Game Theory with Applications to Finance and Marketing, I

Homework 0, to be discussed by TA

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1. Recall Example 7 in *Some Examples.* Suppose that $e = 0$ and M must first choose a face value of debt, $d \in [0, 10]$, before persuading C to reduce the debt obligations from 10 to d. (In Example 7, it is assumed that $d = 9$.) Show that M would not make a renegotiation offer when $x > 2$, and that M would propose $d = 8$ to C when $x \leq 2$.

   Solution. A well-known result from contract theory says that contract renegotiation can never occurs if Pareto efficiency will be attained under the original contract. The idea is that each contracting party can hold on to the original contract, and agreeing to replace the original contract by a new contract must be voluntary. If the original contract can attain Pareto efficiency and yet contract renegotiation occurs which makes one contracting party better off than under the original contract, then some other contracting party must get hurt, which is a contradiction, as the latter contracting party can simply refuse to replace the original contract by the new contract.

   In the current example, when $x > 2+e$, M would invest and productive efficiency would be attained under the original debt contract (where $d = 10$), so that M's payoff and C's payoff are respectively $8+x-e-10$ and 10 under the original contract. By the preceding result from contract theory, there is no room for debt renegotiation when $x > 2+e$. Indeed, if M would propose a new offer d' to C and C would accept it, then C must obtain a payoff greater than or equal to 10, implying that $d' \geq 10$ and M must also invest under the new contract, but then the new contract would make M weakly worse off as $8+x-e-d' \leq 8+x-e-10$.

   Things are different when $e < x \leq 2 + e$, where under the original debt contract making the investment is efficient, but M would rather not invest. (Here we assume that when M feels indifferent about to or not to invest, M chooses to not invest.) In this case, a new debt contract can be proposed to replace the old contract if under the new contract M would invest and d is so chosen that both M and C are weakly better off by replacing the old contract by the new contract. More precisely, recall that M and C would get 0 and 8 respectively under the original debt contract, and hence we must have, for M,

$$8 + x - e - d \geq 0,$$

and for C,

$$d \geq 8.$$

Since $x > e$, any $d \in [8, 8 + x - e]$ will do. In particular, $d = 9$ is one possible choice when $x > 1 + e$. When M alone can determine d, M would optimally choose $d = 8$.[1]

[1] One can reach the same conclusion by solving the sequential game using backward induction. By backward induction, we should first consider the two last-stage subgames. We shall from now on focus on the case where $e = 0$.

- In the subgame where C has rejected M's renegotiation offer, M and C would obtain $\max(8 + x - 10, 0) - e$ and $\min(10, 8 + x)$ respectively if M chooses to invest; and M and C would obtain $\max(8 - 10, 0) = 0$ and $\min(8, 10) = 8$ respectively if M chooses to not invest. Hence M would invest if and only if

  $$e < \max(x - 2, 0) \Leftrightarrow x > 2 + e.$$

  It follows that by rejecting M's offer, C would get $\min(10, 8 + x) = 10$ if $x > 2 + e$ and 8 if $x \leq 2 + e$. Given that $e = 0$, by rejecting M's offer, C would get 10 if $x > 2$ and 8 if $x \leq 2$.

- In the subgame where C has accepted M's renegotiation offer, M and C would obtain $\max(8 + x - d, 0) - e$ and $\min(d, 8 + x)$ respectively if M chooses to invest; and M and C would obtain $\max(8 - d, 0)$ and $\min(8, d)$ respectively if M chooses to not invest. Hence, given d, M would invest if and only if

  $$e < \max(8 + x - d, 0) - \max(8 - d, 0)$$

  $$\Leftrightarrow \begin{cases} e < x, & \text{if } d \leq 8; \\ e < 8 + x - d, & \text{if } 8 < d \leq 8 + x; \\ e < 0, & \text{if } 8 + x < d. \end{cases}$$

2. Consider the following Bertrand game with heterogeneous products, where for $i, j \in \{1, 2\}$, $0 \leq p_i, p_j \leq 1$, and firm i's profit function is

$$\Pi_i(p_i, p_j) = p_i(1 - 2p_i + p_j).$$

Show by iterated deletion of strictly dominated strategies that this game has a unique outcome where both firms price at $\frac{1}{3}$.[3]

Thus after C accepts M's renegotiation offer d, M would choose to not invest if and only if either $10 \geq d > 8 + x$ or $d + e \geq 8 + x \geq d > 8$. In particular, when $e = 0$, following C's acceptance of M's renegotiation offer, M would choose to not invest if and only if $d \in [8 + x, 10]$. Thus by accepting M's offer d, C would get $\min(d, 8) = 8$ if $d \in [8 + x, 10]$ and $\min(8 + x, d) = d$ if $d \in [0, 8 + x)$.

Now, consider M's decision regarding the choice of d. First observe that when $x > 2$ M cannot find a renegotiation offer that is acceptable to C and that raises M's payoff at the same time.[2] Indeed, C can obtain a payoff of 10 by rejecting any renegotiation offer d in this event. Thus we conclude that when $x > 2$ M would not propose any renegotiation offers.

Next, consider the case $x \leq 2$. To induce C to accept the offer d, M must convince C that accepting the offer is better then rejecting it.

- Facing an offer $d \in [0, 8 + x)$, C would accept the offer if and only if $8 \leq d$, and following C's acceptance, M's payoff is $\max(8 + x - d, 0) - e = 8 + x - d$, which is maximized at $d = 8$, yielding for M the optimal payoff x.
- On the other hand, facing an offer $d \in [8 + x, 10]$, C would accept the offer if and only if $8 \leq \min(8, d)$, which is true, and following C's acceptance, M's payoff is $\max(8 - d, 0) = 0$.

Summarizing the above analysis, we conclude that M would not make renegotiation offers when $x > 2$, and that M would propose $d = 8$ to C when $x \leq 2$. In the latter event, M's strategy after proposing $d = 8$ is to invest if C accepts the offer and to not invest if C rejects the offer. Facing the offer $d = 8$, C feels indifferent about accepting or rejecting the offer, but chooses to accept the offer with probability one.

[3]Given any $p_j \in [d_k, u_k]$, show that

$$p_i < \frac{1 + d_k}{4} \Rightarrow \frac{\partial \Pi_i}{\partial p_i} > 0,$$

and

$$p_i > \frac{1 + u_k}{4} \Rightarrow \frac{\partial \Pi_i}{\partial p_i} < 0.$$

That is, each and every $p_i \in [0, \frac{1+d_k}{4})$ is dominated by $p_i = d_{k+1} \equiv \frac{1+d_k}{4}$, and each and every $p_i \in (\frac{1+u_k}{4}, 1]$ is dominated by $p_i = u_{k+1} \equiv \frac{1+u_k}{4}$. Thus define $d_0 = 0$ and

3. Consider firms 1 and 2 competing in the following Cournot game. The inverse demand is

$$p = \tilde{a} - q_1 - q_2,$$

where $\tilde{a}$ is equally likely to take on 4 or 2. Firms can operate at zero costs.

(i) First suppose that the firms compete *after* seeing the realization of $\tilde{a}$. Show that in equilibrium

$$q_1 = q_2 = \frac{\tilde{a}}{3}$$

in state $\tilde{a}$, and each firm obtains the equilibrium payoff

$$\frac{1}{2}[\frac{16}{9} + \frac{4}{9}] = \frac{10}{9}.$$

(ii) Next, suppose instead that the firms must compete *before* learning about the realization of $\tilde{a}$. Show that, given q_j, firm i seeks to

$$\max_{q_i} q_i(E[\tilde{a}] - q_i - q_j).$$

Consequently, in equilibrium

$$q_1 = q_2 = \frac{E[\tilde{a}]}{3} = 1,$$

and each firm gets the payoff

$$(2 - 1 - 1) \cdot 1 = 0$$

in state $\tilde{a} = 2$, and the payoff

$$(4 - 1 - 1) \cdot 1 = 2$$

in state $\tilde{a} = 4$. The firms' common expected profit is then $1 < \frac{10}{9}$.

$u_0 = 1$, and define for all positive integers k, $S^k \equiv [d_k, u_k]$, where $d_k = (\frac{1}{4})^k d_0 + \sum_{j=1}^{k}(\frac{1}{4})^j$ and $u_k = (\frac{1}{4})^k u_0 + \sum_{j=1}^{k}(\frac{1}{4})^j$. Show that the only strategy profile that survives iterated deletion of strictly dominated strategies is the single element contained in $\bigcap_{k=0}^{\infty} S_k = \{\frac{1}{3}\}$. Fill in the details.

(iii) Finally, assume that before choosing the outputs, firm 1 knows the realization of $\tilde{a}$ but firm 2 does not. Show that in this case, the equilibrium output choices are such that

$$q_2 = \frac{6 - q_1(4) - q_1(2)}{4},\ q_1(4) = \frac{4 - q_2}{2},\ q_1(2) = \frac{2 - q_2}{2},$$

so that

$$q_1(4) = \frac{3}{2} > \frac{4}{3} > q_2 = 1 > \frac{2}{3} > q_1(2) = \frac{1}{2}.$$

In equilibrium, firm 1's expected profit is

$$\frac{1}{2}[4 - \frac{3}{2} - 1] \cdot \frac{3}{2} + \frac{1}{2}[2 - \frac{1}{2} - 1] \cdot \frac{1}{2} = \frac{5}{4} > \frac{10}{9},$$

and firm 2's expected profit is

$$\frac{1}{2}[4 - \frac{3}{2} - 1] \cdot 1 + \frac{1}{2}[2 - \frac{1}{2} - 1] \cdot 1 = 1 < \frac{10}{9}.$$

Thus a firm can benefit (suffer) from its superior (inferior) demand information.

(iv) Show that in part (iii), if firm 1 can decide whether or not to let firm 2 know about the true demand state, then firm 1 will share demand information with firm 2 in and only in state $\tilde{a} = 2$.[4] Thus deduce that if firm 2 is rational, then firm 2 knows that the demand is in high state whenever firm 1 is unwilling to share the demand information with firm 2. Thus conclude that giving firm 1 the discretion about whether to share information with firm 2 makes firm 1 worse off. Firm 1 would be better off if it can "commit" to never share demand information with firm 2.

4. This exercise clarifies the concept of common knowledge.

Suppose that there are two players (i.e., $I = \{1, 2\}$) and 5 possible uncertain states of the world, and we denote by $\Omega = \{\omega_i; i = 1, 2, \cdots, 5\}$ the set of possible true states. Let H_i denote player i's information, which is formally a partition of Ω. For example, assume that H_1 is depicted as

[4] The uninformed firm 2 is producing too much in this low-demand state, which forces firm 1 to cut back its production; recall that output choices are strategic substitutes.

$$\boxed{\omega_1|\omega_2,\omega_3|\omega_4,\omega_5}$$

and H_2 is depicted as

$$\boxed{\omega_1,\omega_2|\omega_3|\omega_4|\omega_5}$$

In this example, if the true state is ω_1, then player 1's information would tell him that the true state is exactly ω_1, but player 2's information only shows that the true state is not ω_3, ω_4, or ω_5. If instead the true state is ω_2, then player 1 only knows that the true state is not ω_1, ω_4, or ω_5; and player 2 only knows that the true state is not ω_3, ω_4, or ω_5. In other words, when the true state is ω_j, the element in H_i that contains ω_j is the set of ω's that player i considers possible to be the true state.

Consider three random variables $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$, whose realizations in true state ω are denoted respectively by $x(\omega)$, $y(\omega)$, and $z(\omega)$. Suppose that

	ω_1	ω_2	ω_3	ω_4	ω_5
$x(\cdot)$	5	4	4	3	1
$y(\cdot)$	1	1	0	0	0
$z(\cdot)$	2	2	2	4	5

Assume that the above 3 tables regarding H_1, H_2, and $(\tilde{x}, \tilde{y}, \tilde{z})$ are the two players' common knowledge.[5]

(i) Suppose that the true state is ω_2. Do both players know the realization of $\tilde{x}$? If your answer is yes, do they also know that they both know the realization of $\tilde{x}$? If your answer is again yes, do they know that they both know that they both know the realization of x? Do you think that the realization of $\tilde{x}$ is the two players' common knowledge in state ω_2?

(ii) Suppose that the true state is ω_1. Do both players know the realization of $\tilde{y}$? If your answer is yes, do they also know that they both

[5] To answer the following questions, we obviously must assume that they have *some* common knowledge!

know the realization of $\tilde{y}$? If your answer is again yes, do they know that they both know that they both know the realization of y? Do you think that the realization of $\tilde{y}$ is the two players' common knowledge in state ω_1?

(iii) Suppose that the true state is ω_3. Do both players know the realization of $\tilde{z}$? If your answer is yes, do they also know that they both know the realization of $\tilde{z}$? If your answer is again yes, do they know that they both know that they both know the realization of z? Do you think that the realization of $\tilde{z}$ is the two players' common knowledge in state ω_3?

Solution. In part (i), in state ω_2 player 1 knows the realization of $\tilde{x}$ but player 2 does not.

In part (ii), in state ω_1 both players know the realization of $\tilde{y}$, and player 1 knows that player 2 knows the realization of $\tilde{y}$. Player 2, however, is not sure whether player 1 knows the realization of $\tilde{y}$.

In part (iii), in state ω_3 the realization of $\tilde{z}$ is indeed the two players' common knowledge.