Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 3, due 11/14, 2021.

1. Re-consider Example 9 on page 40 of Lecture 1, Part II. Here we assume that everyone knows that Mr. A *will* sell the equity (or simply, sell the firm) at date 1, and that

$$c = \frac{1}{2},\ \beta = \frac{1}{3}.$$

In the unique NE, Mr. A's equilibrium payoff is equal to ______, and the date-1 transaction price for Mr. A's equity is ______ when $\pi_1 = 0$ and ______ when $\pi_1 = 1$. Mr. A's equity is thus worth ______ *before* the public investors observe π_1.

Solution. According to the discussions for Example 9, there does not exist a pure-strategy equilibrium where Mr. A chooses $e = 0$ with probability one or $e = 1$ with probability one. Thus assume that Mr. A may choose $e = 1$ with probability α, where $0 < \alpha < 1$.

Thus the public investors may observe $\pi_1 = 1$ with probability $\alpha(1-\beta)$ or $\pi_1 = 0$ with probability $1 - \alpha(1-\beta)$, and the date-1 transaction price for Mr. A's equity must be such that

$$E[\pi_1 + \pi_2|\pi_1] = \begin{cases} 1 + 0, & \text{if } \pi_1 = 1; \\ \frac{(1-\alpha)\cdot(0+0)}{1-\alpha+\alpha\beta} + \frac{\alpha\beta\cdot(0+5)}{1-\alpha+\alpha\beta}, & \text{if } \pi_1 = 0. \end{cases}$$

Thus by choosing $e = 0$ Mr. A would obtain the payoff of $\frac{5\alpha\beta}{1-\alpha+\alpha\beta}$, and by choosing $e = 1$ instead Mr. A would obtain

$$\beta \cdot \frac{5\alpha\beta}{1-\alpha+\alpha\beta} + (1-\beta)\cdot 1 - c.$$

For Mr. A to feel indifferent about choosing $e = 0$ or $e = 1$, we must have

$$\alpha = \frac{1-\beta-c}{(1-\beta)(1+4\beta-c)} = \frac{3}{22}.$$

It follows that

$$E[\pi_1 + \pi_2 | \pi_1] = \begin{cases} 1, & \text{if } \pi_1 = 1; \\ \\ \frac{1}{4}, & \text{if } \pi_1 = 0. \end{cases}$$

Since the public investors may observe $\pi_1 = 1$ with probability

$$\alpha(1 - \beta) = \frac{1}{11},$$

Mr. A's equity is worth

$$\frac{1}{11} \cdot 1 + \frac{10}{11} \cdot \frac{1}{4} = \frac{7}{22}$$

before the public investors observe π_1.

2. Consider a linear city where consumers uniformly reside on the unit interval $[0, 1]$, and two firms A and B are located respectively at 0 and at 1 (the left and the right endpoints of the unit interval). Firm j produces a single product j costlessly, $j \in \{A, B\}$, and each consumer may either buy 1 unit of product A, or buy 1 unit of product B, or buy nothing. At each point $t \in [0, 1]$, there exists exactly one consumer, whom we shall refer to as consumer t. Consumer t would obtain a surplus $v - p_A - tc$ if he buys from firm A, and a surplus $v - p_B - (1 - t)c$ if he buys from firm B, where $v > 0$ is the gross utility that consumer t derives from consuming 1 unit of product A or product B, $c > 0$ represents the consumer's round-trip transportation cost per unit distance, and p_j is the retail price chosen by firm j. A consumer gets zero surplus if he buys nothing.

   The game proceeds as follows. The two firms must simultaneously announce retail prices, and upon (costlessly) learning about the two retail prices, consumers must simultaneously decide whether to visit one retailer and make a purchase, or to stay home and buy nothing. We shall concentrate on the pure-strategy Nash equilibria for this game.

   (i) Suppose that (p_A^*, p_B^*) is a pure-strategy equilibrium. Associated with this equilibrium, define for $j = A, B$,

   $T_j \equiv \{t \in [0, 1] : \text{Buying product } j \text{ is an equilibrium best response for consumer } t\}.$

(i-1) Show that for $j \in \{A, B\}$, either $T_j = \emptyset$ or T_j is a closed interval, where in the latter case, for some $t_a, t_b \in [0, 1]$, $T_A = [0, t_a]$ and $T_B = [t_b, 1]$.

(i-2) Show that either $T_A \bigcap T_B = \emptyset$ or there exists a unique $t^* \in [0, 1]$ such that $T_A \bigcap T_B = \{t^*\}$.[1]

(i-3) Show that $T_A \neq \emptyset \neq T_B$.

(i-4) Show that if $T_A \bigcap T_B = \emptyset$ in a pure-strategy NE, then $t_a < t_b$, and consumer t_a gets zero surplus from buying product A and consumer t_b gets zero surplus from buying product B.

(i-5) Suppose that $T_A \bigcap T_B = \emptyset$ in the pure-strategy NE (p_A^*, p_B^*). Show that, given any $\epsilon \in \Re$ with sufficiently small $|\epsilon|$, if given p_B^* firm A deviates and prices at $p_A^* + \epsilon$ instead, then firm A's payoff would become

$$L(\epsilon) \equiv (p_A^* + \epsilon)\frac{1}{c}(v - p_A^* - \epsilon),$$

so that in equilibrium we must have

$$L'(0) = 0,$$

proving that $p_A^* = \frac{v}{2}$. Likewise, show that $p_B^* = \frac{v}{2}$. Show that $(p_A^*, p_B^*) = (\frac{v}{2}, \frac{v}{2})$ is indeed a pure-strategy NE if and only if $0 < v < c$.

(i-6) Suppose that in the pure-strategy NE (p_A^*, p_B^*), $T_A \bigcap T_B = \{t^*\}$, where $0 \leq t^* \leq 1$. Show that this implies that consumer t^* can obtain a non-negative surplus from buying either product, which in turn implies that

$$c \geq \max(p_A^* - p_B^*, p_B^* - p_A^*).$$

Show that given p_B^*, firm A's payoff as a function of p_A is

$$p_A(\frac{p_B^* - p_A}{2c} + \frac{1}{2});$$

and given p_A^*, firm B's payoff as a function of p_B is

$$p_B(\frac{p_A^* - p_B}{2c} + \frac{1}{2}).$$

[1]Note that in the former case, some consumers are left unserved in equilibrium.

Show that in such an equilibrium, necessarily $(p_A^*, p_B^*) = (c, c)$, and this pure-strategy NE exists if and only if $v \geq \frac{3c}{2}$.

(ii) Now, on top of the above extensive game, let us add an earlier stage where firms must decide simultaneously whether to provide a presale service. Providing such a service will cost k_j to firm j, where we assume that $k_B = k > k_A = 0$, and with firm j providing this service, a consumer's gross utility from buying product j rises from v to $v + s$. From now on, we shall assume that the unit transportation cost is

$$c = \frac{1}{2},$$

and that v is very large (so that $T_A \bigcap T_B$ is always non-empty).

Suppose further that

$$0 < k < \frac{s}{3} - \frac{s^2}{9}, \quad s < \frac{3}{2}.$$

Does firm B provide the presale service in the unique SPNE? Does the provision of presale services enhance the profits of the two firms?

Solution. Consider (i-1). We shall prove the assertion for the case $j = A$, and the case $j = B$ can be analogously proved. If T_A is non-empty, then either $T_A = \{0\}$, which is a closed interval, or T_A contains $t > 0$, and in the latter case, we claim that for all non-negative $t' < t$, $t' \in T_A$ also. Indeed, note that $t \in T_A$ if and only if

$$\text{(IR)} \quad v - p_A^* - tc \geq 0;$$

and

$$\text{(IC)} \quad v - p_A^* - tc \geq v - p_B^* - (1 - t)c.$$

If $t > 0$ satisfies both of these inequalities, then so does any t' with $0 \leq t' < t$. Hence T_A must be an interval taking the form of $[0, t_a)$ or $[0, t_a]$. Note that the two functions

$$f(t) \equiv v - p_A^* - tc$$

and

$$g(t) \equiv [v - p_A^* - tc] - [v - p_B^* - (1-t)c]$$

are both continuous in t, and that $t \in T_A$ if and only if $f(t) \geq 0$ and $g(t) \geq 0$. This implies that T_A must be a closed set: let $\{t_n\}$ be a sequence in T_A such that $\lim_{n\to\infty} t_n = t_a$, then we have

$$f(t_a) = f(\lim t_n) = \lim f(t_n) \geq 0, \quad g(t_a) = g(\lim t_n) = \lim g(t_n) \geq 0,$$

proving that $t_a \in T_A$ also. Thus T_A must take the form of $[0, t_a]$.

Consider (i-2). If $T_A \bigcap T_B$ contains t and t' with $t < t'$, then by (IC) we have

$$v - p_A^* - tc = v - p_B^* - (1-t)c$$

and

$$v - p_A^* - t'c = v - p_B^* - (1-t')c,$$

which yields a contradiction.

Consider (i-3). We shall prove the assertion for the case $j = A$, and the case $j = B$ can be analogously proved. Suppose that $T_A = \emptyset$. Note that this can happen only if either

$$(\text{IC}^*)\ p_A^* + tc \geq p_B^* + (1-t)c, \quad \forall t \in [0,1],$$

or

$$(\text{IR}^*)\ p_A^* + tc > v, \quad \forall t \in [0,1].$$

By (IC*) and the fact that $p_B^* \geq 0$, we have

$$p_A^* \geq (1-2t)c,\ \forall t \in [0,1],$$

implying that

$$p_A^* \geq c.$$

By (IR*), we have

$$p_A^* \geq v - tc,\ \forall t \in [0,1],$$

implying that

$$p_A^* \geq v.$$

Note that firm A's equilibrium payoff is zero, and yet given p_B^*, firm A can deviate and price at, for example, $p' = \frac{1}{2}\min(v, c)$, and obtain a strictly positive payoff. Indeed, when firm A announces $p_A = p'$, even if $p_B^* = 0$, consumer t would strictly prefer buying product A to buying product B if

$$t \leq \max(\frac{v}{2c}, \frac{v}{c} - \frac{1}{2}, \frac{1}{2} - \frac{v}{4c}, \frac{1}{4}),$$

implying for firm A a deviation payoff of at least $\frac{p'}{4} > 0$, which contradicts the assumption that p_A^* is firm A's equilibrium best response against p_B^*. Hence in a pure-strategy NE, T_A must be non-empty.

Consider (i-4). In a pure-strategy NE where $T_A \bigcap T_B$ is empty, apparently we must have $t_a < t_b$, which follows directly from (i-1). Now, suppose that consumer t_a gets a strictly positive equilibrium payoff from buying product A; i.e.,

$$v - p_A^* - t_a c > 0.$$

Then for $e > 0$ such that

$$e < t_b - t_a, \quad e < \frac{1}{c}(v - p_A^*) - t_a,$$

obviously consumer $t_a + e$ should also be an element of T_A, which contradicts the definition of t_a! The same argument applies to consumer t_b.

Consider (i-5). By (i-4), we know that for $\epsilon \in \Re$ such that $|\epsilon|$ is sufficiently small, we have $t_a + \frac{\epsilon}{c} < t_b$, so that consumer $t_a + \frac{\epsilon}{c}$ would get a negative surplus if he accepts p_B^* and buys product B. Thus consumer $t_a + \frac{\epsilon}{c}$ would either purchase product A or buy nothing. By changing its price from p_A^* to $p_A^* - \epsilon$, therefore, firm A's sales volume would change from t_a to $t_a + \frac{\epsilon}{c}$, so that its payoff would then change from its equilibrium payoff $L(0)$ to $L(-\epsilon)$. Since $\epsilon = 0$ is an interior optimum for firm A's problem of maximizing L in a small neighborhood around $\epsilon = 0$, we must have $L'(0) = 0$ (the first-order condition), which gives rise to $p_A^* = \frac{v}{2}$. The same analysis can be conducted on the part of firm B. Hence, necessarily, $(p_A^*, p_B^*) = (\frac{v}{2}, \frac{v}{2})$ in such an equilibrium. For such an NE to actually prevail, we must also require that $t_a = \frac{v}{2c} < t_b = 1 - \frac{v}{2c}$, or equivalently that $c > v$.

Consider (i-6). Note that buying either product is an equilibrium best response for consumer t^*, as $t^* \in T_A \bigcap T_B$, and hence

$$v - p_A^* - t^* c = v - p_B^* - (1 - t^*)c,$$

implying that

$$t^* = \frac{p_B^* - p_A^*}{2c} + \frac{1}{2},$$

and for t^* to lie in $[0, 1]$, we must have

$$c \geq \max(p_A^* - p_B^*, p_B^* - p_A^*).$$

Given p_B^*, firm A's payoff, by changing p_A slightly from p_A^*, would be

$$p_A(\frac{p_B^* - p_A}{2c} + \frac{1}{2}),$$

so that its best response p_A^* must satisfy

$$p_A^* = \frac{p_B^* + c}{2}.$$

The same analysis on the part of firm B gives rise to

$$p_B^* = \frac{p_A^* + c}{2}.$$

Thus, necessarily $(p_A^*, p_B^*) = (c, c)$, implying that $t^* = \frac{1}{2}$. It is easy to verify that, indeed, we have

$$c \geq \max(p_A^* - p_B^*, p_B^* - p_A^*).$$

For consumer t^* to obtain a non-negative surplus, we must have $v \geq \frac{3c}{2}$.

Finally, consider part (ii). By backward induction, we must consider 4 subgames when analyzing the two firms' price competition (as in part (i)): (1) the subgame where both firms have spent on service provision; (2) the subgame where neither has chosen to provide the presale service; (3) the subgame where only firm A has spent on service provision; and (4) the subgame where only firm B has spent on service provision.

Now we summarize the subgame equilibria.

- In the subgame where both firms have spent for service provision, $p_A = p_B = c = \frac{1}{2}$, and the firms' equilibrium payoffs are $\Pi_A = \frac{1}{4}$ and $\Pi_B = \frac{1}{4} - k$.
- In the subgame where both firms have chosen not to spend for service provision, $p_A = p_B = c = \frac{1}{2}$, and the firms' equilibrium payoffs are $\Pi_A = \frac{1}{4}$ and $\Pi_B = \frac{1}{4}$.
- In the subgame where only firm A has chosen to spend for service provision, $p_A = \frac{1}{2} + \frac{s}{3}$, $p_B = \frac{1}{2} - \frac{s}{3}$, and the firms' equilibrium payoffs are $\Pi_A = [\frac{1}{2} + \frac{s}{3}]^2$ and $\Pi_B = [\frac{1}{2} - \frac{s}{3}]^2$.
- In the subgame where only firm B has chosen to spend for service provision, $p_A = \frac{1}{2} - \frac{s}{3}$, $p_B = \frac{1}{2} + \frac{s}{3}$, and the firms' equilibrium payoffs are $\Pi_A = [\frac{1}{2} - \frac{s}{3}]^2$ and $\Pi_B = [\frac{1}{2} + \frac{s}{3}]^2 - k$.

By assumption, we have $k < \frac{s}{3} - \frac{s^2}{9}$, and hence in equilibrium even firm B chooses to spend for service provision. This equilibrium is Pareto inefficient for the firms (but not from the perspective of the entire society, because consumers benefit from service provision obviously), since the two firms would be better off if they could coordinate and commit to not spending on service provision. What happens here is that when v is very large, in the symmetric pricing equilibrium the firms' prices depend only on c but not on v, and hence raising v by providing a presale service is totally wasteful from the firms' perspective. In equilibrium, however, out of the fear that it might lose too big a market share to firm A if it did not provide the service, firm B chooses to spend on service provision also.

3. Re-consider the preceding problem, and let $C(\tau)$ denote a consumer's *total* round-trip transportation costs that he must incur in order to visit a seller which is τ distance away from him. Note that $C(\tau) = c\tau$ in the preceding problem. Here, we shall assume that $C(\tau) = s\tau^2$, with the constant $s > 0$.

   Moreover, we assume that the two firms must compete in two stages:

   - In stage 1, the firms simultaneously determine their locations in the unit interval $[0, 1]$, where we let a and $1 - b$ denote A's and B's locations, with $0 \leq a, 1 - b \leq 1$.
   - In stage 2, given (a, b), the firms simultaneously determine their prices p_A and p_B.

   We shall derive the SPNEs for this *multi-stage game with observable actions.* We shall assume that v is sufficiently large so that all consumers located in $[0, 1]$ will be served in equilibrium.

   (i) Suppose that the government would choose (a, b) to minimize the sum of all consumers' transportation costs (and give consumers the product for free).[2] Show that the government would choose $a = b = \frac{1}{4}$.

   (ii) Now, return to the original setting where the two firms would make their own location and pricing decisions.

   (ii-1) Show that given (a, b) such that $a \leq 1 - b$, the two firms would announce the following prices:

$$p_A(a, b) = s(1 - a - b)(1 + \frac{a - b}{3}),\ p_B(a, b) = s(1 - a - b)(1 + \frac{b - a}{3}).$$

   (ii-2) Now, move backwards to consider the stage-1 equilibrium. Show that given b, firm A's payoff is decreasing in a for $a \in [0, 1-b]$; and that

[2] We have assumed that the product can be produced without costs. Thus the government is choosing (a, b) to maximize the social benefit.

given a, firm B's equilibrium payoff is decreasing in b for $1-b \in [a,1]$. Conclude that in equilibrium, one firm is located at the left endpoint of $[0,1]$, and the other at the right endpoint of $[0,1]$.

Solution. Consider part (i). Define $\beta \equiv 1-b$. The government seeks to

$$\min_{0\leq a\leq \beta\equiv 1-b\leq 1} f(a,\beta)\equiv \int_0^{\frac{a+\beta}{2}} s(t-a)^2dt+\int_{\frac{a+\beta}{2}}^1 s(t-\beta)^2dt.$$

By Leibniz rule, we have

$$\begin{aligned}\frac{\partial f}{\partial a} &= \frac{1}{2}\cdot s(\frac{a+\beta}{2}-a)^2+\int_{\frac{a+\beta}{2}}^1 -2s(t-\beta)dt-\frac{1}{2}\cdot s(\frac{a+\beta}{2}-\beta)^2\\ &= \frac{s}{4}(a+\beta)(3a-\beta),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial \beta} &= \frac{1}{2}\cdot s(\frac{a+\beta}{2}-a)^2+\int_0^{\frac{a+\beta}{2}} -2s(t-a)dt-\frac{1}{2}\cdot s(\frac{a+\beta}{2}-\beta)^2\\ &= \frac{s}{4}(2-a-\beta)(3\beta-a-2),\end{aligned}$$

so that

$$\frac{\partial^2 f}{(\partial a)^2}=\frac{s(3a+\beta)}{2},\quad \frac{\partial^2 f}{(\partial \beta)^2}=\frac{s(4-a-3\beta)}{2},\quad \frac{\partial^2 f}{(\partial a)(\partial \beta}=\frac{s(a-\beta)}{2}.$$

It is easy to see that for $a,\beta\in[0,1]$, we have

$$\frac{\partial^2 f}{(\partial a)^2}=\frac{s(3a+\beta)}{2}\geq 0$$

and

$$\frac{\partial^2 f}{(\partial \beta)^2}=\frac{s(4-a-3\beta)}{2}\geq 0.$$

Note that

$$\frac{\partial^2 f}{(\partial a)^2}\cdot\frac{\partial^2 f}{(\partial \beta)^2}-[\frac{\partial^2 f}{(\partial a)(\partial \beta}]^2=h(a;\beta)\equiv s^2[(3a+\beta)-(a+\beta)^2].$$

We claim that h is non-negative for all $a, \beta \in [0,1]$.

Given β, h as a function of a is concave and attains a global maximum at $a_{max} = \frac{3-2\beta}{2}$, which given that $\beta \leq 1$, is non-negative. Moreover, $a_{max} \leq 1$ if and only if $\beta \geq \frac{1}{2}$. Thus when $\beta \leq \frac{1}{2}$, h is increasing in a for all $a \in [0,1]$, implying that

$$h(a,\beta) \geq h(0;\beta) = s^2(\beta - \beta^2) \geq 0, \quad \forall a \in [0,1],\ \forall \beta \in [0, \frac{1}{2}].$$

What if $\beta \geq \frac{1}{2}$? In this case, we have

$$h(0;\beta) = s^2(\beta - \beta^2) \geq 0, \quad h(1;\beta) = s^2(2 - \beta - \beta^2) \geq 0,$$

and since given β, h is concave in a, we have by Jensen's inequality

$$h(a;\beta) = h(a \cdot 1 + (1-a) \cdot 0; \beta) \geq ah(1;\beta) + (1-a)h(0;\beta) \geq 0,\ \forall a \in [0,1],\ \forall \beta \in [\frac{1}{2}, 1].$$

At optimum, either $a = \beta$ or $a \neq \beta$, and in the latter case, we assume without loss of generality that $a < \beta$. It is easy to verify that $a = \beta$ is suboptimal: the total transportation costs can be further reduced by moving a away from the given location of β because some consumers would then visit a rather than β. Now, observe that for $0 \leq a < \beta \leq 1$, we actually have

$$\frac{\partial^2 f}{(\partial a)^2} > 0,\ \frac{\partial^2 f}{(\partial a)^2} \cdot \frac{\partial^2 f}{(\partial \beta)^2} - [\frac{\partial^2 f}{(\partial a)(\partial \beta}]^2 = h(a;\beta) > 0,$$

so that the Hessian of f is positive definite[3] (and hence $f(a,\beta)$ is a strictly convex function of (a,β)) in the region $J \equiv \{(a,\beta) : 0 \leq a <$

[3] A real-valued symmetric matrix

$$\mathbf{A}_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$\beta \leq 1\}$. It follows that (a, β) attains the minimum of f if and only if $(a, \beta) \in J$ and (a, β) satisfies the following first-order condition:

$$\begin{cases} \frac{\partial f}{\partial a} = 0 \Rightarrow (2 - a - \beta)(3\beta - 2 - a) = 0; \\ \\ \frac{\partial f}{\partial \beta} = 0 \Rightarrow (a + \beta)(3a - \beta) = 0. \end{cases}$$

We deduce from $\frac{\partial f}{\partial \beta} = 0$ that either $(a+\beta) = 0$ or $(3a - \beta) = 0$. In the former case, we have from $\frac{\partial f}{\partial a} = 0$ that

$$3\beta - 2 - a = 0 \Rightarrow 3\beta = 2 + a = 2 - \beta \Rightarrow a = -\frac{1}{2} < 0,$$

a contradiction. Hence we conclude from $\frac{\partial f}{\partial \beta} = 0$ that

$$\beta = 3a,$$

so that

$$\frac{\partial f}{\partial a} = 0 \Rightarrow a = \frac{1}{2} \text{ or } \ a = \frac{1}{4},$$

and since $a = \frac{1}{2}$ would imply $\beta = \frac{3}{2}$, we conclude that

$$a = \frac{1}{4}, \ \beta = 1 - b = \frac{3}{4}, \Rightarrow b = \frac{1}{4}.$$

Note that this solution $(a, \beta) = (\frac{1}{4}, \frac{3}{4})$ is indeed contained in J.

Now, consider part (ii-1). Suppose that the two firms are ready to compete in price given (a, b), where $a \leq 1 - b$. (This is an innocuous assumption: we can always rename the two firms by calling the firm located on the left "firm A.") For sufficiently large v, Problem 2 shows

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is positive definite, if and only if the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix}$$

is positive for all $k \in \{1, 2, \cdots, n\}$.

that there must exist $t^* \in [a, \beta]$ that feels indifferent about buying from A or from B, where at t^*, we have

$$v - p_A - s(t^* - a)^2 = v - p_B - s(t^* - \beta)^2.$$

Given p_B, a, and $\beta = 1 - b$, A seeks to

$$\max_{t_A, p_A} p_A t_A$$

subject to

$$p_A = p_B - s(t_A - a)^2 + s(t_A - \beta)^2,$$

so that A's best response is such that

$$t_A = \frac{p_B + s(\beta^2 - a^2)}{4s(\beta - a)}.$$

Similarly, given p_A, a, and $\beta = 1 - b$, B seeks to

$$\max_{t_B, p_B} p_B(1 - t_B)$$

subject to

$$p_B = p_A + s(t_B - a)^2 - s(t_B - \beta)^2,$$

so that B's best response is such that

$$t_B = \frac{p_A + s(a^2 - \beta^2) + 2s(a - \beta)}{4s(a - \beta)}.$$

In equilibrium, we have $t_A = t_B = t^*$, so that

$$p_A + p_B = 2s(\beta - a),$$

which, together with

$$v - p_A - s(t^* - a)^2 = v - p_B - s(t^* - \beta)^2,$$

implies that

$$p_A = \frac{s}{2}[2(\beta - a) + \beta^2 - a^2 + 2(a - \beta)t^*],$$

but since we also have

$$t^* = t_B = \frac{p_A + s(a^2 - \beta^2) + 2s(a - \beta)}{4s(a - \beta)},$$

we conclude that

$$t^* = \frac{2 + a + \beta}{6},$$

implying, by

$$t^* = t_A = \frac{p_B + s(\beta^2 - a^2)}{4s(\beta - a)},$$

that

$$p_B = \frac{s(\beta - a)(4 - a - \beta)}{3} = s(1 - a - b)(1 + \frac{b - a}{3}).$$

Recall that

$$p_A + p_B = 2s(\beta - a),$$

and hence we have

$$p_A = 2s(\beta - a) - p_B = \frac{s(\beta - a)(2 + a + \beta)}{3} = s(1 - a - b)(1 + \frac{a - b}{3}).$$

Finally, consider part (ii-2). Given β, if A wish to be located on the right of B, then A seeks to

$$\max_a \pi_A(a, \beta) \equiv t^* p_A = \frac{s(\beta - a)(2 + a + \beta)^2}{18}.$$

Given β, π_A is decreasing in a, because

$$\frac{\partial \pi_A}{\partial a} = \frac{s}{18}(2 + a + \beta)(\beta - 3a - 2) < 0.$$

Thus if A wishes to be located on the left of B, then A's optimal choice is $a = 0$.

Similarly, one can verify that $\pi_B \equiv (1 - t^*)p_B$, given a, is increasing in β, so that if B wishes to be located on the right of A, then B's optimal choice is $\beta = 1$. It follows that in the unique SPNE, one firm chooses $a = 0$ and the other chooses $b = 0$ (or $\beta = 1$). In equilibrium, we have

$$t^* = \frac{1}{2},\ p_A = s = p_B.$$

Remark. We can interpret the Hotelling street $[0, 1]$ as a continuum of feasible product attributes. For example, 0 can stand for black, and 1 for white, and different points $t \in (0, 1)$ stand for different shades of gray. A consumer located at t regards a product with attribute equal to t as his "ideal point." The consumer's willingness to pay for a product with attribute differing from his ideal point by a distance of τ is equal to $v - C(\tau)$. The two firms must each produce one product, and then compete in price. Interpreted this way, part (ii) shows that the two firms would like to maintain *maximum differentiation* when engaging in new product design (NPD). According to part (i), there is *too much* differentiation in this equilibrium.

4. Now, re-consider the Hotelling model described in Problem 2, but we assume that there is one single buyer *traveling* in the Hotelling main street. The buyer may be traveling from left to right (denoted L-R) or from right to left (denoted (R-L), and these two events are equally likely. We assume that the buyer's intended destination is somewhere on the right of B, if he is traveling in the L-R direction; and that destination is somewhere on the left of A, if he is traveling in the R-L direction. Consumption need may arise at any random location during the buyer's journey. We refer to the same buyer as "buyer t" when his consumption need arises at his real-time location $t \in [0, 1]$. When his consumption need arises, the buyer uses his smart phone and an app to search for nearby sellers, and the app immediately sends an alerting message to A and B, revealing to the two firms the buyer's *real-time location* t, but saying nothing about the buyer's *traveling direction*. Based on this information, the two firms then announce p_A and p_B by simultaneously sending messages to the buyer's smart phone.

   As in Problem 2, the buyer has a unit demand and gross consumption utility v, but the buyer's transportation costs are now different. If the buyer is traveling to the left, then he would incur no additional transportation costs if choosing to visit A; he would incur an additional transportation cost $c(1 - t)$ if choosing to visit B. Similarly, if the buyer is traveling to the right, then he would incur no additional transportation costs if choosing to visit B; he would incur an additional transportation cost ct if choosing to visit A. In addition to transportation costs, the buyer is facing another cost: once the consumption need arises, the buyer becomes *very impatient*, in the sense that the buyer would incur a dis-utility $b\tau$ if he needs to first travel to a seller's outlet, which is τ distance away from his real-time location, before he can start consuming the product at the seller's outlet.

To summarize, the buyer's net utility, denoted by $U_j^d(t)$, depends on his real-time location t, traveling direction $d \in \{\text{L-R, R-L}\}$, and the firm $j \in \{A, B\}$ that serves him, where

$$U_A^{L-R} = v - tc - bt - p_A,\ U_B^{L-R} = v - b(1-t) - p_B,$$
$$U_A^{R-L} = v - bt - p_A,\ U_B^{R-L} = v - (1-t)c - b(1-t) - p_B.$$

Note that, for example, visiting B would not incur additional transportation costs if the buyer is travelling L-R, but his eagerness to get to firm B still generates a (one-way) dis-utility $b(1-t)$. If this buyer chooses to visit retailer A instead, then in addition to the eagerness dis-utility bt, he must also incur an additional round-trip transportation cost ct.

From now on, assume that

$$v = 5,\ b = 2,\ c = 6.$$

(i) Suppose that $t = \frac{1}{2}$. Find a pure- or mixed-strategy Nash equilibrium $(F_A(\cdot), F_B(\cdot))$, where for $j \in \{A, B\}$, $F_j(\cdot)$, as in Example 6 of Lecture 1, Part II, denotes the distribution function for $\tilde{p}_j$.

(ii) Suppose that $t = 0$. Find a pure- or mixed-strategy Nash equilibrium $(F_A(\cdot), F_B(\cdot))$.

Solution. Consider part (i). The following table summarizes buyer $t = \frac{1}{2}$'s net valuation (which equals his gross consumption utility minus the transportation and eagerness costs):

seller/d	R-L	L-R
A	4	1
B	1	4

Note that in the absence of B, A would rather price at 4 and serve only the buyer traveling in direction $d = R - L$: to be able to serve the buyer traveling in direction $d = L - R$, A must price at or below 1, which is a bad idea! The same argument shows that B would price at 4 to serve only the buyer traveling in direction $d = L - R$. It follows that in equilibrium, each firm prices at 4 and serves only the buyer traveling towards its own location.

Consider part (ii). The following table summarizes buyer $t = 0$'s net valuation (which equals his gross consumption utility minus the transportation and eagerness costs):

seller/d	R-L	L-R
A	5	5
B	-3	3

The buyer traveling in direction $d = R - L$ is essentially a loyal customer for A. Thus B's only hope is to serve the buyer traveling in direction $d = L - R$. Without operating costs, B will continue fighting for this buyer until A would price at or below 2; but from A's perspective, pricing at or below $\frac{5}{2}$ and winning the buyer in both traveling directions is worse than pricing at 5 and serving only the buyer traveling in direction $d = R - L$.

It is now easy to see that no pure-strategy equilibrium can exist: in a pure-strategy equilibrium either $p_B > \frac{1}{2}$ or $p_B \leq \frac{1}{2}$. In the former case, A's best response is to price at $p_B + 2 - \epsilon > \frac{5}{2}$ and to serve for sure the buyer traveling in both directions, leaving B with zero equilibrium profit, but B can replace p_B by $p'_B = \frac{1}{2} - \epsilon$ and obtain a positive deviation payoff, a contradiction. In the latter case, A's best response would be to price at 5, but then B would deviate and price slightly below 3 rather than price below $\frac{1}{2}$!

Thus we shall look for a mixed-strategy equilibrium. Note that the buyer traveling in direction $d = R - L$ is A's loyal customer, and the buyer traveling in the other direction is a switcher; firm B does not have a loyal customer. Following Example 6 in Lecture 1, Part II, we conjecture that pricing at 5 is one of A's equilibrium best responses, and $\underline{p}_A$, the greatest lower bound of the support for $\tilde{p}_A$, must also be a best response, which would allow A to win for sure the buyer traveling in both directions. Thus we are guessing that

$$\pi_A = \frac{1}{2} \cdot 5 = \underline{p}_A,$$

where $\frac{1}{2}$ is the probability that the buyer happens to be traveling in direction $d = R - L$.

In competing for the buyer traveling in direction $d = L - R$, B must price below p_A by at least $(5 - 3) = 2$. Thus we conjecture that $\underline{p}_B = \underline{p}_A - 2 = \frac{1}{2}$, and pricing at $\underline{p}_B$ would allow B to win for sure the buyer traveling in direction $d = L - R$, so that $\pi_B = \frac{1}{2} \cdot \underline{p}_B$, where $\frac{1}{2}$ is the probability that the buyer happens to be traveling in direction $d = L - R$. Thus we are also guessing that

$$\pi_B = \frac{1}{2} \cdot \underline{p}_B = \frac{1}{2} \cdot (\frac{5}{2} - 2) = \frac{1}{4}.$$

It follows that

$$\pi_B = \frac{1}{2} \cdot \{x[1 - F_A(x+2)]\},\ \forall x \in [\frac{1}{2}, 3) \Rightarrow F_A(y) = 1 - \frac{2\pi_B}{y-2},\ \forall y \in [\frac{5}{2}, 5),$$

and

$$\pi_A = z\{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot [1 - F_B(z-2)]\},\ \forall z \in [\frac{5}{2}, 5) \Rightarrow F_B(y) = 2 - \frac{2\pi_A}{y+2},\ \forall y \in [\frac{1}{2}, 3).$$

To sum up, we have obtained a mixed-strategy equilibrium where

$$F_A(y) = \begin{cases} 0, & y < \frac{5}{2}; \\ 1 - \frac{1}{2y-4}, & y \in [\frac{5}{2}, 5); \\ 1, & y \geq 5, \end{cases}$$

and

$$F_B(y) = \begin{cases} 0, & y < \frac{1}{2}; \\ 2 - \frac{5}{y+2}, & y \in [\frac{1}{2}, 3); \\ 1, & y \geq 3. \end{cases}$$

It is easy to verify that

$$\lim_{y \uparrow 3} F_B(y) = 1,$$

but pricing at 3 is *not* a pure-strategy best response for B. On the other hand, we have

$$\lim_{y \uparrow 5} F_A(y) = \frac{5}{6} \Rightarrow \Delta F_A(5) = \frac{1}{6},$$

confirming our conjecture that pricing at 5 is indeed one pure-strategy best response for A.

Remark. Note that in part (ii), A does not benefit from serving the buyer traveling in direction $d = L - R$: A would obtain the same payoff if A would commit not to price below 5, which would make B better off. In other words, the mixed-strategy equilibrium is not Pareto optimal for the two firms.