# Game Theory with Applications to Finance and Marketing

## Lecture 1: Games with Complete Information, Part II

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1. This note consists of three contents. Recall that we have introduced in Part I such solution concepts as dominance equilibrium, Nash equilibrium, SPNE, backward induction, and forward induction. In sections 2-15 we shall continue to go over the other relevant equilibrium concepts, such as strong equilibrium, coalition-proof equilibrium, rationalizable strategies, and correlated equilibrium. Then, in sections 16-26, we shall examine a class of pricing games which has been important in the marketing literature. Finally, sections 27-32 introduce the concept of signal-jamming, which has important applications in both marketing and finance.

2. Consider the following strategic game (G1).

| player 1/player 2 | D | C |
|---|---|---|
| D | 0,0 | 0,0 |
| C | 0,0 | 1,1 |

This game has two mixed strategy NE's. In view of Wilson's theorem (1971), this game is quite unusual. Note that (D,D) is an NE where players play weakly dominated strategies in equilibrium. This does not seem reasonable. To get rid of this type of NE's, Selten (1975) proposes the trembling-hand perfect equilibrium in normal form games, which is a refined notion of NE's, aiming at screening out better NE's. To see Selten's idea, note that the reason that (D,D) can become an NE is because players are sure that C will be played by the rival with zero probability. Therefore, if we consider only those strategy profiles which are limits of totally mixed strategy profiles, then (D,D) can be ruled out. Formally, let $\Sigma^0$ be the set of totally mixed strategy profiles, and given any $\epsilon \in \Re_{++}$, $\sigma \in \Sigma^0$ is called an $\epsilon$-perfect equilibrium if $\forall i \in \mathcal{I},\ \forall s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \Rightarrow \sigma_i(s_i) \leq \epsilon.$$

A trembling-hand perfect equilibrium is then a profile $\sigma \in \Sigma$ (which need not be totally mixed!) such that there exists a sequence $\{\epsilon_k; k \in \mathbf{Z}_+\}$ in $\Re_{++}$ and a sequence $\{\sigma_k; k \in \mathbf{Z}_+\}$ in $\Sigma^0$ with (i) $\lim_{k\to\infty} \epsilon_k = 0$; (ii) $\sigma_k$ is an $\epsilon_k$-perfect equilibrium for all $k \in \mathbf{Z}_+$; and (iii) $\lim_{k\to\infty} \sigma_{i,k}(s_i) = \sigma_i(s_i),\ \forall i \in \mathcal{I},\ \forall s_i \in S_i$. It can be shown that a trembling-hand perfect equilibrium must exist for a finite game, and the trembling-hand perfect equilibrium is itself an NE, but the reverse is not true.[1] In particular, the above profile (D,D) is not a trembling-hand perfect equilibrium.

3. Consider the extensive game with two players where player 1 first chooses between L and R, and the game ends with payoff profile $(2, 2)$ if R is chosen, but if instead L is chosen, then player 2 can choose between l and r, with the game ending with payoff profile $(1, 0)$ if r is chosen, and if instead player 2 chooses l, then player 1 can choose between A and B, with the game ending with respectively payoff profiles $(3, 1)$ and $(0, -5)$.

[1] Let us prove that a trembling-hand perfect equilibrium $\sigma$ is an NE. Recall the following definition of NE: a profile $\sigma \in \Sigma$ is an NE if and only if for all $i \in \mathcal{I}$, for all $s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \Rightarrow \sigma_i(s_i) = 0.$$

Note that for all $i \in \mathcal{I}$, for all $s_i, s_i' \in S_i$ such that

$$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i})$$

there exists $K \in \mathbf{Z}_+$ such that

$$k \geq K \Rightarrow u_i(s_i, \sigma_{-i}^k) < u_i(s_i', \sigma_{-i}^k),$$

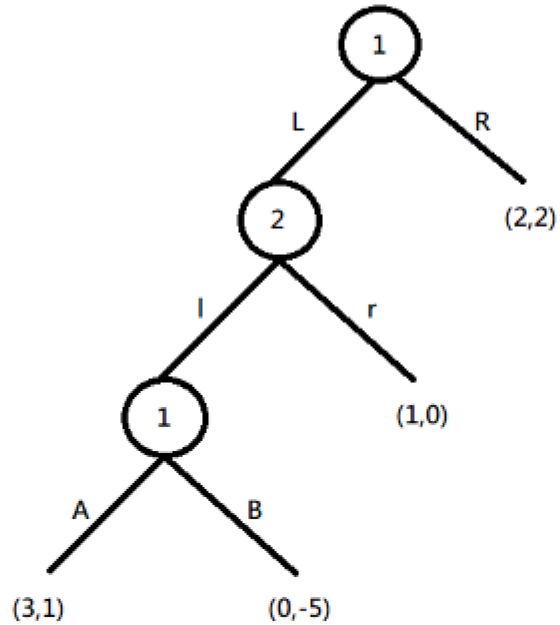
by the fact that $\sigma^k \to \sigma$, and hence for any such $k$, we have

$$\sigma_i^k(s_i) \leq \epsilon_k,$$

implying that

$$0 \leq \sigma_i(s_i) = \lim_{k\to\infty} \sigma_i^k(s_i) \leq \lim_{k\to\infty} \epsilon_k = 0.$$

This shows that a trembling-hand perfect equilibrium is an NE.




This game has a unique SPNE, (L,l,A), but (R,r,B) is a trembling-hand perfect equilibrium in the corresponding (reduced) strategic game: consider letting player 1 play (L,A) with probability $\epsilon^2$ and (L,B) with probability $\epsilon$, where notice that player 2 will optimally respond by playing r (this happens because pro.$((L,B)|L) = \frac{\epsilon}{\epsilon+\epsilon^2}$ is close to one when $\epsilon \downarrow 0$). The problem here is that at the two information sets where player 1 is called upon to take actions, player 1's trembles are correlated. Because of this problem, Selten (1975) argues that we should pay attention to the *agent-normal form*, where player 1 appearing at different information sets is treated as different agents. Then, the trembling-hand perfect equilibria are defined as the trembling-hand perfect equilibria of the agent-normal form, and will be simply referred to as the *perfect equilibria*. With this definition, it can be shown that perfect equilibria are SPNE's.[2]

4. Consider the following strategic game (G2).

| player 1/player 2 | L | M | R |
|---|---|---|---|
| U | 1,1 | 0,0 | -9,-9 |
| M | 0,0 | 0,0 | -7,-7 |
| D | -9,-9 | -7,-7 | -7,-7 |

This game has three NE's, all in pure strategy. These are (U,L), (M,M), and (D,R). In this game, (M,M) becomes a trembling-hand perfect equilibrium![3]

This is unreasonable, for what we did was adding two dominated strategies R and D to (G1) to create the game (G2)! Myerson (1978) proposes a remedy to this situation. Formally, let $\Sigma^0$ be the set of totally mixed strategy profiles, and given any $\epsilon \in \Re_{++}$, $\sigma \in \Sigma^0$ is called an $\epsilon$-proper equilibrium if $\forall i \in \mathcal{I},\ \forall s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \Rightarrow \sigma_i(s_i) \leq \epsilon \sigma_i(s_i').$$

A proper equilibrium is then a profile $\sigma \in \Sigma$ (which need not be totally mixed!) such that there exists a sequence $\{\epsilon_k; k \in \mathbf{Z}_+\}$ in $\Re_{++}$ and a sequence $\{\sigma_k; k \in \mathbf{Z}_+\}$ in $\Sigma^0$ with (i) $\lim_{k\to\infty} \epsilon_k = 0$; (ii) $\sigma_k$ is

[2]In fact, they are also *sequential equilibria* defined by Kreps and Wilson (1982).

[3]To see that this claim is true, given $\epsilon > 0$, consider the following totally mixed strategy profile:

$$\sigma_1^\epsilon(U) = \epsilon,\ \sigma_1^\epsilon(M) = 1 - 2\epsilon,\ \sigma_1^\epsilon(D) = \epsilon,$$

$$\sigma_2^\epsilon(L) = \epsilon,\ \sigma_2^\epsilon(M) = 1 - 2\epsilon,\ \sigma_2^\epsilon(R) = \epsilon.$$

Given player 2's totally mixed strategy $\sigma_2^\epsilon$, player 1's best response is M, not U. This is in sharp contrast to the case where player 2 can use only strategies L and M, where against any totally mixed strategy that player 2 may adopt, U is a better reponse than M from player 1's perspective. The reason is obviously that, in the current strategic game, when player 2 actually adopts R, using M instead of U can save 2 utils for player 1. When player 2 adopts L instead, using U instead of M can only save 1 util for player 1. Now, since player 1 expects player 2 to assign the same probability $\epsilon$ to L and R, player 1 must consider M a better response than U against player 2's totally mixed strategy. Since the above normal-form game is actually a symmetric game, the same argument applies for player 2's comparison about M and L. Thus the above strategy profile indeed defines an $\epsilon$-perfect equilibrium, which converges to (M,M) as $\epsilon \downarrow 0$.

an $\epsilon_k$-proper equilibrium for all $k \in \mathbf{Z}_+$; and (iii) $\lim_{k\to\infty} \sigma_{i,k}(s_i) = \sigma_i(s_i)$, $\forall i \in \mathcal{I}$, $\forall s_i \in S_i$. It can be shown that a proper equilibrium is necessarily a trembling-hand perfect equilibrium (this is obvious; simply observe that $\sigma_i(s_i) \leq \epsilon\sigma_i(s_i') \Rightarrow \sigma_i(s_i) \leq \epsilon$), and hence an NE, but the reverse is not true. In particular, the above game has a unique proper equilibrium (U,L). To see this, consider any $\epsilon$-proper equilibrium $\sigma^\epsilon$, which is by definition totally mixed. Since player 1 would feel indifferent about M and D only if player 2 were expected to use R with probability one, here we conclude that player 1 prefers M to D. This implies that player 1 should assign probabilities

$$(A1) \quad \sigma_1^\epsilon(D) \leq \epsilon\sigma_1^\epsilon(M),$$

which implies that, from player 2's point of view, for $\epsilon > 0$ small enough,

$$\begin{aligned} &u_2(L, \sigma_1^\epsilon) - u_2(R, \sigma_1^\epsilon) \\ &= 10\sigma_1^\epsilon(U) + 7\sigma_1^\epsilon(M) - 2\sigma_1^\epsilon(D) \\ &\geq 10\sigma_1^\epsilon(U) + (7 - 2\epsilon)\sigma_1^\epsilon(M) > 0, \end{aligned}$$

implying that

$$\sigma_2^\epsilon(R) \leq \epsilon\sigma_2^\epsilon(L),$$

which in turn implies that, from player 1's point of view, for $\epsilon > 0$ small enough,

$$\begin{aligned} &u_1(U, \sigma_2^\epsilon) - u_1(M, \sigma_2^\epsilon) \\ &= \sigma_2^\epsilon(L) - 2\sigma_2^\epsilon(R) \\ &\geq (1 - 2\epsilon)\sigma_2^\epsilon(L) > 0, \end{aligned}$$

implying further that

$$(A2) \quad \sigma_1^\epsilon(M) \leq \epsilon\sigma_1^\epsilon(U).$$

By (A1) and (A2), we conclude that $\sigma_1^\epsilon(U) \geq 1 - \epsilon - \epsilon^2$, and hence in any proper equilibrium $\sigma = \lim_{\epsilon\downarrow 0} \sigma^\epsilon$, we have

$$1 \geq \sigma_1(U) = \lim_{\epsilon\downarrow 0} \sigma_1^\epsilon(U) \geq 1.$$

A similar reasoning applies to $\sigma_2(L)$. Hence (U,L) is the unique proper equilibrium of this strategic game.

5. Myerson also proves that any finite strategic game has a proper equilibrium, and hence any finite game has a trembling-hand perfect equilibrium and an NE. Let us sketch Myerson's proof. Note that it suffices to show that for any $\epsilon_k \in (0,1)$, an $\epsilon_k$-proper equilibrium $\sigma^k$ exists, since by the compactness of $\Sigma$, a convergent subsequence of $\{\sigma^k; k \in \mathbf{Z}_+\}$ exists. Thus fix any $\epsilon \in (0,1)$. Define

$$m \equiv \max\{\#(S_i); i = 1, 2, \cdots, I\},$$

where recall that $\#(A)$ is the cardinality of set $A$ (the number of elements of $A$). Define $d \equiv \frac{\epsilon^m}{m}$. For all $i = 1, 2, \cdots, I$, define

$$\Sigma_i^d \equiv \{\sigma_i \in \Sigma_i : \ \sigma_i(s_i) \geq d, \ \forall s_i \in S_i\}.$$

Note that $\Sigma_i^d$ is a non-empty compact subset of $\Sigma_i^0$. Define

$$\Sigma^d \equiv \Pi_{i=1}^I \Sigma_i^d.$$

For all $i = 1, 2, \cdots, I$, consider the correspondence $F_i : \Sigma^d \to \Sigma_i^d$ defined by

$$F_i(\sigma) = \{\sigma_i \in \Sigma_i^d : u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \Rightarrow \sigma_i(s_i) \leq \epsilon \sigma_i(s_i'), \ \forall s_i, s_i' \in S_i\}.$$

Note that given each $\sigma \in \Sigma^d$, $F_i(\sigma)$ is convex and closed. We claim that $F_i(\sigma)$ is also non-empty. To see this, for each $s_i \in S_i$, define $\rho(s_i)$ to be the number of pure strategies $s_i' \in S_i$ with

$$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}).$$

Define

$$\sigma_i'(s_i) \equiv \frac{\epsilon^{\rho(s_i)}}{\sum_{s_i'' \in S_i} \epsilon^{\rho(s_i'')}}, \ \forall s_i \in S_i.$$

By construction, we have $\sigma_i'(s_i) \geq d$, and $\sum_{s_i \in S_i} \sigma_i'(s_i) = 1$. Moreover, it can be verified that $\sigma_i' \in F_i(\sigma)$, showing that $F_i(\sigma)$ is indeed non-empty. Finally, one can verify that $F_i$ is upper hemi-continuous. Define

$$F \equiv \Pi_{i=1}^I F_i : \Sigma^d \to \Sigma^d.$$

Then $F$, inheriting the main properties from the $F_i$'s, is non-empty, convex, and upper hemi-continuous, and hence $F$ has a fixed point by Kakutani's fixed point theorem. A fixed point of $F$ is an $\epsilon$-proper equilibrium. Since $\epsilon \in (0,1)$ was chosen arbitrarily, this proves that we can construct a sequence of $\epsilon_k$-proper equilibria, $\{\sigma^k; k \in \mathbf{Z}_+\}$, and the latter must have a convergent subsequence, of which the limit is exactly a proper equilibrium. This finishes the proof for existence.

6. **Definition 10**. (Aumann, 1959) Given an $I$-person finite strategic game $\Gamma$, a profile $\sigma$ is a *strong equilibrium* if for any $J \subset \{1, 2, \cdots, I\}$, and any $\sigma' \in \Sigma$, there exists $j \in J$ such that $u_j(\sigma) \geq u_j(\sigma'_J, \sigma_{-J})$, where $\sigma'_J$ is the profile $\sigma'$ restricted on the set of players $J$ and $\sigma_{-J}$ is the profile $\sigma$ restricted on the set of players not contained in $J$.

   From now on any nonempty subset of players from the original game is referred to as a *coalition*. Immediately from the above definition, a strong equilibrium is an NE; to see this, just let $J$ be any singleton coalition. Define the set

$$\mathcal{U} \equiv \{(u_1(\sigma), u_2(\sigma), \cdots, u_I(\sigma)) : \ \sigma \in \Sigma\}.$$

   A strong equilibrium, if it exists, must give a profile of payoffs which are not strictly Pareto dominated; this can be seen by taking $J$ to be the entire set of players.[4]

[4]When a strategic game has two NE's with one Pareto-dominating the other, we would naturally think that the former NE is more likely to prevail than the latter NE, but game theorists have argued convincingly that this need not be the case.

Consider the following strategic game taken from Harsanyi and Selten (1988):

| player 1/player 2 | L | R |
|---|---|---|
| U | 9,9 | 0,8 |
| D | 8,0 | 7,7 |

The game has two pure-strategy NE's, which are (U,L) and (D,R), and a mixed-strategy NE where player 1 uses U with probability $\frac{7}{8}$ and player 2 uses L with probability $\frac{7}{8}$. Each player gets 9 in (U,L), 7 in (D,R), and $\frac{63}{8}$ in the mixed-strategy NE. The equilibrium (U,L) Pareto dominates the other two equilibria. By the textbook definition, in the NE (U,L) player 1 is perfectly sure that player 2 will play L and hence player 1 will play U. However, if player 1 is not perfectly sure about player 2's intention to play L, then playing D is less risky than playing U. Indeed, if player 1 suspects that player 2 may play R with a probability greater than $\frac{1}{8}$, then D dominates U from player 1's perspective. Since perfect predictability of the rival's intention is an idealization, (D,R) may make more sense than (U,L) when this game is applied in a real-world scenario.

Thus a strong NE is an NE which is robust against not only unilateral deviations but any coalitional deviations also. The problem with this solution concept is that it asks us to check all possible coalitional deviations, including those coalitional deviations which are themselves unreasonable: given a coalition that might benefit from a joint deviation from the original NE strategy profile, there may be some sub-coalition that can benefit from a joint deviation from this supposed joint deviation of the entire coalition. Thus coalitional deviations must be treated in a logically consistent way; this is where the coalition-proof equilibrium gets into the picture. Intuitively, among the solution concepts of NE, strong NE, and coalition-proof NE, the former is the weakest, and the strong NE is the strongest, so that it can happen that given a game, there exists an NE and a coalition-proof equilibrium, but no strong equilibrium.

7. **Definition 11**. (Bernheim, Peleg, and Whinston, 1987) Suppose that we are given an $I$-person finite strategic game $\Gamma$. Let $\mathbf{J}$ be the set of all feasible coalitions.

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Aumann (1990) points out that (D,R) may remain more reasonable than (U,L) even if the two players have an opportunity to engage in pre-play communication: regardless of player 2's real intention, player 2 has an incentive to convince player 1 that U is better than D. Indeed, player 2 would get either 9 or 8 if player 1 uses U, but player 2 would only get 0 or 7 if player 1 uses D.

Finally, consider the following 3-player game taken from Bernheim, Peleg, and Whinston (1987), where player 1 chooses row, player 2 chooses column, and player 3 chooses between the two bi-matrice, and players are restricted to using only pure strategies:

| player 1/player 2 | L | R | |
|---|---|---|---|
| U | 0,0,10 | -5,-5,0 | A |
| D | -5,-5,0 | 1,1,-5 | |

| player 1/player 2 | L | R | |
|---|---|---|---|
| U | -2,-2,0 | -5,-5,0 | B |
| D | -5,-5,0 | -1,-1,5 | |

The NE (U,L,A) Pareto-dominates the NE (D,R,B). However, given that player 3 chooses bi-matrix A, the two-player game played by players 1 and 2 have two NE's, where (U,L) is Pareto-dominated by (D,R). That is, the Pareto-dominant NE (U,L,A) involves players 1 and 2 play a Pareto-dominated strategy profile!

(i) If $I = 1$, then a profile $\sigma$ is a coalition-proof equilibrium if and only if $u_1(\sigma) \geq u_1(\sigma')$ for all $\sigma' \in \Sigma$.
(ii) Suppose $I \geq 2$ and coalition-proof equilibrium (CPE) has been defined for all $n$-person finite strategic game with $n \leq I - 1$. A profile $\sigma$ is *self-enforcing* for the game $\Gamma$ if for all $J \in \mathbf{J}$ with $\#(J) \leq I - 1$, $\sigma_J$ is a coalition-proof equilibrium in the game $\Gamma/\sigma_{-J}$ (which is the $\#(J)$-person strategic game where everything is as in $\Gamma$ except that players in $-J$ are restricted to playing $\sigma_{-J}$). A *coalition-proof equilibrium* is a self-enforcing profile $\sigma$ such that no other self-enforcing profiles $\sigma'$ can simultaneously provide each and every player in $\Gamma$ a strictly higher payoff than $\sigma$ does.

By definition, a self-enforcing profile must be a Nash equilibrium profile for $\Gamma$ (taking $J$ to be any singleton elements of $\mathbf{J}$), a strong equilibrium must be a Nash equilibrium which is not strictly Pareto dominated by any other strategy profile, and a CPE must be a Nash equilibrium which is not strictly Pareto dominated by another self-enforcing profile. Apparently, a strong equilibrium, if it exists, must be a CPE.

In the three-player game discussed in footnote 4, for example, (D,R,B) is strictly Pareto dominated by (U,L,A), so that (D,R,B) is not a strong equilibrium, but (D,R,B) is nonetheless a CPE; one can verify that (U,L,A) is not self-enforcing.

Thus when $I = 1$, CPE requires only the best response property; and when $I = 2$, self-enforcing profiles coincide with NE profiles, and CPE are equivalent to those NE's which are not Pareto strictly dominated.

With these definitions and discussions in mind, we now consider two problems. First, consider three players A, B, and C, who are to divide one dollar, and each of them must choose a point in the two-dimensional simplex $\{(a, b, c) \in \Re^3_+ : a + b + c = 1\}$. The three players move simultaneously, and if at least two of them pick the same point $(a, b, c)$, then this point will be implemented, in the sense that $a, b$, and $c$ will be the payoffs for A, B, and C respectively; or else, the dollar will be destroyed. We claim that this game has no CPE's. To see this, suppose instead that there were a CPE (denoted $\sigma$) in which the players get expected payoffs $(a, b, c)$, where without loss of generality, $a > 0$. Given $\sigma_1$, players 2 and 3 could jointly deviate in the game $\Gamma/\sigma_1$ by announcing simultaneously $(0, \frac{a}{2}+b, \frac{a}{2}+c)$, for example, thereby having the latter implemented (but this joint deviation must form a Nash equilibrium for players 2 and 3, given player 1's move being fixed!). This arrangement strictly Pareto dominates the original equilibrium profile in the two-person finite strategic game $\Gamma/\sigma_1$ for players 2 and 3, showing that $\sigma$ cannot be self-enforcing. (For $\sigma$ to be self-enforcing, it is necessary that $(\sigma_2, \sigma_3)$ be a CPE in the game $\Gamma/\sigma_1$, which in turn requires that $(\sigma_2, \sigma_3)$ be a Pareto undominated equilibrium in $\Gamma/\sigma_1$.) Thus by definition, $\sigma$ cannot be a CPE, a contradiction.

**Lemma 1**. We say that an $I$-person finite strategic game $\Gamma$ exhibits the unique-NE property if for any $J \in \mathbf{J}$ and any $\sigma_{-J}$, there exists a unique NE in the game $\Gamma/\sigma_{-J}$. A game exhibiting the unique-NE property has a unique CPE.
**Proof**. Note that there can be 1 self-enforcing profile for the game $\Gamma$ if such a profile exists; recall that self-enforcing profiles must be NEs. In this case, the defining property of a CPE that no other self-enforcing profiles strictly Pareto dominates the CPE is automatically satisfied. Thus it suffices to show that for a game exhibiting the unique-NE property, self-enforcing profiles, CPEs and Nash equilibrium profiles are the same. This is obvious if $I = 1$; the player must have a unique optimal pure strategy.

Suppose that $I = 2$ for $\Gamma$. By hypothesis, this game has a unique NE, denoted by $(\sigma_1, \sigma_2)$, and hence a unique self-enforcing profile, which is a CPE obviously.

Suppose that $I \geq 3$ for $\Gamma$ and that it has been proven that for a game with no more than $I - 1$ players and exhibiting the unique-NE property, self-enforcing profiles and CPEs are both equivalent to NE profiles. This game has only 1 NE $\sigma$, and hence no other self-enforcing profiles. Then $\sigma$ is a self-enforcing profile because by assumption the equivalence of self-enforcing profiles, CPEs, and NEs have already been established for $\sigma_J$ in the game $\Gamma/\sigma_{-J}$ (which has no more than $I - 1$ players and exhibits the unique-NE property) for any $J \in \mathbf{J}$. $\|$

Let us give two applications of Lemma 1. First, the prisoners' dilemma discussed in Lecture 1, Part I, apparently exhibits the unique-NE property, and hence has a unique CPE, in which both players choose to confess the crime. This CPE is not a strong equilibrium. (In fact, the same reasoning suggests that any two-player game with a unique NE have a unique CPE.)

Next, consider the Cournot game where $N$ firms producing costlessly a homogeneous good must compete in quantity given that the inverse market demand is (in the relevant region) $p = 1 - \sum_{i=1}^{N} q_i$. This game again exhibits the unique-NE property, and hence has a unique CPE.

8. The last equilibrium concept we shall go over is *rationalizability* (Bernheim, 1984). Define $\Sigma_i^0 = \Sigma_i$. For all natural numbers $n$, define

$$\Sigma_i^n = \{\sigma_i \in \Sigma_i^{n-1} : \exists \sigma_{-i} \in \Pi_{j\neq i} co(\Sigma_j^{n-1}), u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})\ \forall \sigma_i' \in \Sigma_i^{n-1}\}.$$

We call elements in $\bigcap_{n=0}^{+\infty} \Sigma_i^n$ *rationalizable strategies* for player $i$. Intuitively, rational players will never use strategies which are never best responses. Rationalizability extends this idea to fully utilize the assumption that rationality of players is their common knowledge.

9. Let us now develop the notion of rationalizability in detail. Given a game $\Gamma$ in normal form with $I$ players, consider sets $H_i \subset \Sigma_i$ for all $i = 1, 2, \cdots, I$. We shall adopt the following definitions.

   - Let $H_i(0) \equiv H_i$ and define inductively

   $$H_i(t) \equiv \{\sigma_i \in H_i(t-1) : \exists \sigma_{-i} \in \Pi_{j\neq i}\text{co}(H_j(t-1))$$
   $$\ni: u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})\ \forall \sigma_i' \in H_i(t-1)\},$$

   where co$(A)$ is the smallest convex set containing $A$, called the convex hull generated by $A$. Define

   $$R_i(\Pi_{i=1}^{I} H_i) \equiv \bigcap_{t=1}^{\infty} H_i(t).$$

   - A $I$-tuple of sets $(A_1, A_2, \cdots, A_I)$ has the *best response property* if for all $i$, $A_i \subset \Sigma_i$ and for all $i$, for all $\sigma_i \in A_i$, there exists $\sigma_{-i} \in \Pi_{j\neq i}\text{co}(A_j)$ such that $\sigma_i$ is a best response for $i$ against $\sigma_{-i}$.
   - $A_i \subset \Sigma_i$ has the *pure strategy property* if for all $\sigma_i \in A_i$, for all $s_i \in S_i$ such that $\sigma_i(s_i) > 0$, $s_i \in A_i$.
   - A profile $\sigma \in \Sigma$ is rationalizable, if $\sigma_i \in R_i(\Sigma)$ for all $i$.

With these definitions, we have

**Lemma 2**. Suppose that the $I$-tuple of sets $(A_1, A_2, \cdots, A_I)$ is such that for all $i$, $A_i \subset \Sigma_i$ is nonempty, closed, and satisfies the pure strategy property. Then, (a) for all $i$ and all $t \in \mathbf{Z}_+$, $A_i(t)$ is nonempty, closed, and satisfies the pure strategy property; and (b) for some $k \in \mathbf{Z}_+$, $A_i(t) = A_i(k)$ for all $i$ and all $t \geq k$.

**Proof**. By induction, to prove (a), it suffices to show that the statement will be true for $t$ if it is true for $t-1$. By definition, if $\sigma_i \in A_i(t)$, then each $s_i \in S_i$ with $\sigma_i(s_i) > 0$ will too, proving the pure strategy property. To show nonemptiness, note that $\text{co}(A_i(t-1))$ is compact for all $i$, since $A_i(t-1)$ is. By the induction hypothesis, $A_i(t-1)$ is nonempty for all $i$. Since $u_i$ is continuous, the Weierstrass theorem ensures the nonemptiness of $A_i(t)$. Finally, for closedness, note that any convergent sequence $\{ \sigma_i^n \}$ in $A_i(t)$ must have a limit $\sigma_i$ in $A_i(t-1)$, as by the induction hypothesis, $A_i(t-1)$ is closed. Suppose for each $n$, $\sigma_i^n$ is a best response against $\sigma_{-i}^n$ in $\Pi_{j \neq i}\text{co}(A_j(t-1))$. Since the set $\Pi_{j \neq i}\text{co}(A_j(t-1))$ is compact, a subsequence $\{ \sigma_{-i}^{n_k} \}$ converges to some $\sigma_{-i} \in \Pi_{j \neq i}\text{co}(A_j(t-1))$. Now $\sigma_i$ must be a best response against $\sigma_{-i}$ by the continuity of $u_i$. Thus $\sigma_i \in A_i(t)$, showing that $A_i(t)$ is closed.

Finally, consider statement (b). Note that $A_i(t) \neq A_i(t-1)$ only if $\text{co}(A_j(t)) \neq \text{co}(A_j(t-1))$ for some $j \neq i$. By the pure strategy property, this can happen only if some pure strategy $s_j \in A_j(t-1)$ was deleted and was not contained in $A_j(t)$. Since there are only a finite number of pure strategies for any given $j$, this process must stop somewhere.

10. Now we can give the main results regarding the rationalizable set of profiles.

**Proposition 1**. For all $i$, $R_i(\Sigma)$ is nonempty and it contains at least one pure strategy.

*Proof*. Simply let $A_i = \Sigma_i$ and apply lemma 2.

Note that by statement (b) of Lemma 2, the $I$ tuple of sets

$$\{R_1(\Sigma), R_2(\Sigma), \cdots, R_I(\Sigma)\}$$

has the best response property.

**Proposition 2**. Define for all $i$,

$$E_i \equiv \{\sigma_i \in \Sigma_i : \text{for some } I\text{-tuple } \{A_1, A_2, \cdots, A_I\}$$

$$\text{with the best response property,} \quad \sigma_i \in A_i\}.$$

Then, $E_i = R_i(\Sigma)$ for all $i$.

The proof of Proposition 2 is left as an exercise.

11. Because of proposition 2, we can show that

**Proposition 3**. Every NE, denoted $\sigma$, is rationalizable.

*Proof.* The $I$-tuple of sets $\{\ \{\sigma_1\},\ \{\sigma_2\},\ \cdots,\ \{\sigma_I\}\ \}$ satisfies the best response property and $\sigma_i \in \{\sigma_i\},\ \forall i$, so that proposition 2 implies that for all $i$, $\sigma_i \in R_i(\Sigma)$.

12. An important connection between the rationalizable set of profiles and the profiles surviving the iterated strict dominance is now given. In general, the former is contained in the latter.

**Proposition 4**. In two-person finite games, the two concepts coincide.

*Proof.* Suppose that $\sigma_i$ is not a best response to any element of $\Sigma_j$; i.e. for each $\sigma_j \in \Sigma_j$ there exists $b(\sigma_j) \in \Sigma_i$ such that

$$u_i(b(\sigma_j), \sigma_j) > u_i(\sigma_i, \sigma_j).$$

Call the original game $\Gamma$, and construct a zero-sum game $\Gamma_0$ as follows. The new game has the same set of players and pure strategy spaces, but the payoffs are defined as

$$u_i^0(\sigma_i', \sigma_j) \equiv u_i(\sigma_i', \sigma_j) - u_i(\sigma_i, \sigma_j)$$

for all $(\sigma_i', \sigma_j) \in \Sigma$, and

$$u_j^0(\sigma_i', \sigma_j) = -u_i^0(\sigma_i', \sigma_j).$$

This game has an NE in mixed strategy. Let it be $(\sigma_i^*, \sigma_j^*)$. For any $\sigma_j \in \Sigma_j$, we have

$$u_i^0(\sigma_i^*, \sigma_j) \geq u_i^0(\sigma_i^*, \sigma_j^*) \geq u_i^0(b(\sigma_j^*), \sigma_j^*)$$

$$> u_i^0(\sigma_i, \sigma_j^*) = 0,$$

proving that $\sigma_i$ is strictly dominated by $\sigma_i^*$. Thus a strategy for player $i$ that can never be a best response against player $j$'s strategy must be strictly dominated from player $i$'s point of view. Define for the purpose of iterated deletion of strictly dominated strategies $S_i^0 = S_i$, $\Sigma_i^0 = \Sigma_i$, and for all $t \in \mathbf{Z}_+$,

$$S_i^t \equiv \{s_i \in S_i^{t-1} : \forall \sigma_i \in \Sigma_i^{t-1}, \exists s_{-i} \in S_{-i}^{t-1}, u_i(s_i, s_{-i}) \geq u_i(\sigma_i, s_{-i})\},$$

$$\Sigma_i^t \equiv \{\sigma_i \in \Sigma_i : \sigma_i(s_i) > 0 \Rightarrow s_i \in S_i^t\},$$

$$S_i^\infty \equiv \bigcap_{t \in \{0\} \bigcup \mathbf{Z}_+} S_i^t,$$

and

$$\Sigma_i^\infty \equiv \{\sigma_i \in \Sigma_i : \ \forall \sigma_i' \in \Sigma_i, \exists s_{-i} \in S_{-i}^\infty, u_i(\sigma_i, s_{-i}) \geq u_i(\sigma_i', s_{-i})\}.$$

In terms of these new notations, we have proved that $\Sigma_i^1 = \Sigma_i(1)$ (since a strictly dominated strategy for player $i$ can never be a best response against player $j$'s strategy). However, the above argument can be repeated which shows that $\Sigma_i^\infty = \Sigma_i(\infty)$, so that the two concepts are equivalent.

13. Let us offer another proof to Proposition 4. Fix $j \in \{1, 2\}$. Let $(s_j^1, s_j^2, \cdots, s_j^{\#(S_j)})$ be an enumeration of player $j$'s pure strategies. Let $\#(S_j) = n_j$. For each $\sigma_i \in \Sigma_i$, let

$$x_i(\sigma_i) \equiv (u_i(\sigma_i, s_j^1), u_i(\sigma_i, s_j^2), \cdots, u_i(\sigma_i, s_j^{n_j})),$$

and define the set

$$X_i \equiv \{x_i(\sigma_i) : \sigma_i \in \Sigma_i\}.$$

Then $X_i$ is non-empty, convex, and compact. To see that $X_i$ is convex, note that $x_i : \Sigma_i \to \mathcal{R}$ is linear, and recall that linear image of a convex set is convex.[5] To see that $X_i$ is compact, recall that the linear function $x_i(\cdot)$ is continuous, and continuous image of a compact set is compact. Hence $X_i$ is compact because $\Sigma_i$ is compact.

If $\sigma_i$ is not strictly dominated, we claim that $x_i(\sigma_i)$ is a boundary point of $X_i$.[6] Suppose not. Then there would exist $r > 0$ such that $B(x_i(\sigma_i), r) \subset X_i$. This implies the existence of some $\sigma_i' \in \Sigma_i$ such that $x_i(\sigma_i') \in B(x_i(\sigma_i), r)$ and $x_i(\sigma_i') - x_i(\sigma_i) \in \Re^{n_j}_{++}$, showing that $\sigma_i$ is strictly dominated by $\sigma_i'$ from player $i$'s perspective, a contradiction! Next, define

$$Y_i \equiv \{y - x_i(\sigma_i) : y \in X_i\}.$$

It follows that $Y_i$ is a translation of $X_i$, implying that $Y_i$ is nonempty, convex, and compact. Since $x_i(\sigma_i)$ is a boundary point of $X_i$, we conclude that $0 \in \Re^{n_j}$ is a boundary point of $Y_i$. Consider the nonempty $Z \subset \mathcal{R}^{n_j}$ defined by

$$Z \equiv \{z \in \mathcal{R}^{n_j} : z \gg 0_{n_j \times 1}\},$$

where $z \gg 0$ means that for all $k = 1, 2, \cdots, n_j$, the $k$-th element of $z$, denoted $z_k$, is strictly positive. Note that $Z \bigcap Y_i = \emptyset$. Moreover, $Z$ is convex. One version of the separating hyperplane theorem implies the presence of some non-zero vector $p \in \mathcal{R}^{n_j}$ such that $p'y \leq 0 \leq p'z$ for all $y \in Y_i$ and $z \in Z$. Now we claim that for all $k = 1, 2, \cdots, n_j$, the $k$-th element of $p$, denoted $p_k$, is non-negative. To see this, suppose that $p_k < 0$ for some $k$. This implies that for some $l$, $p_l > 0$ (so that $n_j \geq 2$). Let $m$ be the largest $l$ with $p_l > 0$. Pick $z^* \in Z$ such that $z_k^* > (n_j - 1)p_m$ and $z_q^* = 1$ for all $q \neq k$. It follows that, for this $z^*$, $p'z^* < 0$, which is a contradiction.

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[5] For any $\sigma_i, \sigma_i' \in \Sigma_i$ and any $\lambda \in [0, 1]$,

$$\lambda x_i(\sigma_i) + (1-\lambda) x_i(\sigma_i')$$
$$= \lambda(u_i(\sigma_i, s_j^1), u_i(\sigma_i, s_j^2), \cdots, u_i(\sigma_i, s_j^{n_j})) + (1-\lambda)(u_i(\sigma_i', s_j^1), u_i(\sigma_i', s_j^2), \cdots, u_i(\sigma_i', s_j^{n_j}))$$
$$= x_i(\lambda\sigma_i + (1-\lambda)\sigma_i'),$$

and since $\lambda\sigma_i + (1-\lambda)\sigma_i' \in \Sigma_i$, $x_i(\lambda\sigma_i + (1-\lambda)\sigma_i') \in X_i$.

[6] A point $x \in A \subset \mathcal{R}^m$ is a boundary point of $A$ if for all $r > 0$, $B(x, r) \bigcap A \neq \emptyset \neq B(x, r) \bigcap A^c$.

Thus we have shown the existence of a positive vector $p \in \mathcal{R}^{n_j}$, of which not all elements are zero, such that $p$ defines a hyperplane (or a functional) separating the sets $Z$ and $Y_i$. We can normalize this functional by letting $p$ be such that $\sum_{k=1}^{n_j} p_k = 1$, so that $p$ is a legitimate mixed strategy for player $j$. Given $p$, since $p'y \leq 0$ for all $y \in Y_i$, we have shown that $\sigma_i$ is a best response of player $i$ to player $j$'s mixed strategy $p$. As in the first proof for proposition 4, this argument can be iterated to show that the set of profiles surviving iterated strict dominance is included in the set of rationalizable profiles, so that the two solution concepts coincide in two-player finite strategic games.

14. The above proof for proposition 4 fails if $I > 2$ because not all prob. distributions over $S_{-i}$ are products of independent prob. distributions over $S_j$, for all $j \neq i$.[7] (Recall that an NE in mixed strategy assumes independent randomization among players.) However, the equivalence between the two concepts stated in Proposition 4 is restored if players'

[7]For example, suppose that $I = 3$ and consider player $i = 3$. Suppose that player 1 and player 2 have strategy spaces $S_1 = \{U, D\}$ and $S_2 = \{L, R\}$ respectively. When player 1 may adopt U with probability $p$ and player 2 may adopt L with probability $q$, player 3 is facing $\sigma_{-3}(U, L) = pq$, $\sigma_{-3}(U, R) = p(1-q)$, $\sigma_{-3}(D, L) = (1-p)q$, and $\sigma_{-3}(D, R) = (1-p)(1-q)$. When player 1 may adopt U with probability $p'$ and player 2 may adopt L with probability $q'$, player 3 is facing $\sigma_{-3}(U, L) = p'q'$, $\sigma_{-3}(U, R) = p'(1-q')$, $\sigma_{-3}(D, L) = (1-p')q'$, and $\sigma_{-3}(D, R) = (1-p')(1-q')$. However, the convex combination

$$\frac{1}{2}\begin{bmatrix} pq \\ p(1-q) \\ (1-p)q \\ (1-p)(1-q) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} p'q' \\ p'(1-q') \\ (1-p')q' \\ (1-p')(1-q') \end{bmatrix},$$

which is also a probability distribution over $S_{-3}$, cannot be written as

$$\begin{bmatrix} p''q'' \\ p''(1-q'') \\ (1-p'')q'' \\ (1-p'')(1-q'') \end{bmatrix}$$

for some $p'', q'' \in [0, 1]$.

randomization can be correlated.

**Definition 12**. Given a game in normal form,

$$G = (\mathcal{I} \subset \Re; \{S_i; i \in \mathcal{I}\}; \{u_i : \Pi_{i\in\mathcal{I}} S_i \to \Re; i \in \mathcal{I}\}),$$

an (objective) correlated equilibrium is a prob. distribution $p(\cdot)$ over $S$ such that for all $i$, for all $s = (s_i, \tilde{s}_{-i}) \in S$ with $p(s) > 0$,

$$E[u_i(s_i, \tilde{s}_{-i})|s_i] \geq E[u_i(s_i', \tilde{s}_{-i})|s_i], \ \forall s_i' \in S_i.$$

Each $p(\cdot)$ can be thought of as a randomization device for which $s \in S$ occurs with prob. $p(s)$, and when $s$ occurs the device suggests player $i$ play $s_i$ without revealing to player $i$ what $s$ is, such that all players find it optimal to conform to these suggestions at all times. Let $\mathcal{P}$ be the set of all possible devices of this sort. Immediately, all NE's in mixed strategy are elements of $\mathcal{P}$.

**Proposition 5**. If $s_i$ is not strictly dominated for player $i$, then it is a best response for some $p(\cdot) \in \mathcal{P}$.

15. Find all (objective) correlated equilibria for the following game:

| Player 1/Player 2 | L | R |
|---|---|---|
| U | $5, 1$ | $0, 0$ |
| D | $4, 4$ | $1, 5$ |

*Solution.* Let the correlated device assigns (U,L), (U,R), (D,L), and (D,R) with respectively probability $a$, $b$, $c$, and $d$. Define the following 4 inequalities (referred to as I,II,III,and IV):

For player 2 to play L, $$\frac{a}{a+c} \cdot 1 + \frac{c}{a+c} \cdot 4 \geq \frac{a}{a+c} \cdot 0 + \frac{c}{a+c} \cdot 5,$$

For player 1 to play U, $$\frac{a}{a+b} \cdot 5 + \frac{b}{a+b} \cdot 0 \geq \frac{a}{a+b} \cdot 4 + \frac{b}{a+b} \cdot 1,$$

For player 1 to play D, $$\frac{c}{c+d} \cdot 4 + \frac{d}{c+d} \cdot 1 \geq \frac{c}{c+d} \cdot 5 + \frac{d}{c+d} \cdot 0,$$

For player 2 to play R, $$\frac{b}{b+d} \cdot 0 + \frac{d}{b+d} \cdot 5 \geq \frac{b}{b+d} \cdot 1 + \frac{d}{b+d} \cdot 4.$$

For $(a,b,c,d)$ to define a correlated equilibrium, when players are told to play (U,L) for instance, I and II should hold. Similarly, when players are told to play (D,L), (U,R), and (D,R), [I,III], [IV, II], and [III,IV] should respectively hold. Simplifying, we have four conditions:

$$a \geq c,\ a \geq b,\ d \geq c,\ d \geq b \Leftrightarrow \min(a,d) \geq \max(b,c).$$

Let the set of correlated equilibria be $A$. Then,

$$A = \{(a,b,c,d) : a+b+c+d = 1, a,b,c,d \geq 0, \min(a,d) \geq \max(b,c).\}.$$

Note that all NE's are contained in $A$.[8]

16. **Example 6**. In a duopolistic industry two firms that produce respectively products A and B are faced with three segments of consumers with unit demand:

| Segment | Population | Valuation for A | Valuation for B |
|---|---|---|---|
| $L_A$ | $\alpha$ | $V$ | 0 |
| $L_B$ | $\beta$ | 0 | $V$ |
| S | $1-\alpha-\beta$ | $v$ | $v$ |

where $0 < \beta \leq \alpha < 1-\beta$ and $0 < v < V$. $L_A$ and $L_B$ are respectively loyal customers to the two firms and $S$ the switchers who regard the two products as perfect substitutes. The two firms have no production costs. The game proceeds as follows. The two firms simultanously announce prices $p_A$ and $p_B$ to maximize expected profits, and then, upon seeing these prices, consumers simultaneously decide whether to purchase 1 unit of A, or 1 unit of B, or make no purchase.

Let $\Pi_i(p_i, p_j)$ be firm $i$'s expected profit when the two firms' prices are

[8] There are 3 NE's for this game, and they are respectively

$$(a,b,c,d) = (1,0,0,0),\ (a,b,c,d) = (0,0,0,1),\ (a,b,c,d) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).$$

One non-NE correlated equilibrium is $(a,b,c,d) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3})$, which generates the equilibrium payoff $\frac{8}{3}$ for both players.

respectively $p_i$ and $p_j$. Define $\alpha_A \equiv \alpha$ and $\alpha_B \equiv \beta$, and we have

$$\Pi_i(p_i, p_j) = \begin{cases} 0, & \text{if } \; p_i \in (V, +\infty); \\ \alpha_i p_i, & \text{if } \; p_i \in (v, V]; \\ (1-\alpha_j)p_i, & \text{if } \; 0 \le p_i \le v < p_j; \\ (1-\alpha_j)p_i, & \text{if } \; 0 \le p_i < p_j \le v; \\ \frac{1}{2}(1+\alpha_i-\alpha_j)p_i, & \text{if } \; 0 \le p_i = p_j \le v; \\ \alpha_i p_i, & \text{if } \; 0 \le p_j < p_i \le v. \end{cases}$$

17. First, we look for a pure strategy NE. Suppose $(p_A, p_B)$ is an equilibrium. There are 3 possibilities: (i) $p_A, p_B > v$; (ii) $p_A, p_B \le v$; and (iii) $\max(p_A, p_B) > v \ge \min(p_A, p_B)$. For case (i), we must have $p_A = p_B = V$, and for this to be an NE, we must require

$$\beta V \ge (1-\alpha)v, \;\; \alpha V \ge (1-\beta)v. \tag{1}$$

When (1) holds, indeed a pure strategy where $p_A = p_B = V$ exists, and in this NE the switchers are unserved. (Show that (1) can be replaced by $\beta V \ge (1-\alpha)v$ alone.)

On the other hand, if (ii) were an NE, then $p_A = p_B$. To see this, suppose that to the contrary $p_i < p_j$. But then firm $j$ could have done better by pricing at $V$! Again, $p_A = p_B$ is not an NE unless $p_A = p_B = 0$: for otherwise the equilibrium price is dominated by a price slightly lower. It is obvious that $p_A = p_B = 0$ is still not an NE, for each firm can at least make a profit greater than or equal to $\beta V$.

Finally, for case (iii) to be an NE, we must have either (iii-a) $p_A = V$, $p_B = v$ or (iii-b) $p_A = v$, $p_B = V$. The conditions that support (iii-a) are

$$\alpha V \ge (1-\beta)v \ge (1-\alpha)v \ge \beta V, \tag{2}$$

and when (2) holds, indeed a pure strategy NE where $p_A = V$ and $p_B = v$ exists. On the other hand, in a pure strategy NE satisfying (iii-b), we must have

$$(1-\beta)v - \alpha V \geq 0 \geq (1-\alpha)v - \beta V \Rightarrow (\alpha-\beta)v \geq (\alpha-\beta)V,$$

which is impossible.

The bottom line here is that, in any pure strategy equilibrium, at least one firm must totally abandon the switchers.

18. Of course, we observe no dealing behavior in a pure strategy NE. Now we look for mixed strategy NE's. For the ease of exposition, we assume from now on that $\alpha = \beta$. (We will return to the general case $\alpha \geq \beta$ in section 24.) Then (2) becomes

$$\frac{\alpha}{1-\alpha} = \frac{v}{V},$$

which cannot hold generically. Thus the only possible generic pure strategy NE of this game occurs when

$$\frac{\alpha}{1-\alpha} \geq \frac{v}{V}.$$

Therefore, we assume that

$$\frac{\alpha}{1-\alpha} < \frac{v}{V}. \tag{3}$$

Condition (3) says that the loyals are not important enough, and so the firms cannot commit to not compete for the switchers.

19. A mixed-strategy NE for this game can be represented by two distribution functions $F_A(\cdot)$ and $F_B(\cdot)$. Recall that $F : \Re \to \Re$ is a *distribution function* (d.f.) if (i) (*right-continuity*) $F(x) = F(x+) \equiv \lim_{y>x, y\to x} F(y)$ for all $x \in \Re$; (ii) $\lim_{x\to-\infty} F(x) = 0$; and (iii) $\lim_{x\to+\infty} F(x) = 1$. Corresponding to a distribution function $F(\cdot)$ is a random variable $\tilde{x}$, such that for all $x \in \Re$, $F(x)$ gives the probability for the event $\{\tilde{x} \leq x\}$. A distribution function is discontinuous at $x \in \Re$ if and only if $F(x) > F(x-) \equiv \lim_{y<x, y\to x} F(y)$, and in that case we say that $x$ is a *point of jump* for $F(\cdot)$, and denote by $\Delta F(x)$ the probability for the event that $\{\tilde{x} = x\}$. It is known that a distribution function can have at most a countably infinite number of points of jump.

20. Given $F_j(\cdot)$, firm $i$'s profit as a function of $p_i$ is

$$\Pi_i(p_i; F_j(\cdot)) \equiv \int_0^{+\infty} \Pi_i(p_i, x) dF_j(x).$$

Given firm $i$'s equilibrium mixed strategy $F_i(\cdot)$ against $F_j(\cdot)$, we know that (i) if $x$ is a point of jump for $F_i(\cdot)$, then $x$ is a (pure-strategy) best response for firm $i$; and (ii) if $F_i(\cdot)$ is continuous and strictly increasing on an interval $(a, b)$, then every point $x \in [a, b)$ is a (pure-strategy) best response for firm $i$.

At first, suppose that $x \in (a, b)$ and $y$ is a better pure-strategy response for firm $i$ against firm $j$'s mixed strategy $F_j(\cdot)$, in the sense that

$$\Pi_i(y, F_j(\cdot)) > \Pi_i(x, F_j(\cdot)),$$

then we claim that for $\epsilon > 0$ small enough,

$$\Pi_i(y, F_j(\cdot)) > \Pi_i(z, F_j(\cdot)),\ \forall z \in [x, x + \epsilon).$$

To see this, note first that $z \geq x$, and hence given $F_j(\cdot)$, $z$ generates an expected sales volume for firm $i$ which is no greater than the expected sales volume generated by $x$. Now, recall that firm $i$'s payoff equals firm $i$'s price times the expected sales volume, and hence for $\epsilon$ small enough,

$$\Pi_i(y, F_j(\cdot)) > \Pi_i(x, F_j(\cdot)) \Rightarrow \Pi_i(y, F_j(\cdot)) > \Pi_i(z, F_j(\cdot)),\ \forall z \in [x, x+\epsilon).$$

This implies that $F_i(\cdot)$ must be flat on $[x, x + \epsilon)$, a contradiction to the fact that $F_i(\cdot)$ is continuous and strictly increasing on the interval $(a, b)$. Hence every point contained in $(a, b)$ is a pure-strategy best response for firm $i$.

Note that $a$ is also a pure-strategy best response for $i$. To see this, note that there exists a decreasing sequence $\{a_n\}$ contained in $(a, b)$ and converging to $a$ such that each $a_n$ is one of firm $i$'s pure-strategy best responses against $F_j(\cdot)$, and $a > a_n - \frac{1}{n}$. Let $q_n$ and $q$ be firm $i$'s expected sales volume when firm $j$ uses $F_j(\cdot)$ and firm $i$ uses respectively $a_n$ and $a$. Since $a_n > a$, we have $0 \leq q_n \leq q \leq 1$. Let $\Pi_i^*$ be firm $i$'s equilibrium payoff, and let $\Pi_i$ be firm $i$'s payoff when using $a$ against $F_j(\cdot)$. Then, for all $n$, we have

$$\Pi_i^* \geq \Pi_i = a \cdot q > (a_n - \frac{1}{n}) \cdot q \geq a_n q_n - \frac{q}{n} = \Pi_i^* - \frac{q}{n},$$

proving that $\Pi_i = \Pi_i^*$ when we pass $n$ to $+\infty$, and hence $a$ is also a best response for firm $i$ against $F_j(\cdot)$.

We emphasize here that $b$ is in general *not* a pure-strategy best response for firm $i$ even though every point contained in $(a, b)$ is.

21. To derive the mixed-strategy equilibria, we shall first establish a series of lemmas.

**Lemma 3**. In equilibrium, $F_A(v), F_B(v) > 0$.

Lemma 3 says that both firms have a positive prob. to choose some price level equal to or less than $v$. To see this, suppose not. Then at least one firm chooses to not serve the switchers in the NE, and that firm must price at $V$ with probability one, which implies that the other firm should price at $v$ with probability one, and that a contradiction arises because the firm pricing at $V$ can price at $v - \epsilon$ and become better off for some $\epsilon > 0$ small enough.

**Lemma 4**. For $i \in \{A, B\}$, $F_i(\cdot)$ is continuous on $(-\infty, v)$.

If Lemma 4 does not hold, then at some price $x < v$ either $F_A(\cdot)$ or $F_B(\cdot)$ has a jump.[9] Thus assume that for firm $i$, at some $x < v$, $\Delta F_i(x) \equiv F_i(x) - F_i(x-) \equiv F_i(x) - \lim_{y \uparrow x} F_i(y) > 0$. In mathematic terms, $x$ is a point of jump for the function $F_i(\cdot)$, which implies that $x$ is a best response of firm $i$ in equilibrium. Let $\Pi_j(x)$ and $\Pi_j(x - \epsilon)$ be firm $j$'s expected profits when adopting respectively the pure strategies

[9] Recall that if $F : \mathcal{R} \to \mathcal{R}$ is increasing, then the only possible discontinuity points are of the first kind: where $F(\cdot)$ has well-defined left-hand and right-hand limits, but the functional values of $F$ need not equal these limits.

$x$ and $x-\epsilon$ against $F_i(\cdot)$. Here we assume that $\Delta F_i(x-\epsilon)=0$.[10] We have

$$\Pi_j(x)=[\frac{1}{2}(1-2\alpha)+\alpha]x\cdot\Delta F_i(x)+[(1-2\alpha)+\alpha]x\cdot[1-F_i(x)]+\alpha x F_i(x-)$$

$$<\Pi_j(x-\epsilon)=[(1-2\alpha)+\alpha](x-\epsilon)\cdot[1-F_i(x-\epsilon)]+\alpha(x-\epsilon)F_i(x-\epsilon)$$

when $\epsilon>0$ is sufficiently small. In fact, as we can easily see, given $x$ and $\epsilon$, there exists some $\delta>0$ small enough such that[11]

$$\Pi_j(y)<\Pi_j(x-\epsilon),\ \forall y\in(x,x+\delta].$$

[10]Since an increasing function can have at most a countably infinite number of points of jump, no matter how small $e>0$ is, finding a point $x-\epsilon$ in the interval $(x-e,x)$ such that $F_i(\cdot)$ does not jump at $x-\epsilon$ is always possible.

[11]Suppose now that $F_j(\cdot)$ has a point of jump at some $x\in(0,v)$. For a tiny $\epsilon\in(0,v-x)$, we know that there is some $x'\in(x-\epsilon,x)$ such that $F_j(\cdot)$ does not jump at $x'$. We shall also consider a point $x''\in(x,x+\epsilon)$. Given $F_j(\cdot)$, let us compare $\Pi_i(x;F_j(\cdot))$, $\Pi_i(x';F_j(\cdot))$, and $\Pi_i(x'';F_j(\cdot))$.

- If $p_i=x$, then firm $i$'s revenues from respectively $L_i$ and $S$ can be summarized as follows:

| The event | The probability | Revenue from $L_i$ | Revenue from $S$ |
|---|---|---|---|
| $\tilde{p}_j<x$ | $F_j(x-)$ | $\alpha x$ | 0 |
| $\tilde{p}_j=x$ | $\Delta F_j(x)$ | $\alpha x$ | $(\frac{1}{2}-\alpha)x$ |
| $\tilde{p}_j>x$ | $1-F_j(x)$ | $\alpha x$ | $(1-2\alpha)x$ |

- If $p_i=x'$, with $x-\epsilon<x'<x$, then firm $i$'s revenues from respectively $L_i$ and $S$ can be summarized as follows:

| The event | The probability | Revenue from $L_i$ | Revenue from $S$ |
|---|---|---|---|
| $\tilde{p}_j<x'<x$ | $F_j(x'-)$ | $\alpha x'$ | 0 |
| $\tilde{p}_j=x'<x$ | 0 | $\alpha x'$ | $(\frac{1}{2}-\alpha)x'$ |
| $x'<\tilde{p}_j<x$ | $F_j(x-)-F_j(x')$ | $\alpha x'$ | $(1-2\alpha)x'$ |
| $x'<\tilde{p}_j=x$ | $\Delta F_j(x)$ | $\alpha x'$ | $(1-2\alpha)x'$ |
| $x'<x<\tilde{p}_j$ | $1-F_j(x)$ | $\alpha x'$ | $(1-2\alpha)x'$ |

- If $p_i=x''$, with $x+\epsilon>x''>x$, then firm $i$'s revenues from respectively $L_i$ and $S$ can be summarized as follows:

| The event | The probability | Revenue from $L_i$ | Revenue from $S$ |
|---|---|---|---|
| $\tilde{p}_j<x$ | $F_j(x-)$ | $\alpha x''$ | 0 |
| $x=\tilde{p}_j<x''$ | $\Delta F_j(x)$ | $\alpha x''$ | 0 |
| $x<\tilde{p}_j<x''$ | $F_j(x''-)-F_j(x)$ | $\alpha x''$ | 0 |
| $x<x''=\tilde{p}_j$ | $\Delta F_j(x'')$ | $\alpha x''$ | $(\frac{1}{2}-\alpha)x''$ |
| $x<x''<\tilde{p}_j$ | $1-F_j(x'')$ | $\alpha x''$ | $(1-2\alpha)x''$ |

We have just reached the conclusion that

$$F_j(x) = F_j(x + \delta);$$

that is, no pure strategies in the interval $(x, x + \delta]$ can be best responses for firm $j$ against firm $i$'s equilibrium strategy $F_i(\cdot)$, and hence firm $j$ will assign zero probability to these pure strategies in equilibrium. However, this implies that the pure strategy $x$ cannot be a best response for firm $i$ against firm $j$'s equilibrium strategy $F_j(\cdot)$: pricing at $x+\frac{\delta}{2}$ is better, for example, because the probability that firm $i$ may win the switchers' patronage is the same whether firm $i$ announces $p_i = x$ or $p_i = x + \frac{\delta}{2}$, but when firm $i$ wins the switchers' patronage it gets a higher revenue by announcing $p_i = x + \frac{\delta}{2}$ instead of $p_i = x$! This is a contradiction, because $p_i = x$, being a point of jump of $F_i(\cdot)$, should be one of firm $i$'s pure-strategy best responses against $F_j(\cdot)$!

**Lemma 5**. In equilibrium, $F_A(v-), F_B(v-) > 0$.

Lemma 5 says that both firms must randomize at some prices strictly lower than $v$. This is a refinement of Lemma 3. To see this, suppose that $F_i(v-) = 0$, so that by the fact that $F_i(\cdot)$ is increasing, we have

$$0 = F_i(v-) \geq F_i(x) \geq 0, \Rightarrow F_i(x) = 0, \forall x < v.$$

Now, if at some $x < v$, $F_j(x) > 0$, then there must be some $y \leq x$ such that $y$ is a pure-strategy best response for firm $j$ against $F_i(\cdot)$, which leads to a contradiction because $\frac{y+v}{2}$ is obviously a better pure-strategy response than $y$ for firm $j$! Thus if firm $i$ has $F_i(v-) = 0$, then firm $j$ must also have $F_j(v-) = 0$, or equivalently, $F_j(x) = 0$ for all $x < v$. But then, Lemma 3 implies that both $F_i(\cdot)$ and $F_j(\cdot)$ must jump at $v$, which implies another contradiction: given $\Delta F_i(v) > 0$, for tiny $\delta > 0$, pricing at $v - \delta$ is strictly better than pricing at $v$ for firm $j$. Thus we

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Given the constant $\Delta F_j(x) > 0$, a direct comparison reveals that

$$\Pi_i(x'; F_j(\cdot)) > \Pi_i(x; F_j(\cdot)) > \Pi_i(x''; F_j(\cdot))$$

when $\epsilon$ is sufficiently close to zero.

conclude that both firms must randomize below $v$!

**Lemma 6**. For $i \in \{A, B\}$, if at $a < v$, $F_i(a) > 0$, then for all $b \in (a, v)$, $F_i(b) > F_i(a)$.

Lemma 6 says that in equilibrium the distribution function must be strictly increasing at all prices that are close to but lower than $v$. More importantly, this says that if firm $i$ randomizes at $p_i < v$, then not only all $x \in (p_i, v)$ are best responses for firm $i$, in equilibrium firm $i$ must randomize over each $x \in (p_i, v)$.

To see that Lemma 6 is true, suppose to the contrary that $a < v$, $F_i(a) > 0$, and yet $F_i(\cdot)$ is flat on some interval $[z, y] \subset [a, v)$, i.e. $F_i(z) = F_i(y) \geq F_i(a) > 0$, $a \leq z < y < v$. By Lemma 4, $F_j(\cdot)$ is continuous over the region $(-\infty, v)$. We claim that firm $i$'s payoff function $\Pi_i$ is continuous in its (pure-strategy) price $p_i$ on $(-\infty, v)$. Indeed, lemma 4 implies that

$$\Delta F_j(p_i) = 0,\ F_j(p_i-) = F_j(p_i),\quad \forall p_i \in (-\infty, v),$$

so that

$$\Pi_i(p_i, F_j(\cdot)) = p_i\{[\frac{1}{2}(1-2\alpha)+\alpha]\cdot\Delta F_j(p_i)+[(1-2\alpha)+\alpha][1-F_j(p_i)]+\alpha F_j(p_i-)\}$$

$$= p_i\{0 + (1-\alpha)[1 - F_j(p_i)] + \alpha F_j(p_i)\},$$

which is indeed continuous in $p_i$. Now given $y$, define

$$x \equiv \inf Z,$$

where the set

$$Z = \{w \in (-\infty, v) :\ F_i(w) = F_i(y)\}$$

is nonempty (for it contains $z$) and it has a lower bound (which is $a$). Hence $Z$ has a greatest lower bound, implying that $x$ is indeed well defined.

By definition, $F_i(p) < F_i(x) = F_i(y)$ for all $p < x$. Now we claim that for each positive integer $n$, there must exist some best response $x_n \in [(1-\frac{1}{n})x, x)$ for firm $i$. Indeed, if this were not the case, then there must exist some positive integer $n$ such that firm $i$ does not randomize over the interval $[(1-\frac{1}{n})x, x)$, but then $F_i(\cdot)$ would have to be flat on the interval $[(1-\frac{1}{n})x, y]$, showing that $x$ could not be the greatest lower bound of the set $Z$.

Note that the sequence of best responses $\{x_n; n \in \mathbf{Z}_+\}$ converges to $x$, and since given firm $j$'s mixed strategy $F_j(\cdot)$, firm $i$'s payoff

$$\Pi_i(p_i, F_j(\cdot))$$

is continuous in $p_i$, firm $i$'s equilibrium profit $\Pi^*$ must be such that

$$\Pi^* = \lim_{n\to\infty} \Pi^* = \lim_{n\to\infty} \Pi_i(x_n, F_j(\cdot)) = \Pi_i(\lim_{n\to\infty} x_n, F_j(\cdot)) = \Pi_i(x, F_j(\cdot)),$$

proving that $x$ is also a best response for firm $i$.

Finally, we shall show that $x$ cannot be an equilibrium best response for firm $i$, and hence a contradiction would arise, which would then allow us to conclude that $F_i(\cdot)$ can never be flat anywhere within the interval $[a, v)$.

To show that $x$ cannot be an equilibrium best response for firm $i$, recall that by assumption, $F_i(\cdot)$ is flat on $[x, y]$. In this case, none of the prices $p_j$ contained in the interval $[x, y)$ can be pure-strategy best responses for firm $j$: pricing at $\frac{p_j+y}{2}$ is a better response than $p_j$ from firm $j$'s perspective! It follows that $F_j(\cdot)$ has to be flat on the interval $[x, y)$, and in particular, $\Delta F_j(x) = 0$. Now, given that $F_j(\cdot)$ must be flat on the interval $[x, y)$, from firm $i$'s perspective, $x$ cannot be a best response, because pricing at $\frac{x+y}{2}$ is better than pricing at $x$! This concludes our proof.

**Lemma 7**. For $i \in \{A, B\}$, there exists $\underline{p}_i \in (0, v)$ such that $F_i(x) = 0$ for all $x \leq \underline{p}_i$ and $F_i(\cdot)$ is positive, strictly increasing and continuous on $(\underline{p}_i, v)$.

Indeed, any price $x < \frac{\alpha V}{1-\alpha}$ is dominated by the price $V$ from firm $i$'s perspective, and hence the set

$$L_i \equiv \{x \in \Re : \ F_i(x) = 0\}$$

is non-empty. Lemma 3 shows that $L_i$ is bounded above by $v$. Thus $\sup L_i$ exists, and we denote it by $\underline{p}_i$. Since $\frac{\alpha V}{1-\alpha} \in L_i$, we have

$$0 < \frac{\alpha V}{1-\alpha} \leq \underline{p}_i.$$

By Lemma 4, and by the fact that $\underline{p}_i$ is a limit point of $L_i$, we must have $F_i(\underline{p}_i) = 0 < F_i(v)$, proving that $\underline{p}_i < v$. It follows from Lemmas 4 and 6 that for each $x \in (\underline{p}_i, v)$, $F_i(\cdot)$ is continuous and strictly increasing on $(x, v)$. Thus $F_i(\cdot)$ is positive, strictly increasing and continuous on $(\underline{p}_i, v)$. This completes the proof for Lemma 7.

22. Equipped with the above lemmas, now we can derive the mixed strategy NE in closed form. Let $\Pi_i$ be firm $i$'s equilibrium expected profit. By lemma 5, we have

$$\Pi_i = p_i\{[1-\alpha][1 - F_j(p_i)] + \alpha F_j(p_i)\},$$

for some $p_i < v$, so that for $i, j \in \{A, B\},\ i \neq j$,

$$F_j(x) = \frac{(1-\alpha) - \frac{\Pi_i}{x}}{1-2\alpha},$$

for all $x \in [\underline{p}_j, v)$.[12]

Immediately, we have

$$\underline{p}_j = \frac{\Pi_i}{1-\alpha}. \tag{4}$$

[12]This follows from Lemma 6 which says that if $p_i < v$ is a best response for firm $i$ then so is $x$, for all $x \in (p_i, v)$.

The monotonicity of $F_j(\cdot)$ on $[\frac{\Pi_i}{1-\alpha}, v)$ allows us to take limit:[13]

$$F_j(v-) = \frac{1-\alpha-\frac{\Pi_i}{v}}{1-2\alpha} < 1,$$

where the inequality follows because if otherwise, then $\Pi_i = v\alpha < \alpha V$, a contradiction. Since all $p_j \in (v, V)$ are dominated by $p_j' = V$ for firm $j$, it follows that either $F_j(\cdot)$ has a point mass at $v$ or at $V$. Note that it is impossible that both $\Delta F_i(v), \Delta F_j(v) > 0$: if this were to happen, then $v$ would be a best response for both firms, but given firm $i$'s strategy, from firm $j$'s perspective $v$ would be dominated by $p_j = v - \epsilon$ for $\epsilon > 0$ small enough, which is a contradiction.

23. Note that $\underline{p}_i$ is a best response for firm $i$. To see this, note that all $p_i \in (\underline{p}_i, v)$ are, and they generate the same expected profits for firm $i$. By letting $p_i \downarrow \underline{p}_i$ and using the fact that $F_j(\cdot)$ is continuous on $(-\infty, v)$, we have that $\underline{p}_i$ also attains $\Pi_i$ and hence is a best response for firm $i$. Next, we claim that $\underline{p}_i = \underline{p}_j$. To see this, suppose instead that $\underline{p}_i > \underline{p}_j$, so that firm $j$ may randomize at, say $p_j \in (\underline{p}_j, \underline{p}_i)$. Note however that $p_j$ is dominated by $p_j + \epsilon$ for $\epsilon > 0$ small enough. From here, using (4), we conclude that $\Pi_i = \Pi_j$ in equilibrium.

24. Now we summarize the equilibrium strategies. First suppose that for one firm $i$, $\Delta F_i(v) > 0$. Then $\Delta F_j(v) = 0$, implying that $\Delta F_j(V) > 0$ and hence $\Pi_j = \alpha V$. It follows that $\Pi_i = \alpha V$ also. Since $v$ is a best response for firm $i$, we must have

$$\Delta F_j(V)v(1-\alpha)+[1-\Delta F_j(V)]v\alpha = \alpha V, \Rightarrow \Delta F_j(V) = \frac{\alpha(V-v)}{v(1-2\alpha)}. \quad (5)$$

Alternatively, we can obtain the same result from

$$\Delta F_j(V) = 1 - F_j(v-). \quad (6)$$

[13] Real-valued monotone functions defined on $\Re$ are regular, in the sense that the left-hand and right-hand limits exist at each and every point in the domain.

In this case, we have

$$F_i(p_i) = \begin{cases} 0, & p_i \leq \frac{\alpha V}{1-\alpha}; \\ \frac{1-\alpha-\frac{\alpha V}{p_i}}{1-2\alpha}, & p_i \in [\frac{\alpha V}{1-\alpha}, v); \\ p^*, & p_i \in [v, V); \\ 1, & p_i \geq V, \end{cases}$$

and

$$F_j(p_j) = \begin{cases} 0, & p_j \leq \frac{\alpha V}{1-\alpha}; \\ \frac{1-\alpha-\frac{\alpha V}{p_j}}{1-2\alpha}, & p_j \in [\frac{\alpha V}{1-\alpha}, v]; \\ \frac{1-\alpha-\frac{\alpha V}{v}}{1-2\alpha}, & p_j \in (v, V); \\ 1, & p_j \geq V, \end{cases} \tag{7}$$

where $p^* \in (\frac{1-\alpha-\frac{\alpha V}{v}}{1-2\alpha}, 1]$.

Next consider the case where $\Delta F_A(v) = \Delta F_B(v) = 0$. In this case, the equilibrium is symmetric, and we have both $F_A(\cdot)$ and $F_B(\cdot)$ characterized by the $F_j(\cdot)$ above.[14]

25. Let $S_A$ and $S_B$ be the supports of equilibrium prices $p_A$ and $p_B$. Then for $i = A, B$, we can interpret $\sup S_i$ as the *regular price* of firm $i$, and any price strictly lower than $\sup S_i$ as a *dealing price*. Now we can compute the dealing frequency for a firm, which is the firm's probability of selecting a dealing price. In the symmetric equilibrium, the dealing frequency for both firms is $\frac{1-\alpha-\frac{\alpha V}{v}}{1-2\alpha}$, which is decreasing in $\alpha$ and $V$ and increasing in $v$, a result rather consistent with our intuition. The *depth of dealing* for both firms in the symmetric equilibrium is defined as

$$V - E[\tilde{p}|\tilde{p} \leq v] = V - \frac{\alpha V \log(\frac{(1-\alpha)v}{\alpha V})}{1 - \alpha - \frac{\alpha V}{v}}. \tag{8}$$

As an exercise, you can examine how the parameters $V, v$, and $\alpha$ may respectively affect the depth of dealing.

[14] It can be verified that in the former equilibrium $v$ is a not a pure-strategy best response for firm $j$, but it is in the latter equilibrium.

26. We can also consider the case where $\alpha > \beta$, $\alpha V < (1-\beta)v$ and $\beta V < (1-\alpha)v$. In fact, one can show that $\alpha(1-\alpha) > (1-\beta)\beta$,[15] and hence $\alpha V < (1-\beta)v$ implies that $\beta V < (1-\alpha)v$. In this case, one can

[15] Since $1-\alpha-\beta > 0$, $\frac{1}{2} - \beta > \alpha - \frac{1}{2}$, the inequality holds when $\alpha \geq \frac{1}{2} > \beta$. If instead $\alpha < \frac{1}{2}$, then the inequality holds obviously.

show that the unique equilibrium in mixed strategy is such that[16]

$$F_A(x) = \begin{cases} 0, & x \leq \underline{p} \equiv \frac{\alpha V}{1-\beta}; \\ \\ \frac{1-\frac{\underline{p}}{x}}{1-\frac{\beta}{1-\alpha}}, & x \in [\underline{p}, v); \\ \\ \frac{1-\frac{\underline{p}}{v}}{1-\frac{\beta}{1-\alpha}}, & x \in [v, V); \\ \\ 1, & x \geq V, \end{cases} \tag{9}$$

[16]Let us demonstrate in detail how to get this equilibrium.

- At first, the supports of $F_A$ and $F_B$ must share the same greatest lower bound $\underline{p}$, because given $\underline{p}_i$, each $p_j < \underline{p}_i$ is strictly dominated by, say, $\frac{p_j+\underline{p}_i}{2}$, for $i \neq j$, $i, j \in \{A, B\}$. Note that $\underline{p} > 0$ because both firms can ensure a strictly positive profit by serving their own loyals only. Since $F_A(\cdot)$ and $F_B(\cdot)$ must be strictly increasing over $[\underline{p}, v)$ and continuous on $(-\infty, v)$, $\underline{p}$ must be an equilibrium best response for both firms.
- Second, in equilibrium at least one firm $i$ must price at $V$ with a strictly positive probability. Define $\overline{p}_j \equiv \sup\{p_j : p_j \leq v\}$. If instead $F_A(v) = F_B(v) = 1$, then we shall demonstrate a contradiction. We first claim that $\overline{p}_A = \overline{p}_B = \overline{p}$: in case $\overline{p}_i < \overline{p}_j \leq v$, then any $p_j \in (\overline{p}_i, \overline{p}_j)$ would be dominated by $\frac{V+p_j}{2}$ from firm $j$'s perspective, a contradiction to the definition of $\overline{p}_j$. Next, we claim that neither $F_A$ nor $F_B$ can have a jump at $\overline{p}$: if $\Delta F_i(\overline{p}) > 0$ then firm $j$ would rather price at $\overline{p} - \epsilon$ than at $\overline{p}$, for some sufficiently small $\epsilon > 0$, and hence $\Delta F_j(\overline{p}) = 0$, but then pricing at $\overline{p}$ is dominated by pricing at, say, $\frac{1}{2}(V + \overline{p})$ from firm $i$'s perspective, which contradicts the assumption that $\Delta F_i(\overline{p}) > 0$! Now, by the fact that $F_A$ and $F_B$ are both continuous at $\overline{p}$, again, $\overline{p}$ must be a best response for firm $i$ (because for some tiny $\delta > 0$ and for all positive integers $n$, every interval $(\overline{p} - \frac{\delta}{n}, \overline{p})$ contains one equilibrium pure-strategy best response $p_i^n$ for firm $i$ (cf. Lemmas 4 and 6) with $\lim_{n\to\infty} p_i^n = \overline{p}$), but it is clear that pricing at $\overline{p}$ is still dominated by, say, pricing at $\frac{1}{2}(V + \overline{p})$ from firm $i$'s perspective! We conclude that a contradiction would always arise unless at least one firm will price at $V$ with a strictly positive probability in equilibrium.
- Third, there is exactly one firm that may price at $V$ with a strictly positive probability. To see this, recall that firm $i$'s equilibrium payoff (expected profit) is $\Pi_i = \alpha_i V$ if with a strictly positive probability it may price at $V$, and that $\underline{p}$ is an equilibrium best response for both firms, implying that $\Pi_A = (1-\beta)\underline{p}$ and $\Pi_B = (1-\alpha)\underline{p}$. Now, if $V$ were an equilibrium best response for both firms, then we would have
$$\alpha V = \Pi_A = (1-\beta)\underline{p}, \ \beta V = \Pi_B = (1-\alpha)\underline{p},$$
implying that
$$\frac{\alpha V}{1-\beta)} = \underline{p} = \frac{\beta V}{1-\alpha} \Rightarrow \alpha(1-\alpha) = \beta(1-\beta),$$
which is a contradiction.
- Fourth, we claim that it is firm A that may price at $V$ with a strictly positive probability. To see this, suppose firm $i$ may price at $V$ with a strictly positive probability in equilibrium. Then, for all $x \in [\underline{p}, \overline{p})$,
$$\Pi_i = \alpha_i V = x\{\alpha_i + (1-\alpha_i-\alpha_j)[1-F_j(x)]\} \Rightarrow F_j(x) = 1 - \frac{\frac{\alpha_i V}{x} - \alpha_i}{1-\alpha_i-\alpha_j},$$

which, by the fact that $F_j(\underline{p}) = 0$, implies that

$$\underline{p} = \frac{\alpha_i V}{1-\alpha_j}.$$

Now, if $i = B$, then we would have

$$\underline{p} = \frac{\beta V}{1-\alpha},$$

so that firm A's equilibrium payoff would become

$$(1-\beta)\underline{p} < \alpha V,$$

which is a contradiction!

There is an intuitive explanation for this result. Recall that when a firm lowers its price from $V$ in order to compete for the switchers, it fails to fully extract its loyal customers' consumer surplus. Thus the firm with a larger loyal base must incur a greater opportunity cost when competing for the switchers.

- The above analysis implies that in equilibrium firm A must price at $V$ with a strictly positive probability, so that firm B's equilibrium payoff is $\frac{\alpha(1-\alpha)V}{1-\beta} > \beta V$, implying that pricing at $\underline{p}$ dominates pricing at $V$ for firm B, proving that $F_B(\overline{p}) = 1$. We claim that $\Delta F_B(\overline{p}) > 0$: otherwise, $F_B(\cdot)$ would be continuous at $\overline{p}$, so that $\overline{p}$, being the limit of an increasing sequence of pure-strategy best responses for firm A, must be an equilibrium best response for firm A also, implying that firm A's equilibrium payoff $\alpha V$ is also equal to $\alpha\overline{p} \leq \alpha v$, a contradiction.
- Now, given $\Delta F_B(\overline{p}) > 0$, we must have $\Delta F_A(\overline{p}) = 0$: pricing at $\overline{p}$ is dominated by pricing at $\overline{p} - \epsilon$ for firm A for sufficiently small $\epsilon > 0$.
- Given that $\Delta F_B(\overline{p}) > 0$, we claim that $\overline{p} = v$: in case $\overline{p} < v$, pricing at $\frac{v+\overline{p}}{2}$ would strictly dominate pricing at $\overline{p}$ from firm B's perspective, a contradiction to $\Delta F_B(\overline{p}) > 0$, which implies that $\overline{p}$ is an equilibrium pure-strategy best response for firm B.
- To sum up, we have

$$\underline{p} = \frac{\alpha V}{1-\beta},$$

$$F_B(x) = 1 - \frac{\frac{\alpha V}{x} - \alpha}{1-\alpha-\beta}, \ \forall x \in [\frac{\alpha V}{1-\beta}, v) \Rightarrow F_B(v-) \equiv \lim_{x\uparrow v} F_B(x) = 1 - \frac{\alpha(V-v)}{(1-\alpha-\beta)v} < 1 = F_B(v),$$

and

$$F_A(x) = 1 - \frac{\frac{\alpha(1-\alpha)V}{(1-\beta)x} - \beta}{1-\alpha-\beta}, \ \forall x \in [\frac{\alpha V}{1-\beta}, v) \Rightarrow F_A(v) = F_A(v-) \equiv \lim_{x\uparrow v} F_A(x) = 1 - \frac{\frac{\alpha(1-\alpha)V}{(1-\beta)v} - \beta}{1-\alpha-\beta} < 1.$$

This finishes our derivation for $F_A$ and $F_B$.

and

$$F_B(x) = \begin{cases} 0, & x \le \underline{p} = \frac{\alpha V}{1-\beta}; \\ \frac{1-\frac{\underline{p}}{x}}{1-\frac{\alpha}{1-\beta}}, & x \in [\underline{p}, v); \\ 1, & x \ge v. \end{cases} \tag{10}$$

What is the effect of an increase in $\alpha$, say, on the equilibrium pricing strategies?

Observe that an increase in $\alpha$ has two direct influences on the configuration of consumers. It implies an increase in the population of firm A's loyals, and it also implies a decrease in the population of the switchers. The former leads to an increase in the equilibrium $\underline{p}$ (because the set of overly low prices dominated by the price $V$ is enlarged, given that $\alpha$ has increased so that there are now more loyals willing to pay $V$), and the latter results in an increase in the density of $F_A(\cdot)$ for all prices below $v$ that might arise in equilibrium.

The former effect is self-evident. Let us examine the latter effect. Suppose that $\alpha_2 > \alpha_1$, and for $i = 1, 2$, $\alpha_i > \beta$, $\alpha_i V < (1-\beta)v$. Fix $h, l$ such that $v > h > l > \underline{p}_2 > \underline{p}_1$. Note that given $i$, firm B is indifferent about $h$ and $l$:

$$h\beta F_A(h,\alpha_i)+h(1-\alpha_i)[1-F_A(h,\alpha_i)] = l\beta F_A(l,\alpha_i)+l(1-\alpha_i)[1-F_A(l,\alpha_i)],\ \forall i = 1,2.$$

Since given $i$, $hF_A(h,\alpha_i) > lF_A(l,\alpha_i)$, we conclude that $h[1-F_A(h,\alpha_i)] < l[1-F_A(l,\alpha_i)]$ so that when $\alpha$ increases from $\alpha_1$ to $\alpha_2$, if firm A's strategy were still $F_A(x,\alpha_1)$, then firm B would strictly prefer $h$ to $l$. This result is not surprising, as an increase in $\alpha$ also implies a reduction in $1-\alpha-\beta$, and given that $\beta$ does not change, firm B now considers lowering the price to win the switchers and giving up the chance of extracting surplus from the loyals more costly than before.

Since only a mixed-strategy equilibrium can exist given $\alpha = \alpha_2$, just like in the case of $\alpha = \alpha_1$, given the new $F_A(x,\alpha_2)$ firm B must again feel indifferent about $h$ and $l$. We demonstrate below that this will require $F_A$ to have a higher density function under $\alpha = \alpha_2$ than under $\alpha = \alpha_1$ over the prices under $v$ that may be chosen in both the cases $\alpha = \alpha_2$ and $\alpha = \alpha_1$.

Note that the above indifference equation can be re-arranged to get

$$F_A(h,\alpha) + l[\frac{F_A(h,\alpha) - F_A(l,\alpha)}{h-l}] = \frac{1}{1-\frac{\beta}{1-\alpha}}.$$

Taking limit on both sides by letting $l \to h$ and assuming that $F_A$ is differentiable on $(\underline{p}, v)$ (which can be verified independently), we have, given $\alpha$,

$$F_A(h,\alpha) + hf_A(h,\alpha) = \frac{1}{1-\frac{\beta}{1-\alpha}}, \quad \forall h \in (\underline{p}, v),$$

where $f_A = F_A'$ is the density function of firm A's equilibrium price. From here, we see two things. Note first that $F_A$ is strictly concave on $(\underline{p}, v)$:[17] the above righ-hand side is independent of $h$, and since $F_A(h,\alpha)$ is increasing in $h$, $f_A$ has to be strictly decreasing in $h$. Second, note that there is a small interval $[\underline{p}, \hat{p})$ such that at every $x$ inside that interval, $f_A(x,\alpha)$ is increasing in $\alpha$. This happens because the above right-hand side, $\frac{1}{1-\frac{\beta}{1-\alpha}}$ is increasing in $\alpha$, and Leibniz rule tells us that the change in $F_A$ at $h$ induced by a change in $\alpha$ can be attributed to a change in $\underline{p}$ (which has a negative effect) and a change in the density function.[18]

It is easy to verify that, indeed, given any $h \in (\underline{p}_2, v)$ we have $f_A(h,\alpha_2) > f_A(h,\alpha_1)$. In fact, by directly differentiating, we have for all $h \in (\underline{p}, v)$,

$$\frac{\partial^2 F_A(h,\alpha)}{\partial h \partial \alpha} = \frac{\partial f_A(h,\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \frac{\frac{\underline{p}}{h^2}}{1-\frac{\beta}{1-\alpha}} > 0.$$

This reflects the need of restoring indifference for firm 2 after an increase in $\alpha$ from $\alpha_1$ to $\alpha_2$. Following that increase, if firm 1 were to use

[17] This can be confirmed easily when $\alpha = \beta$. In fact, we have shown earlier that in the symmetric case where $\alpha = \beta$, $F_A$ coincides with $F_B$ on $(\underline{p}, v)$, and they are concave on this price region.

[18] Note that $F_A(h,\alpha) = \int_{\underline{p}(\alpha)}^h f_A(x,\alpha)dx$. The Leibniz rule says that

$$\frac{\partial F_A(h,\alpha)}{\partial \alpha} = -\underline{p}'(\alpha) f_A(\underline{p}(\alpha),\alpha) + \int_{\underline{p}(\alpha)}^h \frac{\partial f_A(x,\alpha)}{\partial \alpha}(x,\alpha)dx,$$

provided that $f_A$ and $\underline{p}(\alpha)$ are both continuously differentiable in $\alpha$.

$F_A(\cdot, \alpha_1)$, then firm 2 would strictly prefer $h$ to $l$, and so to restore indifference, we need to make sure that under $\alpha_2$, the difference in the probabilities of losing the switchers, $F_A(h, \alpha_2) - F_A(l, \alpha_2)$, is higher than its counterpart $F_A(h, \alpha_1) - F_A(l, \alpha_1)$ under $\alpha_1$. This being true for all $h$ and $l$, we conclude that $f_A$ is higher under $\alpha_2$ than under $\alpha_1$ at all $h \in (\underline{p}_2, v)$.

Now, let us summarize the effect of an increase in $\alpha$ on $F_A$. By directly differentiating, we have for all $x \in (\underline{p}, v)$,

$$\frac{\partial F_A(x, \alpha)}{\partial \alpha} = \frac{1}{[1 - \frac{\beta}{1-\alpha}]^2}\{\frac{\beta}{(1-\alpha)^2}[1 - \frac{\underline{p}(\alpha)}{x}] - \frac{\underline{p}'(\alpha)}{x}[1 - \frac{\beta}{1-\alpha}]\},$$

so that the sign of $\frac{\partial F_A(x,\alpha)}{\partial \alpha}$ is the same as the sign of

$$G(x) \equiv \frac{\beta}{(1-\alpha)^2}[1 - \frac{\underline{p}(\alpha)}{x}] - \frac{\underline{p}'(\alpha)}{x}[1 - \frac{\beta}{1-\alpha}].$$

Note that $G(\cdot)$ is strictly increasing, with $G(\underline{p}) < 0$. Letting $G(x^*) = 0$, we have

$$x^* = \underline{p} + \frac{(1-\alpha)(1-\alpha-\beta)V}{\beta(1-\beta)}.$$

Thus we can conclude that

- If $\min(1, \frac{\alpha}{1-\beta} + \frac{(1-\alpha)(1-\alpha-\beta)}{\beta(1-\beta)}) > \frac{v}{V} > \frac{\alpha}{1-\beta}$ so that the interval $(\underline{p}, v)$ does not contain $x^*$, then at all $x \in (\underline{p}, v)$, we have $\frac{\partial F_A}{\partial \alpha} < 0$.[19]
- If instead $1 > \frac{v}{V} \geq \frac{\alpha}{1-\beta} + \frac{(1-\alpha)(1-\alpha-\beta)}{\beta(1-\beta)}$ so that $x^* \in (\underline{p}, v)$, then $\frac{\partial F_A}{\partial \alpha}(x, \alpha) \leq 0$ if and only if $x \leq x^*$.

Intuitively, as suggested by Leibniz rule, an increase in $\alpha$ results in a decrease in $F_A$ at all $x \in (\underline{p}_1, \underline{p}_2]$, but to restore a mixed equilibrium, as we mentioned above, the density $f_A$ must become higher at all $x \in (\underline{p}_2, v)$. Thus for $x \in (\underline{p}_2, v)$, either $F_A$ becomes higher or it becomes lower under $\alpha_2$, and which one would happen depends on which between the above two opposing effects dominates.

[19]Recall that in the symmetric case, where $\alpha = \beta$, we have shown that $F_A(x) = \frac{1-\alpha-\frac{\alpha V}{x}}{1-2\alpha}$ for all $x \in (\underline{p}, v)$, and indeed $\frac{\partial F_A}{\partial \alpha} < 0$.

27. **Example 7**. Two firms A and B compete in price to serve two consumers 1 and 2. For simplicity, firms incur no production costs. Each consumer may buy 1 unit from either firm A or firm B, or decide not to buy anything. Firms and consumers are risk-neutral, and they seek to maximize respectively expected profits and expected consumer surplus. Consumer 1 regards both firms' products as perfect substitutes, and she is willing to pay 2 dollars for either product A or product B. Consumer 2 is willing to pay 2 dollars for product A, but 5 dollars for product B. The firms simultaneously post prices $p_A$ and $p_B$, and upon seeing the prices, consumers simultaneously decide whether to buy anything, and which firm to buy from.
(i) Show that there exists no pure-strategy equilibrium for this game. (**Hint**: Can there be a pure-strategy NE where firm B prices at $p_B > 2$? Can there be a pure-strategy NE where firm B prices at $p_B \leq 2$?)

(ii) Find a mixed-strategy NE $\{F_A(\cdot), F_B(\cdot)\}$ for this game.[20]

[20] There are no loyal customers in this game. This creates new difficulty. The two consumers differ only in the minimum price differential that is required to induce them to switch from a high-price firm to a low-price firm. I shall provide a few hints about how to derive an equilibrium.

- First verify that $\underline{p}_A \leq \underline{p}_B$.
- There are two possibilities: either $\underline{p}_B > 2$ or $\underline{p}_B \leq 2$. Let us conjecture that the former is the case. This conjecture can be verified after you derive the equilibrium mixed strategies.
- Given the conjecture $\underline{p}_B > 2$, now you can show that $\overline{p}_B = 5$. If instead $\overline{p}_B < 5$, then you can follow the procedure outlined below to obtain a contradiction.
  (a) If $\overline{p}_B < 5$, $\overline{p}_B - 3 < 2$. Then $\Delta F_A(\overline{p}_B - 3) = 0$: if instead $\Delta F_A(\overline{p}_B - 3) > 0$, then $\Delta F_B(\overline{p}_B) = 0$, implying that for firm A pricing at $p_A = \overline{p}_B - 3$ will lose consumer 2 for sure, and it fails to fully extract rent from consumer 1; pricing at $p_A = 2$ would be better. Then how can $\Delta F_A(\overline{p}_B - 3) > 0$?
  (b) Having shown that $F_A(\cdot)$ is continuous at $\overline{p}_B - 3$, you can show that $\overline{p}_B$ has to be a pure-strategy best response for firm B. You should use the following two facts: (1) $\overline{p}_B$ is the right endpoint of an interval in which every point is a pure-strategy best respons for firm B (cf. Lemma 4 and Lemma 6 in Example 6 of Lecture 1, Part II); and (2) $F_A(\cdot)$ is continuous at $\overline{p}_B - 3$.
  (c) Show that $F_A(\cdot)$ must be flat on $[\overline{p}_B - 3, 2)$!
  (d) Because $F_A(\cdot)$ must be flat on $[\overline{p}_B - 3, 2)$, argue that $\overline{p}_B$ is dominated by 5, contradicting the previously established result that $\overline{p}_B$ has to be a pure-strategy best response for firm B.
- Show that $\underline{p}_A = \underline{p}_B - 3$. To see this, note that since $\underline{p}_B > 2$, firm A should try to extract rent from consumer 1, and for that purpose alone, $\underline{p}_A$ should be as close to 2 as possible. However, firm A also wants to compete for consumer 2. But even for the latter purpose, there is no reason for firm A to price below $\underline{p}_B - 3$. Now, you can argue from firm B's perspective that, similarly, $\underline{p}_B$ cannot be less than $\underline{p}_A + 3$.
- Let the equilibrium profits of the two firms be denoted by $\pi_A$ and $\pi_B$. Since $\underline{p}_j$ has to be a pure-strategy best response for firm $j$, and since when using such a lower-bound price, firm $j$ wins consumer 2 for sure, show that $\pi_A = 2\underline{p}_A$ and $\pi_B = \underline{p}_B$.
- Show that $\overline{p}_A = 2$. If instead $\overline{p}_A < 2$, then since firm B does not price above $\overline{p}_A + 3$, this means that $\overline{p}_B < 5$, a contradiction to the result established earlier that $\overline{p}_B = 5$.
- Show that for all $p_A \in [\underline{p}_A, 2)$, and for all $p_B \in [\underline{p}_B, 5)$,

$$F_A(p_A) = 1 - \frac{\pi_B}{p_A + 3}, \quad F_B(p_B) = 2 - \frac{\pi_A}{p_B - 3}.$$

- Using the previous step to conclude that $\Delta F_A(2) > 0$. If instead that $F_A(2-) = 1$, then $\pi_B = 0$, which is impossible because firm B can always price at $\underline{p}_B$ and get a profit $\underline{p}_B > 2$.

28. **Example 8**. Consider the following variant of the duopolistic industry examined in Example 6. Here the main difference is that the two firms are faced with demand uncertainty. Assume that

| Segment | Population | Valuation for A | Valuation for B |
|---|---|---|---|
| $L_A$ | $\tilde{\alpha}$ | 2 | 0 |
| $L_B$ | 2 | 0 | 2 |
| S | $\tilde{\gamma}$ | 1 | 1 |

where the random variables $(\tilde{\alpha}, \tilde{\gamma})$ are such that

$$(\tilde{\alpha}, \tilde{\gamma}) = \begin{cases} (1, \frac{11}{2}), & \text{with probability } \frac{1}{2}; \\ \\ (6, \frac{9}{2}), & \text{with probability } \frac{1}{2}. \end{cases}$$

While firm B is faced with the above-mentioned demand uncertainty, firm A privately learns whether $(\tilde{\alpha}, \tilde{\gamma}) = (1, \frac{11}{2})$ or $(\tilde{\alpha}, \tilde{\gamma}) = (6, \frac{9}{2})$ right before the two firms compete in price. The above information asymmetry is part of the two firms' common knowledge at the beginning of the game. We shall look for a Bayesian equilibrium, which is denoted by $\{F_1(\cdot), F_6(\cdot), G(\cdot)\}$, where $G(\cdot)$ is the uninformed firm B's

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- Show that $\Delta F_B(5) = 0$.
- Show that firm A when pricing at 2 must lose consumer 2 for sure, thereby concluding that $\pi_A = 2$.
- Show by using the previous step that $\underline{p}_A = 1$.
- Show by using the result $\underline{p}_A = \underline{p}_B - 3$ that $\underline{p}_B = 4$.
- Show by using the result $\pi_B = \underline{p}_B$ that $\pi_B = 4$.
- Give a complete characterization for $(F_A(\cdot), F_B(\cdot))$.
- Verify easily that, given the equilibrium $F_B(\cdot)$, pricing outside $[\underline{p}_A, \overline{p}_A]$ cannot be optimal for firm A in equilibrium.
- Verify easily that, given the equilibrium $F_A(\cdot)$, pricing over $(-\infty, 1)$ or over $(2, 4)$ or over $(5, +\infty)$ cannot be optimal for firm B.
- As a last step, verify that for firm B, given the equilibrium $F_A(\cdot)$, neither pricing at $p_B = 2$ nor pricing over $[1, 2)$ can generate a payoff higher than $\pi_B = 4$.

Note that initially there do not exist loyal customers in the game. However, given the strategy $F_B(\cdot)$, consumer 1 becomes a loyal customer for firm A. Loyalty of a consumer is thus endogenously determined by the firms' equilibrium strategies.

mixed pricing strategy, and for $j = 1, 2$, $F_j(\cdot)$ is firm A's equilibrium mixed pricing strategy when firm A's demand information indicates that $\tilde{\alpha} = j$. (A Bayesian equilibrium is nothing but a Nash equilibrium for the three-player game in which the two types of firm A are treated as two different players.)

(i) Show that there exists a unique mixed strategy equilibrium where[21]

$$F_1(x) = \begin{cases} 0, & x < \frac{17}{28}; \\ 1 - \frac{17}{11}[\frac{1}{x} - 1], & x \in [\frac{17}{28}, 1); \\ 1, & x \geq 1, \end{cases}$$

[21] Let us refer to firm A as of type $j$ when $\tilde{\alpha} = j$. Let us define the following notation.

$$V = 2,\ v = 1, \alpha_1 = 1,\ \alpha_6 = 6,\ \gamma_1 = \frac{11}{2},\ \gamma_6 = \frac{9}{2},\ \beta = 2.$$

Thus $\gamma_j$ is the size of switchers when $\tilde{\alpha} = \alpha_j$. Recall that $\beta$ is the commonly known size of firm B's loyal customers. Note that these numerical values imply that

- the type-6 firm A never wants to compete for switchers,

$$\alpha_6 V > [\gamma_6 + \alpha_6]v;$$

- the type-1 firm A would like to serve the switchers if firm B were absent,

$$\alpha_1 V < [\gamma_1 + \alpha_1]v;$$

- firm B does not know firm A's type, but it knows that firm A has more loyal customers than firm B does if and only if $\tilde{\alpha} = \alpha_6$,

$$\alpha_6 > \beta > \alpha_1;$$

- even if firm B fails to capture any switchers in state $\alpha_1$, pricing at $v$ is still better than pricing at $V$ for firm B,

$$\beta V < [\frac{1}{2}\gamma_6 + \beta]v.$$

The Bayesian equilibrium reflects the above facts, and it extends the equilibrium that we obtained for Example 6 in Lecture 1, Part II. Based on the facts listed above, we have the following conjectures.

- First, the type-6 firm A's pricing behavior is obvious.
- Second, in competing with firm A, firm B knows that firm A will price at $V$ whenever it is of type 6. This implies that, as we just pointed out, firm B can at least get $[\beta + \frac{1}{2}\gamma_6]v$ by pricing at $v$!
- Third, firm B knows that it has more loyal customers than the type-1 firm A, and hence in state $\alpha_1$ the presence of switchers will not add to its expected profit! (A lesson that we learn from Example 6 in Lecture 1, Part II, says that the firm with more loyal customers does not benefit from the presence of switchers.) Thus firm B will not get more than $[\beta + \frac{1}{2}\gamma_6]v$ in this game. It must be that $\Pi_B^* = [\beta + \frac{1}{2}\gamma_6]v$.
- Since firm B has more loyal customers than the type-1 firm A, it is firm B's indifference between $\overline{p}_B = v = 1$ and $\underline{p}$ (which is the common price lower bound for firm B and type-1 firm A) that determines $\underline{p}$. In fact, since pricing at $\underline{p}$ is optimal in equilibrium for firm B, it must be that $\underline{p} = \frac{\Pi_B^*}{\beta + E[\tilde{\gamma}]}$.
- Once $\underline{p}$ is correctly conjectured, we can infer $\Pi_1^*$, since pricing at $\underline{p}$ is optimal in equilibrium for the type-1 firm A also

$$F_6(x) = \begin{cases} 0, & x < 2; \\ \\ 1, & x \geq 2, \end{cases}$$

$$G(x) = \begin{cases} 0, & x < \frac{17}{28}; \\ \\ 1 - \frac{\frac{\frac{13}{2}\times\frac{17}{4}}{7x} - 1}{\frac{11}{2}}, & x \in [\frac{17}{28}, 1); \\ \\ 1, & x \geq 1. \end{cases}$$

(ii) Denote by $\Pi_B^*$, $\Pi_1^*$ and $\Pi_6^*$ the equilibrium payoffs (expected profits) for respectively firm B, firm A with $\tilde{\alpha} = 1$, and firm A with $\tilde{\alpha} = 6$. Show that

$$\Pi_B^* = \frac{17}{4},\ \Pi_1^* = \frac{221}{56},\ \Pi_2^* = 12.$$

29. Before we end this note, we shall introduce a class of games with imperfect information, called games of *signal jamming*. We give 5 examples

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- Finally, infer that for all $x \in [\underline{p}, v)$, we must have

$$\Pi_B(x) = xH(x) \equiv x\{\beta + \frac{1}{2}\cdot 1 \cdot \gamma_6 + \frac{1}{2}\cdot [1 - F_1(x)]\cdot \gamma_1\},$$

and

$$\Pi_1(x) = x\{\alpha_1 + \gamma_1[1 - G(x)]\},$$

where $\Pi_B(x)$ is firm B's payoff as a function of firm B's own price $x$, given the other two firms' equilibrium strategies, and $\Pi_1(x)$ is similarly defined. Now, obtain the functions $F_1(\cdot)$ and $G(\cdot)$ on the interval $[\underline{p}, v)$ from the above two equations.

- We must have $\Delta F_1(v) = 0 < \Delta G(v)$. In case $F_1(\cdot)$ has a jump at $v$, then $G(\cdot)$ cannot have a jump at $v$ at the same time, implying that for firm A pricing at $v$ in state $\alpha_1$ would lose the switchers with probability one, and that firm A's equilibrium payoff in state $\alpha_1$ is $\alpha_1 v < \alpha_1 V$, unless firm B would price at $V$ with a strictly positive probability, or, unless $\Pi_B = \beta V$, but in the latter case, we have

$$F_1(v-) = \lim_{x\uparrow v} F_1(x) = \lim_{x\uparrow v}[1 - \frac{\frac{\beta V}{x} - \beta - \frac{1}{2}\gamma_6}{\frac{1}{2}\gamma_1}] < 1 \Leftrightarrow \beta V > [\beta + \frac{1}{2}\gamma_6]v,$$

which is a contradiction. Hence, $F_1(\cdot)$ does not have a jump at $v$, implying that $H(x)$ is a left-continuous function of $x$ for $x \in (\underline{p}, v]$, and since $xH(x) = \Pi_B^*$ is a constant for all $x \in (\underline{p}, v)$, we conclude that $vH(v) = \lim_{x\uparrow v} xH(x) = \lim_{x\uparrow v} \Pi_B(x) = \Pi_B^*$, and hence $v$ attains the equilibrium payoff of firm B, showing that $v$ is indeed a best response for firm B.

Now, follow the above guidance to finish this exercise.

below.

30. **Example 9**. Consider a firm run by an owner-manager Mr. A. At date 1, Mr. A can either costlessly exert a low effort ($e = 0$) or exert a high effort ($e = 1$) by incurring a disutility $c > 0$. Mr. A's effort choice is unobservable to public investors. Let $\pi_1$ and $\pi_2$ be respectively the firm's date-1 and date-2 profits. If Mr. A exerts a low effort, the firm's date-1 and date-2 profits are both zero. If Mr. A exerts a high effort, then the firm's date-1-date-2 profits may be

$$(\pi_1, \pi_2) = \begin{cases} (0, 5), & \text{with probability } \beta; \\ (1, 0), & \text{with probability } 1 - \beta. \end{cases}$$

The firm has no growth opportunities after date 2, and it will be liquidated at the end of date 2. Mr. A and all investors in the financial market are risk-neutral without time preferences.
(i) Suppose that it is common knowledge that Mr. A will never sell shares by the end of date 2. Find a set of conditions on $c$ and $\beta$ ensuring that Mr. A exerts a low effort in equilibrium.
(ii) Suppose that $1 > \beta > \frac{1}{4}$ and $0 < c < 2$. Suppose that, contrary to part (i), it is common knowledge that Mr. A will sell all the equity at date 1 (without distributing any cash dividends at date 1), and that public investors' only date-1 information about the firm is $\pi_1$.

- Find further conditions on $c$ and $\beta$ ensuring that Mr. A exerts a high effort in equilibrium.
- Find further conditions on $c$ and $\beta$ ensuring that Mr. A exerts a low effort in equilibrium.

*Solution.* The answer to part (i) is $c > 1 + 4\beta$.

The answer to part (ii) is $c > 1 - \beta$ if we want to induce Mr. A to exert a low effort, and this answer needs a little explanation.

At first, note that exerting a high effort is Mr. A's equilibrium strategy if it is common knowledge that Mr. A will not sell shares by the end of date 2, because $1 > \beta > \frac{1}{4}$ and $0 < c < 2$.

This can never be Mr. A's equilibrium strategy if it is common knowledge that Mr. A will sell all his shares at date 1. To see this, suppose instead that it were. Then by exerting a high effort, Mr. A expects that $\pi_1 = 0$ (and hence he can get the proceeds $\pi_1 + \pi_2 = 5$ by selling the firm) with probability $\beta$ and $\pi_1 = 1$ (and hence he can get the proceeds $\pi_1 + \pi_2 = 1$ by selling the firm) with probability $1 - \beta$, so that Mr. A's payoff is accordingly

$$5\beta + (1 - \beta) - c = 1 + 4\beta - c.$$

However, by exerting a low effort, Mr. A can make sure that $\pi_1 = 0$, so that he can sell the firm for a price of 5 for sure at date 1. Thus Mr. A will always deviate, which is a contradiction to the supposed equilibrium.

Having shown that exerting a high effort cannot be Mr. A's equilibrium behavior, I must emphasize that this does not imply that exerting a low effort is automatically Mr. A's equilibrium behavior.

Suppose that Mr. A does exert a low effort in equilibrium. Then he gets nothing by selling his firm. However, by exerting a high effort, Mr. A expects that $\pi_1 = 1$ with probability $1 - \beta$, and following such a zero-probability event the public investors have no choice but to believe that the firm is worth at least one dollar (because $\pi_2$ is always non-negative, regardless of Mr. A's choice of effort).[22] Hence if $c < 1 - \beta$, exerting a low effort is not Mr. A's equilibrium behavior either.

We conclude that there are *no* conditions on $c$ and $\beta$ that can support an equilibrium where Mr. A exerts a high effort, and that $c > 1 - \beta$ is the necessary and sufficient condition that supports the equilibrium where Mr. A exerts a low effort.

This exercise delivers the basic idea of signal-jamming. Note that if the market believes that Mr. A will exert a high effort (because it is efficient to do so from the firm's perspective), then Mr. A has an incentive to fool the market and raise his own payoff by exerting a low effort. By definition, a Nash equilibrium is a state of decision-making

[22] The market does not know that Mr. A has deviated, and hence they still price the firm at zero upon seeing zero profit in period 1. Only when the period-1 profit is 1, the market is forced to re-assess the situation, and the market can then infer that this can only happen after Mr. A has exerted a high effort, and hence the market also infers that the period-2 profit is zero. Thus by exerting a high effort, the manager spends $c$, but only gets 1 with prob. $1 - \beta$. For Mr. A's exerting a low effort to be part an NE, we conclude that $c$ must be greater than $1 - \beta$.

by all players, where nobody predicts his rivals' actions incorrectly. Hence in equilibrium, Mr. A cannot have a chance to fool the market, and we have shown that this implies a corporate inefficiency in the current exercise—only exerting a low effort is consistent with a Nash equilibrium of the current game.

31. **Example 10**. Consider a firm run by an owner-manager Mr. A. The firm has 1 share of common stock outstanding. At date 1, Mr. A can either costlessly exert a low effort ($e = 0$) or exert a high effort ($e = 1$) by incurring a (non-monetary) disutility $c > 0$. Mr. A's effort choice is unobservable to public investors. Let $\pi_1$ and $\pi_2$ be respectively the firm's date-1 and date-2 profits. If Mr. A exerts a high effort, the firm's date-1 and date-2 profits are both 2. If Mr. A exerts a low effort, then the firm's date-1-date-2 profits may be

$$(\pi_1, \pi_2) = \begin{cases} (0, 1), & \text{with probability } \beta; \\ (2, 6), & \text{with probability } 1 - \beta. \end{cases}$$

The firm has no growth opportunities after date 2, and it will be liquidated at the end of date 2. Mr. A and all investors in the financial market are risk-neutral without time preferences.

Now suppose that the firm has decided not to distribute cash dividends at date 1, and that

$$\beta = \frac{5}{7}, \quad c = 2.$$

(i) Suppose that it is common knowledge that Mr. A will never sell shares by the end of date 2. What is the date-1 share price of the firm?
(ii) Suppose that, contrary to part (i), it is common knowledge that Mr. A will sell all the equity at date 1, and that public investors' only date-1 information about the firm is $\pi_1$. What is the date-1 share price of the firm?
*Solution.* For part (i), note that Mr. A will maximize the expected value of $\pi_1 + \pi_2$ minus any disutility from exerting efforts. If he exerts a high effort, then his payoff is

$$2 + 2 - 2 = 2.$$

If he exerts a low effort, then his payoff becomes

$$\beta \cdot (0 + 1) + (1 - \beta) \cdot (2 + 6) = 8 - 7\beta = 3.$$

Thus the stock market expects Mr. A to exert a low effort, and hence the date-1 share price is 1 after the stock market observes $\pi_1 = 0$, and the date-1 share price is 8 after the stock market observes $\pi_1 = 2$.[23]

Now consider part (ii). We first show that exerting a low effort is no longer Mr. A's equilibrium behavior. Suppose instead that it were. Then the date-1 share price would be 8 after the stock market sees $\pi_1 = 2$. But then by exerting a high effort, Mr. A could get the payoff

$$-2 + 8 = 6,$$

which is greater than 3, the supposed equilibrium payoff for Mr. A.

We still need to verify if exerting a high effort is Mr. A's equilibrium behavior. If the market believes that it is, then the date-1 share price will be 4 after the stock market sees $\pi_1 = 2$. What if the market sees $\pi_1 = 0$? This is a zero-probability event from the public investors' perspective, but if it does happen, the market has no choice but to believe that Mr. A has exerted a low effort, and hence the market can infer that $\pi_2 = 1$, thereby pricing the firm at 1. Now, would Mr. A deviate and exert a low effort when the stock market believes that he would exert a high effort?

Mr. A's payoff in this supposed equilibrium is $2+2-2 = 2$. By exerting a low effort instead, according to our analysis above, with probability $\beta$ he would get 1, and with probability $1 - \beta$, he would get 4 (which is the firm value that the stock market believes upon seeing $\pi_1 = 2$). Hence by deviation Mr. A gets the payoff

$$\beta \cdot 1 + (1 - \beta) \cdot 4 = \frac{13}{7} < 2.$$

Thus Mr. A has no incentive to deviate. This proves that exerting a high effort is indeed Mr. A's equilibrium behavior. Thus the date-1 share price is, with probability one, 4.

Value maximization results in a moral hazard problem in this exercise, because Mr. A has an incentive to manipulate the market's belief

[23]We have assumed that the firm does not distribute $\pi_1$ as dividends, and so the date-1 cum-dividend price coincides with the date-1 ex-dividend price. If the stock market is open at date 0, then the date-0 price equals $8 - 7\beta = 3$, which is the expected value of the date-1 price, which is random conditional on the public investors' date-0 information.

regarding the unobservable effort. As is common in the moral hazard literature, the equilibrium effort may be higher or lower than the effort level chosen by Mr. A when his effort is observable. We have seen in Example 9 that the managerial effort can be too low in the signal-jamming equilibrium; here, it is too high.

32. **Example 11.** Two firms compete in an industry that extends for two dates ($t = 1, 2$). Firm 2 may be of type $G$ with prob. $b$ and type $B$ with prob. $1 - b$. Neither firm 1 nor firm 2 knows firm 2's type (notice that we are abusing terminology here!). The game proceeds as follows. At $t = 1$, firm 1 can choose either P (prey) or A (accomodate). If firm 1 chooses P, then firm 1's profit is $-c < 0$ and firm 2's profit is $L < 0$ at date 1. If firm 1 chooses A, then firm 1's profit is 0 at date 1; firm 2's profit is $H > 0$ if its type is $G$ and is $L < 0$ if its type is $B$. Firm 1's action at date 1 is firm 1's private information. After seeing its date-1 profit, firm 2 must decide whether to exit at the beginning of date 2. If firm 2 decides to leave, firm 1 gets $M > 0$ and firm 2 gets zero at date 2; if firm 2 decides to stay, then firm 1 again must decide to P or to A at date 2, and the payoffs of the two firms at date 2 are just as described for date 1. Each firm seeks to maximize the sum of its profits at the two dates.

    Find conditions on the parameters sustaining the SPNE's where, respectively, (1) firm 1 preys at date 1 for sure; (2) firm 1 accomodates at date 1 for sure; and (3) firm 1 randomizes over P and A at date 1. *Solution.* Consider the SPNE described in (1). Firm 1 will choose A at date 2 if firm 2 stays. Thus firm 1 chooses P at date 1 only if firm 2 may leave at date 2. This needs

$$bH + (1 - b)L \leq 0.$$

    That is, rationally expecting firm 1's action P at date 1, firm 2's posterior belief is still that it is of type G with probability $b$, and since firm 1 will always choose A at date 2, leaving is optimal for firm 2 if and only if the above inequality holds. Now, if the above inequality holds, firm 1 must find it optimal to prey at date 1, and this requires that

$$-c + M \geq 0 + [b \cdot 0 + (1 - b) \cdot M].$$

This condition says that if firm 1 deviates and chooses A at date 1, then although firm 2 still believes that firm 1 chooses P, after seeing its date-1 profit being $H$, the type-G firm 2 will learn its type immediately and decide to stay at date 2. In that case, firm 1 will have to accomodate at date 2, so that firm 1's date-2 profit is zero. Thus firm 1's deviation at date 1 implies that its date-1 profit is zero rather than $-c$, and that its date-2 profit is 0 with probability $b$ and $M$ with probability $(1-b)$. The above inequality ensures that firm 1's equilibrium payoff $-c+M$ is greater than what it is expected to make by deviation. Thus the SPNE described in (1) exists if and only if $bM \geq c$ and $bH + (1-b)L \leq 0$.[24] Does signal jamming raise the chance that firm 2 exits at date 2? It does, although firm 2 is not fooled by firm 1's date-1 action. As we explained in the above footnote, without firm 1's interference, firm 2 will leave at date 2 with probability $1-b$, but in this equilibrium, firm 2 will leave at date 2 for sure.

Similarly, for the SPNE described in (2), firm 1's date-1 equilibrium action is A, which generates perfect information for firm 2 about the latter's type. Firm 2 will leave at date 2 if and only if it finds out that

[24]One may wonder why firm 2 enters in the first place if $bH + (1-b)L \leq 0$. Even if firm 1 is sure to choose A at both dates, it does not seem worthwhile for firm 2 to enter. This reasoning is correct if firm 1 preys at date 1 in equilibrium, but it is in general in correct; see our solution for part (2). What is involved here is a real option that firm 2 can obtain by entering: by entering the industry and running the risk of getting $L$ at date 1, firm 2 obtains a real option of staying at date 2 if and only if staying is profitable. For example, in case that firm 1 will choose A (which is exactly the equilibrium described in (2)) at date 1, then firm 2's expected profit is not $2 \times [bH + (1-b)L] \leq 0$; rather, it is $b \times 2H + (1-b) \times L$. Returning to the SPNE in (1), although firm 1's signal jamming activities do not fool firm 2, these activities may destroy firm 2's opportunity of getting the aforementioned real option: by choosing P at date 1, which is correctly expected by firm 2 in equilibrium, firm 1 can make sure that firm 2's posterior belief about its own type remains the same at the beginning of date 2. Facing no such real option, firm 2 should really stay out in the first place. Fudenberg and Tirole's model, which we shall review below, suggests that firm 1's preying at date 1, although failing to fool firm 2, usually implies a lower profit for firm 2 at date 1, and thereby discouraging (credibly because it is an equilibrium) firm 2 from entering in the first place as long as entering incurs a fixed cost. In fact, Fudenberg and Tirole consider a situation where firm 1 cannot destroy firm 2's opportunity of getting the real option by preying at date 1. In equilibrium preying may still benefit firm 1 because before entering firm 2 realizes that its profits may be low because firm 1 always wants to manipulate its belief by preying at date 1, and preying (which is credible) lowers both firms' profits.

its type is B. Thus in equilibrium firm 1 gets zero at date 1, and it expects to get $[b \cdot 0 + (1-b) \cdot M]$ at date 2. What happens if firm 1 secretly deviates at date 1 and chooses to prey? Without changing its belief about firm 1's date-1 action, firm 2 will leave for sure after firm 1 preys at date 1. Thus firm 1's payoff following this deviation is $-c+M$. We conclude that the SPNE described in (2) can be sustained if and only if

$$-c + M \leq 0 + [b \cdot 0 + (1-b) \cdot M] \Leftrightarrow c \geq bM.$$

Finally, consider the SPNE described in (3). Let $p > 0$ be the prob. that firm 1 preys at $t = 1$. Then conditional on its date-1 profit being L, firm 2 thinks that its type is $B$ with prob.[25]

$$\frac{1-b}{p + (1-p)(1-b)} = \frac{1-b}{pb + (1-b)} > 1 - b,$$

where the inequality explains why firm 1 would like to prey at date 1. However, in case

$$\frac{pbH + (1-b)L}{pb + (1-b)} > 0,$$

firm 2 would choose to stay at date 2 even if its date-1 profit is $L$, and in that case firm 1 would have no reason to prey at date 1. Therefore for firm 1 to prey at date 1, it is necessary that

$$\frac{pbH + (1-b)L}{pb + (1-b)} \leq 0.$$

In case

$$\frac{pbH + (1-b)L}{pb + (1-b)} < 0,$$

we have

$$pbH + (1-b)L < 0 \Rightarrow p < \frac{(b-1)L}{bH} < 1.$$

When this condition is met, firm 2 will leave at date 2 for sure if and only if its date-1 profit is $L$. On the other hand, firm 1 must feel

[25] To see this, note that firm 2's date-1 profit is always $L$ if its type is B (occurring with probability $1-b$), and its date-1 profit is $L$ with probability $p$ if its type is G (occurring with probability $b$).

indifferent about P and A at date 1. For this to happen, we claim that it is necessary that $c = bM$. To see this, suppose that $c < bM$, then by preying at date 1 firm 1 gets $-c+M$ which is greater than what it makes by accomodating at date 1, which is $0 + (1-b)M$. Thus firm 1 should assign zero probability to action A, a contradiction. The case $c > bM$ is similar. Our conclusion is therefore that when $bH + (1-b)L > 0$ and $c = bM$, the game has a continuum of SPNE where firm 1 randomizes between P and A with respectively prob. $p \in (0, \frac{(b-1)L}{bH})$ and $1-p$ at date 1.

Next, consider the case where

$$\frac{pbH + (1-b)L}{pb + (1-b)} = 0,$$

or

$$p = \frac{(b-1)L}{bH},$$

where $p < 1$ if and only if $bH + (1-b)L > 0$. In this case, a first-period profit $L$ makes firm 2 feel indifferent about leaving and staying. Let $q$ be its probability of leaving. Then firm 1 must feel indifferent about P and A, that is,

$$-c + b \cdot 0 + (1-b) \cdot (q \cdot M + (1-q) \cdot 0)$$

$$= (q \cdot M + (1-q) \cdot 0),$$

or equivalently,

$$q = \frac{c}{bM},$$

where $q < 1$ if and only if $c < bM$. Hence when $bH + (1-b)L > 0$ and $c < bM$, we have a mixed-strategy equilibrium where firm 1 chooses P with probability $\frac{(b-1)L}{bH}$ and firm 2 stays after seening $H$ in period 1, and firm 2 leaves with probability $\frac{c}{bM}$ after seeing $L$ in period 1.

Note that in the above mixed-strategy equilibria, firm 2 does exit more often than in the case where P is not a feasible action for firm 1.

33. **Example 12.** (Riordan, 1985, *Rand Journal of Economics*.) Suppose that there are $n$ firms competing in quantity at dates 1 and 2. Each firm has unit cost $c$, and is facing a capacity constraint (per date) of $k$.

(That is, unit cost jumps to $+\infty$ for any extra quantity over $k$.) The inverse demand (on the relevant range) at date $t$ is

$$p_t = a_t - \sum_{j=1}^{n} q_{jt}.$$

Assume that

$$a_1 = e_1, \;\; a_2 = \rho a_1 + (1-\rho)e_2,$$

where $\rho \in [0,1]$ and $e_1$ and $e_2$ are i.i.d. (identically and independently distributed) with common density $f(\cdot)$ on the support $[e, E]$, where $e > 0$. Define $a \equiv E[e_t]$. Riordan imposes the following simplifying assumption

$$\min(e-c, e+c-a) > k > \frac{E-c}{n} > 0.$$

Each firm maximizes $\pi_{j1} + \delta\pi_{j2}$, where $\delta \in (0,1)$ is a discount factor, and $\pi_{jt} = (p_t - c)q_{jt}$.

Riordan assumes that at the beginning of date 2, firm $j$ only observes $q_{j1}$ and $p_1$, but neither $a_1$ nor the date-1 outputs chosen by other firms. A symmetric SPNE (in pure strategy) is a pair $(q_1^*, \sigma_2^*(p_1))$, such that $\sigma_2^*(\cdot)$ assigns a $q_{j2}$ for firm $j$ given each and every possible realization of $p_1$. (Note that in a symmetric equilibrium, $\sigma_2^*(\cdot)$ must be measurable to the common knowledge information set of all firms, and hence it can only depend on the commonly observable $p_1$.) More precisely, $(q_1^*, \sigma_2^*(p_1))$ must solve the following constrained maximization problem:

$$\max_{(q_1, \sigma_2(p_1)) \in [0,k]^2} E[(p_1 - c)q_1 + \delta(p_2 - c)\sigma_2(p_1)]$$

subject to

$$p_1 = a_1 - (n-1)q_1^* - q_1,$$
$$p_2 = a_2 - (n-1)\sigma_2^*(p_1) - \sigma_2(p_1).$$

Let us solve this problem by backward induction. At the beginning of date 2, a firm with date-1 output $q_1$ upon seeing $p_1$, can infer the intercept of the date-1 inverse demand function (given its belief that other firms chosen $q_1^*$ at date 1):

$$a_1^*(q_1, p_1) = p_1 + (n-1)q_1^* + q_1.$$

Thus this firm believes that the (random) date-2 price would be

$$p_2 = \rho a_1^*(q_1, p_1) + (1-\rho)e_2 - (n-1)\sigma_2^*(p_1) - q_2$$

$$= \rho[p_1 + (n-1)q_1^* + q_1] + (1-\rho)e_2 - (n-1)\sigma_2^*(p_1) - q_2,$$

and we define

$$p_2^*(q_2; p_1, q_1) \equiv E[p_2] = \rho a_1^*(q_1, p_1) + (1-\rho)a - (n-1)\sigma_2^*(p_1) - q_2.$$

At date 2 the firm seeks to

$$\max_{q_2 \in [0,k]} [p_2^*(q_2; p_1, q_1) - c]q_2.$$

The solution is

$$q_2 = \psi^*(p_1, q_1) \equiv \frac{1}{2}[\rho a_1^*(q_1, p_1) + (1-\rho)a - c - (n-1)\sigma_2^*(p_1)]$$

$$= \frac{1}{2}[\rho(p_1 + (n-1)q_1^* + q_1) + (1-\rho)a - c - (n-1)\sigma_2^*(p_1)].$$

In equilibrium, we must have

$$\sigma_2^*(p_1) = \psi^*(p_1, q_1^*),$$

which gives

$$\sigma_2^*(p_1) = \frac{\rho(p_1 + nq_1^*) + (1-\rho)a - c}{n+1}.$$

Using

$$p_1 = e_1 - (n-1)q_1^* - q_1,$$

we have

$$\frac{d\sigma_2^*(p_1)}{dp_1}\frac{dp_1}{dq_1} = -\frac{\rho}{n+1};$$

that is, from each firm's perspective, increasing its date-1 output by one unit will lead to $\frac{\rho}{n+1}$ units of reduction in the date-2 output of each rival. The idea is that imperfect information does not allow a firm to distinguish a reduction in $a_1$ from an expansion of its rivals' date-1 outputs, and since $a_2$ is positively correlated with $a_1$, upon seeing a low $p_1$ the firm must partially attribute this low price to a low realization of $a_1$, and hence reduce its date-2 output accordingly. Realizing this, a firm cannot resist the temptation of expanding its date-1 output as an attempt to mislead its rivals and discourage the latter from choosing high outputs at date 2. Since all firms are expanding outputs (relative to the full-information case), the firms are actually worse off.

Now, consider a firm's date-1 problem. It seeks to

$$\max_{q\in[0,k]} E\{(p_1-c)q+\delta[p_2^*(\psi^*(p_1,q);p_1,q)-c]\psi^*(p_1,q)\},$$

subject to

$$p_1=a-(n-1)q_1^*-q.$$

The first-order condition, upon replacing $p_1$ into the objective function, is

$$[a-c-(n-1)q_1^*-2q]+\delta\rho\frac{n-1}{n+1}E[\psi^*(p_1,q)]=0,$$

where

$$E[\psi^*(p_1,q)]=\frac{a-c}{n+1}-\frac{\rho}{2}\frac{n-1}{n+1}(q_1^*-q).$$

Solving, and then letting $q=q_1^*$, we have

$$q_1^*=\frac{(a-c)[(n+1)^2+\delta\rho(n-1)]}{(n+1)^3},$$

and

$$\sigma_2^*(p_1)=\frac{\rho(p_1+nq_1^*)+(1-\rho)a-c}{n+1}.$$

The bottom line here is that $\sigma_2^*(p_1)$ coincides with the static Cournot symmetric output when $a_1=p_1+nq_1^*$; that is, no firms are fooled by their rivals' output expansion activities. Still, $q_1^*$ is greater than the static Cournot output level: the firms cannot resist trying to fool their rivals at date 1. In this sense, we conclude that with imperfect information about demand and the rivals' outputs, the date-1 efficiency is lower for the firms, but the date-2 efficiency is unchanged. Of course, if we discuss efficiency from the perspective of social benefit, then the date-1 efficiency is actually higher!

34. **Example 13**. (Fudenberg and Tirole, 1986, *Rand Journal of Economics*.) Suppose that two firms are located at the two ends of the Hotelling main street $[0,1]$, facing consumers uniformly located on the

street with a total population of one. Each consumer is willing to pay $v = 2$ dollars for one unit of the product produced by either firm 1 (located at the left endpoint of the Hotelling main street) or firm 2 (located at the right endpoint of the Hotelling main street). Firm 1 has no production costs, but firm 2 is faced with a random fixed cost $\tilde{F}$, which is uniformly distributed on $[0, \frac{1}{2}]$. A consumer located at $t \in [0, 1]$ must pay $t\theta$ and $(1 - t)\theta$ respectivley if he wants to visit firm 1 and firm 2, where we shall assume for simplicity that $\theta = 1$. The firms must compete in price at periods 1 and 2, and they seek to maximize the sum of profits over the two periods.

At the beginning of period 2, firm 2 can only observe its period-1 profit, but not its own fixed cost or firm 1's period-1 price.

To start the analysis, first verify that in the static counterpart of the model, the two firms will set price at $p_1 = p_2 = 1$, and divide the market equally, ending up with profit $\frac{1}{2}$ each. If firm 1 is a monopoly, then it will again choose $p_1 = 1$, enjoying a profit of 1.

With imperfect information, firm 2 will use its period-1 profit to infer its fixed cost, and this encourages firm 1 to *predate*: by lowering $p_1$ in period 1, firm 1 can lower firm 2's period-1 profit, hoping to convince firm 2 that its fixed cost is high and it should leave the market in period 2. If that works, then firm 1 will become a monopolist in period 2, and as we showed just now, its period-2 profit will increase by $\frac{1}{2}$! (Note that as long as firm 2 chooses to stay in period 2, the period-2 equilibrium prices depend on neither the realized $F$ or firm 2's estimate about $F$.)

In equilibrium, firm 1's signaling-jamming attempt can be rationally expected by firm 2, and the latter will not be fooled. In period 1, suppose that firm 1 chooses $p_1^*$ and firm 2 chooses $r_2(p_1^*)$. Then, given $p_1^*$, we have[26]

$$r_2(p_1^*) = \frac{1 + p_1^*}{2}.$$

From firm 1's perspective, if firm 1's period-1 price is $p_1$, then firm 2's period-1 profit $\pi_2$ will be, given that firm 2 will price at $r_2(p_1^*)$ in period

[26] Note that firm 2's period-1 price choice does not affect firm 1's period-2 equilibrium behavior, and hence $r_2(\cdot)$ is firm 2's static reaction function. In the static Hotelling game, when $\theta$ is sufficiently smaller than $v$, firm $i$'s demand function is $D_i(p_i, p_j) = \frac{1}{2} + \frac{p_j - p_i}{2\theta}$, so that firm $i$'s reaction function is $r_i(p_j) = \frac{p_j + \theta}{2}$.

1,

$$\pi_2 = \frac{[1+p_1-r_2(p_1^*)]r_2(p_1^*)}{2} - F,$$

where $F$ is the realized fixed cost of firm 2. However, believing that firm 1 has chosen $p_1^*$, firm 2 would then believe that its realized fixed cost is, given $\pi_2$,

$$\hat{F} \equiv \frac{[1+p_1^*-r_2(p_1^*)]r_2(p_1^*)}{2} - \pi_2.$$

Since firm 2 would get the period-2 payoff of $\frac{1}{2} - \hat{F}$ if firm 2 chooses to stay, firm 2 will optimally leave the market before period 2 if and only if

$$\frac{1}{2} \leq \hat{F} = \frac{[1+p_1^*-r_2(p_1^*)]r_2(p_1^*)}{2} - \pi_2$$

$$\Leftrightarrow \pi_2 \leq \frac{[1+p_1^*-r_2(p_1^*)]r_2(p_1^*)-1}{2}$$

$$\Leftrightarrow \frac{[1+p_1-r_2(p_1^*)]r_2(p_1^*)}{2} - F \leq \frac{[1+p_1^*-r_2(p_1^*)]r_2(p_1^*)-1}{2}$$

$$\Leftrightarrow F \geq \frac{1}{2} + \frac{p_1-p_1^*}{2} \cdot \frac{1+p_1^*}{2},$$

and the latter event occurs with probability

$$2[\frac{1}{2} - (\frac{1}{2} + \frac{p_1-p_1^*}{2} \cdot \frac{1+p_1^*}{2})] = \frac{(p_1^*-p_1)(1+p_1^*)}{2},$$

given that $F$ is uniformly distributed over $[0, \frac{1}{2}]$.

Thus, firm 1's period-1 problem is to

$$\max_{p_1} \frac{p_1(1-p_1+r_2(p_1^*))}{2} + \frac{1}{2} + \frac{1}{2}[\frac{(p_1^*-p_1)(1+p_1^*)}{2}].$$

In the above objective function, $p_1$ is firm 1's period-1 price, $\frac{1-p_1+r_2(p_1^*)}{2}$ is firm 1's period-1 sales volume given that firm 2 will price at $r_2(p_1^*)$ in period 1, the third term ($\frac{1}{2}$) is firm 1's period-2 profit if firm 2 chooses to stay in period 2, and the last term is the probability that firm 2 may leave before period 2 (which is $\frac{(p_1^*-p_1)(1+p_1^*)}{2}$) times the additional profit that firm 1 can make in period 2 in the event that firm 2 chooses to leave before period 2 (which is $\frac{1}{2}$).

In equilibrium, $p_1^*$ must solve the above maximization problem for firm 1. The first-order condition yields

$$p_1^* = \frac{1}{2},$$

which implies that

$$r_2(p_1^*) = \frac{3}{4}.$$

Thus firm 1 does try to lower price in period 1 as an attempt to fool firm 2, but firm 2 is not fooled in equilibrium. In equilibrium, both firm 1 and firm 2 price below 1 in period 1. Again, period 2's pricing equilibrium is unaffected.

Note that we have thus far assumed that firm 2 is already present in period 1. If in addition to the fixed operating cost $F$ that firm 2 needs to incur in each period, firm 2 must spend another fixed cost $G$ to enter before period 1, then since signal-jamming reduces firm 2's period-1 profit, it tends to reduce the likelihood that firm 2 may be present in period 1 also.

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