Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 6

1. (**Retailer's Opportunistic Pricing Behavior and Consumers' Coupon Redemption.**) There are two consumers with unit demand for the product produced by a firm. The firm has no production costs. The two consumers' valuations for the product are respectively H and L. The firm has already issued a cents-off coupon with face value v, and to redeem the coupon the two consumers must incur costs T_H and T_L respectively.[1]

   Assume that
   $$2L - v > H \geq L + v > L > 0,$$
   and that
   $$H - v \geq H - T_H > L - T_L > v - T_L > 0.$$
   The extensive game starts after the firm has alreay chosen v, and it is described as follows.

   - Seeing v, the two consumers must decide independently whether to carry the coupon and redeem it on the shopping day. A consumer with valuation $j \in \{H, L\}$ will incur a cost T_j *before* the shopping day if he decides to carry the coupon till the shopping day. Consumers' decisions about whether to carry the coupon are unobservable to the firm.
   - Then, on the shopping day, the firm must choose a retail price p before consumers arrive.

[1] Therefore consumer H gets a surplus $H - (p - v) - T_H$ if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Similarly, consumer L gets a surplus $L - (p - v) - T_L$ if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Of course, a consumer can always forget about the coupon, and simply make the purchase. In the latter case, consumer H would get a surplus $H - p$ and consumer L would get a surplus $L - p$. Recall that each consumer gets zero surplus if he chooses to make no purchase.

- Then, consumers walk in the store, see p, and decide whether to make a purchase, and if they have carried a coupon till the shopping day, (it is obviously a dominant strategy at this moment) to present the coupon to the firm in order to get a price reduction equal to v.

(i) Show that given that v satisfies the above conditions, this game has a unique Nash equilibrium where consumer H will never redeem the coupon while consumer L and the firm both use mixed strategies in equilibrium; that is, in equilibrium consumer L feels indifferent about redeeming and not redeeming the coupon, and the firm feels indifferent about *two* optimal prices $p_2 > p_1$.[2]

(ii) Now, suppose instead that $2L > H > M$, where

$$M = 2L - kv,$$

with

$$k = \frac{L - v}{L + v} \in (0, 1).$$

Re-consider the above extensive game. Solve for the mixed-strategy NEs.[3]

[2]Note that the redemption cost T_j is already sunk on the shopping day. If the firm expects consumer L to carry the coupon with probability one, then $p = L + v$, so that consumer L will end up with a negative consumer surplus; and if the firm expects consumer L to not carry the coupon with probability one, then $p = L$, so that consumer L actually prefers to carry the coupon before the shopping day. Show that there can be no pure strategy equilibrium. Then, argue that in a mixed strategy equilibrium, the firm randomizes over at most two prices.

[3]Verify that the solution to part (i) is still valid if $H < M$. Show that if $H = M$, then we have a continuum of mixed-strategy NEs, where the firm randomizes over the three prices L, $L+v$, and H, with the probability of pricing at L being $\frac{TL}{v}$, and where consumer L redeems the coupon with probability k. Show that if $2L > H > M$, then in equilibrium the firm randomizes over L and H, with the probability of pricing at L being $\frac{TL}{v}$, and with consumer L redeeming the coupon with probability $\frac{2L-H}{v}$.

Solution. We shall give a detailed analysis for part (i), and then part (ii) can be analyzed analogously.

Consider part (i). Recall that $H \geq L+v$. Because of $T_H \geq v$, redeeming the coupon would reduce consumer H's valuation. Because consumer H would not redeem the coupon, the firm would set $p = H$ if the firm wants to serve consumer H only. Because the firm can set $p = L$ to serve both consumers H and L and because $2L - v > H$, $p = H$ is dominated by $p = L$. (If $p = L$, the worst possible case facing the firm is the situation where L carries a coupon, so that the firm must reimburse an amount v to L, implying the firm gets a revenue $2L - v$, which is still greater than H, which is the revenue from serving H alone.) Note that all $p < L$ are dominated by $p = L$ (both consumers will buy the product at any such p, with or without a coupon). Note that when consumer L sees the price p, the redemption cost has been sunk, and thus all $p \in (L, L + v)$ are dominated by $p = L + v$ (at any such p, consumer L will buy the product if and only if he is carrying a coupon; consumer H will buy the product always if we assume that $H \geq L+v$). When $p > L + v$, only consumer H may buy the product. Because of $2L > H$, all $p > L+v$ are dominated by $p = L$. We conclude that only $p = L$ and $p = L + v$ are un-dominated choices for the seller.

Is there a pure-strategy equilibrium where $p = L$? Given that the firm sets price at L, consumer L will redeem the coupon. But given that consumer L will redeem the coupon, the firm has an incentive to raise the price to $L + v$ (T_L will be sunk when consumer L sees p!). Thus this equilibrium cannot exist.

Is there a pure-strategy equilibrium where $p = L + v$? Given that the firm sets price at $L + v$, consumer L will not redeem the coupon, but given that consumer L does not redeem the coupon in equilibrium, the firm is better off pricing at L. Thus, this pure-strategy equilibrium cannot exist either. We conclude that there exist no pure-strategy equilibria for this game.

Consider the the mixed-strategy equilibrium where the firm sets $p_2 = L + v$ with probability x and $p_1 = L$ with probability $1 - x$, consumer

L redeems the coupon with probability y, and consumer H does not redeem the coupon. Note that at both p_1 and p_2, consumer H always buy the product. Since the firm uses a mixed strategy in equilibrium, it must obtain the same payoff choosing the prices p_1 and p_2. Thus,

$$(1+y)(L+v) - yv = 2L - yv.$$

Solving the equation above, we have $y = \frac{L-v}{L+v}$. Similarly, since consumer L uses a mixed strategy in equilibrium, consumer L is indifferent about redeeming and not redeeming the coupon. Thus,

$$x(-T_L) + (1-x)(v - T_L) = 0.$$

Solving the equation bove, we have $x = 1 - \frac{T_L}{v}$. Therefore, there is a unique mixed-strategy equilibrium with $x = 1 - \frac{T_L}{v}$ and $y = \frac{L-v}{L+v}$.

Now, consider part (ii). Note that by assumption

$$2L > \max(H, M).$$

The equilibria can be classified as follows.

- It is easy to verify that the solution to part (i) remains valid if $L + v \leq H < M$.[4]
- If instead $H < L + v < M$, then there exists a pure-strategy equilibrium where the firm prices at H with probability one.[5] [6]
- If instead $H = M$, then we have a continuum of mixed-strategy NEs, where the firm randomizes over the three prices L, $L+v$, and H, with the probability of pricing at L being $\frac{TL}{v}$, and where consumer L redeems the coupon with probability k.

[4]Note that M is exactly the expected profit that the firm obtains in the mixed-strategy equilibrium obtained in part (i).

[5]Again one can verify that the firm cannot price at either L or $L+v$ in a pure-strategy equilibrium.

[6]In this pure-strategy equilibrium, the firm's expected profit is $2H - v$, because consumer L will redeem the coupon with probability one. The firm does not want to deviate and price at $L+v$, because $2H - v > (L+v) - v = L$.

- Finally, if $H > M$,[7] then in equilibrium the firm randomizes over L and H, with the probability of pricing at L being $\frac{TL}{v}$, and with consumer L redeeming the coupon with probability $\frac{2L-H}{v}$.

2. (**Benefits of Using a Dual-Channel Strategy**.) Consider a single-product manufacturer M that wishes to sell its product to two segments of consumers, referred to as H (the highs) and L (the lows), whose populations are respectively α and $1-\alpha$. Consumers have either unit or zero demand at each date n, for all positive integers $n \geq 1$; see below. H-buyers and L-buyers are respectively willing to pay V and v for 1 unit of the product. For simplicity, M has no production costs.

   At each date n, there is a (different) physical retailer R_n that can sell the product to the consumers on behalf of M. Except for the wholesale price charged by M, R_n can operate without costs. M, R_n's, and the consumers are all risk-neutral.

   The above physical market is modelled as an infinitely repeated game as follows.

   - At date 0, M can decide whether to build and operate its own online channel by spending a cost F.

   - At each date $n = 1, 2, \cdots$, after learning the interactions between M and $\{R_1, R_2, \cdots, R_{n-1}\}$, M and R_n must play the following date-n stage game.

     At the beginning of date n, M and R_n will first learn about whether there is a demand for M's product at date n. Assume that demand exists for sure at date 1, and given that there is a demand at date $n-1 \geq 1$, with probability $1-q_M$ the demand may vanish from date n on.[8] The game ends at the first date the demand for M's

[7] Verify that $M > L + v$ always!

[8] An interpretation is that with probability $1-q_M$ a new brand may emerge at date n and take all the existing customers from M. With this interpretation, q_M measures the strength of M's brand image.

product vanishes in the physical market. On the other hand, if there is a demand at date n, then the date-n stage game proceeds as follows.

- M must first announce its online price P_n and then must offer a wholesale price w_n to the physical retailer R_n, which R_n can either accept or reject. If R_n accepts w_n, then R_n must announce its retail price p_n.
- Then consumers learn about both p_n and P_n (where $p_n = +\infty$ if R_n has rejected w_n and where $P_n = +\infty$ if M did not build its online channel at date 0) and they must simultaneously decide whether to buy from R_n at the price p_n or to buy online from M at the price P_n, or to make no purchase at all. M must incur a unit cost $c > 0$ when selling through its online channel, and all buyers must incur a cost $\lambda > 0$ to buy online.[9]
- In case some consumers have chosen to buy from M at the price P_n, M can decide whether to avoid trade by telling those consumers that the product has been sold out (i.e., to renege its price P_n), or to sell to those consumers at the price P_n as promised. We assume that renege has no direct costs for M.
- Consumers that have tried to purchase online at the price P_n without success (in case M chose to renege P_n) can then decide whether to return to the physical market and purchase from R_n at the price p_n.
- Then the the date-n profits are realized for M and R_n respectively. Then the game moves on to date $n + 1$.

We assume that each R_n seeks to maximize expected profits, M seeks to maximized the sum of its expected profits accruing at all transaction

[9]The assumption that $c, \lambda > 0$ implies that selling the product via the physical channel is more efficient than selling it via the online channel. The assumption that the highs and the lows share the same parameter λ implies that the manufacturer cannot adopt a dual channel strategy to better screen consumers. However, we shall show that a dual channel strategy can still be beneficial to M.

dates, and consumers seek to maximize expected consumer surplus. Assume that

$$V > v > \lambda > 0 > v - c - \lambda, \ 1 > \alpha, q_M > 0. \tag{1}$$

(i) Suppose that $F = +\infty$ and that M and R_n are the same firm, for all n. (This implies that $w_n = 0$ for all n.) Find the equilibrium p_n and M's equilibrium payoff, assuming that

$$v > \alpha(2 - \alpha)V.$$

(ii) Suppose that $F = +\infty$ and that M and R_n are different firms, for all n. Find the equilibrium w_n and p_n, assuming that

$$v > \alpha(2 - \alpha)V.$$

(iii) Suppose that $F = +\infty$ and that R_n are the same physical retailer R at each date n. Like M, R also seeks to maximize the sum of expected profits accruing at all transaction dates. Find the equilibrium w_n and p_n, assuming that

$$\alpha(2 - \alpha)V > v > \alpha V[1 + (1 - \alpha)(1 - q_M)] > \alpha V.$$

Show that this game has an SPNE supported by the trigger strategy, where in equilibrium $w_n = \frac{v - \alpha(1-q_M)V}{1-\alpha(1-q_M)} \in (0, v)$ and $p_n = v$ for all n. Show that if instead R_n are different firms, then we would have $w_n = V = p_n$ in equilibrium.

(iv) Suppose that $F < +\infty$, $v > \alpha(2 - \alpha)V$, and R_n are different firms. Define

$$\Pi_M^0 \equiv \frac{v - \alpha V}{1 - \alpha}[1 + q_M + q_M^2 + \cdots] = \frac{v - \alpha V}{(1 - q_M)(1 - \alpha)}, \tag{2}$$

and

$$q_M^* \equiv \frac{\max[F, c + \lambda - v]}{v + \max[F, c + \lambda - v] - \frac{v - \alpha V}{1 - \alpha}}. \tag{3}$$

Show that this game has an SPNE supported by the trigger strategy such that in equilibrium $w_n = p_n = v$ and $P_n = v - \lambda$ for all n if and only if $q_M \geq q_M^*$.[10]

Solution. Consider part (i). By assumption, there is a vertically integrated M at each and every date n. If M sets $p_n = v$, then M's date-n payoff is v; if M sets $p_n = V$, then M's date-n payoff is αV. Since

$$v > \alpha(2-\alpha)V \Rightarrow v > \alpha V,$$

M's optimal choice is $p_n = v$ and M's date-n equilibrium payoff is v also. In equilibrium the date-0 present value of M is then $\frac{v}{1-q_M}$.

Consider part (ii). In this case, in the subgame where R_n chooses p_n, R_n would choose $p_n = v$ if and only if

$$(v - w_n) \geq \alpha(V - w_n) \Rightarrow w_n \leq \frac{v - \alpha V}{1-\alpha}.$$

Among those w_n that satisfy the above inequality, M's favorite is $w_n = \frac{v-\alpha V}{1-\alpha}$, which yields for M the date-n payoff $\frac{v-\alpha V}{1-\alpha}$. Among those w_n that violate the above inequality, M's favorite is $w_n = V$, which implies that $p_n = V$ also, yielding for M the date-n payoff αV. Thus M would choose $w_n = \frac{v-\alpha V}{1-\alpha}$ over $w_n = V$ if and only if

$$\frac{v - \alpha V}{1-\alpha} \geq \alpha V \Leftrightarrow v \geq \alpha(2-\alpha)V.$$

[10]**Hint**: Show that pricing at $p_n = w_n = v$ is indeed R_n's equilibrium best response as long as R_n believes that M will never renege P_n. Show that consumers facing $p_n = v$ and $P_n = v - \lambda$ will choose to trade with R_n. Show that when $p_n > v$ consumers all wish to trade online at date n, and in the latter event, by reneging at date n, M's payoff from date n on is

$$q_M \Pi_M^0,$$

and by selling to the consumers at P_n as promised, M's payoff from date n on becomes

$$(v - c - \lambda) + \frac{q_M v}{1 - q_M}.$$

We conclude that in equilibrium $w_n = \frac{v-\alpha V}{1-\alpha}$ and $p_n = v$.

Consider part (iii). Call the stage game where M and R can interact for only once (at date 1) $G(1)$. According to part (ii), in equilibrium of $G(1)$ we have $w_1 = V = p_1$. (The same is true then if R_n's are different retailers.) Now, in $G(\infty)$, if M and R were the same firm, then the vertically integrated firm would choose $p_n = v$ at each date n, according to part (i), which would fulfill the *channel efficiency*.[11] This outcome, by part (ii), is not a Nash equilibrium outcome in $G(1)$, given that R and M are not the same firm and given that $v < \alpha(2-\alpha)V$. However, thanks to the fact that R, like M, is also a long-term player in part (iii), we shall show that $p_n = v$ can arise as an SPNE outcome in $G(\infty)$.

Indeed, if in an SPNE M offers w_n and R chooses $p_n = v$ at each date n, then R's equilibrium date-n payoff would be $v - w_n$; and when the trigger strategy is at work, R's date-n payoff would become 0 (because $w_n = p_n = V$). Hence for R to conform to the equilibrium pricing strategy $p_n = v$, we must have

$$\alpha(V - w_n) - (v - w_n) \leq \frac{q_M(v - w_n)}{1 - q_M}$$

$$\Leftrightarrow \alpha(1 - q_M)(V - w_n) \leq v - w_n$$

$$\Leftrightarrow [1 - \alpha(1 - q_M)]w_n \leq v - \alpha(1 - q_M)V,$$

so that, as asserted, M would offer

$$w_n = \frac{v - \alpha(1 - q_M)V}{1 - \alpha(1 - q_M)}.$$

[11] A distribution channel consists of an upstream manufacturer M and a downstream retailer R. Channel efficiency is attained by the distribution channel, if the sum of M's and R's profits are maximized in equilibrium. Apparently, when M and R are the same firm, there is no conflict of interests between the two firms, and the integrated firm's marketing strategy always fulfills channel efficiency.

Now, consider part (iv). Now R_n's are short-term players, unlike in part (iii), and hence we cannot fulfill channel efficiency by simply letting M interact with physical retailers. However, we have removed the assumption that $F = +\infty$ in part (iv), so that M can use a dual-channel strategy to enhance channel efficiency.

We are asked to show that there is an SPNE where M spends F at date 0 and prices online at $P_n = v - \lambda$ at each date $n \geq 1$ as long as q_M is sufficiently large. In this equilibrium, R_n cannot price higher than v if R_n wishes to win the patronage of any consumer: otherwise, a consumer would rather buy online from M. Thus M can set $w_n = v$ accordingly to extract all the surplus from R. However, there is this problem of

$$(v - \lambda) - c = P_n - c < 0,$$

and hence M would like to renege P_n when consumers really come and try to make a purchase online. The one-time gain from reneging is $c - (v - \lambda)$. Following reneging the trigger strategy would be at work, and by our analysis in part (ii), M's date-n payoff with the trigger strategy is

$$\frac{v - \alpha V}{1 - \alpha}.$$

Thus M will not renege at date n when consumers really come and try to buy online if and only if

$$(v - c - \lambda) + \frac{q_M w_n}{1 - q_M} = (v - c - \lambda) + \frac{q_M v}{1 - q_M} \geq 0 + q_M \Pi_M^0$$

$$\Leftrightarrow q_M \geq \frac{c + \lambda - v}{c + \lambda - \frac{v - \alpha V}{1 - \alpha}}.$$

When the above inequality holds, recognizing that M will never renege its online prices, R_n will choose $p_n = v$ at each and every date n, so that no consumers would really come and try to buy online. Finally, M should be willing to spend F at date 0. Thus we require that

$$F \leq \frac{q_M v}{1 - q_M} - q_M \Pi_M^0$$

$$\Leftrightarrow q_M \geq \frac{F}{v + F - \frac{v - \alpha V}{1 - \alpha}}.$$

The above two inequalities can be compactly written as

$$q_M \geq q_M^* = \frac{\max[F, c + \lambda - v]}{v + \max[F, c + \lambda - v] - \frac{v - \alpha V}{1 - \alpha}},$$

since the function

$$h(z) \equiv \frac{z}{v + z - \frac{v - \alpha V}{1 - \alpha}}$$

is strictly increasing in z. This finishes part (iv).

Remark. The marketing literature has shown that a dual-channel strategy may be beneficial for a manufacturer for several reasons.

- First, buyers may be endowed with heterogeneous costs/benefits of visiting a physical or online outlet, and it may be efficient to direct different buyers to purchase at different outlets. For example, a monopolistic manufacturer M is trying to serve two buyers A and B, both willing to spend 10 for M's product. Suppose that trading online is costless for A but prohibitively costly for B (because B is unfamiliar with the internet), and trading at the physical outlet is costless for B but prohibitively costly for A (because A has a high transportation cost). If building online and physical outlets is costless, then M should serve A at its online outlet and B at its physical outlet.
- Second, a dual-channel strategy may allow M to better discriminate buyers, even if doing so may reduce efficiency. Take again the above example, but assume that A is willing to pay 6 for the product instead of 10. Both A and B can trade costlessly in the physical market, but A and B must incur respectively 1 and 10 if they wish to trade online. Efficiency would require that M serve

both A and B at the physical outlet. However, if M can build an online outlet costlessly, then M can set an online price 5 and an offline price 10 and direct A and B to trade respectively at the online and physical outlets. Letting A to trade online is inefficient, but it allows M to identify B at the physical outlet and extract B's surplus.

- Third, a dual-channel strategy can be valuable for imperfectly competitive manufacturers who wish to reduce competition. When a firm builds an online channel, it induces some buyers in the physical market to migrate to the online market, and this may alleviate competition in the physical market.

In addition to these reasons, this exercise provides yet another rationale for the dual-channel strategy: a manufacturer usually must use an independent retailer's service in the physical market, and there is always a conflict of interests between the manufacturer and the physical retailer. If a manufacturer can build an online channel to compete with its physical retailer, then it can prevent the physical retailer from pricing too high and dropping too many low-valuation buyers (which is against the manufacturer's interest). Such an online pricing strategy by the manufacturer is referred to as a "flank-attack" strategy against the physical retailer.

The problem with this "flank-attack" strategy is whether or not it is credible for the manufacturer to set a low online price. An overly low online price may not be able to cover the unit cost. However, if the physical retailer does not believe that the manufacturer can really trade at that low price at its online outlet, then such a pricing strategy is not credible, and the physical retailer would simply ignore it.

This exercise shows that, building on the perfect folk theorem, if the manufacturer has a strong image (i.e., q_M is sufficiently large), then it is credible for M to stand by its promised online price, and this makes the flank-attack strategy work. In equilibrium, M is able to force the physical retailer to cooperate and price low, serving both high- and low-valuation buyers and fulfilling channel efficiency.

We have assumed in part (iv) that R_n's are different firms. We claim that the manufacturer does not gain if the physical retailers are the same firm R, and hence M will choose a dual-channel strategy (together with the above flank-attack pricing strategy) over a single-channel strategy (together with the SPNE pricing strategy supported by the trigger strategy as in part (iii)) if and only if $q_M \geq q_M^*$.

With the single-channel strategy, M can spare F at date 0. When the trigger strategy is at work, R will get

$$v - \frac{v - \alpha V}{1 - \alpha}$$

at each date, so that facing w_n R would not deviate and price above v if and only if

$$\alpha(V - w_n) + \frac{q_M(v - \frac{v-\alpha V}{1-\alpha})}{1 - q_M} \leq \frac{v - w_n}{1 - q_M},$$

implying that M should choose

$$w_n^* = \frac{q_M(\frac{v-\alpha V}{1-\alpha}) + (1 - q_M)(v - \alpha V)}{1 - \alpha(1 - q_M)}$$

$$= \frac{q_M(\frac{v-\alpha V}{1-\alpha}) + (1 - q_M)(1 - \alpha)(\frac{v-\alpha V}{1-\alpha})}{1 - \alpha(1 - q_M)}$$

$$= \frac{[q_M + (1 - q_M)(1 - \alpha)](\frac{v-\alpha V}{1-\alpha})}{1 - \alpha(1 - q_M)}$$

$$= \frac{[1 - \alpha(1 - q_M)](\frac{v-\alpha V}{1-\alpha})}{1 - \alpha(1 - q_M)}$$

$$= \frac{v - \alpha V}{1 - \alpha} < v,$$

which yields for M a date-0 present value of

$$\frac{q_M w_n^*}{1 - q_M}.$$

The intuition is that, the long-lived physical retailer R realizes that it would receive $v - w_n^*$ in each and every period when the trigger strategy is at work, and hence to induce R to cooperate in an SPNE that attains channel efficiency M must offer some w_n that represents a weakly deeper trade promotion than w_n^*, and the optimal w_n that meets this requirement from M's perspective is w_n^* itself. Thus M will adopt a dual-channel strategy in the presence of a forever-lived R if and only if, again,

$$q_M \geq q_M^*.$$

This exercise is taken from Shan-Yu Chou (2014, The Optimal Product-line Extension, Pricing, Targeting, and Online-Channel Strategies for a Manufacturer Facing a Physical Independent Retailer, NTU working paper).

3. (**Borrowing Bank Loan or Issuing Corporate Bond?**) An entrepreneur needs to invest 1 dollar to build a firm at date 0, while he has only $w < 1$ dollars. There are two types of investors in the financial market: households and commercial banks. The entrepreneur and all investors are risk neutral without time preferences; that is, they all seek to maximize expected profits and future cash flows are never discounted. The difference between the two types of investors is that banks have committed to spend a cost $c > 0$ to monitor each borrowing firm's operations, but households do not have the expertise that is required to oversee the firm's operations. Because banks will have to spend on monitoring, either the entrepreneur chooses to borrow $1 - w$ from households only, or he must borrow (at least partially) from banks, and in the latter case, he needs to raise $1 - w + c$, instead of $1 - w$.

   After the entrepreneur gets the funding, he can incur a private cost $\phi \geq 0$ to determine the quality p of the firm's investment project at date 1. For simplicity, suppose that a project of quality p may generate Y dollars with probability p and nothing with probability $1-p$ at date 3, where the constant $Y > 1$. The personal cost ϕ is a function of p, and let us assume that, for some constant $K > Y$,

$$\phi(p) = \frac{K}{2}p^2, \quad \forall p \in [0,1].$$

If a bank lends to the firm at date 0, then it can see p after spending $c > 0$ before date 2. The firm can be liquidated at date 2, and its (non-negative) liquidation value is $L < 1$. Let $\overline{p}$ be the first-best project quality; that is,[12]

$$\overline{p} = \arg \max_{p \in [0,1]} pY - \phi(p).$$

Assume that $\overline{p}Y - \phi(\overline{p}) > 1$.

(A) First suppose that banks do not exist. In this case, the entrepreneur can first decide to or not to offer a financial contract to a household, and in case he does, the household can either accept or reject it. A financial contract, or a corporate bond, is defined as (I, Q, R), such that, according to this contract,

(i) the household needs to give the entrepreneur I dollars at date 0;

(ii) the household (or, the bondholder) can choose to or not to liquidate the firm at date 2;

(iii) the household will get $Q \in [0, L]$ in the event that the firm is liquidated at date 2; and

[12] Verify that $\overline{p} = \frac{Y}{K}$ and hence the assumption $\overline{p}Y - \phi(\overline{p}) > 1$ reduces to $Y^2 > 2K$. Given Y, this last inequality holds for some K if $Y > 2$.

(iv) the household will get $R \in [0, Y]$ in the event that the firm is not liquidated at date 2, and it generates Y at date 3.

Show that there exists $w^* \in (0, 1)$ such that in equilibrium the entrepreneur chooses to borrow from a household and the household is willing to lend $I = 1 - w$ if and only if $w \geq w^*$.

From now on, assume the following numerical values:

$$K = 72, \; Y = 13, \; L = \frac{3}{5}, \; w = \frac{1}{2}.$$

Let π_E be the entrepreneur's expected wealth and p^* the equilibrium project quality. Show that under the optimal corporate bond (I, Q, R),[13]

$$p^* = \frac{1}{8}, \; R = 4, \; \pi_E = \frac{9}{16}.$$

(B) Next suppose that the entrepreneur can only borrow from banks. In this case, the entrepreneur can first decide to or not to offer a financial contract to a bank, and in case he does, the bank can either accept

[13]**Hint**: Since the household must decide to or not to liquidate the firm before knowing the entrepreneur's equilibrium choice p^*, the game between the entrepreneur and the household is a simultaneous game. Given (I, Q, R), and given p^* (which the household did not observe but can conjecture correctly in equilibrium), the household should liquidate the firm if and only if $Q \geq p^* R$. On the other hand, the entrepreneur will choose $p = 0$ if he believes that the household will subsequently liquidate the firm, and he will choose $p^* = \text{argmax}_{p \in [0,1]} \, p(Y - R) - \phi(p)$ if he expects no liquidation. Note that there may exist multiple Nash equilibria for the subgame where the household has already lent to the entrepreneur. Verify that for this subgame it is an equilibrium where the household always liquidates the firm and the entrepreneur always picks $p = 0$. However, the household would not have lent to the entrepreneur in the first place if they expect this bad subgame equilibrium to subsequently prevail. In other words, bond financing is feasible only if the household expects the entrepreneur to pick $p^* > 0$ and only if the household can at least break even (that is, $p^* R \geq 1 - w$).

or reject it. A financial contract, or a bank loan, is defined as (i, q, r), such that, according to this contract,

(i) the bank needs to give the entrepreneur i dollars at date 0;

(ii) the bank can choose to or not to liquidate the firm at date 2, *after* it sees the firm's choice of p at date 1;

(iii) the bank will get $q \in [0, L]$ in the event that the firm is liquidated at date 2; and

(iv) the bank will get $r \in [0, Y]$ in the event that the firm is not liquidated at date 2, and it generates Y at date 3.

Show that the first-best $\overline{p}$ can be attained in equilibrium if $L \geq 1 - w + c$.[14]

Show that, with the above specified numerical values, under the optimal bank loan contract (i, q, r),

$$p^* = \overline{p} = \frac{13}{72}, \ \pi_E = \frac{169}{144} - \frac{1}{2} - c,$$

so that the entrepreneur prefers borrowing bank debt to issuing a corporate bond and bank debt implements $\overline{p}$ if and only if $c \leq \frac{1}{10}$.

[14]**Hint**: The bank can see the entrepreneur's choice p before it decides to or not to liquidate the firm. Thus the game between the entrepreneur and the lending bank is a sequential game. The bank gets $q \leq L$ by liquidating the firm, and the bank gets pr by letting the firm continue its operations. If the contract (i, q, r) is such that $1 - w + c = q = \overline{p}r$, then the bank will liquidate the firm if and only if $p < \overline{p}$. Facing this credible threat, the entrepreneur's best response is $p^* = \overline{p}$, fulfilling the productive efficiency. The only problem is q must not exceed L, and hence such a wisely designed contract is feasible if and only if $L \geq 1 - w + c$.

Solution. Consider part (A). Consider the subgame where the household has already lent $1-w$ to the firm, and the entrepreneur is about to choose p. Assuming that the household will not liquidate the firm subsequently (forward induction!), the entrepreneur seeks to

$$\max_p p(Y-R)-\phi(p),$$

so that

$$p=\frac{Y-R}{K}.$$

At the time the household lends to the entrepreneur, the household rationally expects the entrepreneur to choose $p=\frac{Y-R}{K}$, given R. Hence the entrepreneur should offer the household the contract (I,Q,R) that satisfies

$$Rp=R(\frac{Y-R}{K})=1-w.$$

This implies that either

$$R=\frac{Y+\sqrt{Y^2-4K(1-w)}}{2}$$

or

$$R=\frac{Y-\sqrt{Y^2-4K(1-w)}}{2}.$$

It also follows that either

$$p=\frac{Y+\sqrt{Y^2-4K(1-w)}}{2K}$$

or

$$p=\frac{Y-\sqrt{Y^2-4K(1-w)}}{2K}.$$

Apparently, the entrepreneur prefers to implement a higher p. Thus we conclude that with bond financing we must have

$$p^*=\frac{Y+\sqrt{Y^2-4K(1-w)}}{2K}\Rightarrow R^*=\frac{Y-\sqrt{Y^2-4K(1-w)}}{2}.$$

In equilibrium, by lending $1-w$ at date 0, the household expects to receive

$$p^*R^* = \frac{Y^2 - [Y^2 - 4K(1-w)]}{4K} = 1-w,$$

so that the household does break even.

Note that we have assumed that $\sqrt{Y^2 - 4K(1-w)}$ is well-defined, or equivalently,

$$w \geq w^* \equiv 1 - \frac{Y^2}{4K} > 1 - \frac{Y}{4}.$$

If $w < w^*$, then for all $R \in [0, Y]$,

$$Rp = R(\frac{Y-R}{K}) < 1-w,$$

so that the household can never break even from lending to the entrepreneur.

Note also that, for p^* to be a legitimate probability, p^* must lie between zero and one, which is true:

$$0 < p^* = \frac{Y + \sqrt{Y^2 - 4K(1-w)}}{2K} < \frac{Y + \sqrt{Y^2}}{2K} = \frac{Y}{K} \leq 1.$$

With the specified numerical values, we have

$$p^* = \frac{1}{8},\ R^* = 4,$$

implying that

$$\pi_E = p^*(Y - R^*) - \phi(p^*) = \frac{9}{16}.$$

Now, consider part (B). Under the optimal (i, q, r), the first-best quality $\overline{p} = \frac{13}{72}$ will be implemented if and only if $c \leq \frac{1}{10}$, and in the latter event the total social surplus is

$$\overline{p}Y - \phi(\overline{p}) = \frac{169}{144}.$$

Thus we have

$$\pi_E = \frac{169}{144} - (1 - w) - c = \frac{169}{144} - \frac{1}{2} - c,$$

which is greater than $\frac{9}{16}$ if and only if $c \leq \frac{1}{9}$. We conclude that the entrepreneur prefers borrowing bank debt to issuing a corporate bond and bank debt implements $\overline{p}$ if and only if $c \leq \frac{1}{10}$.

Remark. Forward induction tells us that either bond financing does not work, or following bond financing the bondholders would never force the firm to liquidate early. Thus the entrepreneur is free to choose any p that he likes. Since an overly low p would not allow bondholders to break even, and since the entrepreneur would choose a high p only if the returns generated by the project will mostly be accrued to the entrepreneur, bond financing is feasible if and only if the entrepreneur himself can finance a large portion of the project; that is, w must be sufficiently close to 1. When bond financing is feasible, the chosen p is always lower than its first-best level $\overline{p}$.

Bank financing has the merit that the lending bank can force early liquidation *after seeing the* p *chosen by the entrepreneur*. The lending bank may have an incentive to let the firm continue (because continuation may generate for the bank a higher payoff than liquidation), but it is generally possible to design a contract that induces the bank to choose liquidation over continuation whenever the entrepreneur chooses an overly low p. Given such a bank-loan contract, the entrepreneur knows that choosing an overly low p will result in the firm being liquidated, and he will not get much in the latter event. Thus bank financing can force the entrepreneur to choose a high p. Indeed, we have seen that it is sometimes (i.e. when $L \geq 1 - w + c$) possible to force the entrepreneur to implement $\overline{p}$ using bank financing. However, this productive efficiency comes at a cost: the bank must spend c to do auditing, which could be spared if the project were instead financed by issuing a bond. Given w, when c is small and L large, bank financing is apparently better than bond financing because it ensures the first-best productive efficiency.

This exercise offers an explanation regarding why most firms are bank-financed in a developing economy like Taiwan in the 1950-1990 period. During that time most Taiwanese firms were small in size (and hence relatively easy to monitor, implying a small c) and possessed mostly general-purpose tangible assets (like factories and physical equipments that are valuable in most traditional business lines), which tend to have higher liquidation values L than intangible assets (like human capital or goodwill) or industry-specific tangible assets possessed by, say, a high-tech company. As bank financing tends to dominate bond financing for a developing economy, it also renders an explanation regarding the late development of a bond market in a developing economy.[15]

4. (**Duopolistic Firms Submitting Supply Curves.**) Recall the Cournot game in Example 1 of Lecture 1, Part I. Assume that $c = F = 0$ and the inverse demand in the relevant range is

$$P(Q) = 1 - Q,\ 0 \leq Q = q_1 + q_2 \leq 1.$$

Now consider a similar but different game, where the two firms must simultaneously choose *supply curves* instead of supply quantities.[16] That is, the two firms first choose supply functions at the same time, where for $i = 1, 2$, firm i's supply curve is denoted by $S_i(P)$, and after they submit their supply functions, the product price P^* is determined via the following market-clearing condition

$$S_1(P^*) + S_2(P^*) = D(P^*),$$

[15] This exercise is adapted from Repullo, R., and J. Suarez, 1998, Monitoring, Liquidation, and Security Design, *Review of Financial Studies*, 11, 163-187.

[16] If we interpret the firms as financial institutions selling a stock, and consumers as market makers absorbing the financial institutions' sell orders, then we are assuming in the Cournot game that financial institutions can only submit *market orders*, whereas in the current example, the financial institutions can submit either *market orders* or any number of *limit orders*. This exercise thus shows that large strategic traders in the stock market can manipulate the stock price by submitting limit orders.

where, from the inverse demand function, the demand curve is

$$D(P) = 1 - P.$$

(i) A *linear* Nash equilibrium is one in which both firms submit *linear* supply functions; that is,

$$S_i(P) = a_i + b_i P, \quad i = 1, 2,$$

for some constants $a_1, a_2 \in \Re$ and $b_1, b_2 \geq 0$. In such an equilibrium, expecting firm j to submit the supply curve $S_j(P) = a_j + b_j P$, firm i optimally submits the supply curve $S_i(P) = a_i + b_i P$ in equilibrium. Now, look for a *symmetric linear* Nash equilibrium, where symmetry means $a_1 = a_2 = a$ and $b_1 = b_2 = b \geq 0$.

(ii) Determine if firms are better off in equilibrium because they can choose supply curves rather than supply quantities.

(iii) We have confined attention to equilibria where $b_1, b_2 \geq 0$. What would your answer to part (ii) become if we allow $b_1, b_2 < 0$?[17]

[17]**Hint**: In order to solve the Nash equilibrium, observe that given firm j's supply curve $S_j(\cdot)$, firm i becomes a monopoly facing the following *residual demand curve*

$$D(P) - S_j(P),$$

and all firm i needs to do is to find a point on this residual demand at which its profit is maximized. After such a point is found, *any* supply curve $S_i(\cdot)$ that passes through that point is one optimal supply curve for firm i! However, we cannot just use any supply curve that passes through that point; we need to find for firm i a linear supply curve passing through that point and inducing firm j to optimally adopt $S_j(\cdot)$ in the first place. In particular, if firm i believes that firm j will submit $S_j(P) = a_j + b_j P$, show that firm i should select the point

$$P(a_j, b_j) = \frac{1 - a_j}{2 + 2b_j}$$

on its residual demand curve. Any linear $S_i(\cdot)$ passing through this point on firm i's residual demand curve is *one* best response for firm i, and such a supply curve must satisfy

$$a_i + b_i P(a_j, b_j) = S_i(P(a_j, b_j)) = 1 - a_j - (1 + b_j) P(a_j, b_j),$$

(iv) Show that, in fact, when the firms can submit any linear supply curves (with $a_1, a_2, b_1, b_2 \in \Re$), every non-negative output pair (q_1, q_2) satisfying $0 < q_1 + q_2 < 1$ can be sustained in *some* Nash equilibrium.

Solution. Consider part (i). In order to solve for the Nash equilibrium, observe that given firm j's supply curve $S_j(\cdot)$, firm i becomes a monopoly facing the following *residual demand curve*

$$D_i(P) \equiv D(P) - S_j(P),$$

and all firm i needs to do is to find a point on this residual demand at which its profit is maximized. After such a point is found, *any* supply curve $S_i(\cdot)$ that passes through that point is one optimal supply curve for firm i! However, we cannot just use any supply curve that passes through that point; we need to find for firm i a linear supply curve passing through that point such that one best response of firm j to this linear supply curve is exactly $S_j(\cdot)$.

More precisely, if firm i believes that firm j will submit $S_j(P) = a_j + b_j P$, then firm i seeks to

$$\max_P PD_i(P) \equiv [1 - a_j - (1 + b_j)P]P,$$

so that firm i should select the point

$$P_i(a_j, b_j) = \frac{1 - a_j}{2 + 2b_j}.$$

Now *any* linear $S_i(\cdot)$ satisfying

$$S_i(\frac{1 - a_j}{2 + 2b_j}) = D_i(\frac{1 - a_j}{2 + 2b_j})$$

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so that after you impose symmetry ($a_i = a_j = a$ and $b_i = b_j = b$), you will get a continuum of equilibria, where for each $b \geq 0$, a is determined by

$$a(b) = \frac{1}{3 + 2b} \leq \frac{1}{3}.$$

For part (ii), notice that when $b = 0$, this equilibrium coincides with Cournot equilibrium.

is *one* best response for firm i; that is, any (a_i, b_i) with $b_i \geq 0$ satisfying

$$a_i + b_i P(a_j, b_j) = S_i(P(a_j, b_j)) = 1 - a_j - (1 + b_j)P(a_j, b_j)$$

defines a best response $S_i(P) = a_i + b_i P$ for firm i. Now, imposing symmetry ($a_i = a_j = a$ and $b_i = b_j = b$), we obtain a continuum of symmetric linear equilibria: for each $b \geq 0$, define

$$a(b) = \frac{1}{3 + 2b} \leq \frac{1}{3}.$$

and

$$S_1(P) = S_2(P) = a(b) + bP$$

defines a symmetric linear equilibrium. This finishes part (i).

Now, for part (ii), simply note that when $b = 0$, the equilibrium obtained in part (i) coincides with the Cournot equilibrium. We claim that the firms are worse off in any symmetric linear equilibrium with $b > 0$ than in the Cournot equilibrium. To see this, note that given $b > 0$, the product price in the symmetric linear equilibrium is

$$P_i(\frac{1}{3 + 2b}, b) = \frac{1 - \frac{1}{3+2b}}{2 + 2b} = \frac{1}{3 + 2b} < \frac{1}{3},$$

and both firms choose the following equilibrium output level

$$\frac{1 - P_i(\frac{1}{3+2b}, b)}{2} = \frac{1 + b}{3 + 2b} > \frac{1}{3}.$$

Recall that each firm gets the profit $\frac{1}{9}$ in the Cournot equilibrium. Here, given $b > 0$, a firm's equilibrium profit becomes

$$\frac{1 + b}{(3 + 2b)^2} < \frac{1}{9}.$$

Although here the firms are faced with a larger strategy space than in the Cournot game (because in choosing a supply curve they can choose either $b = 0$ or $b > 0$), they are worse off in any equilibrium with $b > 0$ than with $b = 0$. The idea is that when the firms are trapped in an equilibrium with $b > 0$, they believe that expanding one's output can drive the product price down, which creates an additional benefit of reducing the rival's supply quantity. Hence the firms are encouraged to expand outputs, leading to a loss of profits for both firms.

For part (iii), consider a symmetric linear equilibrium with $-1 < b < 0$. All computations presented above are still valid in such equilibria. However, note that the firms' equilibrium outputs are less than $\frac{1}{3}$, with the equilibrium product price exceeding $\frac{1}{3}$. The firms are now better off than in the Cournot equilibrium. The idea is that in equilibrium each firm realizes that one' output expansion will drive the price down, which *raises* the rival's supply quantity in no time! Hence the firms end up choosing an output level less than $\frac{1}{3}$, leading to a higher profit for each firm in equilibrium.

Finally, consider part (iv). Given any non-negative output pair (q_1, q_2) with $q_1 + q_2 \leq 1$, let $a_1 = 1 - 2q_2$ and $a_2 = 1 - 2q_1$, so that $-1 \leq a_1, a_2 \leq 1$, and pick some $b_1, b_2 > -1$. We claim that

$$S_1(P) = a_1 + b_1 P, \quad S_2(P) = a_2 + b_2 P$$

constitute a Nash equilibrium that supports the pair (q_1, q_2) as a pair of equilibrium outputs. To see this, note that for $i = 1, 2$, from firm i's perspective, its optimal supply quantity is indeed

$$S_i(P(a_j, b_j)) = 1 - a_j - (1 + b_j)P(a_j, b_j) = 1 - a_j - (1 + b_j)[\frac{1 - a_j}{2 + 2b_j}] = \frac{1 - a_j}{2} = q_i.$$

This completes the proof. The bottom line here is that enlarging the firms' common strategy space leads to an uncountably infinite number of new linear equilibria, and many of them are Pareto dominated by the Cournot equilibrium from the two firms' perspective. Pareto improving linear equilibria require that the firms submit supply curves that are negatively sloping.

Remark. This exercise has implications about imperfect competition between dealers that are making market for a security. If $D(\cdot)$ represents the buy limit orders submitted by public investors, and $S_1(\cdot)$ and $S_2(\cdot)$ represent the two dealers' pricing policies, then allowing dealers to give price discounts to attract order flows may actually promote collusive pricing behavior for the two dealers. In reality, market makers in the US securities markets do offer price discounts from time to time,[18] which used to puzzle finance scholars, because the standard theory suggests that a dealer charge a higher price when it has to absorb a large buy order. For a formal theory regarding how dealers' discounting programs may promote collusive pricing behavior, see Biais, B., T. Foucault, and F. Salanié, 1998, Floors, Dealer Markets and Limit Order Markets, *Journal of Financial Markets*, 253-284.

5. (**Hart and Moore Cash Diversion Model with Credit-Default-Swap Contracts.**) Suppose that after making a loan to the entrepreneur, the creditor (denoted by C) can decide whether to purchase a CDS protection (π, z) from a financial institution (referred to as I), where the CDS contract (π, z) states that by paying I a fee z upfront, in the event that the entrepreneur fails to fully repay the loan, the creditor gets to choose either (1) whatever C gets from entering into debt renegotiation with the entrepreneur; or (2) π.[19] We assume that the CDS contract is

[18] According to Reuters (see http://www.reuters.com/article/2013/08/26/us-batsglobalmarkets-directedge-idUSBRE97P0H320130826):

ICE CEO Jeff Sprecher has been critical of a practice by U.S. stock exchanges of giving large rebates on trading fees to attract order flow, calling the practice "ridiculous." He has said he wants to end the practice at NYSE after the takeover. Direct Edge has been one of the most successful stock exchanges in attracting retail order flow through its discounting programs. Joe Ratterman, CEO of BATS Global Markets, said he has no plans to change that.

"Jeff can talk all he wants about how he thinks the equity markets ought to change, and I hope he makes some of those crazy changes he's thinking about to NYSE, because that will accrue to our customers big time," Ratterman said. "I think he should try it."

[19] This exercise is adapted from
Bolton, P., and M. Oehmke, 2011, Credit Default Swaps and the Empty Creditor Problem, *Review of Financial Studies*, 24(8), 2617-2655.
The authors wrote in the paper that:
Formally, the CDS is a promise of a gross payment π (or equivalently a net payment $\pi - L$) by the protection seller to the protection buyer (in our case the lender) if a "credit event" occurs at date 1, against a fair premium z that is paid by the protection buyer to the seller. We assume that a credit event occurs when the firm fails to repay the face value of debt and if upon non-payment *the firm and the creditor fail to renegotiate the debt contract to mutually acceptable terms.*

fairly priced, in the sense that I would make zero expected profits by selling the CDS contract to the creditor.

(i) First suppose that $K = 90$, $w = 30$, $B = K - w$, $y_1 = 100$, $y_2 = 65$, and $L = 30$. What is the entrepreneur's equilibrium payoffs in respectively the absence and presence of CDS trading, assuming that I would require that $\pi \leq \hat{P}$?

(ii) Now we introduce uncertainty. Suppose that there are two equally likely date-1 states, G and B. In state $j \in \{G, B\}$, the project would generate a date-t cash inflow y_t^j, for $t = 1, 2$, if there is no date-1 liquidation. In state G, $y_2^G = Y$, $y_1^G = X$, which is also the firm's date-1 liquidation value. (Liquidation in state G is an all-or-nothing decision.) In state B, liquidation at date 1 is a discrete choice: we require that $1 - f \in \{0, \lambda, 1\}$, where $\lambda \in (0, 1)$ is a constant. If a fraction $(1 - f)$ of the date-1 physical assets is liquidated in state B, the date-1 proceeds from liquidation is $(1-f)L > 0$, and the remaining physical assets will generate a cash inflow $g(1 - f)$ at date 2. Assume that

$$(\Delta) \quad Y > X > 2K > g(0) = y_2^B > K > g(\lambda) + \lambda L$$
$$> y_1^B + \lambda L > y_1^B > g(1) + L = L > 0.$$

(ii-1) Compute E's equilibrium payoff, assuming that there is no CDS trading. Let $\hat{P}_0$ denote the equilibrium face value of debt. Compute $\hat{P}_0$.

(ii-2) Compute E's equilibrium payoff, assuming that there is CDS trading and C can choose a CDS position $\pi^j \in [0, \hat{P}]$ after learning the date-1 state j. Let $\hat{P}_\pi$ denote the equilibrium face value of debt. Compute $\hat{P}_\pi$.

Solution. Consider part (i). First assume that CDS trading is not available. The project would yield 165 for an investment cost of 90, and in a first-best world where complete contracts can be signed it would be financed. However, in a second-best world with contract incompleteness it will not be. The reason is that a contribution of $K - w = 60$ is required from the investor at date 0, but the most that the investor can recover at date 1 is $L = 30$, given that the entrepreneur can make a take-it-or-leave-it offer to the investor during the date-1 renegotiation. Given this, the investor will refuse to finance the project at date 0. The entrepreneur's equilibrium profit is thus zero. Thus the *ex-ante efficiency* is not attained.

Now, suppose that there is CDS trading. It is optimal for C to take a position $\pi = \min(\hat{P}, y_2) = 60$, which then forces the entrepreneur to fully repay 60. The first-best efficiency is thus attained because of CDS trading. The entrepreneur's equilibrium profit is 75. The CDS position strengthens the creditor's bargaining power and forces E to hand over 60 out of y_1, and hence efficiency is enhanced.

Consider part (ii-1). Suppose first that there is no CDS trading.

Since Y is large, E would like to keep the date-1 physical assets intact. Since X is large relative to K, we conjecture that E can afford to keep the date-1 physical assets intact by repaying C the face value of debt $\hat{P}_0$. Thus by lending K to E at date 0, C can get $\hat{P}$ in state G at date 1.

In state B, since $\hat{P}_0 \geq K > y_1^B > L$, E will default at date 1, and C can only get L in the debt renegotiation following default. It follows from C's date-0 break-even condition

$$K = \frac{1}{2} \cdot \hat{P}_0 + \frac{1}{2} \cdot L$$

that

$$\hat{P}_0 = 2K - L,$$

confirming that our conjecture that $X > \hat{P}_0$ is correct. Since there is no date-1 liquidation in equilibrium, the entrepreneur's equilibrium payoff is simply the project's NPV, which is

$$\frac{X + Y + y_1^B + y_2^B}{2} - K.$$

Now, consider part (ii-2). Let z^j be the premium that C pays the CDS counter-party to obtain the protection π^j. If C has a CDS position $\pi^G \in [0, \hat{P}_\pi]$ in state G, then C only gets the payoff $\hat{P}_\pi$ from E at date 1, implying that no credit event would ever occur, and hence $z^G = 0$. C's date-1 payoff is then $\hat{P}_\pi - z^G = \hat{P}_\pi$.

Suppose that C has taken a CDS position $\pi^B \in [0, \hat{P}_\pi]$ in state B. Note that C would get π_B from the CDS seller if and only if either C or E refuses to enter into debt renegotiation or they fail to renegotiate the debt contract to mutually acceptable terms (and we say that *a credit event occurs* in this case). Thus we can deduce the following:

- Suppose that $\pi^B \leq L$. In this case, C would rather get $\max(\pi^B, L) = L$ from E than get π^B from the CDS counter-party, which implies that no credit event would occur, and hence C's cost of entering the position π^B is $z^B = 0$. Thus by taking a CDS position $\pi^B \leq L$, C gets L, just as in the absence of CDS trading.
- Suppose that $\pi^B \in (L, y_1^B]$. In this case, E would get y_1^B if giving up debt renegotiation, and E would get $y_1^B - \pi^B + y_2^B$ if paying π^B to C during renegotiation. Thus E would choose the latter if and only if $\pi^B \leq y_2^B$. Since $y_2^B = g(0) > y_1^B \geq \pi^B$, we know that E would rather repy π^B to C. Since $y_1^B < K \leq \hat{P}_\pi$, we see that C's optimal choice $\pi^B \in (L, y_1^B]$ is $\pi^B = y_1^B$. This implies that no credit event would ever occur, and hence $z^B = 0$. C's date-1 payoff from choosing $\pi^B = y_1^B$ is y_1^B.
- Suppose that $\pi^B \in (y_1^B, y_1^B + \lambda L]$. In this case, E would get y_1^B if giving up debt renegotiation, and E would get $y_1^B + \lambda L + g(\lambda) - \pi^B$ if paying π^B to C during renegotiation. Thus E would choose the latter if and only if $\pi^B \leq \lambda L + g(\lambda)$. Since $y_1^B + \lambda L \leq \lambda L + g(\lambda)$, and since $\pi^B \leq y_1^B + \lambda L$, E would indeed choose to repay π^B to C. Since $\pi^B \leq y_1^B + \lambda L < K \leq \hat{P}_\pi$, C's optimal choice $\pi^B \in (y_1^B, y_1^B + \lambda L]$ is $\pi^B = y_1^B + \lambda L$. This implies that no credit event would ever occur, and hence $z^B = 0$. C's date-1 payoff from choosing $\pi^B = y_1^B + \lambda L$ is $y_1^B + \lambda L$.
- Suppose that $\pi^B \in (y_1^B + \lambda L, \min(\hat{P}_\pi, y_1^B + L)]$. In this case, E would get y_1^B if giving up debt renegotiation, and E would get $y_1^B + L - \pi^B$ if paying π^B to C during renegotiation. Because

$\pi^B > y_1^B + \lambda L > L$, E would give up renegotiation, which is a credit event, and it results in C getting π^B from the CDS counter-party. The CDS counter-party's payoff is then $L - \pi^B + z^B$. For the CDS counter-party to break even, we have $z^B = \pi^B - L$. Thus by choosing any $\pi^B \in (y_1^B + \lambda L, \min(\hat{P}_\pi, y_1^B + L)]$, C's date-1 payoff is $\pi^B - z^B = L$.

- Suppose that $\pi^B \in (\min(\hat{P}_\pi, y_1^B + L), \hat{P}_\pi]$. The same reasoning as in the preceding case where $\pi^B \in (y_1^B + \lambda L, \min(\hat{P}_\pi, y_1^B + L)]$ would show that, by choosing any $\pi^B \in (\min(\hat{P}_\pi, y_1^B + L), \hat{P}_\pi]$, C's date-1 payoff is L.

To sum up, C's equilibrium choice of the date-1 CDS position in state B is $\pi^B = y_1^B + \lambda L$, which is also C's date-1 payoff in state B.[20] It follows from C's date-0 break-even condition

$$K = \frac{1}{2} \cdot \hat{P}_\pi + \frac{1}{2} \cdot (y_1^B + \lambda L)$$

that

$$\hat{P}_\pi = 2K - \pi^B = 2K - y_1^B - \lambda L > K.$$

The entrepreneur's equilibrium payoff in the presence of CDS trading becomes

$$\frac{1}{2}(X + Y - 2K + y_1^B + \lambda L) + \frac{1}{2} g(\lambda).$$

[20]What would happen if we replace condition (Δ) by condition (Δ') below?

$$(\Delta') \quad Y > X > 2K > g(0) = y_2^B > K > y_1^B + \lambda L$$
$$> g(\lambda) + \lambda L > y_1^B > g(1) + L = L > 0.$$

Show that C would choose $\pi^B = g(\lambda) + \lambda L$ instead, which implies that $\hat{P}_\pi$ would become $2K - g(\lambda) - \lambda L$. Thanks to the assumption that the two states G and B are equally likely, the entrepreneur's new payoff under condition (Δ')

$$\frac{1}{2}(X + Y - 2K + g(\lambda) + \lambda L) + \frac{1}{2} y_1^B$$

equals exactly his payoff under condition (Δ).

Remark. In this exercise, CDS trading results in the firm liquidating at date 1 a fraction λ of its physical assets in state B, and hence CDS trading reduces corporate investment efficiency in this exercise. When CDS trading is contractible (which requires that C's CDS position π^j be verifiable in the court of law), E can instead offer a date-0 contract that requires $\pi^G = \pi^B = 0$. In this exercise, the contract in part (ii-1) together with such a clause would be optimal.

In general, CDS trading, even if it is not contractible, may enhance corporate investment efficiency, because it raises the chance that C may approve E's financing request at date 0. This possibility has however been ruled out by condition (Δ).