Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 1

1. (**Competing Platforms, Part I.**) Consider the following extensive game with two competing platforms, I and E, and two segments of platform users, referred to as side 1 and side 2 respectively. Segment i has population equal to one, and is represented by the interval $(2i, 2i+1)$. There exists a one-to-one measure-preserving correspondence $f : (2,3) \to (4,5)$ such that when user $x \in (2,3)$ meets with user $f(x) \in (4,5)$ on either platform I or platform E, the two users receive payoffs u_1 and u_2 respectively.[1] We say that a match occurs for x and his right partner $f(x)$ in the latter event. The two agents receive zero payoffs if they fail to meet with each other. We assume that $u_2 > u_1 > 0$. The platforms can operate without costs, and the timing of relevant events is as follows:

   - Platform I (the incumbent) must first decide whether to remain inactive (so that it gets a zero payoff by doing nothing) or to announce a pair of two-part tariffs $(p_1^I, t_1^I, p_2^I, t_2^I)$, saying that if a side-$i$ user wishes to use platform I then he needs to pay an upfront *registration fee* (or *access fee*, or *subscription fee*) $p_i^I \in \Re$, and in case a match occurs subsequently, then he needs to pay another *transaction fee* $t_i^I \geq 0$.[2]
   - Upon seeing platform I's announcements, platform E (the entrant) can decide whether to remain inactive (so that it gets a zero payoff by doing nothing) or to announce a pair of two-part tariffs $(p_1^E, t_1^E, p_2^E, t_2^E)$.
   - Then users of both sides arrive, and they must decide simultaneously whether to register for both platforms (multi-homing), or

[1] The one-to-one correspondence is measure-preserving if the Lebesgue measure of any set $A \subset (2,3)$ equals the Lebesgue measure of the image set $f(A) \subset (4,5)$. The requirements for f are satisfied when f is piece-wise affine with slope equal to plus or minus one.

[2] If $t_i^I < 0$, then side-i users would have an incentive to forge a phony match in order to collect money from the platform.

just one of the platforms (single-homing) and which one, or to just leave. If a side-i user has registered for one platform k, then he is restricted to searching for his right partner from side j on platform k; but if the side-i user has registered for both platforms, then he can search for his right partner from side j on both platforms. We assume that whenever x and $f(x)$ have both registered for platform k and have initiated the search, platform k can locate them and make the match occur without incurring any costs.

(i) First suppose that platform E is absent, so that platform I is a monopolist.
(i-a) Suppose that platform I cannot verify whether a match really occurs, and hence it has to set $t_1^I = t_2^I = 0$. Show that there are multiple subgame-perfect Nash equilibria: in one equilibrium platform I makes the maximum profits $u_1 + u_2$ by announcing $(p_1^I, p_2^I) = (u_1, u_2)$, but in another equilibrium platform makes profits u_2 by offering $p_1^I = 0$ and $p_2^I = u_2$, and the latter equilibrium is supported by the off-the-equilibrium beliefs that no users would register for platform I if both p_1^I and p_2^I are strictly positive, and that all side-i users would register for platform I if $p_i^I \leq 0$.
(i-b) Suppose instead that platform I can set positive transaction fees. Show that platform I can essentially obtain the maximum profits u_1+u_2 in equilibrium.

(ii) Now, suppose that both platforms are present, but users can use at most one platform. (That is, users must do "single-homing.")
(ii-a) Suppose that the platforms cannot verify whether a match really occurs, and hence they must set zero transaction fees. Show that all users choose to use platform I, with platform I's equilibrium pricing decisions being $p_1^I = -\max(u_1, u_2 - u_1)$ and $p_2^I = u_2$. Show that in equilibrium, platform I's profits are $\min(u_1, u_2 - u_1)$.
(ii-b) Now, suppose instead that platforms can set positive transaction fees. Show that there exists an equilibrium where all users choose to use platform I, with platform I's equilibrium pricing decisions being $p_1^I = -u_2 - u_1$, $t_1^I = u_1$, $p_2^I = 0$, and $t_2^I = u_2$. Show that in this equilibrium, platform I has zero profits.

(iii) Now, suppose that both platforms are present, and users can use both platforms. (That is, users are allowed to do "multi-homing.") Suppose that the platforms cannot verify whether a match really occurs, and hence they must set zero transaction fees. Show that there exists an equilibrium where all users choose to use platform I, with platform I's equilibrium pricing decisions being $p_1^I = p_2^I = 0$. Show that in this equilibrium, platform I has zero profits.

Solution. Consider (i-a). We claim that it is an equilibrium where platform I announces $p_1^I = u_1$ and $p_2^I = u_2$ and all users then choose to register for platform I. Indeed, expecting all side-i users to accept $p_i^I = u_i$ and register for platform I, a side-j user is confident that he will have a perfect match and obtain a payoff of $u_j - p_j^I = 0$ after registering for platform I himself, and hence that all users accept $p_1^I = u_1$ and $p_2^I = u_2$ and register for platform I constitutes a subgame equilibrium after platform I announces $p_1^I = u_1$ and $p_2^I = u_2$. Note that platform I attains its maximum possible profits when announcing $p_1^I = u_1$ and $p_2^I = u_2$, and hence it has no incentives to deviate from these equilibrium pricing decisions.

Next, we claim that it is also an equilibrium where platform I offers $p_1^I = 0$ and $p_2^I = u_2$, under the beliefs that no users would register for platform I if both p_1^I and p_2^I are strictly positive, and that all side-i users would register for platform I if $p_i^I \leq 0$.

To see that the claim is true, observe first that under the stated beliefs, it is indeed a subgame equilibrium that no users would register for platform I after platform I announces $p_1^I > 0$ and $p_2^I > 0$: expecting no side-i users would accept $p_i^I > 0$ and register for platform I, registering for platform I would result in zero chance of having a match and hence a payoff $-p_j^I < 0$ to each and every side-j user. Observe also that registering for platform I is a weakly dominant strategy for side-i users given that $p_i^I \leq 0$. Now, under the above beliefs, platform I can get side-i users on board by announcing $p_i^I = 0$, which, because rationality is users' common knowledge, convinces side-j users that they would obtain payoff $u_j - p_j^I$ if they accept p_j^I and register for platform I. Thus platform I can obtain payoffs u_1 by offering $(p_1^I, p_2^I) = (u_1, 0)$ and payoffs u_2 by offering $(p_1^I, p_2^I) = (0, u_2)$. Since $u_2 > u_1$, platform I's equilibrium best response is to offer $(p_1^I, p_2^I) = (0, u_2)$.

Our TA has made the following observation for us: *it is valuable for platform I to be able to charge a negative registration fee.* Indeed, if we restrict p_1^I and p_2^I to be non-negative, then there always exists an equilibrium where users never register for platform I and platform I gets zero equilibrium payoffs by announcing $p_1^I = p_2^I = 0$. The latter zero-payoff equilibrium, referred to as a *market breakdown* equilibrium, is Pareto inefficient, and its driving force is the indirect network externality pertaining to a two-sided market. TA shows that this inefficient equilibrium would vanish when platform I can announce a negative registration fee: by setting $p_1^I = -\epsilon$ and $p_2^I = u_2 - \epsilon$, platform I can get all side-1 users on board, which convinces side-2 users that they will obtain the payoff $u_2 - p_2^I = \epsilon > 0$ if they accept p_2^I and register for platform. Thus platform I can essentially attain a payoff of $u_2 > 0$ when allowed to offer a negative registration fee, proving that platform I would never announce $p_1^I = p_2^I = 0$ in equilibrium.

Consider (i-b). Now platform I can resolve the issue of multiple equilibria in (i-a) by announcing $p_i^I = -\epsilon$ for one segment of users, say, side i, and in response, all side-i users will be willing to register for platform I even if $t_i^I = u_i$. Thus by offering $(p_i^I, t_i^I) = (-\epsilon, u_i)$, platform I can get all side-i users on board, which convinces side-j users that they would each obtain the payoff $u_j - t_j^I - p_j^I$ if they are willing to accept (p_j^I, t_j^I) and register for platform I. Thus platform I can choose (p_j^I, t_j^I) such that $p_j^I + t_j^I = u_j - \epsilon$ to go along with $(p_i^I, t_i^I) = (-\epsilon, u_i)$ and obtain a payoff $u_1 + u_2 - 2\epsilon$, and as $\epsilon > 0$ can be chosen to be arbitrarily small, platform I can obtain essentially the payoff of $u_1 + u_2$.

Consider (ii-a). We shall refer to an equilibrium where all users register for one platform and no users register for the other platform as a *tipping equilibrium*. Here we shall focus on the tipping equilibrium where all users choose to register for platform I. We use backward induction, starting with the subgame where platform E is about to make pricing decisions given (p_1^I, p_2^I). In this tipping equilibrium, E can induce side-i

users to switch and join platform E instead if and only if

$$0 \cdot u_i - p_i^E > 1 \cdot u_i - p_i^I \Leftrightarrow p_i^E < p_i^I - u_i,$$

where we have emphasized that everyone (including platform E itself) believes that side-i users' probability of having a match is one if they stick to platform I, and their probability of having a match becomes zero if they switch to platform E. Now, given that E has announced $p_i^E < p_i^I - u_i$ to get side-i users on board, a side-j user's payoff from staying with platform I would become $0 \cdot u_j - p_j^I$,[3] so that platform E can induce side-j users to also switch and join platform E by announcing p_j^E that satisfies

$$1 \cdot u_j - p_j^E > \max(0, 0 \cdot u_j - p_j^I) = -\min(0, p_j^I) \Leftrightarrow p_j^E < u_j + \min(0, p_j^I).$$

Thus, given (p_1^I, p_2^I), platform E can attain the payoff

$$\max_{(i,j)\in\{(1,2),(2,1)\}} p_i^I - u_i + u_j + \min(0, p_j^I).$$

We shall assume that platform E would remain inactive (i.e., it would announce no price offers to users) if, given (p_1^I, p_2^I), the above maximum payoff is less than or equal to zero.

Now, return to the stage where platform I is about to set (p_1^I, p_2^I). In the supposed tipping equilibrium, platform I seeks to

$$\max p_1^I + p_2^I$$

subject to

$$p_1^I - u_1 + u_2 + \min(0, p_2^I) \leq 0;$$
$$p_2^I - u_2 + u_1 + \min(0, p_1^I) \leq 0;$$
$$p_1^I \leq u_1;\ p_2^I \leq u_2.$$

First we claim that platform I would announce $p_2^I \geq 0$. Indeed, if instead $p_2^I < 0$, then the above first constraint requires that

$$p_1^I + p_2^I \leq u_1 - u_2 < 0,$$

[3]These users are said to be *stranded* in the language of Farrell and Saloner (1986); see Farrell, J. and G. Saloner, 1986, Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation, *American Economic Review*, 76, 5, 940-955.

but platform I can at least remain inactive! Now, if $p_1^I \geq 0$ also, then by summing up the first two constraints we obtain $p_1^I + p_2^I \leq 0$, but we shall show that platform I can obtain a strictly positive payoff by announcing *some* negative p_1^I. Thus assume that $p_1^I < 0 \leq p_2^I$, and it follows that the above third constraint can be ignored, and we can re-state the remaining constraints as

$$p_1^I \leq -(u_2 - u_1);$$

$$p_2^I \leq u_2 - u_1 - p_1^I;$$

$$p_2^I \leq u_2.$$

Since the objective function is strictly increasing in p_2^I given p_1^I, the optimal p_2^I should make either the second or the last constraint binding; i.e., we have at optimum either

$$p_2^I = u_2 \text{ and } p_1^I \leq -u_1 \Rightarrow p_1^I = \min(-u_1, u_1 - u_2)$$

$$\Rightarrow p_1^I + p_2^I = u_2 + \min(-u_1, u_1 - u_2) = \min(u_1, u_2 - u_1).$$

or

$$u_2 \geq p_2^I = u_2 - u_1 - p_1^I \geq 2(u_2 - u_1)$$

$$\Rightarrow p_1^I + p_2^I = u_2 - u_1 \text{ when } u_1 \geq u_2 - u_1.$$

Summing up the above discussions, we conclude that if $u_2 - u_1 > u_1$, then $2(u_2 - u_1) > u_2$, so that at optimum $p_1^I = -(u_2 - u_1)$ and $p_2^I = u_2$; and if instead $u_1 \geq u_2 - u_1$, then $u_2 \geq 2(u_2 - u_1)$, so that at optimum $p_1^I = -u_1$ and $p_2^I = u_2$. Thus as we have asserted earlier, $p_1^I < 0$ at optimum, and platform I's equilibrium payoff is $\min(u_1, u_2 - u_1) > 0$.

Now, consider (ii-b). Again, we start with the subgame where platform E must make price decisions given $(p_1^I, t_1^I, p_2^I, t_2^I)$. To induce side-$i$ users to deviate and join platform E instead, platform E must offer (p_i^E, t_i^E) such that

$$-p_i^E > u_i - p_i^I - t_i^I,$$

but the key difference here is that platform E can set $t_i^E = u_i$. This makes platform E's effort of stealing side-i users away from platform I less costly than in scenario (ii-a). Having offered

$$p_i^E = -u_i + p_i^I + t_i^I - \epsilon, \; t_i^E = u_i,$$

platform E can also steal side-j users away from platform I by offering (p_j^E, t_j^E) that satisfies

$$u_j - p_j^E - t_j^E > \max(-p_j^I, 0) \Leftrightarrow p_j^E + t_j^E < u_j + \min(p_j^I, 0).$$

Thus, given $(p_1^I, t_1^I, p_2^I, t_2^I)$, platform E can obtain essentially the following payoff by remaining active:

$$\max_{(i,j)\in\{(1,2),(2,1)\}} p_i^I + t_i^I + u_j + \min(0, p_j^I).$$

Now, return to platform I's pricing decisions. Platform I seeks to

$$\max p_1^I + t_1^I + p_2^I + t_2^I$$

subject to

$$p_1^I + t_1^I + u_2 + \min(0, p_2^I) \leq 0;$$
$$p_2^I + t_2^I + u_1 + \min(0, p_1^I) \leq 0;$$
$$p_1^I + t_1^I \leq u_1;\ 0 \leq t_1 \leq u_1;$$
$$p_2^I + t_2^I \leq u_2;\ 0 \leq t_2 \leq u_2.$$

For $(p_1^I, t_1^I, p_2^I, t_2^I)$ satisfying $p_1^I \geq 0$ and $p_2^I \geq 0$, we would have

$$p_1^I + t_1^I \leq -u_2,\ p_2^I + t_2^I \leq -u_1,$$

implying that platform I's payoff is strictly negative. Thus suppose that $p_i^I < 0$, which implies that

$$p_j^I + t_j^I + p_i^I + t_i^I \leq p_j^I + t_j^I + \min(p_i^I, 0) + u_i \leq 0.$$

Thus platform I cannot attain a strictly positive payoff. Note that one way for platform I to attain zero profits is to set $t_i^I = u_i$, $p_2^I = 0$ and $p_1^I = -u_1 - u_2$.

Consider part (iii). Again, we start with the subgame where platform E must make price decisions given (p_1^I, p_2^I). Note that users are allowed to register at both platforms, and hence any negative p_i^E can induce side-i users to adopt platform E as an additional platform. However, to ensure that users would like to meet at platform E after registering at both platforms, platform E must ensure that side-j users strictly prefer platform E to platform I after knowing that side-i users have chosen to register at both platforms; that is, platform E must offer (p_i^E, p_j^E) such that $p_i^E < 0$ (so that side-i users will adopt platform E as their second choice) and, given this fact, side-j users would rather have a match on platform E than on platform I; that is,

$$1 \cdot u_j - p_j^E > 1 \cdot u_j - p_j^I \Leftrightarrow p_j^E < p_j^I,$$

where note that, given side-i users' multi-homing decisions, everyone (including platform E) realizes that side-j users' chance of having a match is one no matter which platform side-j users choose to use to meet with side-i users. Thus platform E's optimal payoff given (p_1^I, p_2^I) is

$$\sup_{\epsilon \in \Re_{++}} (0, -\epsilon + p_1^I - \epsilon, -\epsilon + p_2^I - \epsilon) = \max(0, p_1^I, p_2^I).$$

Now, return to platform I's pricing decisions. Platform I seeks to

$$\max p_1^I + p_2^I$$

subject to

$$\max(0, p_1^I, p_2^I) \leq 0 \Rightarrow p_1^I, p_2^I \leq 0,$$

implying that platform I must choose $p_1^I = p_2^I = 0$!

Remark 1. We have focused on the so-called tipping equilibrium in the above analysis, where all users choose to join one platform. This type of equilibrium can prevail even if we assume that the two platforms act simultaneously, but with simultaneous moves, there may be other equilibria with symmetric equilibrium allocations. In general, when the dominant platform gets a zero payoff in a tipping equilibrium, the two platforms would also get zero payoffs in a symmetric equilibrium. Here we emphasize an unusual symmetric equilibrium with positive payoffs for the two platforms.

Suppose now that in part (iii) the two platforms must announce prices *simultaneously* before users arrive. Then it is an equilibrium where for platform $k \in \{I, E\}$, $p_1^k = 0$, $p_2^k = \frac{u_2}{2}$.[4] [5] In this equilibrium, side-1 users do single-homing and they split equally between the two platforms, but side-2 users do multi-homing.[6] A side-1 user's equilibrium payoff from joining platform k is $u_1 - p_1^k = u_1 > 0$, and a side-2 user's equilibrium payoff from joining both platforms is $u_2 - p_2^I - p_2^E = 0$. A platform k's equilibrium payoff is $\frac{1}{2} \cdot p_1^k + 1 \cdot p_2^k = \frac{u_2}{2}$.

Remark 2. The platforms studied above can be cybermediaries pro-

[4]To check that no platforms would want to deviate unilaterally from the equilibrium pricing decisions, first note that lowering the registration fee for side-2 users is worthless: lowering the registration fee may in general encourage registration but side-2 users have already chosen multi-homing. Second, lowering the registration fee for side-1 users would make them choose multi-homing, but it has no influence on their (and side-2 users') choices regarding the platform on which they would like the match to take place.

[5]This equilibrium would not break down even if platforms must move sequentially. We first argue that, given that platform I announces $p_1^I = 0$ and $p_2^I = \frac{u_2}{2}$, platform E has no incentive to deviate. If instead platform E announces subsequently $p_1^E = -\epsilon$ and $p_2^E = \frac{u_2}{2} - \epsilon$, then side-1 users will do multi-homing, and side-2 users will do single-homing and drop platform I, and thus all matches will occur on platform E. However, since $t_i^E = 0$, platform E actually becomes worse off: platform E's deviation payoff becomes $1 \cdot p_1^E + 1 \cdot p_2^E < \frac{u_2}{2}$! Next, we argue that platform I has no incentive to deviate either if certain off-the-equilibrium beliefs are held by users and the two platforms. Specifically, if platform I deviates and aims at obtaining a payoff higher than $\frac{u_2}{2}$, then following the deviation platform I *must* announce $p_i^I > 0$ for some i. Thus there are two possible deviation announcements for platform I.

- If following the deviation, platform I announces $p_1^I, p_2^I > 0$, then we assume that side-i users will choose single-homing and join the platform k offering the lower p_i^k. In this case platform E can offer $p_i^E = p_i^I - \epsilon$ and make platform I's deviation payoff equal to zero. This removes platform I's incentive to deviate and offer those positive registration fees in the first place.
- Now, if following the deviation platform I offers $p_i^I \leq 0$ and $p_j^I > 0$, then we assume that whenever $p_i^E \leq 0$ all users believe that side-i users will choose multi-homing but side-j users will choose single-homing and join the platform k offering the lower p_j^k. In this case platform E can offer $p_i^E = 0$ and $p_j^E = p_j^I - \epsilon$, and this removes platform I's incentive to deviate and announce $p_i^I \leq 0$ and $p_j^I > 0$ in the first place.

[6]Note that there is no *need* for side-1 users to do multi-homing, once they are sure that side-2 users are doing multi-homing. To have users from both sides doing multi-homing, for *each* side i, some platform k must be offering $p_i^k \leq 0$.

viding online dating services (like eharmony.com or match.com), or online search engines, or they can be e-commerce firms (e.g. Amazon.com) performing informational intermediation between, say, readers and books. Typically the value of an intermediary for a user on one side relates positively to the number of users on the other side, a phenomenon generally referred to as an *indirect network externality.* Platforms exhibiting this property is referred to as a two-sided market. This exercise is adapted from Caillaud and Jullien (2001).[7]

2. (**Competing Platforms, Part II.**) Here we shall modify Problem 1 as follows:

   - For some $\rho \in (0, 1)$, $u_2 = 1 + \rho$ and $u_1 = 1 - \rho$.
   - The platforms must set zero transaction fees and non-negative registration fees.
   - The timing of relevant events is modified as follows:
     - The two platforms simultaneously announce p_1^I and p_1^E.
     - Then side-1 users arrive and they simultaneously choose which platform(s) to use.
     - Then it becomes public information that there are x side-1 users having registered for platform I, y side-1 users having registered for platform E, and z side-1 users having registered for both platforms, where $0 \leq z \leq x, y \leq 1$.
     - Then the two platforms simultaneously announce p_2^I and p_2^E.
     - Then side-2 users arrive and they simultaneously choose which platform(s) to use.

   (i) Show that if all users must adopt single-homing then there are multiple equilibria, where in one equilibrium all side-1 users choose to use platform $k \in \{I, E\}$ with platform k pricing at $p_1^k = (1 - \rho)$ and then $p_2^k = (1 + \rho)$, and in another equilibrium the two platforms announce $p_1^I = p_1^E = 0$ with side-1 users randomly choosing a platform.

[7] Caillaud, B., and B. Jullien, 2001, Competing Cybermediaries, *European Economic Review (Papers and Proceedings)*, 45, 797-808.

(ii) Suppose instead that side-1 users must adopt single-homing but side-2 users are allowed to multi-home. Show that given x, y, z, the two platforms will charge side-2 users respectively $p_2^I = (1+\rho)(x-z)$ and $p_2^E = (1+\rho)(y-z)$, so that, by backward induction, the two platforms announce $p_1^I = p_1^E = 0$ when serving side-1 users.

Solution. Consider part (i), which assumes that $z = 0$. Consider first the subgame where the two platforms must make price offers to side-2 users, given x, y. Note that a platform k will never offer $p_2^k < 0$ at this stage even if negative registration fees are allowed: if it did, it loses money from getting side-2 users on board, without helping attract side-1 users to come join the party (simply because side-1 users' registration decisions have already been made and the platform cannot charge a positive transaction fee). When $x > y$, platform I and platform E will offer the prices $p_2^E = 0$ and $p_2^I = (1+\rho)(x-y)$ and all side-2 users will join platform I; and when $x < y$, platform I and platform E will offer the prices $p_2^I = 0$ and $p_2^E = (1+\rho)(x-y)$ and all side-2 users will join platform E.[8] In case $x = y$, both platforms offer zero registration fees, and side-2 users randomly pick a platform to join.

Now, return to the stage where the two platforms are about to make offers to side-1 users. According to the above analysis, essentially, a side-1 user will have a match only if he joins the platform that the majority of his side-1 fellow users choose to join. Thus if side-1 users believe that they will all join platform k, then this is a self-fulfilling equilibrium as long as $p_1^k \leq u_1$. This leads to a tipping equilibrium where some platform k offers $p_1^k = (1-\rho)$ and the (non-negative) registration fee chosen by its rival is irrelevant. On the other hand, if side-1 users believe that they will all choose the platform k offering the lower p_1^k, and they will randomly choose a platform to join when $p_1^I = p_1^E$, then it is an equilibrium where the two platforms offer $p_1^I =$

[8]In these two cases, the two platforms are like two Bertrand-competitive firms offering heterogeneous goods at zero costs, where buyers' common valuation for firm k's good is v_k, with $0 < v_1 < v_2$, say. In equilibrium, firm 1 will price at zero, and firm 2 will price at $v_2 - v_1$. The idea is that firm 1 can at best offer buyers a surplus of v_1 at zero price, which firm 2 can match by pricing at (or slightly below) $v_2 - v_1$. Thus firm 1 makes no profits, and firm 2's payoff is $v_2 - v_1$.

$p_1^E = 0$ and side-1 users randomly join the two platforms. Apparently, the former two tipping equilibria are efficient (despite having higher prices and platform profits than the symmetric equilibrium), as every user has a match in equilibrium. The latter symmetric equilibrium is not, where for each user a match may occur only with a probability less than one.

Consider part (ii). Define $x_I = x$ and $x_E = y$. The case where $xy = 0 < x^2 + y^2$ is easy; the platform k having $x_k > 0$ will announce $p_2^k = x_k(1+\rho)$ and get all side-2 users on board. Now, suppose that $x, y > 0$, and note that by assumption side-1 users must single-home, so that we have $z = 0$ once again. Given that $x, y > 0 = z$, can it be an equilibrium where side-2 users single-home? The answer is negative: if side-2 users do not register for platform k, then platform k can always announce $0 < p_2^k < x_k(1+\rho)$ to get side-2 users on board and raise its own payoff. Now, observe that side-2 users' valuation for platform k, given that they will also join the other platform, is $(x_k - z)(1+\rho)$, and hence platform k will price at $p_2^k = (x_k - z)(1+\rho)$ if expecting side-2 users to also join the other platform. We conclude that the equilibrium prices are $p_2^I = x(1+\rho)$ and $p_2^E = y(1+\rho)$ and side-2 users will all multi-home whenever $x, y > 0$.

Now, return to the stage where the two platforms are about to make offers to side-1 users. Rationally expecting side-2 users to multi-home whenever $x, y > 0$, side-1 users are confident that a match will occur with probability one no matter which platform they register for. Thus a side-1 user will join the platform k offering the lower p_1^k, regardless of what his side-1 fellow users will do. That is, $x_I = 0$ if $p_1^I > p_1^E$ and $x_E = 0$ if $p_1^I < p_1^E$. For $k, h \in \{I, E\}$, $k \neq h$, platform k's payoff from serving users from both sides is therefore $x_k(p_1^k + 1 + \rho)$ if $p_1^k < p_1^h$ and zero if $p_1^k > p_1^h$. This leads to the equilibrium outcome of $p_1^I = p_1^E = 0$, and facing these registration fees, side-1 users simply joint a platform at random. To sum up, in equilibrium the multi-homing side is facing high registration fees and left with no surplus, while the single-homing side is offered with free access. A platform's expected equilibrium payoff is $\frac{1}{2}(1+\rho)$.[9]

[9] This exercise is adapted from King, S., 2013, Two-sided Markets, *Australia Economic*

Remark. When one side of users single-home, competition for those users will result in a winning platform, which exclusively gains the positive externality associated with those single-homing users. This fight for the positive externality intensifies competition, and can result in zero prices. When one side of users multi-home, multiple platforms can gain the positive externality associated with those multi-homing users at the same time, and this tends to mute the competition for those multi-homing users. These observations were first made in Armstrong (2006),[10] where the author wrote (pp. 669-670):

$\cdots$ *platforms have monopoly power over providing access to their single-homing customers for the multi-homing side. This monopoly power naturally leads to high prices being charged to the multi-homing side.* $\cdots$ *By contrast, platforms do have to compete for the single-homing users, and high profits generated from the multi-homing side are to a large extent passed on to the single-homing side in the form of low (or even zero) prices.*

3. (**Competitive Manufacturers May Make More Profits with Non-integrated Distribution Channels.**) Recall the Cournot game in Example 1 of Lecture 1, Part I. Assume that $c = F = 0$ and the inverse demand in the relevant range is

$$P(Q) = 1 - Q,\ 0 \leq Q = q_1 + q_2 \leq 1.$$

(i) Find the equilibrium profits for the two firms.
(ii) Now suppose that the two manufacturing firms cannot sell their products to consumers directly. Instead, firm i (also referred to as manufacturer i) must first sell its product to retailer R_i. Then retailers R_1 and R_2 then compete in the Cournot game. The extensive game is now as follows.

- The two firms first announce F_1 and F_2 simultaneously, where F_i is the franchise fee that firm i will charge retailer i, which is a fixed cost of retailer i. R_1 and R_2 simultaneously decide to or not

Review, 46, 2, 247-258.

[10]Armstrong, M., 2006, Competition in Two-sided Markets, *Rand Journal of Economics*, 37, 3, 668-691.

to turn down the offers made by the firms. Assume that firm i and retailer R_i both get zero payoffs if F_i gets turned down by retailer R_i.

- Then, after knowing whether F_1 and F_2 get accepted by respectively R_1 and R_2, the two firms announce w_1 and w_2 simultaneously, where w_i is the unit whole price that firm i will charge retailer i.
- Next, in case the firms' offers are both accepted, then given (F_1, F_2, w_1, w_2), the two retailers simultaneously choose q_1 and q_2.

Show that in the unique subgame-perfect Nash equilibrium (SPNE) each manufacturing firm gets a profit of $\frac{10}{81}$. (**Hint**: Backward induction asks you to always start from the last-stage problem, which is the Nash equilibrium of the subgame where R_1 and R_2 play the Cournot game given some (F_1, F_2, w_1, w_2). You can show that the equilibrium (q_1^*, q_2^*) depend on (w_1, w_2) but not on (F_1, F_2), because the latter are fixed costs. Then, you should move backwards to consider the two manufacturers' competition in choosing w_1 and w_2, given some (F_1, F_2). Here assume that the two manufacturers know that different choices of w_1 and w_2 will subsequently affect R_1's and R_2's choices of q_1 and q_2. Finally, you can move to the first-stage of the game, where the two firms simultaneously choose F_1 and F_2.)[11] [12] [13] [14]

Solution. Let us solve the SPNE using backward induction. First consider the subgame where (F_1, F_2, w_1, w_2) are given, and the two retailers are about to choose q_1 and q_2. Retailer i, given q_j, seeks to

$$\max_{q_i} \Pi_i^R(q_i, q_j; w_i, F_i) \equiv q_i(1 - q_i - q_j - w_i) - F_i.$$

The first-order condition gives retailer i's reaction function

$$r_i(q_j; w_i) = \frac{1 - q_j - w_i}{2}, \quad \forall i, j \in \{1, 2\}, \; i \neq j.$$

Thus there is a unique NE in this subgame, which is[15]

[11] This exercise intends to show why employing independent retailers may be a good idea even if using a firm's own outlets can be cheaper. Essentially, employing an independent retailer amounts to delegating the retailer the choice of output, knowing that the retailer, unlike the manufacturer, will be choosing output given a positive unit cost w_i! A higher unit cost credibly convinces the rival retailer that less output will be produced, and with both manufacturers producing less outputs, their profits become higher.

[12] That the two firms are able to first offer F_1 and F_2 and subsequently choose w_1 and w_2 is important in this game. If instead the two manufacturers must offer (F_1, w_1) and (F_2, w_2) to R1 and R2 at the first stage of the game, then given w_j, firm i would choose $w_i = 0$!

[13] We have assumed that the two firms have homogeneous products and the demand is linear. When the two firms' products are differentiated or when the demand functions are not linear, raising the equilibrium product prices by using an independent retailer may reduce a manufacturer's sales volume by too much and hence may or may not be a good idea; see Patrick Rey and Joseph Stiglitz, 1995, The Role of Exclusive Territories in Producers' Competition, *Rand Journal of Economics*, 26, 431-451. See also T. W. McGuire and R. Staelin, 1983, An Industry Equilibrium Analysis of Downstream Vertical Integration, *Marketing Science*, 2, 161-191.

[14] Note that if a manufacturer i sells through more than one retailer in a small district, then *intra-brand competition* between these retailers will lead to the Bertrand outcome where all retailers hired by manufacturer i offer w_i as the retail price—the distribution channel of manufacturer i is essentially vertically integrated! This highlights the importance of assuming that each manufacturer hires exactly ONE independent retailer (a practice referred to as *exclusive territory*) in this exercise.

[15] Why does q_i^* increase with w_j? Again, this results from the fact that q_1 and q_2 are strategic substitutes. A higher w_j means that retailer j is faced with a higher unit cost, and hence q_j ought to be lower, which then implies that retailer i should optimally respond by choosing a higher q_i.

$$(q_1^*(w_1,w_2), q_2^*(w_1,w_2)) = (\frac{1-2w_1+w_2}{3}, \frac{1+w_1-2w_2}{3}).$$

Correspondingly, the two retailers' profits are

$$\Pi_1^R(q_1^*(w_1,w_2), q_2^*(w_1,w_2); w_1, F_1) = \frac{(1-2w_1+w_2)^2}{9} - F_1$$

and

$$\Pi_2^R(q_2^*(w_1,w_2), q_1^*(w_1,w_2); w_2, F_2) = \frac{(1-2w_2+w_1)^2}{9} - F_2.$$

Now, consider the stage where (F_1, F_2) are given and the two manufacturers are about to choose w_1 and w_2. Manufacturer i, given w_j, seeks to

$$\max_{w_i} F_i + w_i q_i^*(w_i, w_j), \quad \forall i,j \in \{1,2\},\ i \neq j.$$

The first-order condition gives

$$w_i = \frac{1+w_j}{4}, \quad \forall i,j \in \{1,2\},\ i \neq j.$$

Note that w_1 and w_2 are indeed strategic complements![16] Thus there is a unique NE in this subgame where the two manufacturers both set the unit wholesale price at $\frac{1}{3}$:

$$w_1^* = w_2^* = \frac{1}{3}.$$

In this equilibrium, for $i = 1, 2$, manufacturer i's profit is

$$F_i + \frac{6}{81}.$$

[16]When manufacturer i expects manufacturer j to choose a higher w_j, it realizes that, keeping its choice w_i unchanged, subsequently the two retailers will choose higher q_i^* and lower q_j^*, which marginally encourages manufacturer i to raise w_i in the first place: the drawback of raising w_i is that it leads to a lower q_i^*, and hence it is less costly to do this when q_i^* rises because of a higher w_j! This explains strategic complementarity between w_i and w_j.

The correspondingly profits of the two retailers are

$$\Pi_1^R(q_1^*(\frac{1}{3},\frac{1}{3}),q_2^*(\frac{1}{3},\frac{1}{3});\frac{1}{3},F_1)=\frac{4}{81}-F_1$$

and

$$\Pi_2^R(q_2^*(\frac{1}{3},\frac{1}{3}),q_1^*(\frac{1}{3},\frac{1}{3});\frac{1}{3},F_2)=\frac{4}{81}-F_2.$$

Now, consider the stage where the two manufacturers are about to choose F_1 and F_2. Manufacturer i's problem is

$$\max_{F_i} F_i+\frac{6}{81}$$

subject to

$$\Pi_i^R(q_i^*(\frac{1}{3},\frac{1}{3}),q_j^*(\frac{1}{3},\frac{1}{3});\frac{1}{3},F_i)=\frac{4}{81}-F_i\geq 0.$$

There is a unique SPNE in this game where $F_1=F_2=\frac{4}{81}$, and hence the two manufacturers' equilibrium profits are both $\frac{10}{81}$.

Remark. We must emphasize here the role of the timing of the game. That the two firms are able to first offer F_1 and F_2 to respectively R1 and R2 and then to subsequently choose w_1 and w_2 is important to the above result. If instead the two manufacturers must offer (F_1,w_1) and (F_2,w_2) to R1 and R2 at the first stage of the game, then given w_j, firm i would like to choose $w_i=0$, because a zero unit wholesale price can serve as a commitment that convinces Rj that Ri would produce more given any quantity q_j (or, simply, Ri's reaction function will be shifted upwards).[17]

[17] In this case, given (F_j,w_j), manufacturer i seeks to

$$\max_{(F_i,w_i)} F_i+w_iq_i^*(w_i,w_j),$$

subject to

$$q_i^*(1-q_i^*-q_j^*-w_i)-F_i\geq 0.$$

Optimality requires that the latter constraint be binding, and hence

$$F_i=q_i^*(1-q_i^*-q_j^*-w_i),$$

This commitment is valuable, because output choices are *strategic substitutes*, which implies that Rj will reduce output q_j if Rj believes that it is faced with a more aggressive reaction function. Consequently, choosing $w_i = 0$ can raise Ri's profit, which in turn implies that, manufacturer i in offering $w_i = 0$, can choose a higher F_i to extract Ri's profit.

In the current setting, however, given that F_1 and F_2 were offered and accepted in the preceding stage, the two firms in choosing w_1 and w_2 would never choose a zero unit wholesale price, because a zero wholesale price would result in *no* additional income for the manufacturer. Indeed, at this stage, as we have shown, regardless of F_1 and F_2, the two firms choose $w_1 = w_2 > 0$. Retailer Ri can infer this fact (as we do) when it must decide whether to accept F_i. This explains why in equilibrium the two manufacturers are able to set $F_1 = F_2 = \frac{4}{81}$.

Note that when a single manufacturer chooses a positive unit wholesale price, it induces its downstream retailer to reduce output (because the unit wholesale price is the retailer's unit cost, and a higher unit cost leads to a lower output choice), which, by the fact that output choices are strategic substitutes, in turn encourages the other retailer to expand output, which hurts the manufacturer's downstream retailer. However, with both manufacturers offering positive unit wholesale prices, the net

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or equivalently, manufacturer i seeks to

$$\max_{w_i \geq 0} q_i^*(1 - q_i^* - q_j^*) \equiv H(w_i; w_j) = \frac{1}{9}(1 - 2w_i + w_j)(1 + w_i + w_j),$$

where the new objective function is simply the profit function facing an otherwise-identical vertically integrated channel (that is, the firm that is both manufacturer i and Ri). Since q_i^* and q_j^* are respectively decreasing and increasing in w_i, it is easy to verify that this new objective function is decreasing in w_i given w_j, and hence we obtain a corner solution $w_i = 0$. Indeed, direct differentiation yields

$$\frac{\partial H}{\partial w_i} = \frac{1}{9}(-4w_i - w_j - 1) < 0, \Rightarrow w_i^* = 0.$$

The same argument applies to manufacturer j as well, and hence when the two manufacturers must offer (F_1, w_1) and (F_2, w_2) to R1 and R2 at the first stage of the game, the latter two retailers behave just like firms 1 and 2 in Example 1 in Lecture 1, Part I (with zero production costs). As can be easily checked, in the current situation, with $w_1 = w_2 = 0$, the two retailers will choose $q_1^* = q_2^* = \frac{1}{3}$.

effect of positive wholesale prices is to induce both retailers to select an output level that is lower than the output level that the two manufacturers would choose in the absence of independent retailers (or, in the case of vertically integrated distribution channels). This lower output level then leads to a higher equilibrium retail price, which raises the sum of the manufacturer's and the retailer's profits in each distribution channel. The sum of profits of the manufacturer and the retailer coincides with the manufacturer's equilibrium profit in the current case, because by assumption the manufacturer can offer a two-part tariff to its downstream dealer, leaving the latter with a zero profit.

4. (**Entry Deterrence by a Monopolistic Incumbent.**) Consider the following extensive game in which firms A and B may compete in quantity at date 1 and date 2. Both firms seek to maximize the sum of expected date-1 and date-2 profits. The inverse demand at date $t \in \{1, 2\}$, in the relevant region, is $P_t = 1 - Q_t$, where P_t is the date-t product price and $Q_t = q_{At} + q_{Bt}$ is the sum of the two firms' supply quantities at date t. Assume that there are no production costs for the two firms.

   - At date 1, originally firm A is the only firm in the industry. Firm A must first choose q_{A1}. Upon seeing firm A's choice q_{A1}, firm B must decide whether to spend a cost $K > 0$ to enter the industry. If K is spent, then B must choose q_{B1}. Then the two firms' date-1 profits π_{A1} and π_{B1} are realized, where $\pi_{B1} = 0$ if firm B decides not to enter the industry.
   - At date 2, if firm B did not enter at date 1, then firm A, the monopolistic firm in the industry, must choose q_{A2}. If, on the other hand, firm B has entered at date 1, then the two firms choose quantities q_{A2} and q_{B2} simultaneously. Then, the two firms' date-2 profits π_{A2} and π_{B2} are realized, where $\pi_{B2} = 0$ if firm B did not enter the industry at date 1.

   Now we solve for the subgame perfect Nash equilibrium for this game.
   (i) Suppose that $K = \frac{1}{5}$. Find the equilibrium q_{A1} and q_{A2}.
   (ii) Suppose that $K = \frac{1}{9} + \frac{1}{25}$. Find the equilibrium q_{A1} and q_{A2}.

(iii) Suppose that $K = \frac{1}{25}$. Find the equilibrium q_{A1} and q_{A2}.[18]

Solution. Let us solve the game by backward induction. Consider the subgame at date 2.

- If both firms exist, it is easy to show (or recall from Lecture 1, part I) that $q_{A2} = q_{B2} = \frac{1}{3} = P_2$, and the corresponding date-2 profit is $\frac{1}{9}$ for each firm.
- If only firm A exists at date 2, then it will get the monopoly profit $\frac{1}{4}$ by producing $q_{A2} = \frac{1}{2} = P_2$.

Now, move backwards to consider the date-1 subgame where q_{A1} has been chosen, and firm B has spent K. In this case, firm B's optimal supply quantity is $\frac{1-q_{A1}}{2}$; recall Lecture 1, part I. This implies that firm B's profit over the two dates is

$$-K + \frac{(1-q_{A1})^2}{4} + \frac{1}{9}.$$

Next, consider the date-1 subgame where q_{A1} has been chosen, and firm B is about to decide whether to spend K. From the preceding analysis, we know that firm B's optimal decision is as follows: spending K if and only if

$$K < \frac{(1-q_{A1})^2}{4} + \frac{1}{9}.$$

Note that we have assumed that firm B will stay out if entering does not generate a positive profit for it.

Now, we can finally consider firm A's choice of q_{A1}.

[18] This exercise explains why a monopolistic firm may not always produce the monopoly output stated in an economics textbook. If firm A insists on producing the monopoly output $\frac{1}{2}$ at date 1, it may induce entry, which would destroy its monopolistic status at date 2. In part (ii), for example, the monopolistic firm may optimally produce more than $\frac{1}{2}$ in order to deter entry. In this sense, *even a monopolistic firm has potential competitors*, and the presence of potential competitors is enough to force the monopolistic firm to produce more, so that its output choice may get closer to the socially efficient output level. For a formal analysis, see Dixit, A., 1980, The role of investment in entry deterrence, *Economic Journal*, 90, 95-106.

- If q_{A1} is such that

$$K \geq \frac{(1-q_{A1})^2}{4} + \frac{1}{9},$$

then firm B will not enter at date 1, and hence firm A's profit over the two dates is

$$q_{A1}(1-q_{A1}) + \frac{1}{4}.$$

- If q_{A1} is such that

$$K < \frac{(1-q_{A1})^2}{4} + \frac{1}{9},$$

then firm B will enter at date 1, and hence firm A's profit over the two dates becomes

$$q_{A1} \times \frac{1-q_{A1}}{2} + \frac{1}{9}.$$

Note that in either of the two cases considered above, in the absence of the constraint involving K, firm A's unconstrained optimal date-1 supply quantity must maximize $q_{A1}(1-q_{A1})$; that is, the unconstrained optimal supply quantity is $\frac{1}{2}$, which is the optimal supply quantity for a monopolistic firm.

Thus we can summarize firm A's optimal date-1 output policy as follows.

(a) If

$$K \geq \frac{(1-\frac{1}{2})^2}{4} + \frac{1}{9} = \frac{1}{16} + \frac{1}{9},$$

then firm B would not enter when firm A chooses its unconstrained optimal supply quantity $q_{A1} = \frac{1}{2}$. Thus it is indeed optimal for firm A to choose $q_{A1} = \frac{1}{2}$, and it follows that $q_{A2} = \frac{1}{2}$ also. In this case we say that firm A's date-1 output policy *blocks the entry* of firm B.

(b) If

$$K < \frac{(1-1)^2}{4} + \frac{1}{9} = \frac{1}{9},$$

then firm B will still enter even if firm A chooses $q_{A1} = 1$ (which minmaxes firm B at date 1), and in this case firm A's optimal date-1 output strategy is $q_{A1} = \frac{1}{2}$, which leads to $q_{B1} = \frac{1-q_{A1}}{2} = \frac{1}{4}$, so that firm A's profit over the two dates is $q_{A1} \times P_1 + \frac{1}{9} = \frac{1}{8} + \frac{1}{9}$. In this case we say that firm A's date-1 output policy *accomodates the entry* of firm B.

(c) If

$$\frac{1}{16} + \frac{1}{9} > K \geq \frac{1}{9},$$

then firm B will enter if and only if $K < \frac{(1-q_{A1})^2}{4} + \frac{1}{9}$, where note that the right-hand side is strictly decreasing in q_{A1} for $q_{A1} \in [0,1]$. Thus firm A's date-1 output q_{A1} determines whether firm B will enter, and the higher q_{A1} is, the less likely that the constraint $K < \frac{(1-q_{A1})^2}{4} + \frac{1}{9}$ may hold. We say in this case that firm A's date-1 output policy *deters the entry* of firm B, if firm B does not enter in equilibrium. Firm A's optimal date-1 output that results in firm B entering the industry has been solved above, which is $q_{A1} = \frac{1}{2}$, and firm A's payoff from accomodating the entry is correspondingly $\frac{1}{8} + \frac{1}{9}$. On the other hand, firm A's optimal date-1 output that induces firm B to not enter can be obtained by solving the following maximization program:

$$\text{(P)} \quad \max_{q_{A1} \in [0,1]} q_{A1}(1 - q_{A1}) + \frac{1}{4}$$

subject to

$$K \geq \frac{(1 - q_{A1})^2}{4} + \frac{1}{9},$$

and at optimum the above constraint must be binding: if not, then the optimal q_{A1} would equal $\frac{1}{2}$, which, by the fact that $K < \frac{1}{16} + \frac{1}{9}$, would induce rather than deter B's entry. Thus firm A's optimal date-1 output is

$$q_{A1}^* = 1 - \sqrt{4(K - \frac{1}{9})} \in (\frac{1}{2}, 1].$$

We claim that, indeed, choosing this entry-deterring output is better than choosing $q_{A1} = \frac{1}{2}$ to accomodate entry. To see this, recall that by accomodating firm A's payoff is $\frac{1}{8}+\frac{1}{9}$, which is less than $\frac{1}{4}$, the payoff that firm A would obtain by choosing $q_{A1}=1$ to deter B's entry. Note that the date-1 output choice $q_{A1}=1$ is feasible but is generally suboptimal; it is optimal (i.e., $q^*_{A1}=1$) only when $K=\frac{1}{9}$. Thus we conclude that choosing q^*_{A1} to deter entry at date 1 is indeed the optimal strategy for firm A given that $\frac{1}{16}+\frac{1}{9}>K\geq\frac{1}{9}$.

To sum up, our solutions for parts (i)-(iii) are as follows.

- (i) For $K=\frac{1}{5}=\frac{25}{125}>\frac{25}{144}=\frac{1}{16}+\frac{1}{9}$, entry is blocked, and we have $q_{A1}=q_{A2}=\frac{1}{2}$.
- (ii) For $K=\frac{1}{9}+\frac{1}{25}$, which lies between $\frac{1}{9}$ and $\frac{1}{16}+\frac{1}{9}$, entry is deterred, and $q_{A1}=\frac{3}{5}$ and $q_{A2}=\frac{1}{2}$.
- (iii) For $K=\frac{1}{25}<\frac{1}{9}$, entry can only be accomodated, and hence $q_{A1}=\frac{1}{2}$ and $q_{A2}=\frac{1}{3}$.

This exercise explains why a monopolistic firm may not always produce the monopoly output stated in an economics textbook. Observationally firm A is a monopolistic firm at date 2, but this could be a consequence of its non-monopolistic output choice q^*_{A1}: if it insists on producing the monopoly output $\frac{1}{2}$, it may induce entry at date 1, which would destroy its monopolistic status at date 2. In part (ii), for example, the monopolistic firm must produce at $\frac{3}{5}>\frac{1}{2}$ in order to deter entry. In this sense, *even a monopolistic firm has potential competitors*, and the presence of potential competitors is enough to force the monopolistic firm to produce more, so that its output choice may get closer to the socially efficient output level. See the formal analysis in Dixit, A., 1980, The role of investment in entry deterrence, *Economic Journal*, 90, 95-106.

5. (**Signal Jamming and Cournot Competition**) Consider firms 1 and 2 that engage in Cournot competition at $t=1$ and $t=2$, facing random demand functions at both periods. The inverse demand function at $t=1$ is

$$\tilde{p}_1=\tilde{a}-q_1-q_2,$$

where $\tilde{a}$ is a positive random variable with $E[\tilde{a}] = 1$ and q_j is firm j's output level at $t = 1$. The inverse demand function at $t = 2$ is

$$\tilde{p}_2 = \tilde{b} - Q_1 - Q_2,$$

where $\tilde{b}$ is a positive random variable and Q_j is firm j's output level at $t = 2$. Each firm seeks to maximize the sum of expected profits over the two periods. That is, both firms are risk-neutral without time preferences.

The game proceeds as follows.

- At the beginning of $t = 1$, both firms must simultaneously make output choices q_1 and q_2 without seeing the realization of $\tilde{a}$.
- At the beginning of $t = 2$, after knowing q_j and the realization p_1 of $\tilde{p}_1$, firm j must choose Q_j. The two firms make output choices at the same time, without seeing the realization of either $\tilde{a}$ or $\tilde{b}$. At this time, firm j does not see q_i that was chosen by its rival, firm i.

(i) First assume that $\tilde{b}$ and $\tilde{a}$ are independently and identically distributed. Solve the equilibrium output choices $(q_1^*, q_2^*, Q_1^*, Q_2^*)$ in the unique SPNE.

(ii) Ignore part (i). Now assume instead that $\tilde{b} = \lambda\tilde{a}$, where $\lambda < 2$ is a constant known to both firms. Solve the unique symmetric SPNE.[19]

[19] Comparing part (i) to part (ii), we see that both firms make lower expected profits at date 1 in part (ii). This happens because in part (ii) firms cannot resist the temptation of expanding outputs as means of manipulating their rivals' beliefs about the realization of $\tilde{a}$. By secretly expanding its output q_i, firm i wants to make its rival j believe in a lower realization of $\tilde{a}$, which implies a lower demand (whose intercept is $\lambda\tilde{a}$) at date 2, and if firm i succeeds in making its rival believe in a lower date-2 demand, then it can benefit from choosing a higher date-2 output Q_i given that its rival will on average choose a lower output Q_j. In equilibrium this incentive is correctly recognized by its rival j, but the incentive to manipulate a rival's beliefs still changes the two firms' date-1 profits.

(iii) Do the two firms get higher date-1 expected profits in part (ii) or in part (i)? Why?

(iv) Suppose that $\lambda = 1$. Do the two firms get higher date-2 expected profits in part (ii) or in part (i)? Why?[20]

[20]**Hint**: Verify that $(q_1^*, q_2^*, Q_1^*, Q_2^*) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in part (i). For part (ii), let $(q^*, Q^*(p_1, q))$ denote the unique symmetric SPNE, where both firms choose q^* at $t = 1$, and both choose $Q^*(p_1, q)$ after choosing q at $t = 1$ and subsequently learning that the realization of $\tilde{p}_1$ is p_1. Then *in equilibrium*, $\tilde{p}_1 = \tilde{a} - 2q^*$, or $\tilde{a} = \tilde{p}_1 + 2q^*$. At the beginning of $t = 2$, given the realization p_1 of $\tilde{p}_1$ and its own output choice q_i at $t = 1$, and given that firm j does not deviate from its equilibrium strategy, firm i knows that $\tilde{a} = p_1 + q_i + q^*$. Moreover, firm i knows that that firm j would believe that $\tilde{a} = p_1 + 2q^*$ and seek to maximize

$$\max_Q [\lambda(p_1 + 2q^*) - Q^*(p_1, q^*) - Q]Q,$$

where note that firm j does not know firm i has chosen q_i rather than q^*. That is, firm i believes that firm j would choose the Q that satisfies

$$Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1, q^*)}{2},$$

which has to be $Q^*(p_1, q^*)$ also. Hence firm i believes that firm j would choose

$$Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.$$

Firm i, knowing that it has chosen q_i rather than q^* at $t = 1$, seeks to maximize the following date-2 profit:

$$\max_Q [\lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q]Q,$$

so that given (p_1, q_i), firm i's optimal date-2 output level is

$$Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1 + 2q^*)}{3}}{2},$$

which yields for firm i the following date-2 profit

$$\frac{1}{4}[\frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i]^2.$$

At $t = 1$, expecting firm j to choose q^*, firm i seeks to

$$\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4}E[(\frac{2\lambda \tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],$$

which is concave in q_i because $\lambda < 2$. Show that the optimal q_i must satisfy the first-order condition for this maximization problem; that is,

$$1 - q^* - 2q_i + \frac{\lambda}{6}(\frac{2\lambda E[\tilde{p}_1]}{3} + \frac{\lambda q^*}{3} + \lambda q_i) = 0,$$

or using $E[\tilde{p}_1] = 1 - q_i - q^*$, and $q_i = q^*$ in equilibrium, show that

$$q^* = \frac{1}{3} + \frac{\lambda^2}{27}.$$

Show that then $Q^*(p_1, q^*) = \frac{\lambda \tilde{a}}{3}$.

Solution. Consider part (i). Since $\tilde{b}$ and $\tilde{a}$ are independent, the two firms do not care about their date-2 decisions Q_1 and Q_2 when they engage in the date-1 Cournot competition. Being risk-neutral, given q_j, firm i seeks to

$$\max_{q_i} q_i(E[\tilde{a}] - q_i - q_j) = q_i(1 - q_i - q_j),$$

so that this game has the same equilibrium as the Cournot game presented in Example 1 of Lecture 1, Part I. That is, in equilibrium ,

$$q_1^* = q_2^* = \frac{1}{3}.$$

Similarly, at date 2, given Q_j, firm i seeks to

$$\max_{Q_i} Q_i(E[\tilde{b}] - Q_i - Q_j) = Q_i(1 - Q_i - Q_j),$$

so that this game also has the same equilibrium as the Cournot game presented in section 11 of Lecture 1, Part I. That is, in equilibrium ,

$$Q_1^* = Q_2^* = \frac{1}{3}.$$

This finishes part (i).

Now, for part (ii), let $(q^*, Q^*(p_1, q))$ denote the unique symmetric SPNE, where both firms choose q^* at $t = 1$, and both choose $Q^*(p_1, q)$ after choosing q at $t = 1$ and subsequently learning that the realization of $\tilde{p}_1$ is p_1. Then *in equilibrium*, $\tilde{p}_1 = \tilde{a} - 2q^*$, or $\tilde{a} = \tilde{p}_1 + 2q^*$. At the beginning of $t = 2$, given the realization p_1 of $\tilde{p}_1$ and its own output choice q_i at $t = 1$, and given that firm j does not deviate from its equilibrium strategy, firm i knows that $\tilde{a} = p_1 + q_i + q^*$. Moreover, firm i knows that that firm j would believe that $\tilde{a} = p_1 + 2q^*$ and seek to maximize

$$\max_{Q}[\lambda(p_1 + 2q^*) - Q^*(p_1, q^*) - Q]Q,$$

where note that firm j does not know firm i has chosen q_i rather than q^*. That is, firm i believes that firm j would choose the Q that satisfies

$$Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1, q^*)}{2},$$

which has to be $Q^*(p_1, q^*)$ also. Hence firm i believes that firm j would choose

$$Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.$$

Firm i, knowing that it has chosen q_i rather than q^* at $t = 1$, seeks to maximize the following date-2 profit:

$$\max_Q [\lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q]Q,$$

so that given (p_1, q_i), firm i's optimal date-2 output level is

$$Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1+2q^*)}{3}}{2},$$

which yields for firm i the following date-2 profit

$$\frac{1}{4}[\frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i]^2.$$

At $t = 1$, expecting firm j to choose q^*, firm i seeks to

$$\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4}E[(\frac{2\lambda\tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],$$

which is concave in q_i because $\lambda < 2$. It follows that the optimal q_i must satisfy the first-order condition for this maximization problem; that is,

$$1 - q^* - 2q_i + \frac{\lambda}{6}(\frac{2\lambda E[\tilde{p}_1]}{3} + \frac{\lambda q^*}{3} + \lambda q_i) = 0,$$

or using $E[\tilde{p}_1] = 1 - q_i - q^*$, and $q_i = q^*$ in equilibrium, we have

$$q^* = \frac{1}{3} + \frac{\lambda^2}{27}.$$

It follows that $Q^*(p_1, q^*) = \frac{\lambda\tilde{a}}{3}$.

Now, consider part (iii). Comparing part (i) to part (ii), we see that both firms make lower expected profits at date 1 in part (ii). This happens because in part (ii) firms cannot resist the temptation of expanding outputs as means of manipulating their rivals' beliefs about the realization of $\tilde{a}$. By secretly expanding its output q_i, firm i wants to make its rival j believe in a lower realization of $\tilde{a}$, which implies a lower demand (whose intercept is $\lambda\tilde{a}$) at date 2, and if firm i succeeds in making its rival believe in a lower date-2 demand, then it can benefit from choosing a higher date-2 output Q_i given that its rival will on average choose a lower output Q_j. In equilibrium this incentive is correctly recognized by its rival j, but the incentive to engage in signal-jamming still changes the two firms' date-1 profits. Both firms are worse off in part (ii), because of a lower product price resulting from output expansion ($q^* > \frac{1}{3}$).

Finally, consider part (iv). Note that in part (ii)

$$E[Q^*(p_1, q^*)] = \frac{\lambda E[\tilde{a}]}{3} = \frac{E[\tilde{a}]}{3} = \frac{1}{3},$$

where recall that $\frac{1}{3}$ is the two firms' date-2 output choice in part (i). Signal-jamming does not fool any player in equilibrium (that is, both firms can infer correctly the realized $\tilde{a}$ from the realized date-1 price), but in part (ii), since $\tilde{a} = \tilde{b}$, the two firms' common date-2 output choice depends on the realization of $\tilde{a}$. This is in sharp contrast with part (i), where $\tilde{a}$ and $\tilde{b}$ are independent, so that the firms' date-2 output choices can never depend on the realized $\tilde{a}$. Now, since in part (ii) each firm's date-2 expected profit is a *convex* function of its date-2 output $Q^*(p_1, q^*)$, and since $Q^*(p_1, q^*)$ is a mean-preserving spread of the firms' date-2 output choice (which is $\frac{1}{3}$) in part (i), the two firms actually obtain higher expected date-2 profits in part (ii) than in part (i). Indeed, each firm gets the following expected date-2 profit in part (i),

$$\frac{1}{3}(E[\tilde{b}] - \frac{1}{3} - \frac{1}{3}) = \frac{1}{9},$$

but in part (ii) its expected date-2 profit becomes

$$E[\frac{\tilde{a}}{3}(\tilde{a} - \frac{\tilde{a}}{3} - \frac{\tilde{a}}{3})] = \frac{E[\tilde{a}^2]}{9} > \frac{(E[\tilde{a}])^2}{9} = \frac{1}{9},$$

where the inequality follows from Jensen's inequality and the fact that the function $h(z) = z^2$ is strictly convex. Thus the two firms make higher expected date-2 profits in part (ii) than in part (i).