# Game Theory with Applications to Finance and Marketing

## Lecture 3: Static Games with Incomplete Information

1. This note consists of two parts. In the first part, we review some basic theorems about mutual knowledge and common knowledge. There are two results in this part that are relevant in finance: (1) Aumann's theorem, which says that rational people endowed with the same prior beliefs cannot agree to disagree about their posterior beliefs after they receive private information; and (2) Milgrom and Stokey's *no-trade* theorem, which says that rational people cannot trade simply because they have received different information. In the second part we then examine a series of static games with incomplete information, and solve for the Bayesian equilibrium for each of them.

2. **Part I.**

3. **Definition 1.** Given a game, an event is the players' *mutual knowledge* in state $\omega$ if every player knows it in state $\omega$, and an event is the players' *common knowledge* in state $\omega$ if in state $\omega$ every player knows it, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it, and so on. Anything which is not common knowledge is some player's *private information.* A player's private information is also called his *type*; see Harsanyi (1967-68).

4. Let us make the above definition more formal, and derive several important results. Consider a (strategic or extensive) game with uncertainty and with a finite set $I$ of players. To model player $i$'s (interim) information structure under uncertainty, we assume that there is a unique finite sample space $\Omega$, so that player $i$'s information structure is represented by a probability space $(\Omega, \mathcal{F}_i, P_i)$.[1] Recall that an information partition $H$ of $\Omega$ is a collection of subsets (or events) of $\Omega$, such that

[1] It is generally assumed that player $i$'s ex-post information structure is $(\Omega, 2^\Omega, P_i)$, meaning that player $i$ will eventually observe the true state $\omega \in \Omega$. It is usually assumed also that player $i$'s ex-ante information structure is $(\Omega, \{\Omega, \emptyset\}, P_i)$, meaning that player $i$ starts with trivial information sets. Recall that $\{\{\Omega, \emptyset\}, \mathcal{F}_i, 2^\Omega\}$ is an *information filtration*, that completely describes the dynamic fashion in which the uncertainty facing player $i$ is resolved.

elements of $H$ are pairwise disjoint and the union of all elements in $H$ is exactly $\Omega$. Since $\Omega$ is finite, for each $\mathcal{F}_i$, there exists uniquely a finest information partition $H_i$ contained in $\mathcal{F}_i$ such that if $F \in \mathcal{F}_i$, then $F$ can be represented as a union of elements of $H_i$. Conversely, each $H_i$ uniquely generates a $\sigma$-algebra $\mathcal{F}_i$, for which we write $\mathcal{F}_i = \sigma(H_i)$; see my notes in *Stochastic Processes*. Thus given that $\Omega$ is finite, there is a one-to-one correspondence between $\mathcal{F}_i$ and $H_i$, and hence we can represent player $i$'s information structure by $(\Omega, H_i, P_i)$.

5. Given $(\Omega, H_i, P_i)$, for each state $\omega \in \Omega$, there exists $h_i(\omega) \in H_i$ such that $\omega \in h_i(\omega)$. We call $h_i(\omega)$ *player i's information set in state* $\omega$. To see why this terminology is adopted, imagine an extensive game where nature moves first by choosing an element of $\Omega$, and then player $i$ gets to move. As in Lecture 1, Part 1, when player $i$ moves, he knows in which information set he is standing, but he does not know precisely on which node in the information set he is standing. If the true state is $\omega$, meaning that nature has taken the *action* $\omega$, then player $i$ only knows that nature has selected some element in $h_i(\omega)$, but unless $h_i(\omega)$ is a singleton, player $i$ does not know exactly which state in $h_i(\omega)$ nature has selected. Thus $h_i(\omega)$ is indeed player $i$'s information set, which we defined in Lecture 1, Part 1, when we first introduced the notion of a game tree.

6. An event is any element of $\sigma(H_i)$. We say that *an event $E$ occurs in state* $\omega$ if and only if $\omega \in E$. For example, if the true state is $\omega$, then the event $E = \{\omega, \omega'\}$ (which represents the statement that *the true state is either* $\omega$ *or* $\omega'$) is true. Hence we say that $E = \{\omega, \omega'\}$ occurs in state $\omega$. For the same reason, $E = \{\omega, \omega'\}$ also occurs in state $\omega'$. Now by the definition of $h_i(\omega)$, we have $\omega \in h_i(\omega)$, and hence the event $h_i(\omega)$ occurs in state $\omega$. We shall say that *player i knows event $E$ in state* $\omega$ if $h_i(\omega) \subset E$. This definition is easy to undertand: by the definition of $h_i(\omega)$, in state $\omega$, player $i$ knows that the true state is some element of $h_i(\omega)$, and if $h_i(\omega) \subset E$, then player $i$ knows that the true state is some element of $E$, and hence player $i$ knows that event $E$ occurs; on the other hand, if $\omega' \in h_i(\omega) \bigcap E^c$, then in state $\omega$, player $i$ cannot be sure if $E$ occurs. (By the same reasoning, we shall say that *all the players*

*know event* $E$ *in state* $\omega$ if $\bigcup_{i\in I} h_i(\omega) \subset E$.) It follows that (i)

$$h_i(\omega) \subset F, \quad \omega' \in h_i(\omega) \Rightarrow \text{player } i \text{ knows } F \text{ in state } \omega';$$

and (ii) (*axiom of knowledge*)

$$h_i(\omega) \subset F, \quad \omega' \in h_i(\omega) \Rightarrow \omega' \in F \Rightarrow F \text{ occurs in state } \omega'.$$

7. Now let us denote the event (or the set of states where) *player* $i$ *knows* $E$ by $K_i(E) = \{\omega | h_i(\omega) \subset E\}$. The event that *everyone knows* $E$ is denoted by $K_I(E) = \{\omega | \cup_{i\in I} h_i(\omega) \subset E\}$. The event that everyone knows that everyone knows $E$ is denoted by

$$K_I^2(E) = \{\omega | \cup_{i\in I} h_i(\omega) \subset K_I(E)\}.$$

Recursively defining

$$K_I^{n+1}(E) = \{\omega | \cup_{i\in I} h_i(\omega) \subset K_I^n(E)\}, \ \forall n \in \mathbf{Z}_+,$$

and

$$K_I^\infty(E) = \bigcap_{n\in\mathbf{Z}_+} K_I^n(E),$$

we see that, by the *axiom of knowledge*, the sequence of events $\{K_I^n(E); n \in \mathbf{Z}_+\}$ converges decreasingly to $K_I^\infty(E)$. We then define an event $E$ as *common knowledge* in state $\omega$ if and only if $\omega \in K_I^\infty(E)$.

8. Given $\{H_i; i \in I\}$, the *meet* $\mathcal{M}$ of these information partitions is the finest common coarsening of $\{H_i; i \in I\}$; that is, $\mathcal{M}$ is an information partition of $\Omega$ such that (i) for all $\omega \in \Omega$, and for $i \in I$, $h_i(\omega) \subset M(\omega)$, where $M(\omega)$ is the element of $\mathcal{M}$ that contains $\omega$; and (ii) if $\mathcal{M}'$ is another common coarsening of $\{H_i; i \in I\}$ in the sense of (i), then for all $\omega \in \Omega$, $M(\omega) \subset M'(\omega)$, where $M'(\omega)$ is the element of $\mathcal{M}'$ that contains $\omega$.

For example, suppose that $I = \{1, 2\}$, and $\Omega = \{\omega_i; i = 1, 2, \cdots, 5\}$. Suppose that $H_1$ is depicted as

$$\boxed{\omega_1 | \omega_2, \omega_3 | \omega_4, \omega_5}$$

and $H_2$ is depicted as

$$\boxed{\omega_1, \omega_2 | \omega_3 | \omega_4 | \omega_5}$$

so that $\mathcal{M}$ is

$$\boxed{\omega_1, \omega_2, \omega_3 | \omega_4, \omega_5}$$

Is the event $E \equiv \{\omega_1, \omega_2, \omega_3, \omega_4\}$ the two players' common knowledge in state $\omega_3$? The answer is yes. In state $\omega_3$, player 2 knows for sure that the true state is $\omega_3$, and since $\omega_3 \in E$, player 2 knows $E$. Player 1, at the same time, knows that the true state is either $\omega_2$ or $\omega_3$, and in either case $E$ is true, and hence player 1 knows $E$. Moreover, player 1 knows that player 2 must have seen the event $h_2(\omega_2) = \{\omega_1, \omega_2\}$ or the event $h_2(\omega_3) = \{\omega_3\}$, and either way player 2 can infer that player 1 must have known $E$; and player 2, knowing that $\omega_3$ is the true state, believes that player 1 must have seen $h_1(\omega_3) = \{\omega_2, \omega_3\}$, so that player 1 would infer that player 2 must have seen either $h_2(\omega_2)$ or $h_2(\omega_3)$, and since player 2 knows $E$ in both cases, player 2 knows that player 1 knows that player 2 knows $E$. You can continue verifying that $\omega_3 \in K_I^n(E)$ for all $n \in \mathbf{Z}_+$ in this manner.

On the other hand, $E$ is not the two players' common knowledge in state $\omega_4$. In state $\omega_4$, player 1 sees $h_1(\omega_4) = \{\omega_4, \omega_5\}$, and hence player 1 thinks that state $\omega_5$ is also likely, but in state $\omega_5$, player 2 must have seen $h_2(\omega_5) = \{\omega_5\}$, so that player 2 knows $E$ in state $\omega_4$ but not in state $\omega_5$. By his information, therefore, player 1 cannot tell whether player 2 knows $E$ in state $\omega_4$ or not.

9. The above example shows that by the original definition (involving $K_I^\infty(E)$) it is generally difficult to check if an event is the players' common knowledge in a certain state. However, using $\mathcal{M}$, Theorem 1 provides an easy way to check whether an event $E$ is common knowlege in a state $\omega$.

**Theorem 1**. (Aumann, 1976) *Event $E$ is common knowledge at $\omega$ if and only if $M(\omega) \subset E$.*

We shall use the following two lemmas to prove Theorem 1.
**Lemma 1**. $M(\omega)$ is common knowledge at the true state $\omega$.[2]
*Proof.* Note that, by definition of $M(\omega)$,

$$K_I(M(\omega)) = \{\omega' | \cup_{i\in I} h_i(\omega') \subset M(\omega)\} = M(\omega).$$

It follows that for all $n$, $K_I^n(M(\omega)) = M(\omega)$, and hence $K_I^\infty(M(\omega)) = M(\omega)$. We have

$$\omega \in M(\omega) = K_I^\infty(M(\omega)),$$

and hence $M(\omega)$ is common knowledge in state $\omega$.
**Lemma 2**. If $E' \subset E$, then for all $n \in \mathbf{Z}_+$, $K_I^n(E') \subset K_I^n(E)$.
*Proof.* If in state $\omega$ it is true that

$$\cup_{i\in I} h_i(\omega) \subset E',$$

then, since $E' \subset E$, it is true that in state $\omega$

$$\cup_{i\in I} h_i(\omega) \subset E.$$

This implies that $K_I(E') \subset K_I(E)$. Next, suppose for some positive integer $n$ we have $K_I^n(E') \subset K_I^n(E)$. Then, we have

$$K_I^{n+1}(E') = K_I(K_I^n(E')) \subset K_I(K_I^n(E)) = K_I^{n+1}(E).$$

By mathematic induction, the lemma is proved.
Proof of Theorem 1. Suppose that $M(\omega) \subset E$. Lemma 1 implies that

$$\omega \in M(\omega) = K_I^\infty(M(\omega)),$$

and Lemma 2 implies that for all $n$,

$$K_I^n(M(\omega)) \subset K_I^n(E),$$

which in turn implies that

$$\omega \in K_I^\infty(M(\omega)) = \bigcap_{n=1}^{\infty} K_I^n(M(\omega)) \subset \bigcap_{n=1}^{\infty} K_I^n(E) = K_I^\infty(E),$$

[2] In fact Theorem 1 asserts that $M(\omega)$ is the finest event that can be common knowledge in state $\omega$.

showing that $E$ is common knowledge in state $\omega$.

Conversely, suppose that there exists $\omega' \in M(\omega) \bigcap E^c$, which implies that $\omega'$ is not an element of $E$, but there exists a finite sequence $\{\omega = \omega^1, \omega^2, \cdots, \omega^n\}$ contained in $M(\omega)$ and a corresponding sequence $\{i(1), i(2), \cdots, i(n)\}$ in the set $\{1, 2, \cdots, I\}$ such that,

$$\omega' \in h_{i(n)}(\omega^n),$$

and

$$\forall j = 1, 2, \cdots, n-1, \quad \omega^{j+1} \in h_{i(j)}(\omega^j).$$

That is, at state $\omega$ player $i(1)$ cannot rule out the possibility that the true state is $\omega^2$, and if the latter happens, then he knows that player $i(2)$ cannot rule out the possibility that the true state is $\omega^3$, and player $i(2)$ knows that if the latter happens, then player $i(3)$ cannot rule out the possibility that the true state is $\omega^4$, and so on and so forth. To sum up, at state $\omega$, player $i(1)$ thinks that player $i(2)$ thinks that player $i(3)$ thinks that $\cdots$ that player $i(n-1)$ thinks that player $i(n)$ might think that $\omega'$ could be the true state. Since $E$ does not happen at state $\omega'$, the occurrence of event $E$ cannot be the $I$ players' common knowledge at state $\omega$. $\|$

10. **Definition 2**. An event $E$ is called a *public event*, if everyone knows it whenever it happens. Mathematically, $E$ is a public event, if

$$\omega \in E \Rightarrow \omega \in K_I(E).$$

**Theorem 2**. A public event is common knowledge in state $\omega$ if it happens in state $\omega$.
*Proof.* By its mathematic definition, we deduce that for a public event $E$,

$$E = K_I(E).$$

It follows that $E = K_I^\infty(E)$. If $E$ happens at state $\omega$, then $\omega \in E = K_I^\infty(E)$, and hence $E$ is common knowledge at state $\omega$. $\|$

11. Continue to assume that $\Omega$ is a finite space. In this case, any (real-valued) random variable (r.v.) $x$ has a finite number of possible distinct

outcomes, say $\{x_1, x_2, \cdots, x_J\}$. Apparently, $x$ induces an information partition

$$H_x \equiv \{x^{-1}(\{x_j\}); j = 1, 2, \cdots, J\}$$

on $\Omega$ (cf. the $\sigma$-algebra generated by the r.v. $x$; see my notes in *Stochastic Processes*), where given $j$, the pre-image

$$x^{-1}(\{x_j\}) = \{\omega \in \Omega : x(\omega) = x_j\}.$$

Equivalently, for each state $\omega \in \Omega$, define the *pre-image* of $x(\omega)$ by

$$h_x(\omega) = x^{-1}(x(\omega)),$$

so that $h_x(\omega)$ is the element of $H_x$ that contains $\omega$. By our earlier definition, player $i$ knows $h_x(\omega)$ in state $\omega$ if and only if $h_i(\omega) \subset h_x(\omega)$. In this case, we also say that *player $i$ knows the realization of $x$ in state $\omega$*. Similarly, we say that *the realization of $x$ is the players' common knowledge in state $\omega$* if and only if

$$M(\omega) \subset h_x(\omega).$$

This implies that for all $\omega', \omega'' \in M(\omega)$,

$$x(\omega') = x(\omega'').$$

That is, if the realization of $x$ is the players' common knowledge in state $\omega$, then $x(\cdot)$ does not vary on the set $M(\omega)$.

12. Consider two players $i$ and $j$ with an identical ex-post information structure $(\Omega, 2^\Omega, P)$ but different interim information structures $(\Omega, H_i, P)$ and $(\Omega, H_j, P)$, where $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. (We say the two players have the same *prior beliefs* since $P_i = P_j = P$ on $2^\Omega$.). Given any ex-post distinguishable event $E \in 2^\Omega$, define player $i$'s interim probability of $E$ in state $\omega$ by

$$P(E|h_i(\omega)) \equiv \frac{P(E \bigcap h_i(\omega))}{P(h_i(\omega))}.$$

The following Theorem says that rational people (holding the same prior beliefs) cannot agree to disagree, and it has a direct consequence in the theory of asset trading.

**Theorem 3**. (Aumann, 1976) If it is common knowledge that in state $\omega$, rational players $i$ and $j$ have interim probabilities $q_i$ and $q_j$ for some event $E$, then $q_i = q_j$.

*Proof.* Given event $E$, observe that $P(E|h_i(\omega))$ is an r.v., and by the preceding definition, $q_i = P(E|h_i(\omega))$ is common knowledge of the two players in state $\omega$ only if for all $\omega', \omega'' \in M(\omega)$,

$$q_i = P(E|h_i(\omega')) = P(E|h_i(\omega'')).$$

Recall that $\mathcal{M}$ is the finest common coarsening of $H_i$ and $H_j$, meaning that in state $\omega$, we can write

$$M(\omega) = \bigcup_{\omega' \in M(\omega)} h_i(\omega').$$

Thus we have, for each $\omega' \in M(\omega)$,

$$q_i = \frac{P(E \bigcap h_i(\omega'))}{P(h_i(\omega'))} \Rightarrow P(E \bigcap h_i(\omega')) = q_i P(h_i(\omega')),$$

so that by summing over the distinct $h_i(\omega')$'s with $\omega' \in M(\omega)$, we have

$$P(E \bigcap M(\omega)) = q_i P(M(\omega)).$$

Applying the same arugment to player $j$, we have

$$P(E \bigcap M(\omega)) = q_j P(M(\omega)),$$

and hence we conclude that $q_i = q_j$ (since $P(\omega) > 0$ for all $\omega \in \Omega$). $\|$

13. We are ready to present some examples. The following example is due to Aumann (1976). Suppose that players A and B agree ex-ante that there are four equally likely states, $\{a, b, c, d\}$. Players A and B then receive their own private information to form posteriors about an event $\{a, d\}$. Player A's information allows him to distinguish $\{a, b\}$ from $\{c, d\}$ and player B's information allows him to distinguish $\{d\}$ from $\{a, b, c\}$. These information structures are part of the two players' common knowledge. Now, suppose that the true state is $a$. Compute their posteriors for the event $\{a, d\}$. Are these beliefs part of the two

players' common knowledge in state $a$?
*Solution.* We proceed to solve this problem in three steps. First, we derive the two players' posterior beliefs for event $E$. Second, we show that this is the two players' mutual knowledge. Finally, we show that it is not their common knowledge. Now, the first step. Consider player A. Since the true state is $a$, player A knows that the event $\{a, b\}$ has occurred, but he cannot tell if the true state is $a$ or $b$. He knows that state $d$ is not possible, and hence event $E$ is equivalent to the singleton event $\{a\}$. Obviously, player A's posterior for $E$ is then

$$\frac{\text{prob.}(a)}{\text{prob.}(a) + \text{prob.}(b)} = \frac{1}{2}.$$

On the other hand, player B knows that the event $\{a, b, c\}$ has occurred. Again, state $d$ is not possible to player B. Player B's posterior for event $E$ is hence

$$\frac{\text{prob.}(a)}{\text{prob.}(a) + \text{prob.}(b) + \text{prob.}(c)} = \frac{1}{3}.$$

This concludes the first step. Next, are these posteriors the players' knowledge? We must check if player A knows player B's posterior, and if player B knows player A's posterior. Consider player A. He knows that the true state is either $a$ or $b$. But, in either case, player B should have observed the event $\{a, b, c\}$. Hence, player A knows that player B's posterior is $\frac{1}{3}$. Now consider player B. She knows that the true state is either $a$ or $b$ or $c$. Let us follow her thoughts:

> If the true state is $a$ or $b$, player A should have observed the event $\{a, b\}$ and considered event $E$ equivalent to the event $\{a\}$, and in that case player A's posterior for $E$ would be $\frac{1}{2}$. What if the true state is $c$? In this case, player A should have observed $\{c, d\}$ and considered event $E$ equivalent to $\{d\}$. Once again, player A should have posterior $\frac{1}{2}(= \frac{\text{prob.}(d)}{\text{prob.}(c)+\text{prob.}(d)})$ for event $E$.

We conclude that each player, based on his or her own information, is able to deduce the other person's posterior for $E$, and hence $E$ is their mutual knowledge.

Finally, we show that these beliefs are not common knowledge. Let us start with player A's reasoning about how player B thinks about player A's posterior for $E$. Since player A is sure that the true state is either $a$ or $b$, he knows that player B should conclude that player A's posterior for $E$ is $\frac{1}{2}$ (as in step two). Thus, player A knows that player B knows that player A's posterior for $E$ is $\frac{1}{2}$. (Note that player A certainly knows that B knows B's posterior for $E$.)

Now we consider player B's reasoning about how player A thinks about player B's posterior beliefs. According to her information, player B knows that player A could have seen $\{a, b\}$ or $\{c, d\}$. In the first case, player A would know that player B has seen $\{a, b, c\}$, and hence could infer both players' posterior beliefs for $E$. In the second case, however, player A could have mistakenly considered $d$ likely, and inferred that player B could have seen $\{d\}$ and attached prob. one to $E$. This means that player B thinks that, if the true state is $c$, player A will be unable to determine whether player B's posterior for $E$ is $\frac{1}{3}$ or 1!

14. Two distinct digits $x, y$ were selected from the set $\{2, \cdots, 9\}$. Mr. A learns their product $xy$ privately, and Ms. B learns their sum $x + y$ privately. We shall let these two people take turn to report if they know what the two digits are, starting from Mr. A. However, they are only allowed to report if they know the two digits; they are *not* allowed to report what the two digits are.

Suppose that Mr. A learns that $xy = 18$. Determine the sequence of reports from the two people.
*Solution.* There are two possibilities: either the two digits are (2,9) or (3,6).
(**Case 1**) If the two digits are (3,6), then Mr. A first reports that he does not know, followed by Ms. B's report that she knows, and then by Mr. A's report that he knows also.
(**Case 2**) If the two digits are (2,9), then Mr. A first reports that he does not know, followed by Ms. B's report that she does not know either, which is followed by Mr. A's report that he knows, and then by Ms. B's report that she does not know. Ms. B will then remain puzzled forever.

To see that these will happen as suggested, suppose first that the two digits are (3,6). A thinks that (2,9) is also possible, and so he must first report that he does not know the two digits. However, B knows that the sum of the two digits is 9 in this case, and she thinks that the two digits are either (2,7), or (3,6), or (4,5). If the two digits are not (3,6), A would have known the two digits and could not have reported that he did not know! Thus upon seeing that A admits that he does not know, B learns that the two digits are (3,6). Thus B reports that she knows. On the other hand, A thinks that if the two digits are (2,9), then B would have observed $x+y = 11$, and B would think that both (2,9) and (3,8) are possible outcomes, and that under (2,9) A would have seen $xy = 18$ and under (3,8) A would have seen $xy = 24$, so that A could not figure out the two digits in either case. Thus observing A's report that A does not know really cannot help B to know the two digits, if the two digits are actually (2,9). Thus seeing that B reports that she knows, A learns immediately that the two digits are (3,6) rather (2,9)! Thus A also reports that he knows now.

What if the true digits are (2,9)? Again, A must report that he does not know to start with, for (3,6) is also possible from A's point of view. On the other hand, B must have seen $x + y = 11$, and B thinks that the two digits can be either (2,9), or (3,8), or (4,7), or (5,6). Seeing A's report that A does not know allows B to rule out (4,7) and (5,6). But since both (2,9) and (3,8) are consistent with A's report that A does not know to start with, B must report that she does not know at this stage. However, recall that A knows that the two digits are either (3,6) or (2,9) at the beginning, and A can infer that B would have reported that she knows if the two digits are (3,6). Thus A knows that the two digits are (2,9).

Now, does A's report that A now knows what the two digits are allow B to finally know that the two digits are (2,9)? No. Let us show that (3,8) are also consistent with A's first reporting that he does not know and then reporting that he knows. From B's point of view, (3,8) are possible, and if the two digits are really (3,8), then A would have seen $xy = 24$, and A would think that the two digits are either (3,8) or (4,6), and A would expect B to report that she knows if the two digits were (4,6) (A would think that B would see three possibilities: (2,8), (3,7) and (4,6) and would infer that A would not say that he does not know if the two digits are either (2,8) or (3,7)), and thus failing to see such a report makes A realize that the two digits would be (3,8), and that is why A reports that he knows upon seeing B's report that she does not know. The bottom line is that, upon seeing A's report that A now knows the two digits, B still cannot tell whether the two digits are (2,9) or (3,8). Since A will not contribute any more useful information from this point on, the two digits remain unknown to B forever.

15. **The Milgrom-Stokey (1982) No-trade Theorem.** Suppose that all traders are either risk neutral or risk-averse and they interpret new information in the same way. If the pre-trade allocation is Pareto efficient, and trade is voluntary, then when a trading opportunity appears after traders receive private information, every trader would feel indifferent between accepting or rejecting the trade. If traders are strictly risk-averse then no trade can take place.

16. The following example is taken from Milgrom and Stokey (1982). There are 10 probable states: $\Omega = \{t_1, t_2\} \times \{x_1, x_2, x_3, x_4, x_5\}$. Two risk neutral players are considering taking the following bet: If $t = t_1$ then player 2 pays player 1 one dollar; if $t = t_2$, then player 1 pays player 2 one dollar. Assume that the players have common priors as follows.

| x/t | $t_1$ | $t_2$ |
|---|---|---|
| $x_1$ | 0.2 | 0.05 |
| $x_2$ | 0.05 | 0.15 |
| $x_3$ | 0.05 | 0.05 |
| $x_4$ | 0.15 | 0.05 |
| $x_5$ | 0.05 | 0.2 |

At the time the players make their decisions concerning whether to take the bet, their information partitions are respectively

$$H_1 = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}\},\ H_2 = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5\}\}.$$

Suppose that a player takes the bet only if he senses a strictly positive expected profit. Show that there exists an equilibrium where both players refuse to take the bet.

*Solution.* In state $x_1$,[3] player 2 will reject the bet, and recognizing this, player 1 thinks that if he takes the bet and the bet is eventually executed, then the true state must be $x_2$, and since taking the bet in state $x_2$ is a bad idea for player 1, player 1 should reject the bet in the first place. Thus trade cannot occur in state $x_1$. A similar reasoning applies to state $x_5$ (with the roles of the two players reversed). This implies that trade cannot occur in state $x_5$.

Observe that, since player 1 cannot distinguish $x_1$ from $x_2$, and since player 2 cannot distinguish $x_4$ from $x_5$, player 1 will reject the bet in state $x_2$ and player 2 will reject the bet in state $x_4$.

Now if the true state is $x_2$, then since player 2 knows that player 1 will reject the bet in state $x_2$, player 2 can infer that if he takes the bet, and the bet is eventually executed, then the true state has to be $x_3$, and since betting in state $x_3$ implies zero expected profit, player 2 should simply reject the bet. This implies that trade cannot occur in state $x_2$. A similar reasoning applies to state $x_4$ (with the roles of the two players reversed). It follows that trade cannot occur in state $x_4$ either.

What if the true state is $x_3$? Since player 1 rejects the bet in state $x_4$, he has to reject the bet in state $x_3$. Similarly, since player 2 rejects the bet in state $x_2$, he has to reject the bet in state $x_3$. Thus trade cannot occur in state $x_3$ either.

17. A, B, C, and D are sitting on the stairway between level one and level two, facing level one and wearing the hats E delivered to them. It is common knowledge of A, B, C,and D that E owns four blue hats and three red hats, but none of them saw the hats when E put them on their heads. A is sitting behind B, and B behind C, and C behind D. Thus A is sitting closest to level two, and is able to see the hats worn by B, C, and D; B is able to see the hats worn by C and D; and C is able to see D's hat. In turn, A, B, C, and D are asked to say whether they know the colors of their hats. A said that he did not know, and then B and C both said they did not know. Finally, D said that he knew! Determine the color of D' hat.

[3] I have abused the terminology a little bit here; it is more precise to say that in event $\{(t_1, x_1), (t_2, x_1)\}$. I shall continue adopting this abused terminology for simplicity.

18. Two distinct numbers $x, y$ were selected from the set $\{2, 3, \cdots, 99\}$. Mr. A was told what $xy$ was, and Ms. B was told what $x + y$ was. That A and B would know $xy$ and $x + y$ respectively was A and B's common knowledge. B, based on the $x + y$ she was told, said that she knew for sure that A, based on $xy$, could not deduce what $x$ and $y$ were. A said that B was right, but upon listening to this remark from B, he now knew what $x$ and $y$ were. B, upon hearing A's reply, said that she knew what $x$ and $y$ were too! Find $x$ and $y$. Are they unique?
*Solution.* To demonstrate the idea, we show that the two numbers cannot be $(2, 26)$ or $(3, 57)$ or $(9, 19)$. If the two numbers were $(2, 26)$ or $(9, 19)$, then B must have seen a sum of 28, and B could not have ruled out the possibility that the two numbers were $(23, 5)$, and in the latter case A would have known exactly what the two numbers were. Similarly, if the two numbers were $(3, 57)$, then B would have seen a sum of 60, and B could not have ruled out the possibility that the two numbers were $(53, 7)$, and so B could not have said that A could not possibly knew what the two numbers were.

    So, what are the two numbers? The following useful observations are made by Eric Wu. At first, $x + y \leq 54$. This is true for if $x + y \geq 55$, then B could not rule out the possibility that either $x$ or $y$ was a prime number exceeding 50 (starting from 53), but in the latter case A would have been able to figure out what $x$ and $y$ were. Second, $x$ and $y$ cannot both be prime numbers, for otherwise A would have been able to figure out what $x$ and $y$ were. It follows that

$$x + y \in S \equiv \{11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53\}.$$

    It can be verified that $x + y = 17$ stands as the sole possibility. From here, the two numbers $x$ and $y$ can be solved. The solution is $(x, y) = (4, 13)$.

19. **Part II.**

20. We shall define static and dynamic games with incomplete information and the associated equilibrium concepts. Then we shall solve a series of static games with incomplete information.

21. **Definition 3.** A game where no players have private information (the entire normal form game is player's common knowledge in each and

every possible state) is *a game with complete information*. Otherwise, the game is one with *incomplete information*, or one with *differential information*. A game with incomplete information is depicted as a game with imperfect information where some players when making their moves do not get to observe *nature*'s earlier decisions about players' types.

22. **Definition 4.** A game with incomplete information is one with information asymmetry, if there are only two classes of players in the game, those who have (the same) private information and those who do not. The former will then be referred to as the *informed* players, and the latter the *uninformed* players. In an agency problem, in particular, the informed and uninformed players are respectively referred to as the *agents* and the *principals*.

23. **Definition 5.** An incomplete-information game where the uninformed players can move after observing the informed players' moves is called a *dynamic game*, and otherwise a *static game*.

24. **Classification of games.** So far, we have been able to classify games into 4 groups according to whether they are one-shot (static) or dynamic games, and whether there are privately informed players in the games. The following table summarizes the appropriate equilibrium concepts:

| information/time horizon | static | dynamic |
|---|---|---|
| complete | NE | SPNE |
| incomplete | BE | PBE |

25. **Definition 6.** The equilibrium concept for static games with incomplete information is *Bayesian equilibrium*, which, just like NE, is a set of strategies, one for *each type of each player*, such that if all types of all players play their specified strategies, no one wants to deviate unilaterally. Essentially, we have extended NE to static incomplete-information games, where a player possessing different private information is treated as different players.

26. We can now look at examples. The following example is a variant of Example 1 in Lecture 1, Part 1.

In a Cournot duopoly, firms face the inverse demand $P(q_1 + q_2) = a - q_1 - q_2$. But, only firm 1 knows what $a$ is. Firm 2 only knows that $a$ may be 2 with prob. $\frac{1}{3}$ or 4 with prob. $\frac{2}{3}$. Find a BE for this game. Assume $F = c = 0$ for simplicity.
First observe that firm 1 has two possible types. Firm 2 has only one type (no private information). So, a BE is defined by three strategies $q_1^*(2)$, $q_1^*(4)$, and $q_2^*$. These must form an equilibrium: given others' strategies, mine is optimal for myself. Thus, for firm 2,

$$q_2^* = arg \max_{q_2} \frac{1}{3}[q_2(2 - q_1^*(2) - q_2)] + \frac{2}{3}[q_2(4 - q_1^*(4) - q_2)].$$

Similarly, for firm 1 when $a = 2$,

$$q_1^*(2) = arg \max_{q_1} q_1(2 - q_1 - q_2^*);$$

and for firm 1 when $a = 4$,

$$q_1^*(4) = arg \max_{q_1} q_1(4 - q_1 - q_2^*).$$

Each of the above three maximization problems is concave, and so the FOCs are necessary and sufficient. We have

$$(q_1^*(2), q_1^*(4), q_2^*) = (\frac{4}{9}, \frac{13}{9}, \frac{10}{9}).$$

27. Consider two firms engaged in Cournot competition. The inverse demand is

$$p = 3 - q_1 - q_2.$$

Firm 2 has marginal cost $c_2 = 1$ and firm 1's marginal cost $c_1$ is either 1 or 0, with prob. $\pi$ and $1 - \pi$ respectively. Find the pure strategy BE for this incomplete information game.

28. Consider a more complicated version of the preceding problem. Suppose that firm W and firm L are engaged in a Cournot competition with their unit costs being their private information. For simplicity,

the firms have no fixed costs, The inverse demand is, over the relevant range,

$$p = a - q_W - q_L.$$

Firm L's unit cost can be either $C$ or $c$. Firm W's unit cost can be either $\delta C$ or $\delta c$, where $\delta \in (0,1]$. Suppose that firm W believes that firm L's unit cost is $C$ with probability $h$ and is considered by firm L to have unit cost $\delta C$ with probability $g$. Let $Q_w, q_w$ be the Bayesian equilibrium output levels chosen by respectively the type-$C$ and the type-$c$ firm W. Let $Q_l, q_l$ be the equilibrium output levels chosen by respectively the type-$C$ and the type-$c$ firm L. Verify that in the BE,

$$Q_w = \frac{a + hC + (1-h)c - \frac{\delta(gC+(1-g)c)}{2} - \frac{3\delta C}{2}}{3},$$

$$q_w = \frac{a + hC + (1-h)c - \frac{\delta(gC+(1-g)c)}{2} - \frac{3\delta c}{2}}{3},$$

$$Q_l = \frac{a + \delta[gC + (1-g)c] - \frac{hC+(1-h)c}{2} - \frac{3C}{2}}{3},$$

$$q_l = \frac{a + \delta[gC + (1-g)c] - \frac{hC+(1-h)c}{2} - \frac{3c}{2}}{3},$$

and the corresponding expected profits are, for firm W of type $\delta C$,

$$\Gamma_w = hQ_w[a - Q_w - Q_l - \delta C] + (1-h)Q_w[a - Q_w - q_l - \delta C];$$

for firm W of type $\delta c$,

$$\gamma_w = hq_w[a - q_w - Q_l - \delta c] + (1-h)q_w[a - q_w - q_l - \delta c];$$

for firm L of type $C$,

$$\Gamma_l = gQ_l[a - Q_l - Q_w - C] + (1-g)Q_l[a - Q_l - q_w - C];$$

and for firm L of type $c$,

$$\gamma_l = gq_l[a - q_l - Q_w - c] + (1-g)q_l[a - q_l - q_w - c].$$

29. Consider the first-price sealed-bid auction game as follows. Assume that there are $n$ bidders instead of 2. We shall show that as $n$ tends to infinity, bidders' consumer surplus tends to zero; that is, competition does drive away bidders' profits. Assume that there exists a symmetric Bayesian equilibrium where all bidders use the pure strategy $B(v)$, where we recall that the valuation $v_i$ of bidder $i$ for the indivisible good is his private information, which the auctioneer and the other bidders believe is drawn from the uniform distribution on the interval $[0, 1]$. Being private values, these $v_i$'s are independent. To begin, let us assume that

(i) $B(v)$ is continuously differentiable and strictly increasing, so that its inverse function $v = B^{-1}(b)$ exists and that the probability that $B(v_i) = B(v_j)$ (i.e. the event that two bidders' bids equal in equilibrium) is zero; and

(ii) let $b^*(v) = \text{argmax}_{b\in[0,1]} f(b) = (v - b)[B^{-1}(b)]^{n-1}$, then

$$f'(b^*(v)) = 0;$$

that is, if given his valuation for the good being $v$ and his beliefs that all his rivals will use the strategy $B(\cdot)$, a bidder's optimal bid is $b^*(v)$, then $b^*(v)$ satisfies the so-called first-order condition.

(1) Write down explicitly the equation $f^{-1}(b^*(v)) = 0$. Then, impose the symmetric equilibrium assumption (that is, $b^*(v) = B(v)$) to obtain an ordinary differential equation (ODE).

(2) Guess that the ODE has a solution taking the form of $B(v) = kv$ for some constant $k$. Find $k$. Show that $k$ tends to 1 as $n$ tends to $\infty$, thereby concluding that competition leads to zero profits for the bidders.

(3) Show that assumption (i) is verified by $B(v)$. Show that under $B(v)$, the maximum of $f(b)$ does take place at some $b^*$ which satisfies the first-order condition so that assumption (ii) is also verified.

*Solution.* In part (1), we have[4]

$$f'(b^*(v))$$

[4] Recall that if $f, g : \Re \to \Re$ are both continuously differentiable, and $x = g(y)$ is the inverse function for $y = f(x)$, then at the point $(x, y) = (x, f(x))$, we have $g'(y) = g'(f(x)) = \frac{1}{f'(x)}$.

$$= -[B^{-1}(b^*(v))]^{n-1} + (v - b^*(v))(n-1)[B^{-1}(b^*(v))]^{n-2} \cdot \frac{1}{B'(B^{-1}(b^*(v)))} = 0.$$

Now we impose the symmetry condition: $b^*(v) = B(v)$. Replacing $b^*(v)$ by $B(v)$ in the above first-order condition, we have

$$-v^{n-1} + \frac{(n-1)(v - B(v))v^{n-2}}{B'(v)} = 0,$$

or after re-arranging,

$$B'(v) + B(v)\frac{n-1}{v} = (n-1) \Rightarrow B'(v)v^{n-1} + B(v)(n-1)v^{n-2} = (n-1)v^{n-1}$$

$$\Rightarrow \frac{d}{dv}[B(v)v^{n-1}] = (n-1)v^{n-1} \Rightarrow B(v)v^{n-1} = \frac{n-1}{n}v^n + C$$

$$\Rightarrow B(v) = \frac{(n-1)v}{n} + Cv^{1-n},$$

where $C$ is some constant. Now if $C > 0$, then as $v \downarrow 0$, $B(v) \uparrow \infty$, which cannot be compatible with bidders' IR conditions (bidders must make non-negative profits because participation in the auction is voluntary). What if $C < 0$? Then for $v > 0$ sufficiently small, bidders submit negative bids! We shall assume that the auctioneer accepts only positive bids (which makes sense because it is common knowledge that all bidders have positive valuations for the object). It follows that $C = 0$ and $B(v) = kv$ with $k = \frac{n-1}{n}$, which is part (ii). Apparently, as $n \uparrow \infty$, $k \uparrow 1$, indicating that competition drives away bidders' profits. Finally, in part (iii), we recognize that $B(v)$ is a strictly increasing linear function $v$, and hence is continuous. Its inverse function does exist, and the probability that $B(v_i) = B(v_j)$ is indeed zero because $B(\cdot)$ is continuous and both $v_i$ and $v_j$ are continuous random variables. It remains to check that the first order approach is valid. Note that

$$f(b) = (v - b)[B^{-1}(b)]^{n-1} = (v - b)[\frac{nb}{n-1}]^{n-1}$$

so that

$$f'(b) = -[\frac{nb}{n-1}]^{n-1} + (v - b)[\frac{n}{n-1}]^{n-1}(n-1)b^{n-2}$$

$$= [\frac{n}{n-1}]^{n-1} b^{n-2} \{-b + (n-1)(v-b)\}.$$

We conclude that

$$f'(b) \geq 0 \Leftrightarrow b \leq \frac{(n-1)v}{n},$$

and hence $f(b)$ has a unique peak at $b = \frac{(n-1)v}{n}$.[5] Since $f(b)$ has a uniqe interior maximum at $b = \frac{(n-1)v}{n}$, $f'(b) = 0$ if and only if $b = \frac{(n-1)v}{n}$. Thus the optimal bid does satisfy the bidder's first-order condition, as we have conjectured, although $f(\cdot)$ is not a concave function of $b$![6]

[5]Note that $f$ is quasi-concave, although $f$ may not be concave: for all $r \in \Re$, the upper contour set of $f$, $\{b : f(b) \geq r\}$ is a convex set.

[6]The analysis can be carried over to the case where $v_i$ are i.i.d. with common continuous density function $h$ and distribution function $H$ on some closed interval $[0, w]$, where $h(v) > 0$ for all $v \in [0, w]$. In this case, define $G$ and $g$ as respectively the distribution function and the density function of $Y_1 \equiv \max(v_2, v_3, \cdots, v_n)$. Note that

$$\{Y_1 \leq y\} \Leftrightarrow \{v_2 \leq y\} \bigcap \{v_3 \leq y\} \bigcap \cdots \bigcap \{v_n \leq y\},$$

so that

$$G(y) = \text{prob.}(\{Y_1 \leq y\}) = \text{prob.}(\{v_2 \leq y\}) \times \text{prob.}(\{v_3 \leq y\}) \times \cdots \times \text{prob.}(\{v_n \leq y\}) = [H(y)]^{n-1}.$$

From here, we have

$$g(y) = (n-1)h(y)[H(y)]^{n-2}.$$

Now, bidder 1's payoff from bidding $b$ given his valuation is $v_1 = v$ is

$$f(b) = (v-b)G(B^{-1}(b)),$$

so that

$$f'(b) = -G(B^{-1}(b)) + \frac{(v-b)g(B^{-1}(b))}{B'(B^{-1}(b))}.$$

Assume that the optimal solution here is exactly $B(v)$, and assume that this optimal solution satisfies the first-order condition

$$f'(B(v)) = 0,$$

which implies that

$$G(v)B'(v) + g(v)B(v) = vg(v),$$

or equivalently,

$$\frac{d}{dv}[G(v)B(v)] = vg(v),$$

30. In a private-value auction an indivisible good will be sold to one of the two bidders whose valuations for the good are their private information and are drawn independently from the uniform distribution over the unit interval.
(i) Suppose that the seller has zero valuation for the good, and that a first-price sealed-bid auction is adopted. Find a symmetric Bayesian equilibrium for the bidding game.
(ii) Compare the seller's expected revenue under the first-price sealed-bid auction to her expected revenue under the second-price sealed-bid auction.
*Solution.* From the preceding analysis of the first-price sealed-bid auction, we know that, for part (i), $b_i = \frac{v_i}{2}$, if $v_i$'s are independently and identically distributed (i.i.d.) over the unit interval, they have a uniform distribution. For part (ii), we claim that the seller's expected revenue is the same under the two different auction rules. Under the first-price sealed bid auction, the seller's expected revenue is

$$E[\frac{v_1}{2}|v_1 \geq v_2]pro.(v_1 \geq v_2) + E[\frac{v_2}{2}|v_2 \geq v_1]pro.(v_2 \geq v_1)$$

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which, by the fact that $B(0) = 0$, implies that

$$B(v) = \frac{1}{G(v)} \int_0^v tg(t)dt = E[Y_1|Y_1 < v].$$

It is easy to check that (i) $B(\cdot)$ is strictly increasing; i.e.,

$$w > v_2 > v_1 > 0 \Rightarrow E[Y_1|Y_1 < v_2] > E[Y_1|Y_1 < v_1];$$

(ii) $B(\cdot)$ is continuous; and (iii) given $B(\cdot)$,

$$f'(b) = \frac{g(B^{-1}(b))[v - B^{-1}(b)]}{B'(B^{-1}(b))},$$

so that

$$f'(b) > 0 \Leftrightarrow B(v) > b,$$
$$f'(b) < 0 \Leftrightarrow B(v) < b,$$
$$f'(b) = 0 \Leftrightarrow B(v) = b,$$

verifying that the first-order condition indeed yields the best response for bidder 1 given that the other bidders adopt the bidding strategy $B(\cdot)$.

$$= 2\int_{y=0}^{y=1}\int_{x=y}^{x=1}\frac{x}{2}\cdot 1dxdy = \frac{1}{3}.$$

Under the second-price sealed bid auction, the seller's expected revenue is

$$2\int_{y=0}^{y=1}\int_{x=y}^{x=1} y\cdot 1dxdy = \frac{1}{3}.$$

In fact, this is a result called the *revenue equivalence theorem*, which says that the above two auctions both maximize the seller's expected revenue (and hence optimal for the risk neutral seller) given that bidders are also risk neutral and their valuations are independent drawings from a common absolutely continuous distribution.[7]

31. Let us extend the analysis in the preceding section by allowing non-uniform distributions. Assume that $v_i$'s are i.i.d. with a continuous distribution function $F(\cdot)$ and a strictly positive density function $f(\cdot)$ on the support $[0,1]$.[8] Note that if $f \equiv 1$ then this problem is identical to the one treated in the preceding section.
(i) Consider two realizations $v_i$ and $v_i'$ of bidder $i$'s valuation, with $0 \leq v_i' < v_i \leq 1$. Suppose that bidder $j$ adopts a bidding strategy $B_j(v_j)$ in equilibrium. Show that if $b_i$ and $b_i'$ are bidder $i$'s best response when his valuation is respectively $v_i$ and $v_i'$, then $b_i \geq b_i'$.[9]
(ii) Show that with the current distributional assumption, there is a

[7]The revenue equivalence result for the 4 standard auctions (i.e., English, Dutch, the first-price sealed-bid, and the second-price sealed-bid auctions) holds as long as the seller and the bidders are risk neutral and the latter have i.i.d. valuations. In fact, this remains true even if the latter have correlated valuations that are identically distributed. However, in these latter cases, these equivalent standard auctions may fail to be optimal. Intuitively, when the bidders have i.i.d. but *discrete* valuations, under a standard auction a bidder's IC constraint that prevents him from lying and claiming to have a lower valuation may be strictly satisfied, implying that the seller concedes too much information rent to a high-valuation bidder. When the bidders have correlated valuations, on the other hand, under the optimal auction the seller can essentially extract all the surplus from the bidders, which the seller cannot achieve by adopting one of the standard auctions.

[8]The support $S$ of a real-valued random variable $\tilde{x}$ is the smallest closed subset of $\Re$ such that the event that the realization of $\tilde{x}$ lies in $S$ occurs with probability one.

[9]**Hint**: By bidding $b_i$ rather than $b_i'$, type $v_i$ must yield a higher expected payoff; and by bidding $b_i'$ rather than $b_i$, type $v_i'$ must yield a higher expected payoff. Write down these two IC conditions, with $B_j(v_j)$ being treated as a fixed random variable, and then add up the two IC conditions.

symmetric equilibrium where for $i = 1, 2$, bidder $i$'s bidding strategy is, given any $v_i \in [0, 1]$, to bid

$$B(v_i) \equiv E[v_j | v_j < v_i],$$

where the expectation on the right side uses the distribution function $F(\cdot)$. Note that this solution implies as a special case the solution that we obtain under the assumption of uniform distribution.[10]

[10]**Hint**: Two ways to prove this result. First, start with the conjecture that $B(\cdot)$ is the symmetric equilibrium strategy which is strictly increasing and continuous so that $B'(\cdot) > 0$ exists on $(0, 1)$. Given that bidder $i$ believes that bidder $j$ will use the strategy $B(\cdot)$, write down the first-order derivative $H'(b_i)$ of bidder $i$'s objective function $H(b_i)$, where note that

$$H(b_i) = (v_i - b_i)\text{prob.}(b_i \geq B(v_j)) = (v_i - b_i)\text{prob.}(B^{-1}(b_i) \geq v_j)$$
$$= (v_i - b_i)F(B^{-1}(b_i)).$$

Then, set the derivative equal to zero at the point $b_i = B(v_i)$. This will give rise to the following ordinary differential equation for $B(\cdot)$:

$$F(v_i) = [v_i - B(v_i)]\frac{f(v_i)}{B'(v_i)} \Leftrightarrow \frac{d[F(v_i)B(v_i)]}{dv_i} = v_i f(v_i),$$

showing that

$$F(v_i)B(v_i) = C + \int_0^{v_i} t dF(t),$$

and since it must be that $B(0) = 0$, $C = 0$. Second, you can start with the assumption that bidder $j$ uses the strategy

$$B_j(v_j) \equiv E[z|z < v_j],$$

where $z, v_i, v_j$ are i.i.d. with density function $f(\cdot)$, and then show that bidder $i$'s best response is

$$B_i(v_i) \equiv E[z|z < v_i].$$

To this end, verify that for bidder $j$,

$$B_j'(v_j) = \frac{f(v_i)}{F(v_j)}[v_j - B_j(v_j)] > 0,$$

except at $v_j = 0$. Thus $B_j^{-1}(\cdot) : B_j([0, 1]) \to [0, 1]$ is well-defined, where recall that $B_j([0, 1])$ is the image set of the function $B_j(\cdot)$. Now for bidder $i$'s problem of finding the best response, the first-order derivative of bidder $i$'s objective function becomes

$$H'(b_i) = -F(B_j^{-1}(b_i)) + (v_i - b_i)\frac{f(B_j^{-1}(b_i))}{B_j'(B_j^{-1}(b_i))}.$$

This shows that $H'(\cdot)$ need not be a decreasing function. However, use the above expression $B_j'(v_j) = \frac{f(v_i)}{F(v_j)}[v_j - B_j(v_j)]$ to show that

$$H'(b_i) = F(B_j^{-1}(b_i))[\frac{v_i - b_i}{B_j^{-1}(b_i) - b_i} - 1],$$

so that $H'(b_i) > 0$ (respectively, $H'(b_i) < 0$) if and only if $B_j^{-1}(b_i) < v_i$ (respectively, $B_j^{-1}(b_i) > v_i$).

32. Let us extend the preceding analysis one step further. Now we shall consider valuations which are discrete random variables. More precisely, assume that $v_1$ and $v_2$ are i.i.d. with

$$F(v_i) = \begin{cases} 0, & \text{if } v_i < \underline{v}; \\ \underline{p}, & \text{if } v_i \in [\underline{v}, \overline{v}); \\ 1 = \underline{p} + \overline{p}, & \text{if } v_i \geq \overline{v}. \end{cases}$$

In the above, the constants $\underline{v}, \overline{v}, \underline{p}, \overline{p}$ are such that

$$0 < \underline{v} < \overline{v} < 1, \ 0 < \underline{p} < 1.$$

(i) Show that there exists a symmetric equilibrium for this auction game,[11] where a type-$\underline{v}$ bidder bids $\underline{v}$ with probability one, whereas a type-$\overline{v}$ bidder submits a random bid $\tilde{b}$, where the distribution function of $\tilde{b}$ is

$$G(b) = \begin{cases} 0, & b < \underline{v}; \\ \frac{1}{\overline{p}}[\frac{(\overline{v}-\underline{v})\underline{p}}{\overline{v}-b} - \underline{p}], & b \in [\underline{v}, \overline{p}\overline{v} + \underline{v}\underline{p}); \\ 1, & b \geq \overline{p}\overline{v} + \underline{v}\underline{p}. \end{cases}$$

(ii) Show that in equilibrium, a type-$\underline{v}$ bidder's expected payoff is 0, and a type-$\overline{v}$ bidder's expected payoff is $\underline{p}(\overline{v} - \underline{v})$.

[11] This equilibrium is unique! You are not required to show the uniqueness of equilibrium, but it should be easy for you to see that there can be no pure-strategy Bayesian equilibrium of this game. If there were one such equilibrium, then the analysis in the preceding section shows that the high-valuation bidder $i$ should bid higher than the low-valuation bidder $i$ in equilibrium. Note that this implies that a pure-strategy BE, if it exists, cannot be symmetric: otherwise let $\overline{b} > \underline{b}$ be the equilibrium bids for respectively a type-$\overline{v}$ and a type-$\underline{v}$ bidder, and observe that necessarily $\overline{v} > \overline{b}$, which however implies that each bidder has an incentive to deviate by bidding slightly higher when his valuation is $\overline{v}$. However, an asymmetric pure-strategy BE cannot exist either, for if $\overline{b}_j > \overline{b}_i \geq \underline{b}_i$, then bidder $j$ has an incentive to deviate by bidding slightly lower. Finally, the derivation of the unique mixed strategy equilibrium is rather similar to that of the mixed strategy equilibria for the game in Example 6 of Lecture 1, Part II. Here, you are not asked to derive the mixed strategy BE; the equilibrium is already spelt out for you, and all you need to do is to verify that no bidders of any type can benefit from unilateral deviation from his equilibrium bidding strategy.

(iii) Prove or disprove that in equilibrium, the seller's payoff is

$$\underline{p}^2\underline{v} + (1-\underline{p}^2)\overline{v} - 2\overline{p}\underline{p}(\overline{v}-\underline{v}).$$

33. The preceding sections have considered single auctions. Now we consider a double auction. Consider a seller producing a good and selling it to a buyer. The seller's production cost $\tilde{c}$ is her private information. The buyer's valuation $\tilde{v}$ for the good is his private information. Assume that $\tilde{v}$ and $\tilde{c}$ are identically, independently, and uniformly distributed on $[0, 1]$. The trading mechanism is as follows. The seller and the buyer must simultaneously submit bids $p_s$ and $p_b$. If $p_s > p_b$, then there will be no trade, and each party gets 0 payoff; but if $p_s \leq p_b$, then trade occurs at the transaction price $\frac{p_s+p_b}{2}$.
(i) First assume complete information. That is, the seller and the buyer can both see the realizations $v$ and $c$ before submitting bids. Show that if $v < c$, then there will be no trade in any Nash equilibrium; and if $v \geq c$, then for each $t \in [c, v]$, $p_s = p_b = t$ is a Nash equilibrium.
(ii) Now, return to the original game with incomplete information.

- Show that there exists a *linear* Bayesian equilibrium,[12] where for constants $a, b, \alpha, \beta$,

$$p_s(c) = a + bc, \ \forall c \in [0, 1];$$

and

$$p_b(v) = \alpha + \beta v, \ \forall v \in [0, 1].$$

Verify that $\alpha = \frac{1}{12}$, $a = \frac{1}{4}$, and

$$b = \beta = \frac{2}{3}.$$

[12] We need to show (i) that given that the seller will use the strategy $p_s(c) = a + bc$ for all $c \in [0, 1]$, where $a, b$ are constants, there exist constants $\alpha$ and $\beta$ such that for all $v \in [0, 1]$, the type-$v$ buyer's best response is to submit $\alpha + \beta v$; and (ii) that given that the buyer will use the strategy $p_b(v) = \alpha + \beta v$ for all $v \in [0, 1]$, where $\alpha, \beta$ are constants, there exist constants $a, b$ such that for all $c \in [0, 1]$, the type-$c$ seller's best response is to submit $a + bc$. Of course, this will give rise to 4 equations for the 4 unknowns $a, b, \alpha, \beta$, and you can obtain the exact numerical values of the 4 parameters by solving the 4 equations.

- Verify that with these strategies, whenever trade takes place, both parties get non-negative expected payoffs.
- Verify that incomplete information has resulted in too little trade; that is, there exist pairs $(v, c)$ with $v > c$ such that with these parameter values no trade takes place in equilibrium. Explain.

34. So far we have considerd private-value auctions. We now consider an example where bidders have an unknown common value for the auctioned object.

    In a second-price sealed-bid auction, there are two bidders with i.i.d. signals $x_1, x_2$. The object is indivisible with unknown *common value* $v$. Assume that $x_1, x_2$ and $v$ are *strictly affiliated* absolutely continuous random variables with $\Re$ being their common support, with the defining property that

    $$G(x_1, x_2) \equiv E[v|h(x_1), g(x_2)]$$

    is a strictly increasing continuous function from $\Re^2$ into $\Re$ whenever $h(\cdot)$ and $g(\cdot)$ are strictly increasing continuous function from $\Re$ into $\Re$ (see Milgrom and Weber, 1982, *Econometrica*). Show that there is a continuum of asymmetric equilibrium for this game.
    *Solution*. To solve this problem, first define

    $$H(x_1, x_2) \equiv E[v|x_1, x_2].$$

    Then $H(\cdot, \cdot)$ is a strictly increasing continuous function from $\Re^2$ into $\Re$. Let $\phi(\cdot)$ be any strictly increasing continuous function from $\Re$ into $\Re$. Let bidder $i$'s bidding strategy be $b_i(x_i)$. Then, consider the pair of bidding strategies

    $$b_1(x_1) = H(x_1, \phi(x_1)), \quad b_2(x_2) = H(\phi^{-1}(x_2), x_2).$$

    If we can show that these define a BE, then we are done, for there are a continuum of such strictly increasing functions $\phi(\cdot)$. Given $b_2(\cdot)$, bidder 1's problem is to, ignoring the probability of a tie,

    $$\max_{b_1} E[(v - b_2)1_{[b_1 > b_2]}|x_1],$$

    where the expectation is taken over $x_2$ and $v$. The objective function can be rewritten as

    $$E[E[(v - b_2(x_2))1_{[b_1 > b_2(x_2)]}|x_1, x_2]|x_1]$$

$$= E[(H(x_1, x_2) - H(\phi^{-1}(x_2), x_2))1_{[x_2 < b_2^{-1}(b_1)]}|x_1]$$

$$= \int_{-\infty}^{b_2^{-1}(b_1)} [H(x_1, x_2) - H(\phi^{-1}(x_2), x_2)] f(x_2|x_1) dx_2,$$

where the first equality follows from the law of iterated expectations and $f(\cdot|x_1)$ in the last expression is the conditional density function of $x_2$ conditioned on $x_1$. Note that following the assumption that $H_1 \geq 0$, the integrand in the above integral is positive if and only if $x_2 \leq \phi(x_1)$. Hence, the optimal $b_1^*$ should be such that $b_2^{-1}(b_1^*) = \phi(x_1)$. We have

$$b_1^* = b_2(\phi(x_1)) = H(x_1, \phi(x_1)).$$

One can check that bidder 2 does not want to deviate unilaterally either. The proof is complete. (Note that a generic bidder's optimization is in general not concave!)

35. So far we have applied Bayesian equilibrium analysis to auctions only. Now let us consider other applications, including stock trading, sale of a used car, and designing a bank loan, and so on.

    Consider the following stock trading game. A security (called the "stock") pays a one-time dividend $\tilde{v}$ at time 1, and there are three traders, T, M1, and M2, trading the stock at time 0. The following is their common knowledge:
    (i) All traders are risk neutral (seeking to maximize expected profits);
    (ii) $\tilde{v}$ is uniformly distributed over the unit interval $[0, 1]$;
    (iii) M1 and M2 are "dealers" who must simultaneously post ask and bid prices $(A_1, B_1)$ and $(A_2, B_2)$ before trader T enters the market;
    (iv) Trader T is a public investor. With probability $\pi \in [0, 1]$, T is a speculator who somehow has known the realization of $\tilde{v}$, but with probability $1 - \pi$, T has the same information as M1 and M2 do, and in this case he wants to trade because he has been hit by some liquidity shocks, and we assume that it is equally likely that this type of T may want to buy or sell 1 unit. As in the trading model of Kyle (1985), here we assume that the event that T is a liquidity trader and the event that a liquidity trader may want to sell are both independent of $\tilde{v}$.[13]

[13] Notice that the speculative type of T can choose to buy or sell 1 unit or not to trade, depending on which alternative may bring him the highest expected trading profit, but the liquidity type of T is forced to either sell 1 unit or buy 1 unit.

Since the uninformed M1 and M2 cannot move after they see T's actions, this game is a static game with incomplete information, for which the appropriate equilibrium concept if Bayesian equilibrium (BE). A symmetric BE in pure strategy for this game is

$$\{t(v), (A_1, B_1), (A_2, B_2)\},$$

such that (a) $A_1 = A_2 = A$ and $B_1 = B_2 = B$; (b) given $A$ (the ask price that T is faced with if he wants to buy 1 unit from M1 and M2) and B (the bid price that T is faced with if he wants to sell to M1 and M2), $t(v) \in \{-1, 0, 1\}$ is the trading quantity that maximizes T's expected profit conditional on him being informed and on his observation of the realization $v$ of $\tilde{v}$; (c) M1 and M2 earn zero expected profits by posting $A$ and $B$.

(1) Show that from M1 and M2's perspective, the probability that T is of the liquidity type and T wants to sell at the bid price $B$ is $(1-\pi)\cdot\frac{1}{2}$.

(2) Show that from M1 and M2's perspective, the probability that T is of the speculative type and T wants to sell at the bid price $B$ is $\pi\cdot$ Pro.$(\tilde{v} \leq B)$.

(3) Show that $E[v|\text{T is speculative and wants to sell at } B] = E[v|v \leq B]$.

(4) Show that $E[v|\text{T is not speculative and wants to sell at } B] = E[v] = \frac{1}{2}$.

(5) Show that by posting a bid price $B$, M1 and M2's posterior expected value for $\tilde{v}$ when a transaction occurs at $B$ is

$$E[v|\text{T is ``willing to'' sell at } B]$$

$$= E[v|\text{T is speculative and wants to sell at } B]$$
$$\times\frac{\text{Pro.(T is speculative and wants to sell at } B)}{\text{Pro.(T wants to sell at } B)}$$
$$+E[v|\text{T is not speculative and wants to sell at } B]$$
$$\times\frac{\text{Pro.(T is not speculative and wants to sell at } B)}{\text{Pro.(T wants to sell at } B)}$$
$$= \frac{B}{2}\cdot\frac{\pi B}{\pi B + (1-\pi)\cdot\frac{1}{2}} + \frac{1}{2}\cdot\frac{(1-\pi)\cdot\frac{1}{2}}{\pi B + (1-\pi)\cdot\frac{1}{2}}.$$

(6) Show that for M1 and M2 to earn zero expected profits by posting $B$, it must be that

$$B = E[v|\text{T is "willing to" sell at } B].$$

From here, show that

$$B = \frac{\sqrt{(1-\pi)^2 + 2\pi(1-\pi)} - (1-\pi)}{2\pi}.$$

(7) Show that $0 \leq B \leq \frac{1}{2}$ for all $\pi \in [0,1]$. Show that as $\pi$ increases $B$ decreases, and as $\pi$ tends to 1 (and 0 respectively), $B$ tends to zero (and $E[v]$ respectively).
(8) Similarly, show that

$$A = \frac{-\sqrt{(1+\pi)^2 - 2\pi(1+\pi)} + (1+\pi)}{2\pi}.$$

(9) Show that $A$ is always between $E[v]$ and 1 and as $\pi$ increases $A$ increases. Moreover, $A$ tends to 1 (and $E[v]$ respectively) as $\pi$ tends to 1 (and 0 respectively). Interpret.

36. Mr. A is trying to sell a used car to Ms. B. The car may be worth $H$ or $L$ with prob. $a$ and $1-a$, where $H > L > 0$, and how much it is really worth is Mr. A's private information. Both people seek to maximize expected profits. The game proceeds as follows. Ms. B first names a price, and Mr. A can either accept or reject it. Find a BE of this game.
*Solution.* Since Ms. B has full bargaining power, she will never choose a price higher than $H$. Of course Mr. A never accepts a price lower than $L$. Thus Ms. B can confine her attention to the prices contained in $[L, H]$. We claim that all prices strictly greater than $L$ are weakly dominated from Ms. B's perspective. To see this, note that the type-$H$ Mr. A will never accept Ms. B's offer unless the price chosen by Ms. B is $H$, but offering $H$ always generates non-positive profits for Ms. B. On the other hand, offering something less than $H$ and acceptable to Mr. A implies that the value of the car must be $L$. Thus Ms. B offers the price $L$ in any Bayesian equilibrium. The BE ( a strategy

profile, which assigns one strategy to each type of each player) is thus as follows:
(i) Ms. B offers the price $L$;
(ii) the type-$H$ Mr. A accepts Ms. B's offer if and only if the price chosen by Ms. B is greater than or equal to $H$; and
(iii) the type-$L$ Mr. A accepts Ms. B's offer if and only if the offer is greater than or equal to $L$.

37. Mr. Y, the owner of a firm, wants to borrow 1 dollar from competitive risk neutral banks in order to implement an investment project. The project may be of type G or type B, which is Mr. Y's private information. Banks think that the project is of type G with prob. $a$. A type $j$ project can generate a cash flow $X$ with prob. $\pi_j$ and zero with prob. $(1-\pi_j)$, $j \in \{G, B\}$. Assume that

$$\frac{1}{a\pi_G + (1-a)\pi_B} > X > \frac{1}{\pi_G}.$$

These inequalities say that the type G project has a positive NPV, but on average Mr. Y's project seems to have a negative NPV. The game proceeds as follows. The banks first set their interest rates for the dollar, and then the firm borrows from one of the banks that offer the most favorable terms of trade, or the firm can cease to borrow the dollar. Find a symmetric BE of this game where all banks charge the same interest rate.

*Solution.* There is a continuum of pure strategy Bayesian equilibria. Let $F_i$ equals *one plus the interest rate* quoted by bank $i$, and is referred to as the equilibrium *face value of the debt* offered by bank $i$. Let $F$ be the minimum of these $F_i$'s. Any $F_i$'s such that $F > \frac{1}{\pi_B}$ with both types of the firm not borrowing is an equilibrium, where $F$ equals *one plus the interest rate*, and is referred to as the equilibrium *face value* of the debt.

To see this, observe that in a pure strategy equilibrium, neither type of the firm should borrow. Suppose instead that both types borrow in equilibrium. Then the face value $F$ is determined by the Bertrand equilibrium of banks' competition (this will be referred to as the *zero-profit condition* for competitive investors):

$$1 = a\pi_G \min(X, F) + (1-a)\pi_B \min(X, F).$$

Either $F > X$ or $F \leq X$. Suppose that $F \leq X$. Then the last equation gives $F = \frac{1}{a\pi_G+(1-a)\pi_B} > X$, a contradiction. Thus suppose that $F > X$. But then by the above zero-profit condition, we have $\frac{1}{a\pi_G+(1-a)\pi_B} = X$, which is another contradiction.
Next, suppose that only the firm of type G borrows in equilibrium. Then, the face value $F$ of debt must solve

$$1 = \pi_G \min(X, F) \Rightarrow F = \frac{1}{\pi_G} < X.$$

But then, by borrowing, the firm of type B can get $\pi_B(X - F) > 0$, where 0 is the firm's equilibrium profit if it does not borrow. Hence, the firm of type B will deviate and also borrow, which is a contradiction. Suppose finally that only the firm of type B borrows in equilibrium, but this implies that the firm of type B can get a non-negative profit by borrowing and the lending bank can get a non-negative profit by lending, which is impossible. We thus conclude that the firm never borrows in equilibrium, regardless of the firm's type. Given this observation, the equilibrium proposed at the beginning of this solution is valid. Note that this game has no mixed strategy equilibria where the firm randomizes between borrowing and not borrowing.

38. Mr. X is in a car rental business. For simplicity, he has one car, and for him the value of the car is 1 dollar. Two types of customers may show up, the light users and the heavy users. The values of the rental to these two groups are $L$ and $H$ respectively. Renting the car to the two groups will reduce the value of the car by $l$ and $h$ respectively, from the perspective of Mr. X. Assume that

$$0 < l < L < H < h < 1, \ aL + (1-a)H < al + (1-a)H,$$

where $a$ is the subjective probability Mr. X holds regarding how likely a customer may be a light user. The game proceeds as Mr. X first posts a rent $R$, and then customers can each decide to or not to rent the car at the price $R$.
(i) Suppose that Mr. X can distinguish whether a customer is a heavy user, and he can see the customer before deciding the rent. Find an SPNE.

(ii) Suppose that whether a customer is a light user is the customer's private information. Find a BE.
*Solution.* Consider part (i). If the consumer is a heavy user, the monopolist is better off not renting (equivalently, by charging $R_H > H$). If otherwise, then the price is set at $R_L = L$. Consider part (ii). Note that no consumers will rent the car if $R \in (H, \infty)$, that only the heavy users will rent the car if $R \in (L, H]$, and that all consumers will rent the car if $R \leq L$. Because of the assumed inequalities, setting $R \in (L, H]$ will generate negative expected profits for the Mr. X. On the other hand, $R = L$ dominates all other strategies in $[0, L]$, but even $R = L$ results in an expected loss:

$$L - al - (1-a)h < aL + (1-a)H - al - (1-a)h < 0.$$

We conclude that in any SPNE, Mr. X must choose some $R > H$ to exclude both types of consumers.

This is an example of the lemons problem studied by Akerlof (1970, *QJE*). This simple example shows that adverse selection problem can be so severe that markets are no longer viable. The robustness of this example certainly depends on the set of contractible variables assumed. Here we have assumed in part (ii) that the only contractible variable is the rent $R$. Suppose instead that the ex-post car damages are also contractible (that is, $h$ and $l$ are observable and verifiable to the court of law that is responsible for enforcing the legal contracts). Then ex-ante Mr. X can make $R$ contingent on the ex-post damages: If ex-post the damage is $h$, then the rent is $R = R_H > H$; or else, the rent is $R = R_L = L$. Verify that this contract attracts only the light users in equilibrium, and hence the rental market can continue to function. Thus ex-post damages can be used to screen consumers. When ex-post damages are not contractible, there still may be other contractible variables that can be used to design a self-selection mechanism. In this case, those contractible variables are used as a *screening device*. We shall have more to say about *screening games* in a subsequent Lecture.

So far we have assumed that Mr. X cannot do anything after seeing consumers' actions; that is, the incomplete-information game is *static*. If instead that he can take actions after seeing consumers' moves, then this game becomes dynamic and the consumers can take actions to signal to Mr. X about their types. In that case, the rental market may continue to function in equilibrium, although in general signaling is costly to consumers, and hence the equilibrium may involve another inefficiency. We shall review *signaling games* in a subsequent Lecture.

39. Two players, 1 and 2, are playing the following Bayesian game. Player 1 knows which normal form game he is playing, but player 2 thinks that both normal forms are equally likely.[14] Find a BE.

| 1/2 | a | b |
|---|---|---|
| A | 0,0 | 1,2 |
| B | 2,1 | 0,0 |

| 1/2 | a | b |
|---|---|---|
| A | 0,0 | 0,0 |
| B | 2,1 | 0,0 |

*Solution.* Let us look for pure strategy BE's first. Call the player 1 who knows for sure that he is playing the first normal form game the "type-1" player 1. Similarly, the player 1 who knows that he is playing the second normal form game the "type-2" player 1.
First we ask, "Is there a BE where player 2 plays $a$ with probability one?" Suppose that such a BE exists. Then in the BE, player 2 plays $a$, and given $a$, it can be easily verified that player 1's best response is $B$ regardless of his type. On the other hand, given that both types of player 1 will play $B$, it can be easily verified that $a$ is indeed player 2's best response. Thus such a BE does exist, where both types of player 1 play $B$ and player 2 plays $a$.
Next we ask, "Is there a BE where player 2 plays $b$ with probability one?" If such a BE exists, then in equilibrium player 1 plays $A$ if he is of type 1 and he feels indifferent about $A$ and $B$ if he is of type 2. One can check that given the two types of player 1's strategies, playing $b$ is indeed a best response for player 2. Thus such a BE also exists, where

[14] Asymmetric information about a normal game can always be modelled as asymmetric information about the players' payoff functions. One can show that asymmetric information about the set of players, or about the players' strategy spaces, can always be equivalently modelled as asymmetric information about their payoff functions.

player 2 plays $b$, the type-1 player 1 plays $A$, and the type-2 randomizes in any way over $A$ and $B$.
Finally, let us determine if there are BE's where player 2 randomizes over $a$ and $b$. Suppose that player 2 plays $a$ with prob. $\pi \in (0,1)$. Then the type-2 player 1's best response is $B$ for sure, but the type-1 player 1's best response is $A$ if $\pi < \frac{1}{3}$; $B$ if $\pi > \frac{1}{3}$; and $A$ and $B$ if $\pi = \frac{1}{3}$. On the other hand, player 2 will not feel indifferent about $a$ and $b$ unless the type-1 player 1 also randomizes over $A$ and $B$. Let $\eta$ be the prob. that the type-1 player 1 chooses $A$. It can be easily shown that $\pi = \frac{1}{3}$ and $\eta = \frac{2}{3}$ together with the type-2 player 1's playing B constitutes the unique BE in this remaining case.

40. The Bayesian equilibrium concept allows us to justify mixed-strategy Nash equilibrium in a static game with complete information as the limit of Bayesian equilibria for a sequence of static games with incomplete information; this is called the *purification* of mixed-strategy Nash equilibrium. The following example demonstrates the idea.

    Consider the following game in normal form:

    | Player 1/Player 2 | IN | OUT |
    |---|---|---|
    | IN | $-1,-1$ | $2+t_1,0$ |
    | OUT | $0,3+t_2$ | $0,0$ |

    (i) First assume that $t_1 = t_2 = 0$. Find the mixed strategy equilibria.
    (ii) Next, assume that $t_1$ and $t_2$ are respectively private to players 1 and 2. Assume that both $t_1$ and $t_2$ are independently uniformly distributed over the small interval $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, where $\epsilon > 0$. Find a pure strategy BE for this incomplete information game. What happens to the BE when $\epsilon \downarrow 0$?
    *Solution.* The purpose of this problem is to show that a mixed strategy equilibrium of a game with complete information can be considered the limit of a sequence of Bayesian equilibria in a sequence of games which are obtained by adding smaller and smaller incomplete information to the original complete-information game. Part (i) should be easy. Suppose players 1 and 2 play "In" with respectively probabilities $\pi_1$

and $\pi_2$. Given the opponent's randomizing strategy, a player should feel indifferent about his own pure strategies, i.e.

$$(-1)\pi_2 + 2(1 - \pi_2) = 0,$$

$$(-1)\pi_1 + 3(1 - \pi_1) = 0,$$

and therefore

$$\pi_1 = \frac{3}{4}, \pi_2 = \frac{2}{3}.$$

Now, part (ii). Let us first recall the definition of a Bayesian equilibrium. A BE is composed of, in this case, two mappings which map players' types into actions "In" and "Out".[15] Therefore, the BE is $\{ a_1(t_1), a_2(t_2) \}$, with the common range of the two functions being {In, Out }. Note that, for two types of player 1, $t_1 > t_1'$, (i) if $a_1(t_1') = In$, then $a_1(t_1) = In$; and (ii) if $a_1(t_1) = Out$ then $a_1(t_1') = Out$. This means that, given the distribution of $a_2(t_2)$ induced from the distribution of $t_2$, $a_1(\cdot)$ must be non-decreasing in "In," in the sense that we stated above. The same is true for player 2.

Suppose that player 1 believes that with probability $\pi_2$, $a_2(t_2) =$ "In" and that player 2 believes that with probability $\pi_1$, $a_2(t_1) =$ "In." Then, let $t_1^*$ and $t_2^*$ be respectively the types of players 1 and 2 who are just indifferent about playing "In" and "Out." All $t_1 > t_1^*$ and all $t_2 > t_2^*$ will respectively play "In," according to our discussion in the preceding paragragh. We have, in equilibrium,

$$\pi_1 = \frac{\frac{\epsilon}{2} - t_1^*}{\epsilon},$$

$$\pi_2 = \frac{\frac{\epsilon}{2} - t_2^*}{\epsilon},$$

where

$$(-1)\pi_2 + (2 + t_1^*)(1 - \pi_2) = 0,$$

and

$$(-1)\pi_1 + (3 + t_2^*)(1 - \pi_1) = 0.$$

[15]Mixed strategies are ignored here because we are trying to justify the definition of a mixed strategy equilibrium for the original game using the concept of limiting pure strategy equilibrium.

Rearranging the above four equations, we have

$$\pi_1 = \frac{3}{4} + \frac{\epsilon}{4}(\frac{1}{2} - \pi_1)(1 - \pi_2),$$

and

$$\pi_2 = \frac{2}{3} + \frac{\epsilon}{3}(\frac{1}{2} - \pi_2)(1 - \pi_1).$$

It can be shown that the above equations admit a well-defined solution.[16] Note that $\pi_1$ and $\pi_2$ converge to $\frac{3}{4}$ and $\frac{2}{3}$ respectively as $\epsilon$ tends to zero; compare this to the solution to part (i).

Thus a mixed strategy Nash equilibrium in complete information game can be regarded as the limit of Bayesian equilibria in a sequence of perturbed games with incomplete information where the perturbations converge to zero. As complete information is better considered an idealization, it is hard to object the idea of mixed strategy equilibria on the ground that they require randomizing devices.

41. Consider following two-player simultaneous game with uncertainty, where $x, y$ are independent random variables that uniformly distributed over the interval $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$:

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[16]Re-arranging, we conclude that $\pi_2 \in (0, 1)$ must solve

$$f(\pi_2; \epsilon) \equiv (3\pi_2 - 2)[16 + 4\epsilon(1 - \pi_2)] - \epsilon(1 - 2\pi_2)[2 + \epsilon(1 - \pi_2)] = 0,$$

with $\pi_1$ equal to

$$\frac{\frac{3}{4} + \frac{\epsilon}{8}(1 - \pi_2)}{1 + \frac{\epsilon}{4}(1 - \pi_2)}.$$

Note that $f(\cdot; \epsilon)$ is continuous with

$$f(0; \epsilon) = -32 - 10\epsilon - \epsilon^2 < 0 < 16 + 2\epsilon = f(1; \epsilon),$$

and hence by the intermediate value theorem, there must be some $\pi_2 \in (0, 1)$ solving $f(\pi_2; \epsilon) = 0$. Finally, observe that if $\pi_2 \in (0, 1)$, then

$$\pi_1 = \frac{\frac{3}{4} + \frac{\epsilon}{8}(1 - \pi_2)}{1 + \frac{\epsilon}{4}(1 - \pi_2)} \in (0, 1).$$

| player 1/player 2 | B | S |
|---|---|---|
| B | $(3+x,2)$ | $(1,1)$ |
| S | $(1,1)$ | $(2,3+y)$ |

(a) It is easy to show that when $e=0$, this game has a mixed-strategy Nash equilibrium where player 1 may choose B with probability $\frac{2}{3}$ and player 2 may choose B with probability $\frac{1}{3}$.

(b) Now, suppose that $e>0$ is small, and before the game gets started, player 1 privately learns about $x$ and player 2 privately learns about $y$.

We conjecture that there exist $(x^*,y^*)$ such that in the unique pure-strategy Bayesian equilibrium of the game, player 1 would choose B for sure if $x>x^*$ and S for sure if $x<x^*$; and player 2 would choose B for sure if $y<y^*$ and S for sure if $y>y^*$. It follows that for the type-$x^*$ player 1, he would get

$$(3+x^*)\cdot(\frac{y^*}{e}+\frac{1}{2})+1\cdot(-\frac{y^*}{e}+\frac{1}{2})$$

from choosing B, and he would get

$$1\cdot(\frac{y^*}{e}+\frac{1}{2})+2\cdot(-\frac{y^*}{e}+\frac{1}{2})$$

from choosing S, and he must feel indifferent about choosing B or S. Thus we have

$$(2+x^*)\cdot(\frac{y^*}{e}+\frac{1}{2})=1\cdot(-\frac{y^*}{e}+\frac{1}{2}).$$

By symmetry, we can deduce for the type-$y^*$ player 2 that

$$(2+y^*)\cdot(\frac{x^*}{e}+\frac{1}{2})=1\cdot(-\frac{x^*}{e}+\frac{1}{2}).$$

It follows that $x^*=y^*$, which we denote by $z$, so that $z$ satisfies

$$z^2+z(3+\frac{e}{2})+\frac{e}{2}=0.$$

In the Bayesian equilibrium, given $z$ player 1 believes that player 2 may choose B with probability

$$\pi(e)\equiv\frac{z}{e}+\frac{1}{2}.$$

(c) We claim that $\pi(e)$ converges to $\frac{1}{3}$ as $e$ tends to zero; that is, when the private information is "small," the pure-strategy Bayesian equilibrium looks just like the mixed-strategy Nash equilibrium when the private information is totally absent.

Note that

$$\pi(e) = \frac{e - (3 + \frac{e}{2}) + f(e)}{2e},$$

given that

$$z = \frac{-(3 + \frac{e}{2}) + f(e)}{2},$$

and

$$f(e) = \sqrt{(3 + \frac{e}{2})^2 - 2e}.$$

By Taylor's expansion, we have

$$f(e) \sim f(0) + f'(0)e + \frac{1}{2}f''(0)e^2 + \cdots,$$

where

$$f'(e) = \frac{1 + \frac{e}{2}}{2\sqrt{9 + e + \frac{e^2}{4}}},$$

and

$$f''(e) = \frac{\sqrt{9 + e + \frac{e^2}{4}} - (1 + \frac{e}{2}) \cdot \frac{1+\frac{e}{2}}{\sqrt{9+e+\frac{e^2}{4}}}}{4(9 + e + \frac{e^2}{4})},$$

so that

$$f'(0) = \frac{1}{6},\ f''(0) = \frac{2}{27}.$$

It follows that

$$\pi(e) \sim \frac{\frac{e}{2} - 3 + (3 + \frac{e}{6} + \frac{2e^2}{27})}{2e} = \frac{1}{3} + \frac{e}{54},$$

where in the numerator of $\pi(e)$, we have ignored terms of order $e^n$ with $n \geq 3$ in the Taylor's expansion for $f(\cdot)$. Thus as asserted, $\lim_{e\downarrow 0} \pi(e) = \frac{1}{3}$.

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