_1 \equiv a_j$ and $a_2 \equiv a_h$, and repeat the above argument. We thus obtain an infinite sequence of distinct elements of S^A, which, given that S^A is finite, is a contradiction. We conclude that $V_N^A \subset T_M^A$, and by symmetry, that $T_M^A \subset V_N^A$, so that $V_N^A = T_M^A$. This conclusion applies to player B as well.

[2]First note the following fact: If σ_A strictly dominates a_j when B is restricted to using pure strategies in $S_B \subset S^B$, then for all $b_r \in S_B$, $\sum_{i=1}^m u_A(a_i, b_r)\sigma_A(a_i) - u_A(a_j, b_r) > 0$, implying that $\sigma_A^j \equiv \frac{1}{1-\sigma_A(a_j)}(\sigma_A(a_1), \sigma_A(a_2), \cdots, \sigma_A(a_{j-1}), \sigma_A(a_{j+1}), \cdots, \sigma_A(a_m))$ is also a mixed strategy that dominates strictly a_j. Note that σ_A^j assigns zero probability to a_j. Thus when a_j is strictly dominated, we can always assume that it is dominated by a mixed strategy that assigns positive probabilities only to A's pure strategies that differ from a_j. Now, define a^1 as a_i mentioned above, and if σ'_A is infeasible when A is restricted to using pure strategies in V_N^A, then there is $a^2 \in [V_N^A]^c$ with $\sigma'_A(a^2) > 0$ and a^2 differs from a^1. Repeating this argument, we see that if the afore-mentioned σ^*_A does not exist, then we would obtain an infinite sequence $\{a^n\}$ contained in $S^A \bigcap [V_N^A]^c$, which contradicts the fact that S^A is finite.