Game Theory with Applications to Finance and Marketing, I

Solutions to Homework 7

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1. (**Non-linear Pricing with Countervailing Effects.**) Consider the following version of the screening game considered in section 5 of Lecture 4, where the a θ_2-buyer has reservation utility v, which may differ from zero:

$$\text{Problem (P): } \max_{T_1,T_2,q_1,q_2} T_1 + T_2 - cq_1 - cq_2$$

subject to

$$\theta_1 V(q_1) - T_1 \geq 0, \tag{1}$$

$$\theta_2 V(q_2) - T_2 \geq v, \tag{2}$$

$$\theta_1 V(q_1) - T_1 \geq \theta_1 V(q_2) - T_2, \tag{3}$$

$$\theta_2 V(q_2) - T_2 \geq \theta_2 V(q_1) - T_1. \tag{4}$$

We shall assume that $v \geq 0$, $c = \frac{1}{4}$, $\theta_1 = 3$, $\theta_2 = 4$, and $V(q) = \ln(1+q)$.

A scheme (T_1, T_2, q_1, q_2) satisfying (1)-(4) is called *incentive feasible*. The seller then seeks to find an incentive feasible scheme that generates for him the highest expected profits, and such a solution is called *incentive optimal*, or simply *optimal*, and will be denoted by $(q_2^{**}, q_1^{**}, T_2^{**}, T_1^{**})$. (We have also referred to this solution as the *second-best solution* in Lecture 4.)

We shall use Kuhn-Tucker Theorem to find the optimal scheme for the seller.[1]

[1]**Hint**: It is useful to first make some observations and take simplifying steps. At first, define $x_j \equiv V(q_j)$, so that x_j, just like q_j, is a non-negative real number, and $q_j = h(x_j) \equiv e^{x_j} - 1$. Now, the objective function becomes strictly concave in (x_1, x_2),

(a) Suppose that $v = 0$. Prove or disprove that for the seller, the optimal scheme is such that $q_2^{**} = 15$ and $q_1^{**} = 7$. What are the associated T_2^{**} and T_1^{**}?

(b) Now, assume instead that $v = \ln(9) > 0$. Find the optimal scheme $(q_2^{**}, q_1^{**}, T_2^{**}, T_1^{**})$ for the seller.

(c) Assume now that $v = \ln(13)$. Find the optimal scheme $(q_2^{**}, q_1^{**}, T_2^{**}, T_1^{**})$ for the seller.

Solution. Following the hint and defining $h(x_j) \equiv e^{x_j} - 1$, we can re-write the maximization problem as

$$\text{Problem (P):} \max_{T_1,T_2,x_1,x_2} f(T_1, T_2, x_1, x_2) \equiv T_1 - ch(x_1) + T_2 - ch(x_2)$$

subject to

$$g_1 \equiv T_1 - \theta_1 x_1 \leq 0;$$
$$g_2 \equiv T_2 - \theta_2 x_2 \leq 0;$$
$$g_3 \equiv T_1 - T_2 + \theta_1(x_2 - x_1) \leq 0;$$
$$g_4 \equiv T_2 - T_1 + \theta_2(x_1 - x_2) \leq 0.$$

Let μ_j be the associated Lagrange multiplier for constraint $g_j \leq 0$. Note that

$$Df = \begin{bmatrix} 1 \\ 1 \\ -ch'(x_1) \\ -ch'(x_2) \end{bmatrix}, \quad Dg_1 = \begin{bmatrix} 1 \\ 0 \\ -\theta_1 \\ 0 \end{bmatrix}, \quad Dg_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\theta_2 \end{bmatrix},$$

given (T_1, T_2). Second, we can re-write (1)-(4) as $g_i(T_1, T_2, x_1, x_2) \leq 0$, $i = 1, 2, 3, 4$, and verify that each g_i is affine (and hence convex) in (x_1, x_2). Third, say that the incentive feasible scheme (T_1, T_2, x_1, x_2) makes constraint i *binding* if $g_i(T_1, T_2, x_1, x_2) = 0$, and show that we can disregard an incentive feasible scheme (T_1, T_2, x_1, x_2) that makes neither (1) nor (2) binding. Then classify the set of incentive feasible schemes into two categories: those making (1) binding and those making (2) binding. Then, proceed on your own, following and mimicking the above reasoning.

$$Dg_3 = \begin{bmatrix} 1 \\ -1 \\ -\theta_1 \\ \theta_1 \end{bmatrix}, \ Dg_4 = \begin{bmatrix} -1 \\ 1 \\ \theta_2 \\ -\theta_2 \end{bmatrix}.$$

Consider part (a). It is useful to gain some insights before computing.

- Note that the θ_2-buyer can always pretend to be the θ_1-buyer and take the deal (T_1, x_1), which would allow the θ_2-buyer to obtain a payoff

$$\theta_2 x_1 - T_1 \geq \theta_1 x_1 - T_1 \geq 0,$$

  and the first inequality would be strict if $x_1 > 0$ (or equivalently $q_1 > 0$). Thus $x_1 > 0$ together with $g_4 \leq 0$ would imply $g_2 \leq 0$. That is, if we conjecture that $x_1 > 0$ then removing the second constraint would not alter the optimal solution to (P).

- Following the removal of $g_2 \leq 0$, we can further conjecture that the first constraint $g_1 \leq 0$ must be binding at an optimal solution, for otherwise we could raise T_1 and T_2 by the same tiny positive amount without violating g_1, g_3,and g_4, but this would increase f!

- The removal of $g_2 \leq 0$ and the conjecture that $g_1 = 0$ at optimum now allow us to further conjecture that g_4 must be binding at an optimal solution, for otherwise we could raise T_2 alone by a tiny positive amount without violating the other constraints, but this would increase f!

- Now, following $g_1 = 0 = g_4$ and following the removal of $g_2 \leq 0$, we can re-state $g_3 \leq 0$ as

$$(\theta_1 - \theta_2)(x_2 - x_1) \leq 0,$$

  but this last inequality would not be binding so long as $x_2 > x_1$.

Thus if we conjecture that $x_2 > x_1 > 0$ at optimum then we would also conjecture that[2]

[2]Recall that by Theorem 4 of Lecture M, $\mu_j = 0$ if $g_j < 0$ at optimum or if $g_j \leq 0$ can be removed.

$$\mu_2 = \mu_3 = 0,\ T_1 = \theta_1 x_1,\ T_2 = T_1 + \theta_2(x_2 - x_1) = \theta_2 x_2 - (\theta_2 - \theta_1)x_1.$$

Now, by the fact that f is concave and g_1, g_2, g_3, g_4 are all convex, the sufficiency of Kuhn-Tucker Theorem applies, and hence we only need to find $\mu_1, \mu_4 \geq 0$ such that

$$Df = \mu_1 Dg_1 + \mu_4 Dg_4 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ -ch'(x_1) \\ -ch'(x_2) \end{bmatrix} = \mu_1 \begin{bmatrix} 1 \\ 0 \\ -\theta_1 \\ 0 \end{bmatrix} + \mu_4 \begin{bmatrix} -1 \\ 1 \\ \theta_2 \\ -\theta_2 \end{bmatrix},$$

and if a solution to this system of equations exists and if the solution implies that $x_2 > x_1 > 0$ and $g_2, g_3 < 0$, then we are done. Moreover, because f is strictly concave in (x_1, x_2), the solution would be unique!

Upon solving

$$Df = \mu_1 Dg_1 + \mu_4 Dg_4,$$

we obtain

$$\mu_1 = 2,\ \mu_2 = 1,\ 2\theta_1 - \theta_2 = ch'(x_1),\ \theta_2 = ch'(x_2),$$

so that we have

$$x_1 = \ln(8),\ x_2 = \ln(16) \Rightarrow T_1 = 3\ln(8),\ T_2 = 4\ln(16) - \ln(8).$$

Now, it is easy to verify that at this solution, we indeed have $x_2 > x_1 > 0$ and $g_2, g_3 < 0$!

Remark 1. Note that if the seller were facing the θ_1-buyers alone, then it would be optimal for the seller to offer $(q_1^*, T_1^*) = (11, 3\ln(12))$, which is the so-called *first-best contract* for a θ_1-buyer.[3] Why would the seller choose to sell only $q_1^{**} = 7$ units to θ_1-buyers in the second-best

[3]Verify that, similarly, the *first-best contract* for θ_2-buyers is $(q_2^*, T_2^*) = (15, 4\ln(16))$. However, if this contract is offered by the seller, a θ_1-buyer would never like to take it. For this reason, the seller never has to distort q_2 in the second-best contract (a property referred to as *efficiency at the top*), and this explains why we have $q_2^{**} = q_2^* = 15$.

contract? Recall that the seller cannot tell a θ_2-buyer from a θ_1-buyer, and if the seller offers (q_1^*, T_1^*) to θ_1-buyers, then a θ_2-buyer would be tempted to take the deal (q_1^*, T_1^*), and it would become difficult for the seller to persuade the θ_2-buyer to choose (q_2, T_2) over (q_1, T_1). To make sure that θ_2-buyers would choose (q_2, T_2) over (q_1, T_1), the seller must reduce q_1 to below q_1^*. Of course, this choice of q_1 would reduce the money that the seller can earn from θ_1-buyers, but it would allow the seller to earn much more from θ_2-buyers, and hence is worthwhile.$\|$

Now, consider part (b). Here, note that, other things equal, we must re-define g_2 as $T_2 - \theta_2 x_2 + \ln(9)$ in (P). Again, we shall start with some useful observations.

Now note that the second-best contract obtained for part (a) yields for the type-θ_2 buyers a surplus[4] of $\theta_2 V(q_1^{**}) - T_1^{**} = (\theta_2 - \theta_1)V(q_1^{**}) = \ln(8) < \ln(9)$, and hence that contract violates $g_2 \leq 0$ and becomes infeasible in part (b)!

Thus the seller must fine-tune the above second-best contract obtained in part (a) in order to generate for the type-θ_2 buyers a surplus which is no less than $\ln(9)$. The question is: what is the cheapest way for the seller to attain this goal?

Since $q_1^{**} = 7$ is already overly low compared to $q_1^* = 11$ (recall **Remark 1**), the optimal way to fine-tune the part-(i) second-best contract so that the modified contract can become feasible in part (b) is to raise q_1 from $q_1^{**} = 7$ to $q_1^{**} = 8$ and to lower T_2 accordingly to make $g_4 \leq 0$ remain binding: this will allow the seller to offer enough surplus to the type-θ_2 buyers, and it will also improve the seller's revenue obtained from the type-θ_1 buyer (recall **Remark 1**). Thus we conjecture that the second-best contract in part (b) is such that $q_1^{**} = 8$, $T_1^{**} = 3\ln(9)$, and with $q_2^{**} = q_2^* = 15$ and $T_2^{**} = 4\ln(16) - \ln(9)$.

Now, we apply Kuhn-Tucker Theorem to verify the above conjecture. By conjecture, the seller would leave a θ_2-buyer exactly a surplus of

[4]Recall that $g_4 = 0$ at the optimum in part (a), which means that a θ_2-buyer's surplus, $\theta_2 x_2 - T_2$ is equal to $\theta_2 x_1 - T_1$.

$\ln(9)$, implying that $g_2 = 0$. Although the seller would raise q_1 but there is no reason that the seller would leave a surplus to the θ_1-buyers, and hence we conjecture that, once again, $g_1 = 0$. Moreover, the seller would earn as much money as possible from the θ_2-buyers by raising T_2 until $g_4 = 0$. The only constraint that would not be binding, as we conjecture, is the third constraint, so that we also conjecture that $\mu_3 = 0$.

Thus we conjecture that

$$\mu_3 = 0,\ T_1 = \theta_1 x_1 = \theta_2 x_1 - \ln(9),$$

and require that

$$Df = \mu_1 Dg_1 + \mu_2 Dg_2 + \mu_4 Dg_4$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -ch'(x_1) \\ -ch'(x_2) \end{bmatrix} = \mu_1 \begin{bmatrix} 1 \\ 0 \\ -\theta_1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\theta_2 \end{bmatrix} + \mu_4 \begin{bmatrix} -1 \\ 1 \\ \theta_2 \\ -\theta_2 \end{bmatrix},$$

so that we have

$$T_1 = 3\ln(9),\ x_1 = \ln(9) \Rightarrow h'(x_1) = e^{x_1} = 9,$$

and hence μ_1, μ_2, μ_4 must satisfy

$$\begin{cases} \mu_1 - \mu_4 = 1; \\ \mu_2 + \mu_4 = 1; \\ \frac{-9}{4} = -3\mu_1 + 4\mu_4, \end{cases}$$

implying that

$$\mu_1 = \frac{7}{4},\ \mu_2 = \frac{1}{4},\ \mu_4 = \frac{3}{4},$$

so that

$$h'(x_2) = e^{x_2} = 16(\mu_2 + \mu_4) = 16 \Rightarrow x_2 = \ln(16),$$

which by $g_2 = 0$, implies that

$$T_2 = 4\ln(16) - \ln(9).$$

It is easy to see that $g_3 < 0$, $x_2 > x_1 > 0$, as we conjectured. This finishes part (b).

Now, consider part (c). Here, note that, other things equal, we must re-define g_2 as $T_2 - \theta_2 x_2 + \ln(13)$ in (P). Once again, we shall start with some useful observations.

First recall that the second-best contract obtained for part (b) yields for the type-θ_2 buyers a surplus of $\ln(9)$, and hence that contract violates $g_2 \leq 0$ and becomes infeasible in part (c)!

Thus the seller must fine-tune the above second-best contract obtained in part (i) in order to generate for the type-θ_2 buyers a surplus which is no less than $\ln(13)$. The question is again: what is the cheapest way for the seller to attain this goal?

In part (c), even if we put $q_1 = q_1^* = 11$, the surplus for the type-θ_2 buyer generated by the second-best contract for part (b) would be $(\theta_2 - \theta_1)V(q_1^*) = \ln(12)$, which is still less than $\ln(13)$. Raising q_1 to above q_1^* can only do worse for the seller. Thus for (q_1, T_1), the seller should offer the first-best contract for the θ_1-buyers; and for (q_2, T_2), the seller should offer q_2^* but reduce T_2 sufficiently to generate for the θ_2-buyer a surplus of $\ln(13)$. Thus we conjecture that g_1, g_2 would both be binding, with $g_3 < 0$ and $g_4 < 0$. The part-(c) second-best contract is such that $T_2^{**} = 4\ln(16) - \ln(13)$, $q_1^{**} = q_1^* = 11$, $T_1^{**} = T_1^* = 3\ln(12)$, and $q_2^{**} = q_2^* = 15$.

Now, we apply Kuhn-Tucker Theorem to verify the above conjecture. By conjecture, we have

$$\mu_3 = \mu_4 = 0,\ T_1 = \theta_1 x_1,\ T_2 = \theta_2 x_2 - \ln(13),$$

and we require that

$$Df = \mu_1 Dg_1 + \mu_2 Dg_2$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -ch'(x_1) \\ -ch'(x_2) \end{bmatrix} = \mu_1 \begin{bmatrix} 1 \\ 0 \\ -\theta_1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\theta_2 \end{bmatrix},$$

so that we have

$$\mu_1 = \mu_2 = 1,$$

implying that

$$-\frac{1}{4}e^{x_1} = -3, \; -\frac{1}{4}e^{x_2} = -4,$$

and hence

$$x_1 = \ln(12), \; x_2 = \ln(16) \Rightarrow T_1 = 3\ln(12), \; T_2 = 4\ln(16) - \ln(13).$$

It is easy to verify that at this contract we have $x_2 > x_1 > 0$, $g_3 = \ln(13) - \ln(16) < 0$ and $g_4 = \ln(12) - \ln(13) < 0$, just as conjectured. This finishes part (c).

Remark 2. We did not consider the situation where v is greater than $\ln(13)$. If v gets even larger, and if the seller still wants to serve the θ_2-buyers, then T_2 may have to fall so much that even the θ_1-buyers would like to take (q_2, T_2). In this case, the seller may have to raise q_2 to above $q_2^* = 15$ to discourage the θ_1-buyers from taking (q_2, T_2), and the seller may also have to reduce T_1 in order to persuade the θ_1-buyers to choose (q_1, T_1) over (q_2, T_2). At optimum it may happen that $g_1, g_4 < 0 = g_2 = g_3$, which is in sharp contrast with the case where $v = 0$. Finally, when v tends to infinity, the seller would eventually choose to abandon the θ_2-buyer, and in that case the seller would offer (q_1^*, T^*) together with $(q_2, T_2) = (0, 0)$.

2. (**Chain-store Paradox with Five Entrants.**) Let us modify the game of chain-store paradox in Lecture 4 by assuming 5 entrants instead of 3. Find as many PBE's as possible for this reputation game.

Solution. It is easy to verify that no entrants will ever enter if $x_1 \geq \frac{1}{2}$. There are two remaining cases.

- **Case 1**: $x_1 \in [\frac{1}{4}, \frac{1}{2})$.

  Consider the subgame where E_1 has just entered. It is easy to see that preying E_1 with probability zero is not the sane incumbent's equilibrium behavior: if it were, then the incumbent's action would fully reveal whether the incumbent is sane or not so that the sane incumbent would get zero by not preying, but by preying the sane incumbent would be recognized as the crazy incumbent, which would yield a strictly positive payoff for the sane incumbent.

  Preying E_1 with probability one, on the other hand, is consistent with a PBE: it will induce both E_2 and E_3 to stay out, but after that E_4 and E_5 will enter. More precisely, because of the sane and the crazy incumbents' pooling behavior after E_1 enters, upon seeing E_1 being preyed, the rest 4 potential entrants believe that $x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and according to our analysis for the 4-entrant case in Lecture 4, E_2 would rather stay out, which implies that $x_3 = x_2 \in [\frac{1}{4}, \frac{1}{2})$, so that E_3 would also stay out, according to our analysis for the 3-entrant and 2-entrant cases in Lecture 4. In this *pooling* equilibrium preying E_1 thus yields for the sane incumbent a payoff of $-1 + \frac{3}{4} + \frac{3}{4} > 0$, which is indeed higher than the deviation payoff generated by not preying E_1, which is zero.
  Rationally expecting that both types of the incumbent will prey after E_1 enters, E_1 will stay out for sure. It follows that $x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and hence E_2 will stay out also; according to our analysis for the 4-entrant case in Lecture 4. But then $x_3 = x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and hence E_3 will stay out also, according toour analysis for the 3-entrant and 2-entrant cases in Lecture 4. It follows that $x_4 = x_3 = x_2 = x_1 \in [\frac{1}{4}, \frac{1}{2})$, and hence both E_4 and E_5 will enter, according to our analysis for the 3-entrant and 2-entrant cases in Lecture 4.

  There is no semi-separating equilibrium for the stage where the incumbent interacts with E_1; see below.

- **Case 2**: $x_1 \in (0, \frac{1}{4})$.

It is easy to verify that in equilibrium of the subgame where E_1 has just entered, the sane incumbent must randomize between preying and not preying.[5]

Thus after E_1 enters, the sane incumbent may prey or not prey both with positive probabilities. When E_1 is not preyed, the rest 4 potential entrants know immediately that the incumbent is sane, and hence $x_2 = 0$, which implies that all the rest 4 potential entrants will then enter and the sane incumbent gets zero by not preying E_1. In order for the sane incumbent to get zero continuation payoff by preying E_1, it is necessary that upon seeing E_1 being preyed, $x_2 = \frac{1}{4}$. This fact can be proved by contraposition as follows.

(a) Suppose instead that $x_2 < \frac{1}{4}$ after E_1 is preyed, so that according to our analysis for the 4-entrant case in Lecture 4 E_2 will enter with probability one. If that did happen, then the sane incumbent would get $-1 + 0 < 0$ by preying E_1, so that the sane incumbent would be better off by not preying E_1, a contradiction.

(b) Suppose instead that $x_2 > \frac{1}{4}$ after E_1 is preyed, so that according to our analysis for the 4-entrant case in Lecture 4 E_2 will stay out with probability one, which implies that $x_3 = x_2 > \frac{1}{4}$, and hence E_3 will also stay out. If that did happen, then the sane incumbent would get $-1 + \frac{3}{4} + \frac{3}{4} + 0 + 0 > 0$ by preying E_1, implying that the sane incumbent would not feel indifferent about preying and not preying E_1, which is again a contradiction.

Now, if the sane incumbent preys E_1 with probability $y_1 \in (0, 1)$

[5] Unlike in the previous case, now there can be no pooling behavior on the part of the incumbent after E_1 enters: if instead such an equilibrium did exist, then $x_2 = x_1 < \frac{1}{4}$ and hence E_2 would enter after seeing E_1 being preyed, but according to our analysis for the 4-entrant case in Lecture 4, by preying E_1 the sane incumbent's payoff would be $-1 + 0 < 0$, which is a contradiction because the sane incumbent could be better off not preying E_1.

after E_1 gets in, y_1 must satisfy

$$\frac{x_1}{x_1 + (1 - x_1)y_1} = \frac{1}{4} \Rightarrow y_1 = \frac{3x_1}{1 - x_1}.$$

Note that for y_1 to lie strictly between 0 and 1, it is necessary that $x_1 < \frac{1}{4}$. (*This proves that the sane incumbent cannot randomize between preying E_1 and not preying E_1 in the above* **Case 1**, *and hence the pooling behavior reported there is indeed the unique outcome when the incumbent is facing E_1.*)

What about the equilibrium behavior of E_2, E_3, E_4 and E_5 upon seeing E_1 being preyed? Let E_2 stay out with probability a upon seeing E_1 being preyed, and E_3 stay out with probability b upon seeing E_2 stay out. Show that any $a, b \in (0, 1)$ satisfying

$$3a(1 + b) = 4$$

are now consistent with equilibrium, for then the sane incumbent will indeed feel indifferent between preying E_1 and not preying E_1. For example, $a = 1$ and $b = \frac{1}{3}$ are consistent with a PBE. In this particular equilibrium, after seeing E_1 being preyed, E_2 stays out for sure because $x_2 = \frac{1}{4}$, and following that E_3 enters with probability $\frac{2}{3}$. In equilibrium, E_1 knows that preying will occur with probability $4x_1$ and hence E_1 will enter if and only if $x_1 < \frac{1}{8}$. Now you can summarize the equilibrium path as follows.

(a) If $x_1 \in (0, \frac{1}{8})$, then E_1 enters, and if E_1 is preyed, then in one equilibrium (where $a = 1, b = \frac{1}{3}$) E_2 stays out and E_3 enters with probability $\frac{2}{3}$. Then if E_3 is preyed, then E_4 may or may not enter, and if E_4 does not enter, then E_5 may or may not enter.

(b) If $x_1 \in [\frac{1}{8}, \frac{1}{4})$, then E_1 stays out, E_2 enters, and if and only if E_2 is preyed, then E_3 stays out and E_4 and E_5 may or may not enter.

3. (**Debt Financing and Internet Shopping Agent.**) There are n sellers, each being a local monopolist selling an identical product to $\frac{1}{n}$ local consumers. (Thus the population of all consumers is one.)

Each consumer will buy either zero or one unit of the product, with willingness to pay equal to $r > 0$. Initially, a consumer only knows his local seller, and hence he purchases from that seller at the price r. The sellers have zero costs of production. Thus each seller's profits are $\frac{r}{n}$.

Now, suppose that an ISA (Internet Shopping Agent) emerges, and it announces an access fee κ for consumers and an access fee ϕ for sellers. Given the ISA's announcements, each seller can choose its own price p and then decide whether to spend ϕ to advertise its price p at the ISA; and at the same time, each consumer can decide whether to pay κ to the ISA and go see the price advertisements posted by sellers. Assume that once a consumer pays κ, he will purchase from the seller that posts the lowest price at the ISA (as long as that price does not exceed r), and if no sellers post any prices at the ISA, then he will return to his local seller and make a purchase (as long as the local seller's price does not exceed r).

The ISA is essentially an online platform that we have encountered in Homework 1. It is clear that for $\phi, \kappa > 0$, the game has an equilibrium where no sellers or consumers would pay the access fees, but we shall focus on the equilibrium that is most favorable to the ISA in this exercise.

We shall assume that $n = 2$, and all the local sellers have borrowed the same amount of debt, with face value $D \in [0, \frac{r}{2})$. Assume that there is one shareholder (S) and one creditor (C) at each seller's firm, and other than selling the product, the local seller has no other sources of income (and it possesses no other assets). The shareholder S will choose p and decide whether to pay ϕ. The debt will mature right after sellers and consumers make transactions. Each local seller is protected by limited liability.

The game proceeds as follows. At first, given D, r and $n = 2$, the monopolistic platform announces ϕ and κ. Then sellers and consumers must simultaneously decide whether to pay their access fees, and sellers would have to choose their unit prices at the same time. Then, if a consumer chooses to not pay κ, then he visits his local seller and decides whether to make a purchase; and if he chooses to pay κ, then he visits the ISA to look for the lowest price, and if he finds no price listings, then he returns to his local seller. Then after the sellers-buyers transactions, each shareholder S at seller i must pay $\min(z_i, D)$ to his creditor C, where z_i is seller i's realized profits from the transactions with consumers.

(i) Verify that the following is a symmetric equilibrium when $D = 0$: In equilibrium, given κ, all consumers join the ISA for sure, but given ϕ, a seller is equally likely to, or not to, pay ϕ. In equilibrium $\phi = \frac{r}{4}$, and a seller would price at r when it does not pay ϕ; and its price p after paying ϕ is random, which has the following distribution function:

$$F_0(p) = \begin{cases} 0, & p \leq \frac{r}{2}; \\ \\ 2 - \frac{r}{x}, & x \in [\frac{r}{2}, r]; \\ \\ 1, & x \geq r. \end{cases}$$

Compute the ISA's optimal choice of κ, denoted by κ_0.

(ii) Very that the following is a symmetric equilibrium when $D \in (0, \frac{r}{2})$: In equilibrium, given κ, all consumers join the ISA for sure, but given ϕ, a seller is equally likely to pay or not to pay ϕ. In equilibrium $\phi = \frac{r}{4}$, and a seller would price at r when it does not pay ϕ; and its price p after paying ϕ is random, which has the following distribution function:

$$F_D(p) = \begin{cases} 0, & p \leq \frac{r+D}{2}; \\ \\ 2 - \frac{r-D}{x-D}, & x \in [\frac{r+D}{2}, r]; \\ \\ 1, & x \geq r. \end{cases}$$

Compute the ISA's optimal choice of κ, denoted by κ_D.

(iii) Suppose that $D = 0$. Does the presence of the ISA benefit or hurt the local sellers? Compare the case with $D = 0$ to the case with $D \in (0, \frac{r}{2})$. Does the ISA benefit or hurt sellers *more* because of debt financing?

(iv) Suppose that $D \in (0, \frac{r}{2})$. Is debt default-free when the ISA is absent? Is debt default-free when the ISA is present?

Solution. It is clear that part (i) is a special case of part (ii). Thus let us verify the above assertions in part (ii). Suppose that the other seller would follow the equilibrium strategy prescribed above, a seller spending ϕ and then pricing at $\frac{r+D}{2}$ would win every consumer's patronage for sure[6] and obtain profits $\frac{r+D}{2} - \phi = \frac{r+2D}{4}$, implying that the payoff for the borrowing shareholder is $\frac{r+D}{2} - D - \phi = \frac{r-2D}{4}$. On the other hand, if a seller does not spend ϕ, then with probability $\frac{1}{2}$ it would get nothing[7]; and with probability $\frac{1}{2}$ it would get the profits $\frac{r}{2}$, so that the borrowing shareholder's payoff is again $\frac{1}{2} \cdot \frac{r-2D}{2} = \frac{r-2D}{4}$. Thus the borrowing shareholder is willing to spend ϕ with probability $\frac{1}{2}$. It is easy to verify that given the rival seller follows the prescribed mixed strategy, a borrowing shareholder, after spending ϕ, feels indifferent about all the prices $p \in [\frac{r+D}{2}, r]$. The borrowing shareholder has no reasons to price below $\frac{r+D}{2}$ or above r.

Note that in the equilibrium that is most favorable to the ISA, κ must be chosen to make the consumers' participation constraint binding. If a consumer chooses to not participate (or to not pay κ), then his payoff is

$$\alpha \cdot \int_{\frac{r+D}{2}}^{r} (r-z) dF_D(z) + (1-\alpha) \cdot 0;$$

and if he chooses to participate instead, then he would get

$$-\kappa + \alpha^2 \int_{\frac{r+D}{2}}^{r} \int_{\frac{r+D}{2}}^{r} [r - \min(x,y)] dF_D(x) dF_D(y)$$

$$+2\alpha(1-\alpha) \cdot \int_{\frac{r+D}{2}}^{r} (r-z) dF_D(z) + (1-\alpha)^2 \cdot 0,$$

so that we have in equilibrium, given $D \in [0, \frac{r}{2})$,

$$\kappa_D = \alpha^2 \int_{\frac{r+D}{2}}^{r} \int_{\frac{r+D}{2}}^{r} [r - \min(x,y)] dF_D(x) dF_D(y) + (\alpha - 2\alpha^2) \cdot \int_{\frac{r+D}{2}}^{r} (r-z) dF_D(z),$$

[6]This happens because either the rival seller would not spend ϕ or it would but then its price would exceed $\frac{r+D}{2}$ with probability one.

[7]When its rival seller chooses to spend ϕ and advertise a price $p \in [\frac{r+D}{2}, r]$, which by assumption, captures all consumers' patronage.

which, with $\alpha = \frac{1}{2}$, is equal to

$$\kappa_D = \frac{1}{4}\int_{\frac{r+D}{2}}^{r}\int_{\frac{r+D}{2}}^{r}[r - \min(x,y)]dF_D(x)dF_D(y).$$

There is no need to get a closed-form solution for the above integral.

For part (iii) and (iv), it is easy to verify that the presence of an ISA hurts a seller (as $\frac{r+2D}{4} < \frac{r}{2}$),[8] but when the sellers are partially financed by debt, they are hurt less.[9]

For part (iv), note that a seller can fully repay its debt in the absence of an ISA, since $D \leq \frac{r}{2}$. The presence of an ISA not only reduces a seller's expected profits, it also implies that the same amount of debt, D, is no longer default-free. In both the event that the seller spends ϕ and the event that the seller does not, the seller's profits may fall short of D with a positive probability.

4. (**Should Competitive Physical Retailers Go Mobile?**) Consider a Hotelling city, where consumers uniformly reside along the unit interval $[0,1]$, and two firms A and B are located respectively at the left and the right endpoints of the unit interval. Firm j produces a single product j without costs, $j \in \{A, B\}$, and each consumer may either buy 1 unit of product A, or buy 1 unit of product B, or buy nothing.

   At each point $t \in [0,1]$, there exists exactly one consumer, whom we shall refer to as consumer t. Consumer t would obtain a surplus $v - p_A - tc$ if he buys from firm A, and a surplus $v - p_B - (1-t)c$ if he buys from firm B, where $v > 0$ is the gross utility that consumer t derives from consuming 1 unit of product A or product B, $c > 0$ represents the consumer's round-trip transportation cost per unit distance,

[8] This can be seen by letting $D = 0$, where both sellers choose to participate in the ISA and then price below r with a positive probability. This is another version of the prisoner's dilemma: given that its rival seller plans to join the ISA and that all consumers are prepared to visit the ISA, a seller has no better choice than join the ISA also.

[9] The presence of debt implies that (the shareholder of) a seller would never price too low: an overly low price may benefit the debtholders by increasing the seller's profits in the event of default, which however does not benefit the shareholders.

and p_j is the retail price chosen by firm j. A consumer gets zero surplus if he buys nothing.

We shall remodel the game by allowing the two firms to first decide whether to engage in mobile marketing before starting price competition. In the following, we assume that v is very large so that in equilibrium all consumers will be served. We assume that if a buyer t can obtain the same consumer surplus from buying from either firm, then *it is equally likely that he may buy from either firm.* We shall use the following lemma:

Lemma 1 *Consider a simultaneous game where firms 1 and 2 endowed with the same product are competing in price to serve a single buyer with unit demand. Let v denote the buyer's willingness to pay. Suppose that $v > c_2 > c_1 > 0$, where $c_j > 0$ is firm j's unit cost of operation. This game, referred to as a* Bertrand game with differential costs*, has a continuum of payoff-equivalent equilibria $(p_1, \tilde{p}_2)$, where $p_1 = c_2$, and the distribution function for firm 2's random price $\tilde{p}_2$, denoted by $F_2(\cdot)$, is such that*[10]

$$\frac{x - c_2}{x - c_1} \leq F_2(x), \ \forall x \in (c_2, v].$$

In any such equilibrium, firm 1 gets to serve the buyer for sure.

The new game consists two stages and it proceeds as follows.

- At stage 1, the firms must simultaneously announce whether or not they would adopt uniform pricing in stage 2. A firm that makes such a commitment is said to have adopted U in stage 1. A firm that chooses to not adopt U in stage 1 would learn about each consumer's location t separately before making a price offer in stage 2, and such a firm is said to have adopted D (denoting price discrimination) in stage 1.

[10] In words, firm 2's equilibrium mixed strategy must ensure that firm 1 does not wish to price at anywhere higher than c_2. This requires that $F_2(\cdot)$ be first-order stochastically dominated by some benchmark distribution function.

- At stage 2, the firm(s) that have adopted U at stage 1 must simultaneously announce a unit price that applies to all buyers; let p_i denote the announced uniform price if firm i has chosen to adopt U. Upon seeing these announced uniform prices, the firm(s) that chose to adopt D at stage 1 must simultaneously announce a price schedule that specifies a (probably different) price for each distinct buyer $t \in [0,1]$; let $p_j(\cdot)$ denote such a price schedule if firm j has chosen to adopt D.[11]
- Then, simultaneously, each and every buyer $t \in [0,1]$ learns about the two firms' price offers, and decides whether to buy 1 unit from firm A or to buy 1 unit from firm B, or not to make any purchase.

(i) Show that if the stage-1 outcome is (U,U), then at stage 2 both firms would announce the uniform unit price c and obtain profit $\frac{c}{2}$.

(ii) Suppose that the stage-1 outcome is (D,D). Show that at stage 2,

- when $t < \frac{1}{2}$, firm A would announce $p_A(t) = c(1-t) - ct$ with probability one and firm B would adopt a mixed strategy $\tilde{p}_B$ whose distribution function $F_B(\cdot)$ is such that $F_B(0) = F_B(0+) = 0$, $F_B(v - c(1-t)) = 1$, and $p[1 - F_B(p)] \leq p_A(t)$ for all $p \in (0, v - c(1-t)]$;
- when $t > \frac{1}{2}$, firm B would announce $p_B(t) = ct - c(1-t)$ with probability one and firm A would adopt a mixed strategy $\tilde{p}_A$ whose distribution function $F_A(\cdot)$ is such that $F_A(0) = F_A(0+) = 0$, $F_A(v - ct) = 1$, and $p[1 - F_A(p)] \leq p_B(t)$ for all $p \in (0, v - ct]$; and
- when $t = \frac{1}{2}$, both firms would announce $p_A(t) = p_B(t) = 0$.

Conclude that, before learning about the realization of t, each firm gets expected profit $\frac{c}{4}$.[12]

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[11] We have in mind the following scenario: a firm can use some free app to identify each and every buyer's real-time location as long as every buyer keeps using a smart phone all the time (which we assume). A firm that has adopted U has committed to give up this opportunity of offering different prices to individual buyers.

[12] **Hint**: Although the firms do not have production costs, they must ultimately re-

(iii) Show that if the stage-1 outcome is (U,D), then at stage 2 firm A would announce the uniform price $p_A = \frac{c}{2}$ and firm B would announce the price schedule $p_B(t) = \max[p_A + ct, c(1-t)] - c(1-t)$, so that buyers located in $[0, \frac{1}{4}]$ would buy from firm A and the rest buyers would buy from firm B. Show that firm A's profit is $\frac{c}{8}$ and firm B's profit is $\frac{9c}{16}$.[13]

(iv) Show that in any pure-strategy equilibrium the stage-1 outcome must be (D,D). Conclude that mobile geographic targeting reduces profits, but the two firms must engage in mobile geographic targeting (a form of prisoners' dilemma).[14]

Solution. Part (i) follows directly from Problem 2 of Homework 3. Part (ii) follows directly from Lemma 1.

To gain some intuition, we can look at the case where the two firms compete in the *delivered price* (or *FOB, free-on-board price*) rather than in the *mill price*, assuming that the firms, instead of a buyer, must pay the transportation cost and make the delivery to the buyer; here we assume that the firms and the buyer would incur the same amount of transportation cost.[15]

Thus let $P_A(t)$ and $P_B(t)$ denote the delivered prices charged to buyer t by firms A and B. Given t, firms A and B must respectively incur a unit cost ct and $c(1-t)$ to make the delivery. Since the two firms are selling a homogeneous product, this game given t is a Bertrand game with differential costs. Thus we can quote Lemma 1.

imburse a buyer's transportation costs tc or $(1-t)c$. Thus given t, the two firms essentially become Bertrand-competitive firms with different unit costs. In this case, the low-cost firm must win the buyer's patronage with probability one; see Lemma 1.

[13]**Hint**: Given t and p_A, if $p_A + ct > (1-t)c$, then firm B can and will find a price p_B to win buyer t's patronage. Thus when pricing uniformly at p_A, firm A knows that only those buyers with $p_A + ct \leq (1-t)c$ would ultimately buy from firm A.

[14]We have assumed that getting the app service to identify a buyer's location is costless. Hence conclude that a very high price for the app service can benefit the firms.

[15]Think of a pizza man, who delivers a pizza at a delivered price equal to the mill price plus the actual transportation cost for making that delivery. Note that we have assumed in Problem 2 of Homework 3 that a buyer must pay for his own transportation cost.

According to Lemma 1, if $ct < c(1-t)$, then $P_A(t) = c(1-t)$, and firm B must adopt a mixed pricing strategy $\tilde{P}_B(t)$ whose realizations are higher than $c(1-t)$ with probability one; and similarly, if $ct > c(1-t)$, then $P_B(t) = ct$, and firm A must adopt a mixed pricing strategy $\tilde{P}_A(t)$ whose realizations are higher than ct with probability one. At $t = \frac{1}{2}$, so that the two firms are faced with the same unit cost, they must price at $P_A(\frac{1}{2}) = P_B(\frac{1}{2}) = ct = c(1-t) = \frac{c}{2}$ and each gets zero profits. Now, we can recover the equilibrium mill prices by simply noting that $P_A(t) = p_A(t) + ct$ and $P_B(t) = p_B(t) + c(1-t)$.

Note that in equilibrium firm A wins buyer t's patronage for sure if $t < \frac{1}{2}$, and firm B wins buyer t's patronage for sure if $t > \frac{1}{2}$. To compute firm A's equilibrium expected profit, note that

$$\int_0^{\frac{1}{2}} p_A(t)dt = c \cdot (t - t^2)|_0^{\frac{1}{2}} = \frac{c}{4}.$$

Similarly, to compute firm B's equilibrium expected profit, note that

$$\int_{\frac{1}{2}}^1 p_B(t)dt = c \cdot (t^2 - t)|_{\frac{1}{2}}^1 = \frac{c}{4}.$$

Consider part (iii). Given t and p_A, if $p_A + ct > (1-t)c$, then firm B, which operates at zero costs, can and will find a price p_B to win buyer t's patronage. Thus when pricing uniformly at p_A, firm A knows that only those buyers with $p_A + ct \leq (1-t)c \Leftrightarrow t \leq \frac{c-p_A}{2c}$ would ultimately buy from firm A. Hence at the beginning of stage 2 firm A seeks to

$$\max_{p_A} (\frac{c - p_A}{2c}) p_A$$

so that firm A would price at $p_A = \frac{c}{2}$ and obtains a volume of sales equal to $\frac{1}{4}$, implying that firm A's equilibrium profit is $\frac{c}{8}$.

On the other hand, firm B would lose to firm A when facing any buyer $t \in [0, \frac{1}{4})$. One best response for firm B when facing buyer $t = \frac{1}{4}$ is obviously $p_B(t) = 0$. When facing any buyer $t > \frac{1}{4}$, firm B would price just slightly below $p_A + ct - (1-t)c = c(2t - \frac{1}{2})$ and win the buyer's patrongage for sure. (Strictly speaking, firm B does not have a well-defined best response in this subgame!) Consequently, firm B's equilibrium expected profit is essentially

$$\int_{\frac{1}{4}}^{1} \frac{c}{2}(4t-1)dt = \frac{c}{2} \cdot (2t^2 - t)|_{\frac{1}{4}}^{1} = \frac{c}{2} \cdot (1 + \frac{2}{16}) = \frac{9c}{16}.$$

Consider part (iv). It is clear that (U,D) is inconsistent with a pure-strategy equilibrium: firm A would get $\frac{c}{8}$ under (U,D), but it would get $\frac{c}{4}$ if it would deviate and adopt D in stage 1 instead! For the same reason, (D,U) is inconsistent with a pure-strategy equilibrium either.

It is also clear that (U, U) is inconsistent with a pure-strategy equilibrium: firm B would get $\frac{c}{2}$ under (U, U), but it would get $\frac{9c}{16}$ if it would deviate and adopt D in stage 1 instead!

Finally, it is obvious that (D,D) is consistent with equilibrium, and it is the stage-1 outcome of the unique pure-strategy equilibrium of the game.

Remark. Allowing the players to have a larger strategy space may lead to a lower equilibrium payoff for each and every player—we have learned this fact since Lecture 1. Here, allowing the players to engage in third-degree price discrimination results in each and every player getting a lower equilibrium payoff, and although the players would become better off if they could both commit to not performing third-degree price discrimination, each of them has an incentive to renege on such a commitment. This is one version of the prisoners' dilemma that we discussed in Lecture 1.

To compare what happens in stage 2 under respectively (U,U) and (D,D), note that under (U,U) firm i cannot serve firm j's local buyers with a very low price, because pricing uniformly in this way would mean to give up the chance to extract surplus from one firm's own local buyers. (This reasoning has been emphasized in Problem 1 as well.) Under (D,D), because firm i can serve different buyers at different prices, firm i can offer a very low price to firm j's local buyers without having to worry about firm i's own local buyers. This forces firm j to defend its turf by offering a much lower price. In equilibrium, price discrimination forces both firms to charge lower prices to their own local buyers, but without changing a firm's equilibrium turf: firm A is still serving buyers at $t < \frac{1}{2}$ and firm B is still serving buyers at $t > \frac{1}{2}$. Hence both firms end up having lower profits under (D,D) than under (U,U).

5. (**A CSV Model with Inefficient Mergers.**) Consider two firms, called firm 1 and firm 2, run by risk-neutral entrepreneurs A_1 and A_2 respectively. Both firms are penniless. Each firm is endowed with one investment project at date 0, which needs a date-0 capital outlay of $I = \frac{18}{100}$, which the firm must raise from (one of) the competitive risk-neutral investors. Nobody in this exercise has time preferences. For $j = 1, 2$, firm j's project will generate date-1 cash flow $\tilde{z}_j \geq 0$, whose realization z_j is costlessly observable to only entrepreneur j. An investor must spend a cost c for state verification if he also wishes to learn this realization, and following state verification, z_j will become public information at date 1. The game proceeds as follows.

   - At date 0, the two entrepreneurs A_1 and A_2 can decide whether to have their firms merged into a new firm, and if they decide to do so, then each of them will hold $\frac{1}{2}$ of the new firm's equity. The merger itself would be inefficient. It creates no synergistic gain; instead, if the new firm can raise $2I = \frac{36}{100}$ at date 0, then the new firm's date-1 cash flow generated by its investment project (which the new firm can choose to abandon) will be equal to $z_1 + z_2 - \delta$, where $\delta > 0$ is a constant.

   - If a merger has taken place at date 0, then at date 0 the new firm must either abandon its investment projects or try to raise $2I$ by offering an optimal financial contract to (one of) the competitive investors, and the chosen investor can either accept or reject the contract. The game ends at date 0 if the new firm's offer gets turned down by investors, and in that case each entrepreneur's payoff is zero; otherwise, the investor provides the amount $2I$ of cash to the new firm, and the game moves on to date 1.

- If there has been no merger at date 0, then at date 0 each firm j can either abandon its investment project or try to raise I by offering its own financial contract to (one of) the competitive investors. The game ends at date 0 for firm j, if A_j's offer gets turned down by investors, and in that case A_j's payoff is zero; otherwise, the chosen investor must provide the amount I of cash to firm j, and the game moves on to date 1.

- Suppose that the game continues at date 1. Then the cash flows are realized and the firm(s) and the contracting investor(s) must then act according to the date-0 contract(s) regarding state verification and profits-sharing.

Suppose that

$$c = \frac{7}{20}, \quad z_1 = z, \quad z_2 = 1 - z,$$

where z is uniformly distributed over the unit interval $[0, 1]$.

(i) Suppose that the two firms have chosen to not undergo the merger. What is the equilibrium face value of debt chosen by entrepreneur A_1? What is entrepreneur A_2's equilibrium payoff (computed at date 0)?

(ii) Suppose that the two firms have completed the merger. In this subgame, the new firm will implement its investment projects if and only if $\delta \leq \delta^*$. What is δ^*?

(iii) In equilibrium a merger of the two firms occurs at date 0 if and only if $\delta \leq \delta^{**}$. What is δ^{**}?

Solution. Consider part (i). Let the face value of debt be F_1 in A_1's offer. We must solve the following binding IR_B condition:

$$\frac{18}{100} = I = (E[z|z < F_1] - c)\text{prob.}(z < F_1) + F_1\text{prob.}(z \geq F_1)$$

$$= (\frac{F_1}{2} - c)F_1 + F_1(1 - F_1)$$
$$= -\frac{F_1^2}{2} + (1 - c)F_1$$
$$= -\frac{F_1^2}{2} + \frac{13}{20}F_1$$
$$\Rightarrow F_1 = \frac{18}{20} \text{ or } \frac{8}{20},$$

so that $F_1 = \frac{2}{5}$. Entrepreneur A_1's payoff from offering a debt contract with face value equal to $F_1 = \frac{1}{2}$ is

$$E[z_1] - I - c \cdot \text{prob.}(z_1 < F_1)$$
$$= \frac{1}{2} - \frac{18}{100} - \frac{7}{20} \cdot \frac{2}{5} = \frac{9}{50}.$$

By symmetry, this is also entrepreneur A_2's equilibrium payoff.

Consider part (ii). If the two firms choose to merge into a new firm, then by undertaking its investment projects, the new firm's date-1 cash flow will be equal to $z_1 + z_2 - \delta = 1 - \delta$ for sure if the new firm can raise $2I = \frac{36}{100}$ at date 0. The new firm will go ahead with these investment projects if and only if

$$\delta \leq \delta^* = \frac{64}{100}.$$

This finishes part (ii).

Finally, consider part (iii). If the condition stated in part (ii) holds, then entrepreneur A_1's payoff after going through the merger and costless borrowing would be

$$\frac{1}{2}[\frac{64}{100} - \delta],$$

which exceeds $\frac{9}{50}$, the payoff that entrepreneur A_1 would obtain if firm 1 remains stand-alone, if and only if

$$\delta \leq \delta^{**} = \frac{28}{100} = \frac{7}{25}.$$

Remark. More generally, corporate hedging is beneficial when the firm is faced with a CSV problem. Without such a problem, individual investors can hedge their own risks, and corporate hedging cannot do a better job. In the presence of a CSV problem, corporate hedging can raise the lowest possible outcome of the firm's date-1 cash flow, which reduces the expected dead-weight cost that the borrowing firm and the lending investor have to spend on earnings verification. Here, in this exercise, corporate hedging takes a very costly form: going through a merger with another firm whose date-1 cash flow (if it gets financed) will be highly negatively correlated with the current firm's date-1 cash flow.