Game Theory with Applications to Finance and Marketing, I

Solutions to the Final Exam

Name: __________ ID: __________

Write your solutions into the following table:

Question No.	No. 1	No. 2	No. 3	No. 4	No. 5
Your Solution	4	$\frac{3}{2}$	$\frac{33}{8}$	$\frac{37}{16}$	0
Question No.	No. 6	No. 7	No. 8	No. 9	No. 10
Your Solution	2	$\frac{25}{8}$	$\frac{23}{16}$	$\frac{3}{2}$	$\frac{1}{4}$
Question No.	No. 11	No. 12	No. 13	No. 14	No. 15
Your Solution	0	$\frac{\theta_2}{k}$	$\frac{\pi_2\theta_2^2}{2k}$	20	48
Question No.	No. 16	No. 17	No. 18	No. 19	No. 20
Your Solution	219	457	13	84	44
Question No.	No. 21	No. 22	No. 23	No. 24	No. 25
Your Solution	46	S	48	$\frac{4}{5}$	$\frac{26}{5}$
Question No.	No. 26	No. 27	No. 28	No. 29	No. 30
Your Solution	$\frac{3}{10}$	$\frac{16}{5}$	$\frac{3}{5}$	$\frac{22}{5}$	1
Question No.	No. 31	No. 32	No. 33	No. 34	No. 35
Your Solution	2	4	$\frac{7}{8}$	—	—

1. (**The Hart and Moore Cash Diversion Model with CDS Trading.**) Consider the following Hart and Moore cash diversion model with CDS trading.

   The penniless entrepreneur (who is also the single shareholder) of firm I needs to raise $F > 0$ at date 0 in order to undertake a two-period investment project. There are two equally likely unverifiable date-1 states, G and B. The project has zero liquidation value at date 2 in either state.

   (a) In state G, the investment project generates no date-1 cash inflow, but a fraction $\delta \in [0, 1]$ of the firm's production capacity can be liquidated to generate a verifiable date-1 cash inflow of δX, where $X > 0$. Then, at date 2, the firm can use the remaining capacity to generate an unverifiable cash inflow $(1-\delta)X$. Note that in state G, liquidation and continuation at date 1 are equally efficient.

   (b) In state B, the project generates an un-verifiable date-1 cash inflow y_1, and then a fraction λ of the firm's production capacity can be liquidated to generate a verifiable date-1 cash inflow of λL, where $\lambda \in \{0, \frac{3}{4}, 1\}$. Thus, unlike in state G, in state B the date-1 liquidation is a discrete choice. Then, at date 2, the firm can obtain profits $y_2(\lambda)$ using the remaining capacity to produce q_I units of product Y, where $q_I \leq (1 - \lambda)a$, with $a > 0$ being a constant.

   Let us be more specific about $y_2(\lambda)$. At the beginning of date 2, a rival firm E can first observe firm I's date-2 liquidation decision λ and then decide whether to spend a set-up cost $K_E > 0$ to enter the market for product Y.

   - If firm E chooses to enter, then firm I must first announce a supply quantity $q_I \leq (1-\lambda)a$, and upon seeing q_I, firm E then announces its supply quantity q_E, and the price for product Y is then determined via the following inverse demand function for product Y,

     $$p = a - q_I - q_E,$$

     where $a > 0$ is the constant defined above. Thus firm I (the incumbent) and firm E (the entrant) are respectively the leader and the follower in the date-2 Stackelberg game of output competition.

- If instead firm E chooses to stay out at the beginning of date 2, then firm I becomes monopolistic in the market for product Y, and it alone will then announce a profit-maximizing supply quantity $q_I \leq (1-\lambda)a$ at date 2.
- In either case, there are no production costs for firm I and firm E at date 2.

We shall consider both the scenario that there is CDS trading and the scenario that there is no CDS trading. In either scenario, we assume that firm I has only one creditor at date 1. Assume the following parameter values:

$$F = \frac{41}{16},\ K_E = 2,\ L = 1,\quad a = 4,\quad X = \frac{9}{2},\quad y_1 = \frac{5}{4}.$$

Again, the date-0-date-1 period for firm I is the familiar Hart and Moore cash diversion model, and we assume with Hart and Moore that firm I can make a take-it-or-leave-it offer to its creditor in date-1 debt renegotiation, and with CDS trading, the creditor can use his CDS position, which is not contractible at date 0, to enhance his bargaining power against firm I in state B.[1]

Let D_0 and D_π denote respectively the equilibrium face values of debt in the absence and presence of CDS trading.

(i) It can be shown that $y_2(0) =$ __No. 1__ and $y_2(\frac{3}{4}) =$ __No. 2__.

(ii) First suppose that there is no CDS trading. In this case, the face value of firm I's debt D_0 is equal to __No. 3__, and the entrepreneur (of

[1]Recall that, whether there is CDS trading or not, a date-0 feasible financial contract for firm I is simply a short-term debt maturing at date 1, which gives the creditor the right to liquidate the firm at date 1 in (and only in) the event that firm I fails to fully repay the creditor. When there is CDS trading, by purchasing a CDS protection (z, π), the creditor must pay z to the CDS seller upfront, and the creditor will receive π from the CDS seller at date 1 if and only if firm I fails to fully repay the creditor and upon non-payment firm I and the creditor fail to renegotiate the debt contract to mutually acceptable terms. We again assume that the creditor can make its purchase of (z, π) contingent on the date-1 state $j \in \{G, B\}$, but his CDS position is not contractible. We also assume that the CDS seller would break even by selling CDS protection (z, π) to the creditor.

firm I) has an equilibrium payoff equal to _No. 4_. In state B, firm E's date-2 payoff is equal to _No. 5_.[2]

(iii) Now, suppose instead that there is CDS trading. Suppose that firm I's creditor can determine his CDS position $\pi \in [0, D_\pi]$ after learning the date-1 state. In state B, $\pi =$ _No. 6_. The face value of firm I's debt D_π is equal to _No. 7_,[3] and the entrepreneur (of firm I) has an equilibrium payoff equal to _No. 8_. In state B, firm I's date-2 profit is equal to _No. 9_ and firm E's date-2 payoff is equal to _No. 10_.

Solution. Consider part (i). Given q_I and λ, firm E would produce $\frac{a-q_I}{2}$ if firm E chooses to enter, which implies a post-entry payoff of $(a - q_I - q_E)q_E = \frac{(a-q_I)^2}{4}$ for firm E. Firm I can choose $q_I \leq (1-\lambda)a$ to either deter entry, or to accommodate entry.

- To deter entry, firm I's problem is to

  $$\max_{q_I} q_I(a - q_I - q_E)$$

  subject to

  $$q_I \leq (1-\lambda)a, \quad q_E = 0, \quad \frac{(a-q_I)^2}{4} \leq K_E.$$

  We can re-write firm I's maximization problem as

  $$\max_{q_I} q_I(a - q_I)$$

  subject to

  $$4(1 - \frac{\sqrt{2}}{2}) = a - 2\sqrt{K_E} \leq q_I \leq (1-\lambda)a,$$

  which is feasible only when $\lambda = 0$. When $\lambda = 0$, to deter entry, firm I would choose $q_I = \frac{a}{2}$, which yields for firm I the payoff of $\frac{a^2}{4} = 4$.

[2] Conjecture that $L < D_0 < X$ and then verify.
[3] Conjecture that $y_1 + L < D_\pi < X$ and then verify.

- On the other hand, to accommodate entry, firm I's problem is to

$$\max_{q_I} q_I(a - q_I - q_E)$$

subject to

$$q_I \leq (1-\lambda)a, \quad q_E = \frac{a - q_I}{2}, \quad \frac{(a-q_I)^2}{4} \geq K_E.$$

We can re-write firm I's maximization problem as

$$\max_{q_I} \frac{1}{2} q_I(a - q_I)$$

subject to

$$q_I \leq 4\min(1 - \frac{\sqrt{2}}{2}, 1-\lambda) = \begin{cases} 4 - 2\sqrt{2}, & \text{if } \lambda = 0; \\ 1, & \text{if } \lambda = \frac{3}{4}; \\ 0, & \text{if } \lambda = 1. \end{cases}$$

Thus to accommodate entry, firm I's best choice of output is

$$q_I = \begin{cases} 4 - 2\sqrt{2}, & \text{if } \lambda = 0; \\ 1, & \text{if } \lambda = \frac{3}{4}; \\ 0, & \text{if } \lambda = 1, \end{cases}$$

and with the best output choice firm I's payoff from accommodating entry is

$$\frac{1}{2} q_I(a - q_I) = \begin{cases} 8(\sqrt{2} - 1), & \text{if } \lambda = 0; \\ \frac{3}{2}, & \text{if } \lambda = \frac{3}{4}; \\ 0, & \text{if } \lambda = 1, \end{cases}$$

Summing up the above discussions, we conclude that entry will be deterred if and only if $\lambda = 0$, and that

$$y_2(1) = 0, \; y_2(\frac{3}{4}) = \frac{3}{2}, \; y_1(0) = \max[8(\sqrt{2}-1), 4] = 4.$$

Consider part (ii). We conjecture that $L < D_0 < X$. Under this conjecture, the creditor receives D_0 in state G. Since in state B the entrepreneur of firm I has all bargaining power against the creditor in debt renegotiation following default, and since $y_1 > L$, the entrepreneur would pay the creditor L out of y_1 and then choose $\lambda = 0$. The creditor's date-0 break-even condition then gives rise to

$$D_0 = 2F - L = \frac{33}{8},$$

confirming our conjecture that $L < D_0 < X$. The entrepreneur of firm I has an equilibrium payoff equal to

$$\frac{1}{2} \cdot [y_2(0) + y_1] + \frac{1}{2} \cdot X - F = \frac{37}{16}.$$

Consider part (iii). First we conjecture that $y_1 + L < D_\pi < X$ so that in state G firm I would fully repay its debt in the absence of CDS trading, which in turn implies that CDS trading plays no role in state G.

Consider the subgame of date-1 debt renegotiation in state B. How does the position π in CDS affect the creditor's payoff in this subgame?

- A position $\pi \leq L$ in CDS is equivalent to $\pi = 0$, and hence with such a π the creditor gets only L.
- Suppose that the creditor has chosen $\pi \in (L, y_1]$. The entrepreneur of firm I would get y_1 if he gives up renegotiation, and he would get $y_1 - \pi + y_2(0)$ if repaying the creditor π during renegotiation and choosing $\lambda = 0$. Since $y_2(0) = \frac{a^2}{4} = 4 > y_1 \geq \pi$, firm I would rather repay π to the creditor than give up renegotiation. Thus the creditor's optimal choice of $\pi \in (L, y_1]$ is y_1.
- Suppose that the creditor has chosen $\pi \in (y_1, y_1 + \frac{3}{4}L]$. The entrepreneur of firm I would get y_1 if he gives up renegotiation, and he would get $y_1 + \frac{3}{4}L - \pi + y_2(\frac{3}{4})$ if repaying the creditor π during renegotiation and choosing $\lambda = \frac{3}{4}$. Note that

$$y_2(\frac{3}{4}) = \frac{3}{2} > \frac{5}{4} = y_1 \Rightarrow \frac{3}{4}L - \pi + y_2 > \frac{3}{4}L - \pi + y_1 \geq 0,$$

and hence firm I would choose repaying the creditor π over giving up renegotiation, which in turn implies that the creditor's optimal choice of $\pi \in (y_1, y_1 + \frac{3}{4}L]$ is $y_1 + \frac{3}{4}L = 2$.

- Suppose that the creditor has chosen $\pi \in (y_1 + \frac{3}{4}L, y_1 + L]$. The entrepreneur of firm I would get y_1 if he gives up renegotiation, and he would get $y_1 + L - \pi + y_2(1)$ if repaying the creditor π during renegotiation and choosing $\lambda = 1$. Since $\pi > L$, firm I would give up renegotiation, and hence given $\pi \in (y_1 + \frac{3}{4}L, y_1 + L]$ a credit event would take place for sure from the CDS seller's perspective. Since the CDS seller would eventually get L back, the creditor must pay the CDS seller $\pi - L$ when taking the position π in CDS. Thus the creditor's payoff from taking any position $\pi \in (y_1 + \frac{3}{4}L, y_1 + L]$ would be $\pi - (\pi - L) = L$.
- Suppose that the creditor has chosen $\pi \in (y_1 + L, D_\pi]$. In this case, firm I would certainly give up renegotiation, and hence a credit event would take place for sure from the CDS seller's perspective. Since the CDS seller would eventually get L back, the creditor must pay the CDS seller $\pi - L$ when taking the position π in CDS. Thus the creditor's payoff from taking any position $\pi \in (y_1 + \frac{3}{4}L, y_1 + L]$ would again be $\pi - (\pi - L) = L$.

Summarizing the above discussions, the creditor would optimally take the position $\pi = y_1 + \frac{3}{4}L = 2$ in state B, which results in firm I choosing $\lambda = \frac{3}{4}$ and obtaining a date-2 profit of $y_2(\lambda) = \frac{3}{2}$. The date-2 equilibrium payoff is $\frac{9}{4} - K_E = \frac{1}{4}$ for firm E.

It follows that

$$D_\pi = 2F - \pi = \frac{41}{8} - 2 = \frac{25}{8},$$

confirming our conjecture that $y_1 + L < D_\pi < X$. Thus the entrepreneur of firm I has an equilibrium payoff equal to

$$\frac{1}{2} \cdot (X - D_\pi) + \frac{1}{2} \cdot \frac{a}{4} \cdot (a - \frac{a}{4} - \frac{a - \frac{a}{4}}{2}) = \frac{23}{16}.$$

Remark. Owing to the fact that output choices are strategic substitutes, CDS trading encourages entry by firm E in state B by reducing firm I's date-2 capacity. Once again, if the creditor's CDS position were contractible, the optimal contract would restrict his CDS position in state B, simply because liquidation is more costly in state B than in state G.

In general, there may be a large number of different date-1 states, and allowing CDS trading in some date-1 states may help enhance the chance that the creditor would approve financing at date 0. Thus the restriction to be placed on the creditor's CDS position had better be state-dependent. In reality, the states of nature that are relevant for efficient contracting may be un-verifiable, but then it would be infeasible to enforce a state-dependent restriction on a trader's CDS position.

2. (**The Optimal Product Line Design with Advance Selling.**) A seller is about to produce two product items (q_1, q_2) to serve one buyer that has two possible types, represented as θ_1 and θ_2. The buyer has unit demand, and he derives gross consumption utility $\theta_j q$ from consuming a product of quality $q \in \Re_+$ when his type is θ_j. The seller can trade with the buyer at either date 0 or date 1. The buyer will privately learn about his own type at date 1. At date 0, it is the seller and buyer's common knowledge that with probability π_j the buyer may be of type θ_j, where

$$0 < \theta_1 < \frac{\pi_2}{\pi_1}(\theta_2 - \theta_1).$$

To produce 1 unit of a product of quality q the seller must incur a cost of $\frac{kq^2}{2}$, where $k > 0$ is a constant.

(i) (**Spot Selling.**) Suppose that the seller and the buyer can only trade at date 1, after the buyer has privately learned about his own type. In this case, by the revelation principle, the seller seeks to

$$\max_{q_1, T_1, q_2, T_2} \pi_1(T_1 - \frac{kq_1^2}{2}) + \pi_2(T_2 - \frac{kq_2^2}{2})$$

subject to

$$\theta_1 q_1 - T_1 \geq 0;$$

$$\theta_2 q_2 - T_2 \geq 0;$$

$$\theta_1 q_1 - T_1 \geq \theta_1 q_2 - T_2;$$
$$\theta_2 q_2 - T_2 \geq \theta_2 q_1 - T_1;$$
$$q_1, q_2 \geq 0, \quad T_1, T_2 \in \Re.$$

At optimum, we have $q_1 =$ No. 11 , $q_2 =$ No. 12 , and the seller's payoff is No. 13 .

(ii) (**Advance Selling.**) Now, suppose that the seller wishes to trade with the buyer at date 0. In this part, we shall assume that the buyer is risk neutral but the seller is risk averse, and the seller would derive utility $u(w)$ when his wealth is w after trading with the buyer. Assume that the seller's initial wealth is zero, and $u(w) = \sqrt{w}$. Assume the following parameter values also:

$$\pi_1 = \pi_2 = \frac{1}{2},\ \theta_1 = 5,\ \theta_2 = 12,\ k = \frac{1}{4}.$$

Given (q_1, T_1, q_2, T_2), define

$$w_j \equiv T_j - \frac{kq_j^2}{2},\ j = 1, 2,$$

so that w_j is the seller's realized wealth after serving the type-θ_j buyer; and define

$$v_j \equiv \theta_j q_j - T_j,\ j = 1, 2,$$

so that v_j is the type-θ_j buyer's consumer surplus.

Now, consider the following set of incentive feasible contracts:

$$\mathcal{C} \equiv \{(q_1, T_1, q_2, T_2) :\ w_1 = w_2 = w^*,\ \pi_1 v_1 + \pi_2 v_2 = 0,\ q_1, q_2 \geq 0,\ T_1, T_2 \in \Re.\}.$$

We shall look for the optimal contract that solves the following maximization problem for the seller:

$$\max_{(q_1, T_1, q_2, T_2) \in \mathcal{C}} u(w^*)$$

subject to

$$v_1 \geq \theta_1 q_2 - T_2;$$
$$v_2 \geq \theta_2 q_1 - T_1.$$

At optimum, we have $q_1 = \underline{\text{No. 14}}$, $q_2 = \underline{\text{No. 15}}$, $T_1 = \underline{\text{No. 16}}$, $T_2 = \underline{\text{No. 17}}$, and the seller's payoff (i.e., $u(w^*)$) is equal to $\underline{\text{No. 18}}$.

Solution. For part (i), we conjecture that the two constraints IR_1 and IC_2 will be binding at optimum, so that the seller seeks to

$$\max_{q_1,q_2\geq 0} f(q_1,q_2) \equiv \pi_1(\theta_1 q_1 - \frac{kq_1^2}{2}) + \pi_2[\theta_2 q_2 - q_1(\theta_2-\theta_1) - \frac{kq_2^2}{2}],$$

where, thanks to the assumption that

$$0 < \theta_1 < \frac{\pi_2}{\pi_1}(\theta_2 - \theta_1),$$

f is decreasing in q_1, and hence we have $q_1 = 0$ at optimum. It follows that θ_2 would be the lowest possible type faced with by the seller, and hence the seller can fully extract the type-θ_2 buyer's surplus. It follows that the seller would optimally choose $q_2 = \frac{\theta_2}{k}$, the first-best quality level for the type-θ_2 buyer, and the seller would price it at $T_2 = \theta_2 q_2 = \frac{\theta_2^2}{k}$ accordingly. The seller's payoff at optimum is then

$$\pi_1 \cdot 0 + \pi_2 \cdot [T_2 - \frac{kq_2^2}{2}] = \frac{\pi_2\theta_2^2}{2k}.$$

Consider part (ii). Let us first make a few observations. Here, efficient risk sharing would require that $w_1 = w_2$: the seller is risk averse, but the buyer is not, and hence it is efficient to let the buyer alone bear all the risk. Moreover, to maximize the seller's welfare, the optimal solution must make the buyer's ex-ante IR constraint binding; that is, we must have $\pi_1 v_1 + \pi_2 v_2 = 0$. Thus, intuitively, the optimal contract that the seller would choose must be an element of $\mathcal{C}$!

Now, given that the seller would get a constant payoff w^*, the type-θ_j buyer's surplus can be represented as the social benefit minus w^*,

$$v_j = \theta_j q_j - \frac{kq_j^2}{2} - w^*, \quad j = 1,2.$$

It follows that the type-θ_j buyer would wish the seller to choose the first-best quality level $q_j^{FB} = \frac{\theta_j}{k}$. Indeed, if the seller offers $q_1 = q_1^{FB}$

and $q_2 = q_2^{FB}$, then the type-θ_1 buyer would prefer q_1^{FB} to q_2^{FB}, and the type-θ_2 buyer would prefer q_2^{FB} to q_1^{FB}, so that the two IC constraints would automatically be satisfied.

Based on the above observations, we require the three unknowns (w^*, T_1, T_2) to simultaneously satisfy the following three equations:

$$T_1 - \frac{k}{2}(q_1^{FB})^2 = w^* = T_2 - \frac{k}{2}(q_2^{FB})^2,$$

and

$$\pi_1[\theta_1 q_1^{FB} - \frac{k}{2}(q_1^{FB})^2 - w^*] + \pi_2[\theta_2 q_2^{FB} - \frac{k}{2}(q_2^{FB})^2 - w^*] = 0.$$

Solving, we obtain

$$q_1 = q_1^{FB} = 20, \quad q_2 = q_2^{FB} = 48, \quad w^* = \frac{\pi_1\theta_1^2 + \pi_2\theta_2^2}{2k(\pi_1 + \pi_2)} = 169,$$

$$T_1 = w^* + \frac{\theta_1}{2k} = 219, \quad T_2 = w^* + \frac{\theta_2}{2k} = 457.$$

The seller's payoff at optimum is then

$$u(w^*) = \sqrt{169} = 13.$$

Remark. This exercise shows that ex-ante contracting can attain the first-best solution even when the seller is risk averse, if the buyer's surplus is linear in q.[4] Things are different with ex-post contracting, where, knowing that the buyer has already known his own type, the seller would typically like to screen the buyer: by distorting q_1 downward the seller would be able to reduce the surplus conceded to the type-θ_2 buyer, and the latter benefit typically outweighs the loss caused by the former. Consequently, the second-best solution under ex-post contracting typically differs from the first-best solution.

3. (**More on Advance Selling.**) S owns a ticket to a concert that has been scheduled for date 2. Let X denote the ticket. S is considering selling the ticket to B (an early buyer) at date 1, or wait till date 2 and

[4] This result may cease to hold if the buyer exhibits risk aversion with respect to q, or if the buyer is faced with financial constraints.

then sell the ticket to C (a late buyer). If S keeps the ticket, his own valuation for the date-2 concert is 24. B's valuation v for the concert is random, which may take on 84, 36, or 18 with equal probability. B will privately learn about the realization of v at date 2, right before the concert gets started. C's valuation for the concert is 18, but S will not be able to locate C before date 2, right before the concert gets started. We shall compare three selling formats below for S, assuming that the three persons are all risk neutral without time preferences.

(a) (**Date-2 spot selling**.) S can announce a price p_2, and upon seeing p_2, B and C can simultaneously express willingness to buy. All willing buyers will get the same chance to get X by paying p_2 to S. In case no willing buyer exists, S will keep X and go to the concert by himself.

(b) (**Date-1 advance selling**.) S can make a price offer p_1 to the early buyer B, and B can either accept or reject the offer. Whether trade takes place at date 1 or not, there will be *no* transaction for product X at date 2.

(c) (**Date-1 advance selling with date-2 resale.**) The same as in trading format (b), but whoever keeps X at the beginning of date 2, can either go to the concert on his own, or announce a price p_2' to the other two persons before the concert gets started. Again, in the latter event, all willing buyers have equal chance to get X by paying p_2' to the seller.

(i) Under trading format (a), the equilibrium price $p_2 =$ No. 19 , and S has equilibrium payoff No. 20 .

(ii) Under trading format (b), the equilibrium price $p_1 =$ No. 21 .

(iii) Under trading format (c), the buyer during the date-2 resale must be (answer S, B, or C) No. 22 , and the equilibrium payoff for S is equal to No. 23 .

Solution. Consider part (i). Since S would not offer a price over 84 or below 18, and since S's own valuation is 24, we only need to compare the price 84 to the price 36.

By setting $p_2 = 84$, S's payoff is

$$\frac{1}{3} \cdot 84 + \frac{2}{3} \cdot 24 = 44.$$

By setting $p_2 = 36$, S's payoff is

$$\frac{2}{3} \cdot 36 + \frac{1}{3} \cdot 24 = 32.$$

Thus the equilibrium price choice for S is $p_2 = 84$ and S obtains a payoff of 44.

Consider part (ii). B's expected valuation for X is

$$\frac{84 + 36 + 18}{3} = 46$$

at date 1, and hence S will offer B the price $p_1 = 46$, and obtain a payoff of 46.

Consider part (iii). Now, with the date-2 resale opportunity, B can expect to sell X back to S (never to C!) at date 2 in (and only in) the event that $v = 18$, after B has already obtained X at date 1. Thus with resale B is willing to accept any price less than or equal to 48 at date 1. Recognizing this fact, S will then offer B the price 48 at date 1.

Remark. Why does trade format (b) benefit S more than trade format (a)?[5]

This happens because when S offers p_2 in spot selling S knows that B would have already known his valuation for X, and S would rather offer a high price to bet on the event that B's valuation is 84 than offer a low price to bet on the event that B's valuation is 36. In other words, S's intention to extract more rent from B when B's valuation is 84 destroys the opportunity of completing an efficient trade with B when B's valuation is actually 18!

[5]Finance students would know that other things equal, advance selling has the advantage of getting the cash inflow early. The assumption that no one has time preferences removes this advantage, and allows us to focus on the strategic aspects of advance selling.

With advance selling at date 1, on the other hand, S knows that when p_1 is offered to B, B, just like S, does not know the exact realization of v. Thus S can offer a "bundle," saying that X will be sold to B no matter which realization of v will come up. This way, S ensures that he can sell X to B even if B's valuation is 36.

The cautious reader must have discovered that, selling X to B for sure also has a problem: selling X to B is inefficient when B's valuation is 18, less than S's valuation 24!

However, the efficiency loss from completing an inefficient trade when B's valuation is 18, which is $24 - 18 = 6$, is less than the efficiency gain from completing an otherwise lost trade when B's valuation is 36, which is $36 - 24 = 12$! This explains why S benefits more from trade format (b) than trade format (a), even if trade format (b) is less than perfect.

Now it becomes clear why trade format (c) is better than trade format (b). The resale opportunity provides a remedy for the problem pertaining to (b)! When B learns that his valuation for X is 18, he would be happy to sell X back to S at the price of 24. With rational expectations (or with backward induction), both S and B can see how the resale opportunity would change B's valuation for X at date 1, and hence S can benefit from raising p_1 from 46 to 48.

Finally, note that the resale opportunity in (c) can be replaced by a returns policy, and the latter, like advance selling, is an important topic in marketing theory.[6] More precisely, (b) can attain the same trading efficiency as (c) if we can append to (b) the following refund policy: B can get a refund of 24 if B returns X (in good conditions) to S at date 2.

[6] For a formal theory of advance selling, see Shugan, S., and J. Xie, 2005, Advance-selling as a Competitive Marketing Tool, *International Journal of Research in Marketing,* 22, 351-373. For returns policy, see Anderson, E., K. Hansen, and D. Simester, 2009, The Option Value of Returns: Theory and Empirical Evidence, *Marketing Science*, 28, 405-423.

Note also that we have assumed in (c) that B has all bargaining power against S during resale at date 2. One can show that, as long as S and B have rational expectations, assuming any other allocation of bargaining power for the date-2 resale does not alter our conclusion.

4. (**Equity Financing versus Debt Financing.**) At date 0, firm X, which is penniless and is currently run by a single stockholder, Mr. X, needs to raise x dollars in order to operate. The firm can issue either common stock or standard debt to raise the money from Bertrand competitive investors. The firm has all bargaining power against investors at date 0, and investors and Mr. X are all risk neutral without time preferences. The game proceeds as follows.

   - At date 0, given x, Mr. X can choose either to raise x dollars by issuing common stock or debt. In case of stock financing, an investor giving x dollars to the firm will obtain a fraction a of the firm's date-1 cash flow z; and in case of debt financing, the investor will get a fixed cash inflow F in case $F \leq z$, or the investor will get z in case $F \geq z$. With all bargaining power, Mr. X will determine either a or F for the deal.
   - The game ends with zero payoffs for everyone in case no financing succeeds at date 0. Otherwise, Mr. X invests the x dollars, and then must choose an effort level $q \in [0, 1]$. Choosing q incurs a negative payoff $-\frac{5q^2}{2}$ to Mr. X at date 0, but it implies that at date 1, either $z = 6$ with probability q or $z = 2$ with probability $1 - q$.
   - Then, at date 1, z is realized. In case of stock financing, Mr. X will get $(1 - a)z$, and the funding investor will get az. In case of debt financing, Mr. X will get $\max(z - F, 0)$ and the funding investor (the debtholder) will get $\min(z, F)$.

   (i) Suppose that $x = 2$. In case of debt financing, the equilibrium $q =$ No. 24__, and the firm value is No. 25__. In case of stock financing, the equilibrium $q =$ No. 26__, and the firm value is No. 27__.

   (ii) Now, suppose that $x = \frac{13}{5}$. In case of debt financing, the equilibrium $q =$ No. 28__, and the firm value is No. 29__.

Solution. Let us first consider debt financing. We use backward induction.

At the subgame where Mr. X has raised x dollars by issuing debt with face value F, Mr. X seeks to, given F,

$$\max_{0\leq q\leq 1} -\frac{5q^2}{2} + \max(6-F,0)q + \max(2-F,0)(1-q),$$

so that the optimal choice of q is such that

$$q = \begin{cases} 0, & \text{if } F \geq 6; \\ \frac{6-F}{5}, & \text{if } 6 > F \geq 2; \\ \frac{4}{5}, & \text{if } 2 > F. \end{cases}$$

Thus given F, the date-0 debt value is

$$E[\min(\tilde{z},F)] = \begin{cases} 2, & \text{if } F \geq 6; \\ \frac{6-F}{5}\cdot F + \frac{F-1}{5}\cdot 2, & \text{if } 6 > F \geq 2; \\ F, & \text{if } 2 > F, \end{cases}$$

and Mr. X's payoff is

$$E[\max(\tilde{z}-F,0)] - \frac{5q^2}{2} = \begin{cases} 0, & \text{if } F \geq 6; \\ \frac{6-F}{5}\cdot(6-F) - \frac{5}{2}(\frac{6-F}{5})^2, & \text{if } 6 > F \geq 2; \\ \frac{4}{5}\cdot(6-F) + \frac{1}{5}\cdot(2-F) - \frac{5}{2}(\frac{4}{5})^2, & \text{if } 2 > F, \end{cases}$$

Mr. X, when choosing F, would maximize his own payoff, subject to the constraint that the date-0 debt value must be greater than or equal to x. Since any $F \geq 6$ would result in zero payoff for Mr. X, and any $F < 2$ would result in a date-0 debt value less than x (which equals 2 in part (i) and $\frac{13}{5}$ in part (ii)), Mr. X must focus on $F \in [2,6)$. Mr. X's problem of choosing F is then

$$\max_{2\leq F<6} \frac{(6-F)^2}{10},$$

subject to

$$\frac{6-F}{5}\cdot F+\frac{F-1}{5}\cdot 2\geq x.$$

Since the objective function decreases with F over the interval $[2,6)$, the optimal choice of F is the smaller root of the equation

$$\frac{6-F}{5}\cdot F+\frac{F-1}{5}\cdot 2=x,$$

which is $F=2$ when $x=2$. (The other root is $F=6$.) This implies that $q=\frac{4}{5}$ so that the date-0 firm value is

$$E[\tilde{z}]=q\cdot 6+(1-q)\cdot 2=\frac{4\cdot 6+1\cdot 2}{5}=\frac{26}{5}.$$

Note that when $x=\frac{13}{5}$ instead, the two rootes of the equation

$$\frac{6-F}{5}\cdot F+\frac{F-1}{5}\cdot 2=x=\frac{13}{5}$$

are $F=3$ and $F=5$, and once again Mr. X would choose the smaller root $F=3$, which implies that $q=\frac{3}{5}$, leading to a firm value of

$$E[\tilde{z}]=q\cdot 6+(1-q)\cdot 2=\frac{3\cdot 6+2\cdot 2}{5}=\frac{22}{5}.$$

Now, consider equity financing. At the subgame where Mr. X has raised x dollars by issuing equity (so that Mr. X would keep a fraction $1-a$ of the ownership), Mr. X seeks to

$$\max_{0\leq q\leq 1} -\frac{5q^2}{2}+(1-a)[q\cdot 6+(1-q)\cdot 2],$$

so that the optimal choice of q is

$$q=\frac{4-4a}{5},$$

which generates for Mr. X the payoff

$$-\frac{5}{2}\cdot(\frac{4}{5})^2(1-a)^2+(1-a)[2+4\cdot\frac{4}{5}(1-a)],$$

so that the date-0 value of the investor's shares is equal to

$$a[q\cdot 6+(1-q)\cdot 2]=a[2+4\cdot\frac{4}{5}(1-a)],$$

which must equal $x=2$ in equilibrium for part (i). It follows that

$$a=1 \text{ or } \frac{5}{8}.$$

Thus $a=\frac{5}{8}$ (why?), implying that $q=\frac{3}{10}$, and hence the date-0 value of the firm is

$$2+4q=\frac{16}{5}.$$

Remark. In this problem, riskless debt can be issued when $x=2$, which leads to the maximum firm value, while equity financing always involves a dead-weight loss.

What happens here is that the CEO (Mr. X) must incur a personal cost ($-\frac{5q^2}{2}$) when expending an effort q to run the firm, and he tends to under-expend the effort. Indeed, he would choose the value-maximizing $q=\frac{4}{5}$ only when the marginal benefit generated by q for the entire firm coincides with the marginal benefit for himself, which would be true when, and only when, riskless debt (debt with face value $F\leq 2$) is issued.

5. (**Chain-store Paradox with Five Entrants.**) Recall the game of chain-store paradox discussed in section 34 of Lecture 4, and we focus here on the special version of the game with five entrants that appears in Homework 7. Let z_i be the equilibrium payoff for E_i, and $\underline{i}^*$ be i^*'s lowest possible outcome such that the event $\{i^*=\underline{i}^*\}$ may occur with a positive probability, where, as in Lecture 4, i^* is the smallest i such that the type of the incumbent becomes publicly known in equilibrium at the stage where the incumbent interacts with E_i.[7]

[7] We write $i^*=+\infty$ if the incumbent's type remains its private information at the time that the game ends.

(i) Suppose that $x_1 = \frac{1}{16}$. Then $\underline{i}^* =$ <u>No. 30</u>.

(ii) Suppose that $x_1 = \frac{3}{16}$. Then $\underline{i}^* =$ <u>No. 31</u>.

(iii) Suppose that $x_1 = \frac{3}{8}$. Then $\underline{i}^* =$ <u>No. 32</u> and $\sum_{j=1}^{5} z_j =$ <u>No. 33</u>.

Solution. It is straightforward that, according to the *Solutions to Homework 7*, $\underline{i}^* = 1$ in part (i), $\underline{i}^* = 2$ in part (ii), and $\sum_{j=1}^{5} z_j = \frac{7}{8}$ and $\underline{i}^* = 4$ in part (iii).