Game Theory with Applications to Finance and Marketing, I

Lecture 5: Various Applications of Game Theory in Finance, and Mechanism Design

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1. This note consists of three parts. In part I, we shall review the costly-state-verification (CSV) model of debt financing and give several applications. In part II. we shall discuss how risky short-term debt and long-term debt may respectively affect a borrowing firm's status in product market competition. In part III, we introduce the concept of Nash implementation, and talk about the subjects to be covered in *Game Theory with Applications to Finance and Marketing, II.*

2. (**Part I.**) Consider the following simplified version of the CSV model studied in Gale and Hellwig (1985, *RES*). Entrepreneur A must raise $I > 0$ dollars from a competitive bank B at date 0 in order to implement an investment project, which yields a random cash inflow (interchangeably, profit) $\tilde{z}$ at date 1. Assume that A and B are both risk-neutral without time preferences, and A has all bargaining power against B at date 0 when they sign a financial contract (because there are many banks competing with B at date 0). Assume also that $\tilde{z}$ is uniformly distributed over the unit interval $[0, 1]$. (The results will remain valid if instead $\tilde{z}$ has a positive density over its support, which is a compact interval.) At date 1, only A gets to see the realization of $\tilde{z}$ (called the *earnings state* or the *realized profit*), but if B wants, B can spend $c > 0$ (to hire a CPA) to find out the true profit (and to produce legal evidence for that true profit). Assume that $I + c < E[\tilde{z}] = \frac{1}{2}$; that is, the investment project has a positive NPV.

   A date-0 incentive-feasible financial contract is a tuple

$$\mathcal{C} \equiv \{R_0(\hat{z}), R_1(\hat{z}, z), d(\hat{z});\ \forall z, \hat{z} \in [0, 1]\}$$

such that, given contract $\mathcal{C}$ has been signed at date 0,
(1) A must first make an earnings report $\hat{z} \in [0,1]$ at date 1;
(2) B will then spend c to find out A's true profit (equivalently, to verify A's earnings state), if and only if it is specified in $\mathcal{C}$ that $d(\hat{z}) = 1$, where $\forall \hat{z} \in [0,1]$, $d(\hat{z})$ equals either 0 or 1;
(3) A must repay B the amount $R_0(\hat{z})$ if A has reported a profit $\hat{z}$ such that $d(\hat{z}) = 0$;
(4) A must repay B the amount $R_1(\hat{z}, z)$ if A has reported a profit $\hat{z}$ such that $d(\hat{z}) = 1$, and A's true profit is instead z (which will be revealed to the public after B spends c);
(5) (Condition LL) $0 \leq R_1(\hat{z}, z) \leq z, \quad 0 \leq R_0(\hat{z}) \leq \hat{z}$;
(6) (Condition IC_A) for all $z \in [0,1]$, reporting $\hat{z} = z$ is A's date-1 optimal strategy; and
(7) (Condition IR_B) B can at least break even by accepting contract $\mathcal{C}$.

Some explanations are in order. In plain words, an incentive-feasible contract must specify when B will send a CPA to audit A's profit at date 1, and this decision d is contingent on A's profit report $\hat{z}$. Moreover, if given A's profit report $\hat{z}$, B must audit A's profit according to the contract, then the true profit z will become known, and in that case A's repayment to B can depend on both A's report $\hat{z}$ and the true profit z; this explains the function $R_1(\cdot, \cdot)$. On the other hand, if given A's profit report $\hat{z}$, B should not audit A's profit according to the contract, then since the true profit remains A's private information, the contract can only specify a repayment $R_0(\cdot)$ that is contingent only on A's profit report $\hat{z}$. Furthermore, the repayment R_1 cannot exceed the true profit z, and in case of no audit, the repayment cannot exceed the profit report $\hat{z}$; these are referred to as the *limited-liability constraint* for A. Note that R_1 and R_0 are also required to be non-negative; these are called *limited-liability constraint* for B. The latter says that B is not obliged to lending more money to the firm at date 1. These limited-liability constraints are written as Condition LL in (5). Next, note that we have required that A must always truthfully report the date-1 profit under an incentive-feasible contract. This is Condition IC_A, and it originates from the *revelation principle* in contract theory, which says that for each contract that makes d, R_0, and R_1 contingent on some verifiable messages there exists a contract that makes d, R_0, and R_1 contingent on $\hat{z}$ and z only and that induces truthtelling as A's best response in each and every true state z (which gives rise to the constraint IC_A), where equivalence means that the two contracts yield the same payoff for A. Finally, an incentive-feasible contract must ensure that B is willing to accept it in the first place, and this is stated as Condition IR_B.

To sum up, in designing a financial contract, we can always confine our attention to the set of *incentive-feasible contracts* defined above. Among these incentive feasible contracts, A's favorite contracts (which may not be unique) will be termed *incentive efficient*, or simply the *optimal financial contracts*. Now let us characterize an optimal financial contract.

Step 1. Suppose that $\mathcal{C}$ is incentive-feasible. Then $R_0(\hat{z}_1) = R_0(\hat{z}_2)$ for all $\hat{z}_1, \hat{z}_2$ such that $d(\hat{z}_1) = d(\hat{z}_2) = 0$.

Proof. Suppose that $d(\hat{z}_1) = d(\hat{z}_2) = 0$ but, say, $R_0(\hat{z}_1) > R_0(\hat{z}_2)$. Then given that the true profit is $\hat{z}_1$, A strictly prefers to lie and report $\hat{z}_2$. This is a contradiction to IC_A. ‖

Step 2. If $\mathcal{C}$ is incentive-feasible, then there exists some $z \in [0, 1]$ such that $d(z) = 1$.

Proof. Suppose that $d(z) = 0$ for all $z \in [0, 1]$. Then according to Step 1, A must make the same repayment regardless of the true profit z. Condition LL implies that, by reporting $z = 0$, A does not have to repay anything. This violates IR_B, since B, after paying $I > 0$ at date 0, can get nothing back at date 0. This shows that $\mathcal{C}$ is not incentive-feasible, a contradiction. ‖

Step 3. Suppose that $\mathcal{C}$ is incentive-feasible. Let $F = R_0(\hat{z})$ for all $\hat{z}$ such that $d(\hat{z}) = 0$. Then $F \geq R_1(z, z)$ for all z such that $d(z) = 1$.

Proof. This assertion is vacuously true if $d(z) = 1$ for all $z \in [0, 1]$. So, let us assume that there exists z' with $d(z') = 0$ and let $F = R_0(z')$. If for some z with $d(z) = 1$ we have $R_1(z, z) > F$, then when the true profit is z, A strictly prefers to lie and report profit z', which violates IC_A, a contradiction. ‖

To sum up, the first three steps have shown that if $\mathcal{C}$ is incentive-feasible with F being the repayment made by A to B in the event of $d = 0$, then $R(z) \leq \min(z, F)$ for all $z \in [0, 1]$, where $R(z) = R_1(z, z)$ if $d(z) = 1$ and $R(z) = R_0(z)$ if $d(z) = 0$. A contract $\mathcal{C}$ that specifies $R_1(z, z) = z$ and $K = [0, F)$ is referred to as a standard debt (SD) contract, which is uniquely defined by the face value of debt, F. Note that a SD contract is incentive feasible, and with such a contract, $R(z) = \min(z, F)$ for all $z \in [0, 1]$. The remaining steps will establish that no other contracts can outperform an optimal SD contract. That is, with respect to the costly state verification problem, standard debt contracts are optimal contracts.

Step 4. Suppose that $\mathcal{C}$ is incentive-feasible. Let $F = R_0(z)$ for all z with $d(z) = 0$. Define the event

$$K \equiv \{z \in [0, 1] : \ d(z) = 1\}.$$

Then $[0, F) \subset K$. That is, we must have $d(z) = 1$ whenever $z < F$.

Proof. Suppose that $z < F$ and yet $d(z) = 0$. Then we would have $F = R_0(z) \leq z < F$, a contradiction. ‖

Step 5. Suppose that $\mathcal{C}$ is incentive-efficient. Then IR_B is binding under $\mathcal{C}$.

Proof. By definition, $\mathcal{C}$ is incentive-efficient if and only if it solves the following maximization problem:

$$\max_{K,F,R_1(\cdot,\cdot)} \int_K [z - R_1(z,z)]dz + \int_{[0,1]\setminus K} [z - F]dz$$

subject to

$$(\text{IR}_B) \ \int_K R_1(z,z)dz + F[1 - \text{prob.}(K)] - c\ \text{prob.}(K) \geq I;$$

$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)],\ \forall z, z' \in K;$$

$$0 \leq F \leq z,\ \forall z \in [0,1] \setminus K.$$

Suppose instead that IR_B is not binding under $\mathcal{C}$. Then for sufficiently small $e > 0$, for all $z', z \in K$, we can replace F and $R_1(z',z)$ by respectively $F(1-e)$ and $R_1(z',z)(1-e)$ and ensure that the new contract is still incentive feasible. Since A would be better off offering B the new contract rather than $\mathcal{C}$, by definition $\mathcal{C}$ cannot be incentive efficient. Thus we have a contradiction. ‖

Step 6. $\mathcal{C}$ is incentive-efficient if and only if it solves the following minimization program:

$$\min_{K,F,R_1(\cdot,\cdot)} \text{prob.}(K)$$

subject to

$$\text{prob.}(K) = \frac{1}{F+c}[\int_K R_1(z,z)dz + F - I],$$

$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)],\ \forall z, z' \in K;$$

$$0 \leq F \leq z,\ \forall z \in [0,1] \setminus K.$$

Proof. By Step 5, $\mathcal{C}$ is incentive-efficient only if IR_B is binding, which implies that

$$\int_K R_1(z,z)dz + F[1 - \text{prob.}(K)] = c\ \text{prob.}(K) + I,$$

and hence we can re-write the A's objective function as

$$\int_K [z - R_1(z,z)]dz + \int_{[0,1]\setminus K} [z - F]dz = E[z] - I - c \text{ prob.}(K).$$

It follows that $\mathcal{C}$ is incentive efficient if and only if it solves the following maximization problem:

$$\max_{K,F,R_1(\cdot,\cdot)} E[z] - I - c \text{ prob.}(K)$$

subject to

$$\text{prob.}(K) = \frac{1}{F+c}[\int_K R_1(z,z)dz + F - I],$$
$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)],\ \forall z, z' \in K;$$
$$0 \leq F \leq z,\ \forall z \in [0,1] \setminus K.$$

Now that $E[z]$, I, and $c > 0$ are all constants, we can further re-write Mr. A's optimization problem as

$$\min_{K,F,R_1(\cdot,\cdot)} \text{prob.}(K)$$

subject to

$$\text{prob.}(K) = \frac{1}{F+c}[\int_K R_1(z,z)dz + F - I],$$
$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)],\ \forall z, z' \in K;$$
$$0 \leq F \leq z,\ \forall z \in [0,1] \setminus K.$$

This completes the proof. $\|$

Define F^* as the unique solution to

$$g(F) \equiv -\frac{1}{2}F^2 + (1-c)F - I = 0.$$

Then, $0 < F^* = (1-c) - \sqrt{(1-c)^2 - 2I} < 1 - c$.

Step 7. Recall that a standard debt contract can be uniquely defined by its face value F.

- A standard debt contract with face value F has prob.$(K) = F$.
- A standard debt contract with face value F is incentive feasible if and only if $F^* \leq F \leq 1$.
- The SD contract with face value F^* dominates all other SD contracts, and will be denoted by $\mathcal{C}^*$. Under $\mathcal{C}^*$, IR_B is binding.
- B's payoff from holding a standard debt with face value F is strictly increasing in F on the interval $[F^*, 1-c)$, and it is strictly decreasing in F on the interval $(1-c, 1]$.

Proof. It is easy to see that a standard debt contract satisfies A's IC constraints and the limited liability constraints if and only if $0 \leq F \leq 1$. To satisfy B's IR constraint, B's payoff from accepting the standard debt contract, which is

$$\int_K R_1(z,z)dz + F[1 - \text{prob.}(K)] - c\ \text{prob.}(K) - I$$

$$= \int_0^F zdz + F[1-F] - cF - I$$
$$= -\frac{1}{2}F^2 + (1-c)F - I = g(F),$$

must be non-negative; that is, a SD contract with face value F would make IR_B binding if and only if $g(F) = 0$.

Since we have assumed that the net present value of the project is positive,

$$E[\tilde{z}] = \frac{1}{2} > c + I,$$

we have

$$(1-c)^2 - 2I = c^2 + 2[\frac{1}{2} - (c+I)] > 0,$$

implying that $g(F) \geq 0$ if and only if

$$F^* = (1-c) - \sqrt{(1-c)^2 - 2I} \leq F \leq (1-c) + \sqrt{(1-c)^2 - 2I}.$$

However, we have

$$\frac{1}{2} - c - I > 0 \Rightarrow 1 - 2c - 2I > 0 \Rightarrow (1-c)^2 - 2I > c^2$$

$$\Rightarrow \sqrt{(1-c)^2 - 2I} > c \Rightarrow (1-c) + \sqrt{(1-c)^2 - 2I} > 1.$$

Since $F > 1$ would violate A's IR constraint, we conclude that a standard debt contract is incentive feasible if and only if $F^* \leq F \leq 1$.

The last assertion now follows from the fact that the concave function $g(F)$ attains its maximum at $F = 1 - c$. $\|$

Step 8. Suppose that $\mathcal{C}$ is incentive efficient. Then under $\mathcal{C}$ the event

$$E = K \bigcap [F, 1]$$

is a zero-probability event.

Proof. Suppose that E occurs with a positive probability under the incentive efficient contract $\mathcal{C}$, and we shall demonstrate a contradiction.

First note that under $\mathcal{C}$, prob.$(K) = F+$prob.(E).

Consider a standard debt contract $\mathcal{C}'$ with face value equal to the constant function $R_0(\cdot)$ under the incentive efficient contract $\mathcal{C}$. Then under $\mathcal{C}'$ B receives F if $z \in [F, 1]$ and $z - c$ if $z \in [0, F]$. However, under $\mathcal{C}$, B receives $R_1(z) - c \leq z - c$ if $z \in [0, F]$, F if $z \in [F, 1] \bigcap K^c$, and $R_1(z) - c \leq F - c$ if $z \in [F, 1] \bigcap K$. That is, B gets weakly more from accepting $\mathcal{C}'$ than accepting $\mathcal{C}$. Since B must break even when accepting the incentive efficient contract $\mathcal{C}$, the SD contract $\mathcal{C}'$ is also incentive feasible, which, by Step 7, implies that $F \in [F^*, 1]$.

Note that the standard debt contract $\mathcal{C}^*$ is incentive feasible and it makes IR_B binding, and it has a probability of state verification equal to $F^* \leq F < F+$prob.(E), so that by Step 6, $\mathcal{C}$ cannot be incentive efficient, a contradiction! $\|$

Note that Step 4 and Step 8 together imply that under an incentive feasible contract, prob.$(K) = F \equiv R_0(\cdot)$.

Step 9. Suppose that $\mathcal{C}$ is incentive efficient. Then under $\mathcal{C}$ the event

$$G = \{z \in [0, F] : \; R_1(z, z) < z\}$$

is a zero-probability event, where $F = R_0(\cdot)$.

Proof. Suppose instead that G may occur with a positive probability under the incentive efficient contract $\mathcal{C}$. We shall demonstrate a contradiction.

First note that under $\mathcal{C}$, by Step 6, prob.$(K) = F$.

Consider the standard debt contract $\mathcal{C}'$ with face value F. Because G may occur with a positive probability, B gets strictly more from accepting $\mathcal{C}'$ than accepting $\mathcal{C}$. Since $\mathcal{C}$ is incentive efficient, the SD contract $\mathcal{C}'$ is also incentive feasible, and under $\mathcal{C}'$, IR_B is not binding. This implies, by Step 7, that $F \in (F^*, 1]$.

Recall that the standard debt contract $\mathcal{C}^*$ with face value F^* is incentive feasible and it makes IR_B binding, and it has a probability of state verification equal to F^*. The SD contract $\mathcal{C}'$ and the incentive efficient contract $\mathcal{C}$ both share a probability of state verification equal to F. Since $F > F^*$, by Step 6, $\mathcal{C}$ cannot be incentive efficient, a contradiction! $\|$

Thus Steps 8 and 9 together have established that an incentive efficient contract in the current model is "essentially" a standard debt contract.

Remark. We have assumed that the contracting parties can only commit to a deterministic state-verification policy (i.e., d must equal either zero or one). Consequently, IC_A may hold as an inequality under the optimal contract. In fact, verifying with probability one is un-necessary for the purpose of inducing truth-telling. Mookherjee and Png (1989, *Quarterly Journal of Economics*) show that the optimal state verification policy would be stochastic whenever stochastic policies are feasible; see the example in the next section. We have also assumed that the entrepreneur is risk-neutral. Winton (1994, *Review of Financial Studies*) shows that when the entrepreneur is risk-averse, standard debt is no longer optimal as it results in the entrepreneur bearing too much risk. The optimal contract would leave the entrepreneur with a positive payoff even in the event of state verification. If we interpret state verification as the event of bankruptcy, then Winton's theory is consistent with the empirical fact that shareholders tend to get a positive payoff in the event of bankruptcy.

3. (**Example 1**.) (**CSV Model without Commitment Power**) Reconsider the following modified CSV model. As in section 2 of *Corporate Financing Decisions and Product Market Competition*, at date 0 A must offer a contract to B to raise I, but only A gets to see the realized profit z at date 1. In this exercise, we assume that z is equally likely to take on 3 or 0, and that

$$I = 1, \quad c = \frac{1}{5}.$$

(i) Suppose first that A can commit in the date-0 contract either $d = 0$ or $d = 1$. In this case, let F be the face value of the equilibrium debt contract that A offers B at date 0. Then $F =$? Compute A's equilibrium payoff.

(ii) Now, suppose instead that A can commit in the date-0 contract any $d \in [0, 1]$. In this case, let F' be the face value of the equilibrium debt contract that A offers B at date 0. Then $F' =$? Compute A's equilibrium payoff.

(iii) Now, suppose instead that the date-0 contract must give investor B the right to verify the earnings state at date 1, but state verification occurs at date 1 when and only when it is in B's interest to verify the profit at date 1. That is, the date-0 contract cannot commit to any d that is inconsistent with B's preferences at date 1. Moreover, assume that B can obtain the true profit z whenever state verification proves that $\hat{z} \neq z$; that is, the date-0 contract imposes the maximum penalty for lying. Let F'' be the face value of the equilibrium debt contract that A offers B at date 0. Then $F'' =$? Compute A's equilibrium payoff.[1]

[1]**Hint**: At date 1, A must first report $\hat{z} \in \{0, 3\}$. Given F'', it can be shown that

Solution. For part (i), recall that F must satisfy B's binding IR constraint:

$$\frac{1}{2}F - \frac{1}{2}c = I \Rightarrow F = \frac{11}{5}.$$

In this case, A's payoff is equal to

$$\frac{1}{2}(3 - F) = \frac{2}{5}.$$

Consider part (ii). First note that in part (i), when $z = 3$, by telling the truth A would get

$$3 - \frac{11}{5} = \frac{4}{5},$$

but A's payoff would drop to zero if A chose to lie and to trigger state verification. That is, A's IC constraint is strictly satisfied under the contract in part (i). This is wasteful, because A would still be willing to tell the truth if the probability d following $\hat{z} = 0$ were reduced slightly. Note that we have assumed that this cannot be done in part (i), where A can only commit to a deterministic policy of state verification.

- B will never verify if A has reported $\hat{z} = 3$; and
- given that B may verify with probability d after A has reported $\hat{z} = 0$, A will always tell the truth when $z = 0$.

Thus suppose that A may lie about the true profit $z = 3$ with probability a, and B may verify with probability d following A's report $\hat{z} = 0$. It can be shown that in equilibrium A must feel indifferent about telling the truth or lying when $z = 3$, and B must feel indifferent about spending or not spending c after A reports that $\hat{z} = 0$. It follows that, given F'', we must have

$$\begin{cases} 3 - F'' = 3(1 - d); \\ \\ 0 = -c + \frac{\frac{1}{2} \cdot a}{\frac{1}{2} \cdot a + \frac{1}{2} \cdot 1} \cdot 3, \end{cases}$$

Now, since any $d \in [0,1]$ is allowed in part (ii), A can do better by offering a date-0 contract (F', d) such that both IR_B and IC_A are binding. Solving the following system of equations simultaneously,

$$\begin{cases} \frac{1}{2}F' - \frac{1}{2}dc = I; \\ \\ 3 - F' = 3(1-d), \end{cases}$$

we obtain

$$d = \frac{I}{\frac{3}{2} - \frac{1}{10}} = \frac{5}{7}, \quad F' = \frac{3}{\frac{3}{2} - \frac{1}{10}} = \frac{15}{7}.$$

In this case, A's equilibrium payoff is

$$\frac{1}{2}(3 - F') = \frac{3}{7}.$$

Finally, consider part (iii). Given F'', what would happen in the date-1 subgame where A has reported $\hat{z}$? It can be shown that

- B will never verify if A has reported $\hat{z} = 3$; and
- given that B may verify with probability d after A has reported $\hat{z} = 0$, A will always tell the truth when $z = 0$.

Thus suppose that A may lie about the true profit $z = 3$ with probability a, and B may verify with probability d following A's report $\hat{z} = 0$. It can then be shown that in equilibrium

$$0 < a, d < 1;$$

that is, A must feel indifferent about telling the truth or lying when $z = 3$, and B must feel indifferent about spending or not spending c upon seeing A's report that $\hat{z} = 0$.

It follows that, given F'', we must have

$$\begin{cases} 3 - F'' = 3(1-d); \\ \\ 0 = -c + \frac{\frac{1}{2}\cdot a}{\frac{1}{2}\cdot a + \frac{1}{2}\cdot 1} \cdot 3, \end{cases}$$

so that

$$d = \frac{F''}{3}, \quad a = \frac{\frac{1}{10}}{\frac{1}{2}(3 - \frac{1}{10})} = \frac{1}{14}.$$

Now, at date 0, F'' must be the solution to the following maximization problem:

$$\max_F \frac{1}{2}[3a(1-d) + (1-a)(3-F)]$$

subject to

$$d = \frac{F''}{3}, \quad a = \frac{\frac{1}{10}}{\frac{1}{2}(3 - \frac{1}{10})} = \frac{1}{14},$$

and the following IR_B:[2]

$$\frac{1}{2}(1-a)F'' - I \geq 0.$$

At optimum, IR_B must be binding, and hence we obtain

$$F'' = \frac{3 - \frac{1}{5}}{\frac{3}{2} - \frac{1}{5}} = \frac{28}{13}.$$

[2]B no longer expects to incur a state-verification cost! At the state verification date, upon seeing $\hat{z} = 0$, B either chooses to not verify the state (which costs nothing), or B may verify and yield a return which is expected to exactly cover B's verification cost. These two actions are equally good, and hence B will randomize over them upon seeing the earnings report $\hat{z} = 0$.

It follows that A's equilibrium payoff becomes

$$\frac{1}{2}(3 - F'') = \frac{11}{26}.$$

Remark. In this exercise, A's payoff in part (ii) is the highest, and his payoff in part (i) is the lowest. This is quite natural: B will break even under the optimal contract obtained in part (i) or part (ii) or part (iii), but the expected cost for state verification differs over the three scenarios.

Part (ii) allows A to directly minimize that expected cost, and hence it maximizes A's payoff. Via the mixed-strategy equilibrium of the date-1 subgame, part (iii) can implement a probability of state verification lower than $d = 1$, but since in equilibrium state verification may occur with probability $ad = \frac{1}{14} \times \frac{28}{39}$ even in the true state $z = 3$, A's optimal payoff in part (iii) is less than in part (ii). Indeed, since in part (iii) state verification may occur in the state $z = 3$, in order for B to break even in both part (ii) and part (iii), we must have $F'' > F'$, and hence A's payoff in part (iii), which is $\frac{1}{2}(3 - F'')$ is less than A's payoff in part (ii), which is $\frac{1}{2}(3 - F')$.

4. (**Part II.**) Now we review a few game-theoretic models of debt financing with imperfectly competitive firms.

5. (**Example 2**.) (**Risky Short-term Debt May Lead to More Aggressive Cournot Competition**.)
Consider two firms 1 and 2 engaging in Cournot competition at date 1. They produce the same product and face the following inverse demand:

$$P = k(1 - q_1 - q_2),$$

where $k > 0$ is a constant. Firm 2's unit production cost is kc, and firm 1's unit production cost is $k\tilde{c}_1$ which is equally likely to be $k(c+d)$ and $k(c-d)$. Firm 1 must choose q_1 before seeing the realization of $\tilde{c}_1$. We assume that

$$1 > c + d > c > c - d > 0.$$

Firms are risk-neutral without time preferences.
(i) Suppose that firms seek to maximize expected profits. Find the

Nash equilibrium.
(ii) Suppose that firm 1 has borrowed some debt prior to date 1. Let the face value of firm 1's debt be F, with

$$(\Lambda)\ \ \max(k[\frac{1-c-4d}{3}][\frac{1-c+2d}{3}], k[\frac{(1-c)^2}{9}-\frac{(1-c)d}{3}]) < F < k[\frac{1-c+2d}{3}]^2.$$

Assume that each firm seeks to maximize its equity value, and investors are all risk-neutral without time preferences. Find a Nash equilibrium in which firm 1 does not always default on its debt.

Solution. It is useful to consider a situation where firm i's unit cost is c_i. It is easy to show that firm i's reaction function is

$$R_i(q_j) = \frac{1-q_j-c_i}{2}, \quad i,j = 1,2,$$

so that in equilibrium

$$(\Gamma)\ \ q_i^* = \frac{1+c_j-2c_i}{3}, \quad i,j=1,2.$$

In part (i), note that $E[\tilde{c}_1] = c$, and since both firms are risk-neutral (so that only the expected unit cost matters), in equilibrium both firms produce the same output, which is

$$q_1^* = q_2^* = \frac{1-c}{3}.$$

Consider part (ii). Let $\tilde{\Pi}_j$ denote firm j's profit. Then by assumption firm 1 seeks to maximize

$$E[\max(\tilde{\Pi}_1 - F, 0)],$$

where

$$\tilde{\Pi}_1 = kq_1(1-q_1-q_2-\tilde{c}_1).$$

Since firm 2's profit $kq_2(1-q_1-q_2-c)$ is strictly concave in q_2, firm 2 will adopt a pure strategy q_2^* in equilibrium. In an equilibrium where firm 1 does not default on its debt in the low-cost state, there are two possibilites: either firm 1 never defaults on its debt, or it defaults on its debt only in the high-cost state. Observe that in either case, firm 1 must also adopt a pure strategy q_1^* in equilibrium. Hence there are two possibilities regarding the equilibrium (q_1^*, q_2^*): either

$$kq_1^*(1 - q_1^* - q_2^* - c - d)q_1^* \leq F < kq_1^*(1 - q_1^* - q_2^* - c + d)q_1^*$$

or

$$kq_1^*(1 - q_1^* - q_2^* - c - d)q_1^* > F$$

must be true. In the former case, given q_2^*, firm 1's equilibrium best response is

$$q_1^* = \frac{1 - q_2^* - c + d}{2},$$

which together with firm 2's reaction function

$$q_2^* = \frac{1 - q_1^* - c}{2}$$

implies that

$$(q_1^*, q_2^*, p^*) = (\frac{1 - 2(c - d) + c}{3}, \frac{1 - 2c + (c - d)}{3}, \frac{k[1 + c + (c - d)]}{3}),$$

and in this equilibrium, firm 1's profit is

$$kq_1^*(1 - q_1^* - q_2^* - c - d)q_1^* = k[\frac{1 - c - 4d}{3}][\frac{1 - c + 2d}{3}]$$

if its realized unit cost is $c + d$; and firm 1's profit is

$$kq_1^*(1 - q_1^* - q_2^* - c + d)q_1^* = k[\frac{1 - c + 2d}{3}]^2$$

if its realized unit cost is $c - d$. Condition (Λ) implies that this equilibrium does exist. In the latter case, firm 1 will never default on its debt in equilibrium, and hence the equilibrium is as stated in part (i). If the latter equilibrium exists, then firm 1's equilibrium profit would be

$$k[\frac{(1 - c)^2}{9} - \frac{(1 - c)d}{3}]$$

when its realized unit cost is $c + d$, but condition (Λ) implies that firm 1 must default on its debt when its realized unit cost is $c + d$, a contradiction to the conjecture that firm 1 never defaults on its debt in equilibrium. Hence the latter equilibrium does not exist.[3]

Our conclusion is thus that, under condition (Λ), in equilibrium firm 1's objective function becomes

$$\frac{1}{2}[kq_1(1 - q_1 - q_2^* - c + d) - F] + \frac{1}{2} \cdot 0.$$

Note carefully that, given q_2^*, firm 1 has essentially become a firm with unit cost $c - d$! In equilibrium, firm 2 must reduce its output to below $\frac{1-c}{3}$, because firm 2 realizes that firm 1 has become more aggressive than in part (i) in choosing its output, and, expecting this, firm 2's best response to cut back on its own output. Indeed, in the SPNE we have, using the formulae in (Γ),

$$q_1^{**} = \frac{1 + c - 2c + 2d}{3} = \frac{1 - c + 2d}{3} > \frac{1 - c}{3} = q_1^*,$$

and

$$q_2^{**} = \frac{1 + c - d - 2c}{3} = \frac{1 - c - d}{3} < \frac{1 - c}{3} = q_2^*.$$

To sum up, by issuing risky short-term debt, firm 1 gains a competitive advantage, since it behaves as if its unit cost were low with probability

[3]Condition (Λ) is needed here because without such a condition, this game may not have a (mixed- or pure-strategy) Nash equilibrium. The following is a numerical example. Suppose that

$$k = 1, \; c = d = \frac{1}{10}, \; F = \frac{109}{1800}.$$

In equilibrium, either firm 1 defaults on its debt in the high-cost state, or it does not. But one can verify that neither is consistent with equilibrium. Indeed, if firm 2 expects firm 1 to default on its debt in the high-cost state, then firm 2 expects firm 1 to expand output, and hence firm 2 must cut back on its own output, resulting in firm 1 having a realized profit in the high-cost state exceeding F. On the other hand, if firm 2 expects firm 1 to never default on its debt, then the two firms will choose the same output level, leading to firm 1 having a realized profit in the high-cost state which is less than F. The non-existence of equilibrium is not surprising, because a game where players' strategy spaces are not finite sets may not have a Nash equilibrium. Existence of equilibrium requires certain convexity conditions.

one, and expands output accordingly. By the fact that the two firms' output choices are strategic substitutes, this forces its rival to produce less and concede in market share. Note that to have this commitment value, firm 1's debt must be risky: firm 1 must get nothing when its unit cost is high and the profit that it makes cannot fully repay the debt. Note that $\text{var}[\tilde{\Pi}_1]$ rises because of firm 1's risky debt, which is the *asset substitution* effect pointed out in Jensen and Meckling (1976). That risky short-term borrowing may induce a firm to expand output and compete more aggressively was first pointed out by Brander and Lewis (1986, *AER*).

6. (**Example 3**.) (**Long-term Debt May Promote Collusion**.)
Now, assume that the two firms in Example 2 also compete at date 0, and that firm 1 borrows debt at date 0 when it is penniless. Again, the debt with face value F will be due at date 2. It is now referred to as a long-term debt. At date 0, each firm can spend a cost h on promotion. If neither promotes, each firm gets v at date 0. If exactly one firm promotes, then that firm gets $2v - h$, leaving the other firm with zero profit. If both promote, then each gets $v - h > 0$.

To make things interesting, let us assume that

$$(\Delta) \quad \frac{kd(2-2c-d)}{9} > v - h,$$

and that

$$(\Sigma) \quad F-(2v-h) < \max(k[\frac{1-c-4d}{3}][\frac{1-c+2d}{3}], k[\frac{(1-c)^2}{9} - \frac{(1-c)d}{3}])$$

$$< F - (v-h) < k[\frac{1-c+2d}{3}]^2.$$

Note that, were the date-1 competition not existent, the two firms would compete at date 0 with their only concerns being their date-0 profits. It is clear that they are playing a version of the game called the prisoner's dilemma, where the only Nash equilibrium is the one in which both promote.

Since firms are rational, and since they know that they have to compete again at date 1, they seek to maximize the sum of profits over dates 0 and 1 when taking date-0 actions. Condition (Σ) says that if firm 1 does not make enough profits at date 0, it will panic because the long-term debt will default at date 2 in the event that $\tilde{c}_1 = c + d$, and hence it will expand output, which will hurt firm 2. Realizing this, firm 2 must "make firm 1 look good" at date 0, so that at the beginning of date 1, firm 1 knows that its debt will never default at date 2, which will then induce firm 1 to choose a low output, thereby raising firm 2's output and date-1 profit. Condition (Δ) says that by conceding at date 0 and then gaining at date 1, firm 2 will become better off. Hence in the SPNE, firm 2 does not promote at date 0, leaving firm 1 with a profit $(2v - h)$, which results in a symmetric date-1 output choice $q_i^* = \frac{1-c}{3}$ as in part (i) of Example 5.

Remark. That long-term borrowing can mitigate competition before the debt maturity date gets close was first pointed out by Glazer (1994, *JET*). We have examined in Example 3 the case where only firm 1 is financially leveraged.

Now consider the case where both firms have issued long-term debt with identical face value, and where both firms have unit cost $k\tilde{c}_1$. If the two firms compete only at date 1, as in Example 2, then risky short-term debt would make both of them worse off. This happens because both firms would behave as if their unit cost were sure to be $c - d$! (This is like a prisoner's dilemma, where in equilibrium each firm has a lower value; recall that firm value is the sum of debt value and equity value.) On the other hand, If these firms compete at both dates 0 and 1, as in Example 3, then debt actually benefit both of them: each firm would want to make the other firm look good at date 0 in order to boost its own date-1 profit. Consequently these two firms would be able to avoid the inefficient date-0 outcome of the prisoner's dilemma. Moreover, the high profits they make at date 0 also allow them to avoid competing aggressively at date 1! Note that without long-term borrowing, the two firms would be trapped in an inefficient outcome at date 0, although their date-1 equilibrium output choices would be as efficient as in the presence of long-term borrowing.

7. (**Example 4.**) **The Bolton-Scharfstein (1990) CSV model**.

Firm B needs F dollars to operate in the product market at respectively date 0 and date 1. Profits are generated at respectively date 1 and date 2. Firm B has no cash initially, and it has to borrow from an investor who has all bargaining power against firm B. Profits are only observable to firm B and verifying profits is prohibitively costly for the investor. The revelation principle implies that the contract-design problem between B and the investor can be modeled as a direct revelation game with no loss of generality. In the direct revelation game, the repayment of the financial contract only depends on the firm's report of profit. Assume that at each date t ($t = 1, 2$) the profit of B can be either π_1 (with probability θ) or π_2, with $\pi_2 > \pi_1$, $\overline{\pi} \equiv E(\pi) = \theta\pi_1 + (1 - \theta)\pi_2 > F$, and $\pi_1 < F$. Also, assume all parties are risk neutral with no time preference.

(i) Show that if firm B operates for only one period, the investor will refuse to lend F. (**Hint**: If the investor does, B will *always* report $\pi_1 < F$.)

Because of (i), we now suppose that B operates for two periods and that $\pi_2 - \pi_1 < F$. The financing is assumed to proceed as follows. At date 0, the investor lends F to B. Then at date 1, B reports its date-1 profit $\pi_i \in \{\pi_1, \pi_2\}$. If B reports its date-1 profit to be π_i, then it has the obligation of paying the investor R_i at date 1. After this repayment is made, with probability β_i the debt is renewed. In case the debt is renewed at date 1, then the investor gives F to B at date 1, and at date 2, B reports its profit π_j. The second period repayment is denoted by R_{ij} if at date 1 firm B has reported π_i and at date 2 it reports π_j.

The game proceeds as follows. The investor first decides to or not to lend at date 0. If lending is the decision, then the investor offers a financial contract $(R_1, R_2, \beta_1, \beta_2, R_{11}, R_{12}, R_{21}, R_{22})$ to B, and B can either accept or reject.[4] Such a contract specifies only variables that

[4]This implies that the borrower may be a small firm, which lacks bargaining power when negotiating the loan contract with a large bank.

can be subsequently observed by both contracting parties and can be verified by the court of law (so that the latter can enforce it). When specifying these contract variables, the investor must make sure that B will accept (accepting generates for B a utility higher than otherwise), which is called B's individual rationality condition (IR condition). The investor must also make sure that B will truthfully report its profits (truthtelling is better than lying), which is called B's incentive compatibility condition (IC condition). Finally, the repayment R_i and R_{ij} must really be affordable by B when the true profts are respectively π_i and π_j at dates 1 and 2. This is called the limited liability condition (LL condition).

Any contract satisfying these three conditions is said to be *feasible*. The investor wants to find a feasible contract that maximizes her own expected utility. The solution is called an *optimal contract* (because such a contract is Pareto optimal within the set of feasible contracts). Thus, when deciding to lend at date 0, the investor's optimal contract problem is

$$\max_{\beta_i, R_i, R^i} -F + \theta[R_1 + \beta_1(R^1 - F)] + (1-\theta)[R_2 + \beta_2(R^2 - F)],$$

subject to
(IC at date 1) $\pi_2 - R_2 + \beta_2(E(\pi) - R^2) \geq \pi_2 - R_1 + \beta_1(E(\pi) - R^1)$;
(LL at date 1) $\pi_i \geq R_i$,
(LL at date 2) $\pi_i - R_i + \pi_1 \geq R^i, i = 1, 2$;
(IR at date 0) $\theta[\pi_1 - R_1 + \beta_1(E(\pi) - R^1)] + (1-\theta)[\pi_2 - R_2 + \beta_2(E(\pi) - R^2)] \geq 0$.
Note that in the above, we have used the fact that R_{ij} must be independent of π_j in order to satisfy B's second period IC condition, and we have written R_{ij} as R^i. This is also why we did not impose B's IC condition at date 2.

(ii) Show that under optimal contract, firm B always tells the truth when reporting the second-period profit. (**Hint**: Like the reasoning in part (i), if the repayment were dependent on the second-period report, B would always report π_1 in the second period, violating B's IC constraint.)

(iii) Show that the optimal contract $(R_1^*, \beta_1^*, R_2^*, \beta_2^*)$ is $(\pi_1, 0, E(\pi), 1)$ if

$$\theta F + (1-\theta)E(\pi) > \pi_1,$$

and $(\pi_1, 1, \pi_1, 1)$ if otherwise. (**Hint**: Show that the above IC condition has to be binding at optimum. Thus,

$$\pi_2 - R_2 + \beta_2(E(\pi) - R^2) = \pi_2 - R_1 + \beta_1(E(\pi) - R^1).$$

Replace this equality into the objective function and note that the objective function becomes strictly increasing in β_2. This implies that the objective function does not depend on R_2 and R^2 separately; rather, it depends on $R_2 + R^2$ only (and similarly for the constraints.) Thus, there is no loss to set $R^{2*} = \pi_1$. Also, the objective function is increasing in both R_1 and R^1. Finally, note that the objective function is decreasing in β_1 if and only if

$$\theta F + (1-\theta)E(\pi) > \pi_1.$$

Depending on whether this inequality holds, the optimal contract can be fully solved.)

(iv) Show that the investor lends F to B at date 0 if and only if

$$F < \frac{(\pi_1 + (1-\theta)E(\pi))}{2-\theta}.$$

Up to now, we have assumed that $\pi_2 - \pi_1 < F$, and so refinancing at date 1 is necessary for firm B to continue its business in the second period.
(v) Show that, if instead,

$$\min(\pi_2 - F, F) \geq \pi_1,$$

then the investor refuses to lend at date 0 even if B can operate for two periods.

Solution. Consider part (i). Apparently, B will *always* report $\pi_1 < F$ in this one-period setting, and expecting this, investors never want to lend to B in the first place.

Consider part (ii). As in part (i), if the second-period repayment were made dependent on the second-period profit report in a non-trivial way, then B will always report π_1 in the second period, violating B's second-period IC constraint. Thus, the optimal contract requires that R_{ij} be independent of π_j.

Consider part (iii). First it can be proved that the above IC condition must be binding at optimum. Thus,

$$\pi_2 - R_2 + \beta_2(E(\pi) - R^2) = \pi_2 - R_1 + \beta_1(E(\pi) - R^1).$$

Replace this equality into the objective function and note that the objective function becomes strictly increasing in β_2. This means that $\beta_2^* = 1$, which in turn implies that the objective function does not depend on R_2 and R^2 separately; rather, it depends on $R_2 + R^2$ only (and the same is also true for the constraints). Thus, there is no loss to set $R^{2*} = \pi_1$. Also, the objective function is increasing in both R_1 and R^1. Thus, $R_1 = R^1 = \pi_1$, according to LL. Finally, note that the objective function is decreasing in β_1 if and only if

$$\theta F + (1-\theta)E(\pi) > \pi_1,$$

and in this case, $\beta_1^* = 0$. On the other hand, when

$$\theta F + (1-\theta)E(\pi) \leq \pi_1,$$

it is optimal to set $\beta_1^* = 1$.

Next, consider part (iv). We need to show that the investor lends F to B at date 0 if and only if

$$F < \frac{(\pi_1 + (1-\theta)E(\pi))}{2-\theta}.$$

This follows from the fact that the investor's expected profit is

$$\pi_1 - F + (1-\theta)(E(\pi) - F),$$

which cannot be negative.

Finally, assume in part (v) that

$$\min(\pi_2 - F, F) \geq \pi_1.$$

We need to show that the investor would refuse to lend to B at date 0 even if B can operate for two periods. To see that this is so, note that if the above inequality holds, B will always report π_1, with no concern about whether he will get refinancing. This happens because, by the above assumption, once B gets to operate for one period, B will collect enough money to cover the second-period F. Recognizing this fact, investors will not lend to B in the first place. This is one version of the *free cash flow* problem discussed in Jensen (1986, *American Economic Review*). There, Michael Jensen points out that there may be substantial benefits resulting from a voluntary reduction of a firm's internal funds (by buying shares back, paying dividends, or repaying existing debts).

8. (**Part III.**) **Nash Implementation**.

9. A set of I players are facing uncertain states of nature contained in the sample space Θ, and they must make a collective choice for each state $\theta \in \Theta$. Let A denote the set of feasible social choices, @ and a a typical element of A. Suppose that each player $i \in \{1, 2, \cdots, I\}$ can separately see the realized θ, but θ is not verifiable in the court of law, and hence is not contractible. Moreover, for each θ, there exists a subset $f(\theta) \subset A$ which consists of all the social choices acceptable to a central planner. The central planner, who is not one of the I players, would like to design a game form for the I players, such that the set of Nash equilibrium outcomes of that game form in state θ coincides with $f(\theta)$. (Here we assume that participating in the game form is mandatory for each player, so that we do not need to impose an IR condition.) We shall refer to $f(\cdot)$ as a social choice rule (SCR) or a social choice correspondence (SCC).

   A game form is a pair (g, S), where S is the common strategy space for each and every player, and $g : S^I \Rightarrow A$ maps each strategy profile $s \equiv (s_1, s_2, \cdots, s_I)$ into a social choice. Note that player i's payoff function is θ-dependent, and hence the game form (g, S), failing to fully describe the payoff function for each player, is not a normal form game. However, for each $\theta \in \Theta$, (g, S, θ) is indeed a normal-form game. Let $s^*(\theta)$ be one pure-strategy Nash equilibrium of the game (g, S, θ). Let $E_g(\theta)$ be the set of all pure-strategy Nash equilibria in state θ. Then, define $g(E_g(\theta)) \equiv \{g(s^*) : s^* \in E_g(\theta)\}$ as the set of equilibrium social choices. We say that (g, S) *fully implements* f in Nash equilibrium if and only if $g(E_g(\theta)) = f(\theta)$ for all $\theta \in \Theta$. We say that f is Nash implementable if we can find at least one game form (g, S) that fully implements f.

10. **Example 5.** Two women, Amy and Beth, carry one baby to the king, and each of them claims to be the mother of the baby. There are two possible states: the mother is either Amy (state α) or Beth (β). Thus let $\Theta = \{\alpha, \beta\}$. For the king, there are 4 feasible actions: to give the baby to Amy (a); to give it to Beth (b); to cut the baby in half and let each woman take one half (c); or to let both women and the baby die (d). Can you find a game form to fully implement the social choice rule f satisfying $f(\alpha) = a$ and $f(\beta) = b$?

11. **Theorem N1.** (Maskin 1977; Maskin 1999) f is Nash implementable only if f is monotonic.[5]

    Proof. Define the lower contour set at a for agent i in state θ by

    $$L^i(a, \theta) \equiv \{b \in A : aR^i(\theta)b\}.$$

    We shall prove Theorem N1 by contraposition. Recall from section 17 that f is not monotonic if and only if there exist $\theta, \phi \in \Theta$ and $a \in A$ such that for all $i = 1, 2, \cdots, I$,

    $$L^i(a, \theta) \subset L^i(a, \phi),$$

[5] Maskin, E., 1977, Nash Equilibrium and Welfare Optimality, MIT working paper. Maskin, E., 1999, Nash Equilibrium and Welfare Optimality, *Review of Economic Studies*, 66, 23-38.

and yet $a \in f(\theta) \setminus f(\phi)$.[6] We show that in this case no game forms (g, S) can fully implement f in Nash strategy.

Suppose instead that there were such a game form (g, S). Then there exists some Nash equilibrium s^* for the game (g, S, θ) such that $g(s^*) = a \in f(\theta)$. That is, for agent i, given his rival agents would play s^*_{-i} in state θ,

$$a = g(s_i^*, s_{-i}^*) R^i(\theta) g(s_i, s_{-i}^*), \ \forall s_i \in S^i,$$

$$\Rightarrow g(s_i, s_{-i}^*) \in L^i(a, \theta) \subset L^i(a, \phi), \ \forall s_i \in S^i,$$

but then s_i^* continues to be agent i's best response against s_{-i}^* in state ϕ! As this is true for all agents i, we conclude that s^* would also arise as a pure-strategy Nash equilibrium in state ϕ. But then $a \in g(E_g(\phi)) \setminus f(\phi)$, showing that (g, S) does not fully implement f in Nash equilibrium.

12. To state our next result, we introduce the notion of no veto power. An SCC f satisfies (weak) no veto power if for all $i \in \{1, 2, \cdots, I\}$, for all $\theta \in \Theta$, and for all $a \in A$,

$$L^j(a, \theta) = A, \forall j \neq i \Rightarrow a \in f(\theta).$$

In words, if a is top ranked by all agents $j \neq i$ in state θ, then $a \in f(\theta)$ whether agent i likes a or not. (Agent i has no veto power!)

13. **Theorem N2.** (Maskin 1977; Repullo 1987[7]) Suppose that $I \geq 3$, and that f is monotonic and satisfies no veto power. Then f is Nash implementable.

Proof. The proof is by construction of a canonical game form (g, S) which fully implements f. Define for all i, $S^i = \Theta \times A \times \mathbf{Z}_+$, where $\mathbf{Z}_+$ denotes the set of positive integers, and define $g : S \to A$ as follows:

[6]An equivalent definition for f being monotonic is this: for any $\theta, \phi \in \Theta$ and $a \in A$ with $a \in f(\theta) \setminus f(\phi)$, there must exist agent i and some $b \in A$ such that $b \in L^i(a, \theta) \setminus L^i(a, \phi)$, or such that $aR^i(\theta)b$ but $bP^i(\phi)a$.

[7]Repullo, R., 1987, A Simple Proof of Maskin's Theorem on Nash Implementation, *Social Choice and Welfare*, 4, 39-41.

(a) If s is such that there exists $i \in \{1, 2, \cdots, I\}$ such that $s_i = (\eta, a_i, k_i)$ and for all $j \neq i$, $s_j = (\theta, a, k)$ with $a \in f(\theta)$, then

$$g(s) = \begin{cases} a_i, & \text{if } \; a_i \in L^i(a, \theta); \\ \\ a, & \text{otherwise.} \end{cases}$$

(b) If s is such that (a) does not apply, then $g(s) = a_i$ where i is an agent announcing the highest k_i, with ties being broken by selecting among the agents announcing the highest k_i the person with the smallest i.

We shall show first that $f(\theta) \subset g(E_g(\theta))$ for all $\theta \in \Theta$, and then that $g(E_g(\theta)) \subset f(\theta)$ for all $\theta \in \Theta$.

- $f(\theta) \subset g(E_g(\theta))$ for all $\theta \in \Theta$.

  Given any $a \in f(\theta)$, define for all i, $s_i = (\theta, a, 1)$. Then s is such that (a) holds, and if agent i alone would like to deviate and to implement another a_i, he must choose some $a_i \in L^i(a, \theta)$, and hence he has no incentive to make unilateral deviations. Thus $a \in g(E_g(\theta))$, and this being true for all $\theta \in \Theta$ and for all $a \in f(\theta)$, we conclude that $f(\theta) \subset g(E_g(\theta))$ for all $\theta \in \Theta$.
- $g(E_g(\theta)) \subset f(\theta)$ for all $\theta \in \Theta$

  Let $s \in E_g(\theta)$, and we shall show that $g(s) \in f(\theta)$. Suppose θ is the true state. We take cases.
  - Suppose that s is such that $s_i = (\eta, a, k)$ $\forall i \in \{1, 2, \cdots, I\}$, with $a \in f(\eta)$, so that $g(s) = a$.

    For all $i \in \{1, 2, \cdots, I\}$, if agent i wishes to deviate unilaterally from s, then according to (a) above, agent i must announce some $s_i' = (\phi, a_i, k_i)$ with $a_i \in L^i(a, \eta)$. Since s is a Nash equilibrium in the true state θ, agent i weakly prefers the equilibrium outcome $a = g(s)$ to a_i in the true state θ, and this implies that

$$a_i \in L^i(a, \eta) \Rightarrow a_i \in L^i(a, \theta),$$

and this being true for all $i \in \{1, 2, \cdots, I\}$, we conclude that $a \in f(\theta)$ since f is monotonic.

– Suppose that s is such that $s_i = (\eta, a, k)$ $\forall i \in \{1, 2, \cdots, I\}$, with $a \notin f(\eta)$.

  In this case, by (b), any agent i can deviate and announce $s_i' = (\phi, a_i, k')$, where $k' > k$, so that the outcome a_i rather than $g(s)$ would be implemented. Since s is a Nash equilibrium in the true state θ, it must be that

$$g(s) R^i(\theta) a_i, \ \forall a_i \in A,$$

  or equivalently,

$$L^i(g(s), \theta) = A,$$

  and with this being true for each single agent i, we conclude that $g(s) \in f(\theta)$ by the fact that f satisfies no veto power.

– Suppose that s is such that there exist $i \neq j$, $s_i \neq s_j$.

  In this case, thanks to the fact that $I \geq 3$, some agent $h \notin \{i, j\}$ can implement any $a_h \in A$ by announcing an integer k_h exceeding k_n for all $n \neq h$. Since s is a Nash equilibrium in the true state θ, it must be that

$$L^h(g(s), \theta) = A, \ \forall h \notin \{i, j\}.$$

  Moreover, it is impossible that $s_h = s_i$ and $s_h = s_j$, simply because $s_i \neq s_j$. Suppose that $s_h \neq s_i$. Then we can repeat the above argument and conclude that

$$L^j(g(s), \theta) = A.$$

  It follows that $g(s)$ is top ranked in state θ by all agents $n \neq i$, so that $g(s) \in f(\theta)$ by the fact that f satisfies no veto power.

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