Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 4

1. An acquirer (Mr. A) is attempting to take over a target firm, T. Firm T is all-equity financed, and it has three shareholders, each holding 1 share of the firm's equity (so that the firm has 3 shares of common stock outstanding). The current value of firm T is zero (as a normalization). If Mr. A is able to obtain 2 or more than 2 shares, then the takeover will succeed, and the value of firm T will become 18 in that event.

   Suppose that Mr. A has announced a share price $p = 3$ to the three shareholders of firm T, saying that Mr. A is willing to buy as many shares from them as possible at the price p. The three shareholders must simultaneously and independently decide whether to sell his share to Mr. A at the price p.

   This game has a unique symmetric NE, where each target shareholder may sell his share to Mr. A (or tender his share) with probability π, where $\pi =$ ___A___, and hence Mr. A's takeover attempt may succeed with probability ___B___. In this NE, each target shareholder's equilibrium payoff is ___C___, and Mr. A's expected profit is ___D___.

   Solution. We claim that this game has no pure-strategy NE's. Indeed, if $\pi = 0$ in the symmetric NE, then the takeover attempt fails for sure, but then a target shareholder's equilibrium payoff would be zero, while he can deviate unilaterally by selling his share and obtain a payoff of $p = 3$, which is a contradiction.

   Similarly, if $\pi = 1$, then the takeover would succeed for sure, and a target shareholder's equilibrium payoff would be $p = 3$, while he can deviate unilaterally and obtain a payoff of 6 by keeping his share till the takeover is completed, which is another contradiction!

   We conclude that a symmetric NE must involve each target shareholder using a mixed strategy; i.e., we must have $0 < \pi < 1$.

   In this symmetric NE, a target shareholder can get $p = 3$ by tendering his share for sure, and for him to feel indifferent about tendering and not tendering his share, he must believe that without tendering his own share, the takeover attempt may still succeed with probability $\frac{p}{\frac{18}{3}} = \frac{1}{2}$. Recall that for the takeover attempt to succeed, Mr. A must obtain at least 2 shares. Thus, without tendering his own share, a target shareholder must believe that the takeover attempt may still succeed with probability π^2. Thus we have

$$\frac{1}{2} = \pi^2 \Rightarrow \pi = \frac{\sqrt{2}}{2}.$$

It follows that the probability that Mr. A's takeover attempt may succeed is

$$\begin{aligned} &\text{prob.(at least two target shareholders would tender shares)} \\ &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \pi^2(1-\pi)^1 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \pi^3(1-\pi)^0 \\ &= 3\pi^2(1-\pi) + \pi^3 = \frac{3-\sqrt{2}}{2}, \end{aligned}$$

so that Mr. A's takeover attempt generates a total surplus of

$$[3\pi^2(1-\pi) + \pi^3] \cdot (18 - 0) = 27 - 9\sqrt{2},$$

from which the three target shareholders together take away $3p = 9$,[1] and hence Mr. A ends up with an expected profit of

$$27 - 9\sqrt{2} - 9 = 18 - 9\sqrt{2}.$$

Now, let us derive the optimal p for Mr. A. Note that

$$p = 6\pi^2$$

[1]Note that from a target shareholder's perspective, tendering and not tendering his own share are both equilibrium best responses. Thus his equilibrium payoff equals the payoff of tendering his share, which is $p = 3$.

and Mr. A's expected profit from offering the share price $p = 6\pi^2$ is

$$(18 - 0)[3\pi^2(1 - \pi) + \pi^3] - 3p = 36f(\pi),$$

where

$$f(x) = x^2(1 - x).$$

It is easy to verify that

$$f''(x) \leq 0 \Leftrightarrow x \geq \frac{1}{3};$$

$$f'(x) > 0 \Leftrightarrow 0 < x < \frac{2}{3};$$

$$f'(x) = 0 \Leftrightarrow x \in \{0, \frac{2}{3}\};$$

and

$$f(0) = f(1) = 0,\ f(\frac{2}{3}) = \frac{4}{27}.$$

Thus $f(x)$ has a unique maximum at $\frac{2}{3}$ over the unit interval $[0, 1]$. Mr. A would optimally offer the share price

$$p^* = 6 \times (\frac{2}{3})^2 = \frac{8}{3},$$

which generates for Mr. A the expected profit of $36f(\frac{2}{3}) = \frac{16}{3} > 18 - 9\sqrt{2}$. The likelihood that the takeover attempt may succeed is $\frac{20}{27}$.

Remark. For a formal analysis of the above tender offer game, see Holmström, B., and B. Nalebuff, 1992, To the Raider Goes the Surplus? A Re-examination of the Free-rider Problem, *Journal of Economics & Management Strategy*, 1, 1, 37-62. Upon discarding the assumption that there is a continuum of target shareholders, the authors show that a raider can typically obtain half of the synergistic gain from finishing an acquisition, even in the presence of the target shareholders' free-rider problem.

2. At date 0, two banks B1 and B2 are competing to lend 1 dollar to firm X, which is penniless and is currently run by a single stockholder, Mr. X. The two banks and Mr. X are all risk neutral without time preferences. The game proceeds as follows.

   - At date 0, B1 and B2 must simultaneously offer a loan contract to firm X. Bank Bj's contract states that if it lends the dollar to firm X at date 0, then firm X must repay F_j at date 1, and if firm X fails to do so, then bank Bj can take over all the firm's date-1 assets (but nothing further).
   - Given F_1 and F_2, Mr. X can either reject both offers, or accept one of them. The game ends at date 0 if Mr. X rejects both offers, and all the three players get zero payoffs in that event.
   - In case Mr. X has chosen to accept F_j at date 0, then Mr. X must make an investment decision at date 0. Firm X is facing a continuum of investment projects at date 0 and Mr. X must decide for the firm which project to invest. Each project requires a one-dollar outlay at date 0. The set of feasible projects is denoted by the open interval $(0, 1)$. If Mr. X chooses project $\theta \in (0, 1)$, then the firm's date-1 (random) cash flow $\tilde{x}(\theta)$ is such that

$$\tilde{x}(\theta) = \begin{cases} 0, & \text{with probability } \theta; \\ 9\theta, & \text{with probability } 1 - \theta. \end{cases}$$

   - Now, at date 1, if Mr. X has chosen project θ at date 0, then as the sole stockholder of firm X, Mr. X has the date-1 payoff of $E[\max(\tilde{x}(\theta) - F_j, 0)]$, and the lending bank B$j$ has the date-1 payoff of $E[\min(\tilde{x}(\theta), F_j)] - 1$.

   (i) Suppose first that firm X has 1 dollar but no other assets at date 0, so that it does not need to borrow from either of the two banks. In this case Mr. X would optimally choose $\theta =$ ___E____, which implies that the date-0 value of firm X is equal to ___F____.

   (ii) Now, we return to the case where firm X is penniless at date 0. Imagine that Mr. X has accepted the offer $F_j = F$ from bank B$_j$.

Then given F, the date-0 equity value of firm X is equal to $\underline{\quad G \quad}$ and the date-0 value of the bank debt is equal to $\underline{\quad H \quad}$. It follows that in the unique SPNE, at date 0, both banks must make the same offer $F_1 = F_2 = F^*$, where $F^* = \underline{\quad I \quad}$, and after accepting F^* Mr. X would optimally choose $\theta = \underline{\quad J \quad}$.

Solution. Part (i) is easy. Mr. X seeks to maximize the date-0 value of firm X; i.e.,

$$\max_{\theta \in (0,1)} V(\theta) \equiv (1-\theta) 9\theta,$$

so that we have

$$\theta^* = \frac{1}{2},\ V(\theta^*) = \frac{9}{4}.$$

Now, consider part (ii). Consider the date-0 subgame where Mr. X has accepted an offer F (so that $F < 9$; why?) and Mr. X is about to determine θ. Mr. X seeks to

$$\max_{\theta \in (0,1)} (1-\theta)(9\theta - F),$$

subject to $\theta \geq \frac{F}{9}$. This latter constraint does not bind at the optimal solution, which we denote by $\theta(F)$. In fact, for the unconstrained problem, the necessary and sufficient first-order condition gives Mr. X's reaction function:

$$\theta(F) = \frac{F+9}{18} > \frac{F}{9}.$$

Note that the higher θ is, the greater the probability of failure, but also the greater the date-1 cash inflow in the event of success. Mr. X's reaction function shows that a higher face value of debt encourages Mr. X to choose a project with a higher θ. Given F and $\theta(F)$, Mr. X's payoff is exactly the date-0 equity value

$$S(F) \equiv [1 - \theta(F)][9\theta(F) - F] = \frac{9-F}{18} \cdot \frac{9-F}{2} = \frac{(9-F)^2}{36},$$

which, by the fact that F cannot exceed 9, is strictly *decreasing* in F.

Now, consider the stage where the two banks must compete in offering F_1 and F_2 at date 0. It is necessary that the two banks obtain zero expected profits by offering some $F_1 = F_2 = F$ at date 0. That is, the Bertrand outcome must prevail at date 0. Hence we require that the date-0 value of the bank debt, denoted by $D(F)$, be such that

$$1 = D(F) \equiv [1 - \theta(F)]F = \frac{9F - F^2}{18}.$$

It follows that

$$(9 - F)F - 18 = 0 \Rightarrow F = 3 \text{ or } \ 6.$$

This implies that we have at most two SPNE's. In one SPNE, $F = 6$ and $\theta(6) = \frac{5}{6}$, and in the other SPNE, $F = 3$ and $\theta(3) = \frac{2}{3}$.

Now we claim that $F = 6$ is inconsistent with the date-0 equilibrium. In fact, given that the other bank chooses to offer $F = 6$, one bank can deviate and offer $F' = 6 - \epsilon$ with sufficiently small $\epsilon > 0$, and Mr. X will choose this latter offer F' over $F = 6$. To see this, note that $\theta(F)$ is increasing in F, and $S(F)$ and $D(F)$ are both decreasing in F at $F = 6$. By deviating and offering $F' = 6 - \epsilon$, a bank can ensure that $1 - \theta(F') > 1 - \theta(F)$, so that the project may succeed with a higher probability, which, at $F = 6$, results in both $S(F') > S(F)$ and $D(F') > D(F)$. Hence it is not an equilibrium where both banks offer $F = 6$ at date 0.

The same problem does not arise at $F = 3$ because $D'(3) > 0 > S'(3)$. If given its rival bank offers $F = 3$ a bank wishes to make a deviation offer F' that can be accepted by Mr. X, then F' must be less than 3, so that $D(F') < D(3) = 1$, implying that the bank cannot break even after making this unilateral deviation. Thus we conclude that in equilibrium both banks must offer $F^* = 3$, and Mr. X then chooses $\theta(F^*) = \frac{2}{3}$.

Remark. Note that at date 0 the borrowing firm has a firm value (the sum of the equity value and debt value) equal to $2 < \frac{9}{4}$, where recall that $\frac{9}{4}$ is the date-0 value of the borrowing firm's all-equity counterpart. The difference $\frac{9}{4} - 2 = \frac{1}{4}$ is referred to as an *agency cost* resulting from debt financing. Intuitively, after borrowing risky debt, Mr. X (on behalf of the shareholders) has an incentive to carry out an investment project with excessive risk, even though this move may reduce the firm value. This incentive problem is referred to as the *asset substitution problem.*

3. Consider the following extensive game where an entrepreneur must choose one among three feasible borrowing methods and determine his effort level at date 0, and then interact with the creditor at date 1.

   - At date 0, the penniless entrepreneur must raise $I > 0$ by (1) issuing to households a long-term arm's-length debt (i.e., corporate bond) maturing at date 2 with face value D_a; or (2) borrowing short-term bank debt maturing at date 1 with face value D_1; or (3) borrowing long-term bank debt maturing at date 2 with face value D_2. The entrepreneur has the right to determine whether to liquidate the firm at date 1 if (1) or (3) is chosen, but this liquidation right is given to the lending bank if (2) is chosen. The entrepreneur will remain penniless prior to date 2. From now on, we shall assume for simplicity that regardless of the financing method chosen by the entrepreneur, the firm will have a single debtholder.
   - There are four possible states of nature, denoted by $\omega_1, \omega_2, \omega_3, \omega_4$. Define the events

     $$E = \{\omega_1, \omega_2, \omega_3\}, \quad E^c = \{\omega_4\}.$$

     The entrepreneur's date-0 effort is denoted by q, which is also the probability that event E may occur. More precisely, after raising $I > 0$, the entrepreneur must choose an effort level $q \in [0, 1]$, and given his choice q, event E may then occur at date 1 with probability q, but the entrepreneur must then incur a dis-utility $\phi(q) = \frac{kq^2}{2}$, where $k > 0$ is a constant. The public would know at date 1 whether or not event E has occurred, but given that E has occurred, the public would not know which state contained in E is the true state. On the other hand, we refer to E^c as the *failure* event because when event E^c occurs, the entrepreneur's investment project would generate no cash flows at any date, and the firm has zero liquidation value at both date 1 and date 2.

- In event E the investment project would generate cash L at date 1 if it is liquidated at date 1, and the investment project would generate cash $x_j + y_j$ at date 2 in state $\omega_j \in E$ if it is not liquidated at date 1. Here, for $j = 1, 2, 3$, x_j and y_j denote respectively the date-2 verifiable and un-verifiable cash flows generated by the investment project in state ω_j. Note that the entrepreneur can divert completely y_j to his own use, and hence a debtholder can only be repaid at date 2 from the verifiable cash flow x_j. Note that the proceeds L from liquidating the firm in event E are verifiable, and hence L can be shared by the debtholder and the entrepreneur.[2]

- When event E occurs, the true state is equally likely to be ω_1, ω_2, or ω_3. In event E,
  (a) under the arm's-length debt, the firm is supposed to repay the debtholder $\min(L, D_a)$ at date 1 if the firm is liquidated at date 1;
  (b) under the long-term bank debt, the firm is supposed to repay the lending bank $\min(L, D_2)$ at date 1 if the firm is liquidated at date 1; and
  (c) under the short-term bank debt, the firm is supposed to repay the lending bank $\min(x_j, D_1)$ in state x_j at date 2 if the firm is not liquidated at date 1. We said "supposed to" because under short-term or long-term bank debt, the entrepreneur and the bank can always renegotiate and come up with a new face value of debt if they agree to do so.

- If the entrepreneur has chosen to issue long-term or short-term bank debt at date 0, then the true state $\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$ would become the bank and the entrepreneur's common knowledge at date 1, but ω remains unknown to the public. If the entrepreneur has chosen to issue arm's-length debt at date 0, then ω would

[2]To fulfill operating efficiency in state ω_j, the firm should be liquidated at date 1 if $L \geq x_j + y_j$, and to be continued till date 2 if $L \leq x_j + y_j$. The firm may not follow this efficiency criterion since (1) the debtholder would ignore y_j when given the liquidation right; (2) the households, unlike a bank, are by assumption unable to engage in debt renegotiation with the entrepreneur; and (3) the entrepreneur will remain penniless at date 1 and hence cannot afford to bribe the debtholder and talk the debtholder into implementing the efficient liquidation decision.

become the entrepreneur's private information. (This assumption captures the fact that banks are equipped with expertise that households do not have.)

- If under the date-0 contract the firm's date-1 liquidation/continuatoin decision in state ω_j does not attain efficiency (i.e., the firm's decision results in a cash flow $z_j < \max(L, x_j + y_j)$) then the entrepreneur and the debtholder would want to replace the date-0 contract by a new contract that can re-store efficiency. We assume that whenever debt renegotiation occurs, the debtholder and the entrepreneur would share the renegotiation gain $\max(L, x_j + y_j) - z_j$ equally. However, we assume that it is impossible for the entrepreneur to renegotiate an arm's-length debt with the lending households.
- We assume that *the efficient markets hypothesis* holds. That is, when the entrepreneur issues debt to raise I at date 0, an investor purchasing the debt, whether it is a bank or a household, would just break even.

We shall assume the following parameter values:

$$x_1 = 9, \quad x_2 = x_3 = 0,$$

$$y_1 = 0, \quad y_2 = 1, \quad y_3 = 5,$$

$$I = \frac{2}{3}, \ L = 2, \ k = 6.$$

(i) Consider the benchmark case where the entrepreneur is endowed with $I > 0$ himself at date 0. What is his choice of q?[3] No. 1 .

(ii) Suppose that the penniless entrepreneur decides to issue the arm's-length debt at date 0. Then the optimal D_a= No. 2 ,[4] which then

[3]In this case, the entrepreneur's date-1 liquidation decision will be efficient: in state ω_j he will liquidate the firm if and only if $L \geq x_j + y_j$.

[4]You should make the conjecture that in equilibrium $x_1 > D_a, D_2 > I$ and $x_1 > D_1 > L$, and then verify that these conjectures are correct.

induces the entrepreneur to choose $q = \underline{\text{ No. 3 }}$.

(iii) Suppose instead that the penniless entrepreneur decides to issue short-term bank debt at date 0. Then the optimal $D_1 = \underline{\text{ No. 4 }}$, which then induces the entrepreneur to choose $q = \underline{\text{ No. 5 }}$.

(iv) Suppose that the penniless entrepreneur decides to issue long-term bank debt at date 0. Then the optimal $D_2 = \underline{\text{ No. 6 }}$, which then induces the entrepreneur to choose $q = \underline{\text{ No. 7 }}$.

Solution. Consider part (i). Efficiency requires that the firm be liquidated in and only in state ω_2.[5] Thus at date 0 the entrepreneur seeks to

$$\max_{q\in[0,1]} \frac{q}{3}(x_1 + L + y_3) - \phi(q),$$

and hence his optimal choice is the *first-best* effort level

$$q^{FB} = \frac{x_1 + L + y_3}{3k} = \frac{8}{9}.$$

Consider part (ii). Let us conjecture that $x_1 > D_a > I$, and hence the entrepreneur would always let the project continue at date 1 when event E takes place. Thus at date 0 the entrepreneur seeks to

$$\max_{q\in[0,1]} \frac{q}{3}[(x_1 - D_a) + y_2 + y_3] - \phi(q),$$

and hence his optimal choice is

$$q_a \equiv \frac{(x_1 - D_a) + y_2 + y_3}{3k}.$$

Since the arm's-length debtholder must break even under the optimal choice of D_a, we have

$$I = \frac{q_a}{3} \cdot D_a \Rightarrow 9kI = D_a[(x_1 - D_a) + y_2 + y_3]$$

[5] Efficiency requires that at date 1 the firm be liquidated in state ω_j if and only if $L \geq x_j + y_j$.

$$\Rightarrow 36 = 15D_a - D_a^2 \Rightarrow D_a = 12 \text{ or } 3.$$

To satisfy $x_1 > D_a > I$, as conjectured, we have $D_a = 3$, implying that

$$q_a = \frac{(x_1 - D_a) + y_2 + y_3}{3k} = \frac{(9-3)+1+5}{3 \cdot 6} = \frac{2}{3}.$$

Note that date-1 continuation in state ω_2 is inefficient, but by assumption, renegotiation is out of the question under the arm's-length debt.

Consider part (iii). Again, we conjecture that $x_1 > D_1 > L$. The inside bank would liquidate the firm in states ω_2 and ω_3, but not in state ω_1. Thus at date 0 the entrepreneur seeks to

$$\max_{q \in [0,1]} \frac{q}{3}[(x_1 - D_1) + 0 + 0] - \phi(q),$$

and hence his optimal choice is

$$q_1 \equiv \frac{(x_1 - D_1)}{3k}.$$

Since the inside bank must break even under the optimal choice of D_1, we have

$$I = \frac{q_1}{3} \cdot [D_1 + 2L] \Rightarrow D_1 = 5 \text{ or } 0.$$

To satisfy $x_1 > D_1 > L$, as conjectured, we have $D_1 = 5$, implying that

$$q_1 = \frac{(x_1 - D_1)}{3k} = \frac{2}{9}.$$

Note that it is inefficient that the lending bank chooses to liquidate the firm in state ω_3, but no renegotiation can occur: the entrepreneur is penniless at date 1, and he cannot commit to hand over y_3 to the inside bank at date 2.

Finally, consider part (iv). Again, we conjecture that $x_1 > D_2 > I$. Hence the entrepreneur would always like to let the project continue at date 1 when event E takes place. However, continuation in state ω_2 is inefficient, and hence renegotiation would take place, so that D_2 would be replaced by a new face value of debt $D' = (1-\mu)(L - y_2)$, where $\mu = \frac{1}{2}$ and $1 - \mu$ measure respectively the bargaining power of the entrepreneur and the lending bank during a debt renegotiation at date 1. With this new face value of debt, the entrepreneur would then prefer to liquidate the firm and fulfill efficiency at date 1. Thus at date 0 the entrepreneur seeks to

$$\max_{q \in [0,1]} \frac{q}{3}[(x_1 - D_2) + (y_2 + \mu(L - y_2)) + y_3] - \phi(q),$$

and hence his optimal choice is

$$q_2 \equiv \frac{(x_1 - D_2) + [y_2 + \mu(L - y_2)] + y_3}{3k}.$$

Since the lending bank must break even under the optimal choice of D_2, we have

$$I = \frac{q_2}{3} \cdot [D_2 + (1 - \mu)(L - y_2)]$$

$$\Rightarrow 4D_2^2 - 60D_2 + 113 = 0$$

$$\Rightarrow D_2 = \tfrac{15+4\sqrt{7}}{2} \text{ or } \tfrac{15-4\sqrt{7}}{2}.$$

To satisfy $x_1 > D_2 > I$, as conjectured, we have

$$I < D_2 = \frac{15 - 4\sqrt{7}}{2} \sim 2.2085.$$

It follows that

$$q_2 = \frac{4 + \sqrt{7}}{9} \sim 0.7384.$$

Remark. Debt contracts must deal with multiple incentive problems in this exercise. Ideally, a *first-best* debt contract must simultaneously attain three objectives:
(1) it must allow the creditor to break even so that the firm can succeed in fund-raising at date 0;
(2) it must make sure that the entrepreneur would exert the first-best effort (i.e., the q^{FB} obtained in part (i)) at date 0, so that event E

would occur at date 1 with the "right" probability; and
(3) it must liquidate the firm at date 1 when and only when liquidation would generate higher cash inflows than continuation during the date-1-date-2 period.

Judging from the above criteria, short-term bank debt leads to inefficient liquidation in state ω_3, which cannot be fixed by date-1 renegotiation because of the entrepreneur's wealth constraint. Long-term bank debt results in efficient liquidation/continuation decisions at date 1, but it allows the lending bank to extract a rent from the entrepreneur in state ω_2, which induces the entrepreneur to choose an effort below the first-best level at date 0. The arm's-length debt is passive at date 1 because by assumption the households are uninformed about ω and cannot participate in date-1 debt renegotiation, and thus it results in inefficient continuation in state ω_2.

The assumption that the three states $\omega_1, \omega_2, \omega_3$ are equally likely in event E makes each of the above issues important to the entrepreneur's date-0 choice of q. Since the creditor must break even in equilibrium, the entrepreneur's equilibrium payoff increases with q. Comparing q_a, q_1 and q_2, we conclude that the penniless entrepreneur's best choice is to issue long-term bank debt, which, given the assumed parameter values, ensures a highest choice of q, which is 0.7384.

4. Consider a Hotelling main street denoted by the unit interval $[0, 1]$. The population of consumers is 1, and consumers are uniformly distributed along the Hotelling main street. We shall refer to the consumer located at $x \in [0, 1]$ by "consumer x."

   In the following stage game $G(1)$, there are two firms (firm 0 and firm 1) producing a homogeneous product (called product Y) and trying to sell it to the consumers living along the Hotelling main street. Consumers have unit demand. For $i = 0, 1$, firm i is located at the point $i \in [0, 1]$. For all $x \in [0, 1]$, consumer x must incur a transportation cost x if he wants to visit firm 0, and he must incur a transportation cost $1 - x$ if he wants to visit firm 1. Consumers' common valuation for product Y is v. Let p_i be the unit price for product Y charged by firm i. Then consumer x will buy from firm 0 if and only if

$$\text{(IR)} \quad v - p_0 - x \geq 0 \text{ and (IC)} \quad v - p_0 - x \geq v - p_1 - (1 - x).$$

Similarly, consumer x will buy from firm 1 if and only if

$$\text{(IR)} \quad v - p_1 - (1 - x) \geq 0 \text{ and (IC)} \quad v - p_0 - x \leq v - p_1 - (1 - x).$$

Given (p_0, p_1), if there exists $x^* \in [0, 1]$ such that

$$v - p_0 - x^* = v - p_1 - (1 - x^*) \geq 0,$$

then x^* is firm 0's sales volume, and $1 - x^*$ is firm 1's sales volume. Assume that firm i's unit production cost is c_i, where $c_1 = 1$ and $c_0 = 0$.

The above stage game $G(1)$ proceeds as follows.

- Firm 0 and firm 1 must simultaneously announce p_0 and p_1.
- Then, given (p_0, p_1), consumers must simultaneously decide whether to buy 1 unit of product Y from firm 0, or to buy 1 unit of product Y from firm 1, or not to buy anything.
- Then profits are realized for the two firms, and the game ends.

(i) Suppose that $v = \frac{5}{4}$. Find the Nash equilibrium (p_0^*, p_1^*) for the stage game $G(1)$.[6] What is firm 0's equilibrium sales volume? What is firm 0's equilibrium profit Π_0? What is firm 1's equilibrium profit Π_1?

(ii) Now, suppose that $v = 4$,[7] and consider the infinitely repeated version $G(\infty)$ of $G(1)$, where firm j's discount factor is $\rho_j \in (0, 1)$. Show that there exist ρ_0^* and ρ_1^* such that whenever $\rho_1 \geq \rho_1^*$ and $\rho_0 \geq \rho_0^*$, there exists an SPNE for $G(\infty)$ sustained by the trigger strategy, where in each and every period the two firms' equilibrium prices

[6] Since v is very small, we conjecture that in equilibrium (i) the two firms are local monopolists; and (ii) some consumers are left unserved.

[7] Since v is rather large, we conjecture that in equilibrium of $G(1)$ no consumers are left unserved.

are $(p_0, p_1) = (\frac{13}{4}, \frac{15}{4})$.[8] Compute ρ_1^* and ρ_0^*.

Soluton.

Consider part (i). Since v is very small, we conjecture that the two firms are local monopolists in equilibrium. That is, given p_0^*, firm 0's sales volume is x with $v - p_0^* - x = 0$, so that firm 0's equilibrium profit is

$$f(p_0^*) \equiv (v - p_0^*)(p_0^* - c_0),$$

which must satisfy

$$f'(p_0^*) = 0 \Rightarrow p_0^* = \frac{v + c_0}{2} = \frac{5}{8},$$

implying that in equilibrium firm 0's sales volume is

$$v - p_0^* = \frac{5}{4} - \frac{5}{8} = \frac{5}{8},$$

and firm 0's profit is

$$f(p_0^*) = \frac{5}{8} \cdot (\frac{5}{8} - 0) = \frac{25}{64}.$$

Similarly, we can derive

$$p_1^* = \frac{v + c_1}{2} = \frac{9}{8},$$

implying that in equilibrium firm 1's sales volume is

$$v - p_1^* = \frac{5}{4} - \frac{9}{8} = \frac{1}{8},$$

and firm 1's profit is

$$(p_1^* - c_1) \cdot \frac{1}{8} = \frac{1}{64}.$$

[8] Conjecture that in each and every period all consumers are served in the SPNE of $G(\infty)$.

As conjectured, in equilibrium consumers located in the interval $[0, \frac{5}{8}]$ purchase from firm 0, and consumers located in the interval $[\frac{7}{8}, 1]$ purchase from firm 1. Consumers located in $(\frac{5}{8}, \frac{7}{8})$ are left unserved.

Consider part (ii). Since v is large relative to other parameters, we conjecture that in equilibrium of $G(1)$ all consumers are served. There must exist some $x \in [0, 1]$ such that consumers located at x feel indifferent about purchasing from firm 0 or purchasing from firm 1. That is, given the equilibrium prices (p_0, p_1) for $G(1)$,

$$v - p_0 - x = v - p_1 - (1 - x) \Rightarrow x = \frac{p_1 - p_0 + 1}{2}.$$

It follows that given p_1, p_0 is the solution to

$$\max_p \frac{p_1 - p + 1}{2} \cdot (p - c_0),$$

and that given p_0, p_1 is the solution to

$$\max_p \frac{p_0 - p + 1}{2} \cdot (p - c_1),$$

so that we have

$$p_0 = \frac{p_1 + c_0 + 1}{2}, \ p_1 = \frac{p_0 + c_1 + 1}{2},$$

implying that

$$p_0 = \frac{4}{3}, \ p_1 = \frac{5}{3}.$$

It follows that firm 0's equilibrium sales volume is

$$\frac{p_1 - p_0 + 1}{2} = \frac{2}{3},$$

and firm 1's equilibrium sales volume is

$$\frac{p_0 - p_1 + 1}{2} = \frac{1}{3}.$$

In equilibrium, consumers located in the interval $[0, \frac{2}{3}]$ purchase from firm 0, and consumers located in the interval $(\frac{2}{3}, 1]$ purchase from firm 1.[9] Firm 0's equilibrium profit is

$$\frac{2}{3} \cdot (\frac{4}{3} - 0) = \frac{8}{9},$$

and firm 1's equilibrium profit is

$$\frac{1}{3} \cdot (\frac{5}{3} - 1) = \frac{2}{9}.$$

Now, we claim that in the above-mentioned SPNE for $G(\infty)$, given the two firms' equilibrium prices $(p_0, p_1) = (\frac{13}{4}, \frac{15}{4})$ all consumers are served in each and every period. Indeed, consider a consumer located at $\frac{3}{4}$. This consumer would obtain a surplus of

$$4 - \frac{13}{4} - \frac{3}{4} = 0$$

if he chooses to purchase from firm 0, and he would obtain

$$4 - \frac{15}{4} - \frac{1}{4} = 0$$

if he chooses to purchase from firm 1. All consumers in the interval $[0, \frac{3}{4})$ are then better off purchasing from firm 0 and obtaining a positive surplus than purchasing from firm 1. Similarly, all consumers in the interval $(\frac{3}{4}, 1]$ are better off purchasing from firm 1 than from firm 0. In this SPNE, firm 0's per-period profit is

$$(\frac{13}{4} - 0) \cdot \frac{3}{4} = \frac{39}{16},$$

and firm 1's per-period profit is

$$(\frac{15}{4} - 1) \cdot \frac{1}{4} = \frac{11}{16}.$$

Now, let us determine ρ_0^*. We claim that if firm 0 wishes to deviate from the above SPNE in period n, then its optimal deviation is to price

[9] Consumers located at $\frac{2}{3}$ may purchase from firm 1 instead, but that is immaterial.

at $\frac{11}{4}$, which generates a sales volume of 1 and a period-n profit of $\frac{11}{4}$. To see this, note that the optimal deviating price q for firm 0 must solve the following maximization problem:

$$\max_q \ (q - c_0) \min[1, v - q, \frac{\frac{15}{4} + q + 1}{2}]$$

$$= \begin{cases} q, & \text{if } \ q \in [0, \frac{11}{4}]; \\ q(\frac{19}{8} - \frac{q}{2}), & \text{if } \ q \in [\frac{11}{4}, \frac{13}{4}]; \\ q(4 - q), & \text{if } \ q \geq \frac{13}{4}. \end{cases}$$

The solution is $q = \frac{11}{4}$.

Now, by the trigger strategy, following its deviation in period n, firm 0 will lose $(\frac{39}{16} - \frac{8}{9})$ in each and every period $m \geq n + 1$. Thus firm 0 will not deviate in period n if and only if

$$\frac{11}{4} - \frac{39}{16} = \frac{5}{16} \leq \frac{\rho_0(\frac{39}{16} - \frac{8}{9})}{1 - \rho_0}$$

$$\Leftrightarrow \rho_0 \geq \rho_0^* \equiv \frac{45}{268}.$$

Similarly, from firm 1's perspective, if if firm 1 wishes to deviate from the above SPNE in period n, then its optimal deviation is to price at $\frac{21}{8}$, which generates a sales volume of $\frac{13}{16}$ and a period-n profit of $\frac{13}{16}(\frac{21}{8} - 1) = \frac{169}{128}$. To see this, note that the optimal deviating price q for firm 1 must solve the following maximization problem:

$$\max_q \ (q - c_1) \min[1, v - q, \frac{\frac{13}{4} - q + 1}{2}]$$

$$= \begin{cases} (q - 1), & \text{if } \ q \in [0, \frac{9}{4}]; \\ (q - 1)(\frac{17}{8} - \frac{q}{2}), & \text{if } \ q \in [\frac{9}{4}, \frac{15}{4}]; \\ (q - 1)(4 - q), & \text{if } \ q \geq \frac{15}{4}. \end{cases}$$

The solution is $q = \frac{21}{8}$.

Now, by the trigger strategy, following its deviation in period n, firm 1 will lose $(\frac{11}{16} - \frac{2}{9})$ in each and every period $m \geq n+1$. Thus firm 1 will not deviate in period n if and only if

$$\frac{169}{128} - \frac{11}{16} = \frac{81}{128} \leq \frac{\rho_1(\frac{11}{16} - \frac{2}{9})}{1 - \rho_1}$$

$$\Leftrightarrow \rho_1 \geq \rho_1^* \equiv \frac{729}{1265}.$$

This finishes part (ii).

Remark. The theory of infinitely repeated games allows us to formally define brand image and reputation for competitive firms. This exercise shows that these concepts are related to firm-specific characteristics such as the unit cost of production (c_i) and product quality (as reflected by the magnitude of v).