Game Theory with Applications to Finance and Marketing, I

Lecture 5: Further Applications in Finance, Mechanism Design, and Entry Deterrence

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1. This note consists of five parts. In part I, we shall review the costly-state-verification (CSV) model of debt financing and give several applications. In part II. we shall discuss how risky short-term debt and long-term debt may respectively affect a borrowing firm's status in product market competition. In part III, we discuss several topics in stock trading, including initial public offering (IPO), friendly merger, hostile takeover, tender offer, the free-rider problem, the determination of equilibrium bid and ask prices in an over-the-counter market, and the implications of different trading mechanisms for risk-sharing efficiency and the equilibrium transaction costs. In part IV, we review a reputation game with entry deterrence. Finally, in part V, we introduce the concept of Nash implementation, and talk about the subjects to be covered in *Game Theory with Applications to Finance and Marketing, II.*

2. (**Part I.**) Consider the following simplified version of the CSV model studied in Gale and Hellwig (1985, *RES*). Entrepreneur A must raise $I > 0$ dollars from a competitive bank B at date 0 in order to implement an investment project, which yields a random cash inflow (interchangeably, profit) $\tilde{z}$ at date 1. Assume that A and B are both risk-neutral without time preferences, and A has all bargaining power against B at date 0 when they sign a financial contract (because there are many banks competing with B at date 0). Assume also that $\tilde{z}$ is uniformly distributed over the unit interval $[0, 1]$. (The results will remain valid if instead $\tilde{z}$ has a positive density over its support, which is a compact interval.) At date 1, only A gets to see the realization of $\tilde{z}$ (called the *earnings state* or the *realized profit*), but if B wants, B can spend

$c > 0$ (to hire a CPA) to find out the true profit (and to produce legal evidence for that true profit). Assume that $I + c < E[\tilde{z}] = \frac{1}{2}$; that is, the investment project has a positive NPV.

A date-0 incentive-feasible financial contract is a tuple

$$\mathcal{C} \equiv \{R_0(\hat{z}), R_1(\hat{z}, z), d(\hat{z});\ \forall z, \hat{z} \in [0, 1]\}$$

such that, given contract $\mathcal{C}$ has been signed at date 0,
(1) A must first make an earnings report $\hat{z} \in [0, 1]$ at date 1;
(2) B will then spend c to find out A's true profit (equivalently, to verify A's earnings state), if and only if it is specified in $\mathcal{C}$ that $d(\hat{z}) = 1$, where $\forall \hat{z} \in [0, 1]$, $d(\hat{z})$ equals either 0 or 1;
(3) A must repay B the amount $R_0(\hat{z})$ if A has reported a profit $\hat{z}$ such that $d(\hat{z}) = 0$;
(4) A must repay B the amount $R_1(\hat{z}, z)$ if A has reported a profit $\hat{z}$ such that $d(\hat{z}) = 1$, and A's true profit is instead z (which will be revealed to the public after B spends c);
(5) (Condition LL) $0 \le R_1(\hat{z}, z) \le z, \quad 0 \le R_0(\hat{z}) \le \hat{z}$;
(6) (Condition IC_A) for all $z \in [0, 1]$, reporting $\hat{z} = z$ is A's date-1 optimal strategy; and
(7) (Condition IR_B) B can at least break even by accepting contract $\mathcal{C}$.

Some explanations are in order. In plain words, an incentive-feasible contract must specify when B will send a CPA to audit A's profit at date 1, and this decision d is contingent on A's profit report $\hat{z}$. Moreover, if given A's profit report $\hat{z}$, B must audit A's profit according to the contract, then the true profit z will become known, and in that case A's repayment to B can depend on both A's report $\hat{z}$ and the true profit z; this explains the function $R_1(\cdot, \cdot)$. On the other hand, if given A's profit report $\hat{z}$, B should not audit A's profit according to the contract, then since the true profit remains A's private information, the contract can only specify a repayment $R_0(\cdot)$ that is contingent only on A's profit report $\hat{z}$. Furthermore, the repayment R_1 cannot exceed the true profit z, and in case of no audit, the repayment cannot exceed the profit report $\hat{z}$; these are referred to as the *limited-liability constraint* for A. Note that R_1 and R_0 are also required to be non-negative; these are called *limited-liability constraint* for B. The latter says that B is not obliged to lending more money to the firm at date 1. These limited-liability constraints are written as Condition LL in (5). Next, note that we have required that A must always truthfully report the date-1 profit under an incentive-feasible contract. This is Condition IC_A, and it originates from the *revelation principle* in contract theory, which says that for each contract that makes d, R_0, and R_1 contingent on some verifiable messages there exists a contract that makes d, R_0, and R_1 contingent on $\hat{z}$ and z only and that induces truthtelling as A's best response in each and every true state z (which gives rise to the constraint IC_A), where equivalence means that the two contracts yield the same payoff for A. Finally, an incentive-feasible contract must ensure that B is willing to accept it in the first place, and this is stated as Condition IR_B.

To sum up, in designing a financial contract, we can always confine our attention to the set of *incentive-feasible contracts* defined above. Among these incentive feasible contracts, A's favorite contracts (which may not be unique) will be termed *incentive efficient*, or simply the *optimal financial contracts*. Now let us characterize an optimal financial contract.

Step 1. Suppose that $\mathcal{C}$ is incentive-feasible. Then $R_0(\hat{z}_1) = R_0(\hat{z}_2)$ for all $\hat{z}_1, \hat{z}_2$ such that $d(\hat{z}_1) = d(\hat{z}_2) = 0$.

Proof. Suppose that $d(\hat{z}_1) = d(\hat{z}_2) = 0$ but, say, $R_0(\hat{z}_1) > R_0(\hat{z}_2)$. Then given that the true profit is $\hat{z}_1$, A strictly prefers to lie and report $\hat{z}_2$. This is a contradiction to IC_A. $\|$

Step 2. If $\mathcal{C}$ is incentive-feasible, then there exists some $z \in [0, 1]$ such that $d(z) = 1$.

Proof. Suppose that $d(z) = 0$ for all $z \in [0, 1]$. Then according to Step 1, A must make the same repayment regardless of the true profit z. Condition LL implies that, by reporting $z = 0$, A does not have to repay anything. This violates IR_B, since B, after paying $I > 0$ at date 0, can get nothing back at date 0. This shows that $\mathcal{C}$ is not incentive-feasible, a contradiction. ‖

Step 3. Suppose that $\mathcal{C}$ is incentive-feasible. Let $F = R_0(\hat{z})$ for all $\hat{z}$ such that $d(\hat{z}) = 0$. Then $F \geq R_1(z, z)$ for all z such that $d(z) = 1$.

Proof. This assertion is vacuously true if $d(z) = 1$ for all $z \in [0, 1]$. So, let us assume that there exists z' with $d(z') = 0$ and let $F = R_0(z')$. If for some z with $d(z) = 1$ we have $R_1(z, z) > F$, then when the true profit is z, A strictly prefers to lie and report profit z', which violates IC_A, a contradiction. ‖

To sum up, the first three steps have shown that if $\mathcal{C}$ is incentive-feasible with F being the repayment made by A to B in the event of $d = 0$, then $R(z) \leq \min(z, F)$ for all $z \in [0, 1]$, where $R(z) = R_1(z, z)$ if $d(z) = 1$ and $R(z) = R_0(z)$ if $d(z) = 0$. A contract $\mathcal{C}$ that specifies $R_1(z, z) = z$ and $K = [0, F)$ is referred to as a standard debt (SD) contract, which is uniquely defined by the face value of debt, F. Note that a SD contract is incentive feasible, and with such a contract, $R(z) = \min(z, F)$ for all $z \in [0, 1]$. The remaining steps will establish that no other contracts can outperform an optimal SD contract. That is, with respect to the costly state verification problem, standard debt contracts are optimal contracts.

Step 4. Suppose that $\mathcal{C}$ is incentive-feasible. Let $F = R_0(z)$ for all z with $d(z) = 0$. Define the event

$$K \equiv \{z \in [0, 1] : \; d(z) = 1\}.$$

Then $[0, F) \subset K$. That is, we must have $d(z) = 1$ whenever $z < F$.

Proof. Suppose that $z < F$ and yet $d(z) = 0$. Then we would have $F = R_0(z) \leq z < F$, a contradiction. ‖

Step 5. Suppose that $\mathcal{C}$ is incentive-efficient. Then IR_B is binding under $\mathcal{C}$.

Proof. By definition, $\mathcal{C}$ is incentive-efficient if and only if it solves the following maximization problem:

$$\max_{K,F,R_1(\cdot,\cdot)} \int_K [z - R_1(z,z)]dz + \int_{[0,1]\setminus K} [z - F]dz$$

subject to

$$(\text{IR}_B)\ \int_K R_1(z,z)dz + F[1 - \text{prob.}(K)] - c\ \text{prob.}(K) \geq I;$$

$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)],\ \forall z, z' \in K;$$

$$0 \leq F \leq z,\ \forall z \in [0,1] \setminus K.$$

Suppose instead that IR_B is not binding under $\mathcal{C}$. Then for sufficiently small $e > 0$, for all $z', z \in K$, we can replace F and $R_1(z',z)$ by respectively $F(1-e)$ and $R_1(z',z)(1-e)$ and ensure that the new contract is still incentive feasible. Since A would be better off offering B the new contract rather than $\mathcal{C}$, by definition $\mathcal{C}$ cannot be incentive efficient. Thus we have a contradiction. ||

Step 6. $\mathcal{C}$ is incentive-efficient if and only if it solves the following minimization program:

$$\min_{K,F,R_1(\cdot,\cdot)} \text{prob.}(K)$$

subject to

$$\text{prob.}(K) = \frac{1}{F+c}[\int_K R_1(z,z)dz + F - I],$$

$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)],\ \forall z, z' \in K;$$

$$0 \leq F \leq z,\ \forall z \in [0,1] \setminus K.$$

Proof. By Step 5, $\mathcal{C}$ is incentive-efficient only if IR_B is binding, which implies that

$$\int_K R_1(z,z)dz + F[1 - \text{prob.}(K)] = c \text{ prob.}(K) + I,$$

and hence we can re-write the A's objective function as

$$\int_K [z - R_1(z,z)]dz + \int_{[0,1]\setminus K} [z - F]dz = E[z] - I - c \text{ prob.}(K).$$

It follows that $\mathcal{C}$ is incentive efficient if and only if it solves the following maximization problem:

$$\max_{K,F,R_1(\cdot,\cdot)} E[z] - I - c \text{ prob.}(K)$$

subject to

$$\text{prob.}(K) = \frac{1}{F+c}[\int_K R_1(z,z)dz + F - I],$$

$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)], \ \forall z, z' \in K;$$

$$0 \leq F \leq z, \ \forall z \in [0,1] \setminus K.$$

Now that $E[z]$, I, and $c > 0$ are all constants, we can further re-write Mr. A's optimization problem as

$$\min_{K,F,R_1(\cdot,\cdot)} \text{prob.}(K)$$

subject to

$$\text{prob.}(K) = \frac{1}{F+c}[\int_K R_1(z,z)dz + F - I],$$

$$0 \leq R_1(z,z) \leq \max[z, F, R_1(z',z)], \ \forall z, z' \in K;$$

$$0 \leq F \leq z, \ \forall z \in [0,1] \setminus K.$$

This completes the proof. ‖

Define F^* as the unique solution to

$$g(F) \equiv -\frac{1}{2}F^2 + (1-c)F - I = 0.$$

Then, $0 < F^* = (1-c) - \sqrt{(1-c)^2 - 2I} < 1 - c$.

Step 7. Recall that a standard debt contract can be uniquely defined by its face value F.

- A standard debt contract with face value F has prob.$(K) = F$.
- A standard debt contract with face value F is incentive feasible if and only if $F^* \leq F \leq 1$.
- The SD contract with face value F^* dominates all other SD contracts, and will be denoted by $\mathcal{C}^*$. Under $\mathcal{C}^*$, IR$_B$ is binding.
- B's payoff from holding a standard debt with face value F is strictly increasing in F on the interval $[F^*, 1-c)$, and it is strictly decreasing in F on the interval $(1-c, 1]$.

Proof. It is easy to see that a standard debt contract satisfies A's IC constraints and the limited liability constraints if and only if $0 \leq F \leq 1$. To satisfy B's IR constraint, B's payoff from accepting the standard debt contract, which is

$$\int_K R_1(z,z)dz + F[1 - \text{prob.}(K)] - c \text{ prob.}(K) - I$$

$$= \int_0^F zdz + F[1-F] - cF - I$$
$$= -\frac{1}{2}F^2 + (1-c)F - I = g(F),$$

must be non-negative; that is, a SD contract with face value F would make IR$_B$ binding if and only if $g(F) = 0$.

Since we have assumed that the net present value of the project is positive,

$$E[\tilde{z}] = \frac{1}{2} > c + I,$$

we have

$$(1-c)^2 - 2I = c^2 + 2[\frac{1}{2} - (c+I)] > 0,$$

implying that $g(F) \geq 0$ if and only if

$$F^* = (1-c) - \sqrt{(1-c)^2 - 2I} \leq F \leq (1-c) + \sqrt{(1-c)^2 - 2I}.$$

However, we have

$$\frac{1}{2} - c - I > 0 \Rightarrow 1 - 2c - 2I > 0 \Rightarrow (1-c)^2 - 2I > c^2$$

$$\Rightarrow \sqrt{(1-c)^2 - 2I} > c \Rightarrow (1-c) + \sqrt{(1-c)^2 - 2I} > 1.$$

Since $F > 1$ would violate A's IR constraint, we conclude that a standard debt contract is incentive feasible if and only if $F^* \leq F \leq 1$.

The last assertion now follows from the fact that the concave function $g(F)$ attains its maximum at $F = 1 - c$. $\|$

Step 8. Suppose that $\mathcal{C}$ is incentive efficient. Then under $\mathcal{C}$ the event

$$E = K \bigcap [F, 1]$$

is a zero-probability event.

Proof. Suppose that E occurs with a positive probability under the incentive efficient contract $\mathcal{C}$, and we shall demonstrate a contradiction.

First note that under $\mathcal{C}$, prob.$(K) = F+$prob.(E).

Consider a standard debt contract $\mathcal{C}'$ with face value equal to the constant function $R_0(\cdot)$ under the incentive efficient contract $\mathcal{C}$. Then under $\mathcal{C}'$ B receives F if $z \in [F, 1]$ and $z - c$ if $z \in [0, F]$. However, under $\mathcal{C}$, B receives $R_1(z) - c \leq z - c$ if $z \in [0, F]$, F if $z \in [F, 1] \bigcap K^c$, and $R_1(z) - c \leq F - c$ if $z \in [F, 1] \bigcap K$. That is, B gets weakly more from accepting $\mathcal{C}'$ than accepting $\mathcal{C}$. Since B must break even when accepting the incentive efficient contract $\mathcal{C}$, the SD contract $\mathcal{C}'$ is also incentive feasible, which, by Step 7, implies that $F \in [F^*, 1]$.

Note that the standard debt contract $\mathcal{C}^*$ is incentive feasible and it makes IR_B binding, and it has a probability of state verification equal to $F^* \leq F < F$+prob.(E), so that by Step 6, $\mathcal{C}$ cannot be incentive efficient, a contradiction! $\|$

Note that Step 4 and Step 8 together imply that under an incentive feasible contract, prob.$(K) = F \equiv R_0(\cdot)$.

Step 9. Suppose that $\mathcal{C}$ is incentive efficient. Then under $\mathcal{C}$ the event

$$G = \{z \in [0, F] : \ R_1(z, z) < z\}$$

is a zero-probability event, where $F = R_0(\cdot)$.

Proof. Suppose instead that G may occur with a positive probability under the incentive efficient contract $\mathcal{C}$. We shall demonstrate a contradiction.

First note that under $\mathcal{C}$, by Step 6, prob.$(K) = F$.

Consider the standard debt contract $\mathcal{C}'$ with face value F. Because G may occur with a positive probability, B gets strictly more from accepting $\mathcal{C}'$ than accepting $\mathcal{C}$. Since $\mathcal{C}$ is incentive efficient, the SD contract $\mathcal{C}'$ is also incentive feasible, and under $\mathcal{C}'$, IR_B is not binding. This implies, by Step 7, that $F \in (F^*, 1]$.

Recall that the standard debt contract $\mathcal{C}^*$ with face value F^* is incentive feasible and it makes IR_B binding, and it has a probability of state verification equal to F^*. The SD contract $\mathcal{C}'$ and the incentive efficient contract $\mathcal{C}$ both share a probability of state verification equal to F. Since $F > F^*$, by Step 6, $\mathcal{C}$ cannot be incentive efficient, a contradiction! $\|$

Thus Steps 8 and 9 together have established that an incentive efficient contract in the current model is "essentially" a standard debt contract.

Remark. We have assumed that the contracting parties can only commit to a deterministic state-verification policy (i.e., d must equal either zero or one). Consequently, IC_A may hold as an inequality under the optimal contract. In fact, verifying with probability one is un-necessary for the purpose of inducing truth-telling. Mookherjee and Png (1989, *Quarterly Journal of Economics*) show that the optimal state verification policy would be stochastic whenever stochastic policies are feasible; see the example in the next section. We have also assumed that the entrepreneur is risk-neutral. Winton (1994, *Review of Financial Studies*) shows that when the entrepreneur is risk-averse, standard debt is no longer optimal as it results in the entrepreneur bearing too much risk. The optimal contract would leave the entrepreneur with a positive payoff even in the event of state verification. If we interpret state verification as the event of bankruptcy, then Winton's theory is consistent with the empirical fact that shareholders tend to get a positive payoff in the event of bankruptcy.

3. (**Example 1**.) At date 0 A must offer a contract to B to raise I, but only A gets to see the realized profit z at date 1. In this exercise, we assume that z is equally likely to take on 3 or 0, and that

$$I = 1, \quad c = \frac{1}{5}.$$

(i) Suppose first that A can commit in the date-0 contract either $d = 0$ or $d = 1$. In this case, let F be the face value of the equilibrium debt contract that A offers B at date 0. Then $F =$? Compute A's equilibrium payoff.

(ii) Now, suppose instead that A can commit in the date-0 contract any $d \in [0, 1]$. In this case, let F' be the face value of the equilibrium debt contract that A offers B at date 0. Then $F' =$? Compute A's equilibrium payoff.

(iii) Now, suppose instead that the date-0 contract must give investor B the right to verify the earnings state at date 1, but state verification occurs at date 1 when and only when it is in B's interest to verify the profit at date 1. That is, the date-0 contract cannot commit to any d

that is inconsistent with B's preferences at date 1. Moreover, assume that B can obtain the true profit z whenever state verification proves that $\hat{z} \neq z$; that is, the date-0 contract imposes the maximum penalty for lying. Let F'' be the face value of the equilibrium debt contract that A offers B at date 0. Then $F'' =$? Compute A's equilibrium payoff.[1]

Solution. For part (i), recall that F must satisfy B's binding IR constraint:

$$\frac{1}{2}F - \frac{1}{2}c = I \Rightarrow F = \frac{11}{5}.$$

In this case, A's payoff is equal to

$$\frac{1}{2}(3 - F) = \frac{2}{5}.$$

Consider part (ii). First note that in part (i), when $z = 3$, by telling the truth A would get

$$3 - \frac{11}{5} = \frac{4}{5},$$

[1]**Hint**: At date 1, A must first report $\hat{z} \in \{0, 3\}$. Given F'', it can be shown that

- B will never verify if A has reported $\hat{z} = 3$; and
- given that B may verify with probability d after A has reported $\hat{z} = 0$, A will always tell the truth when $z = 0$.

Thus suppose that A may lie about the true profit $z = 3$ with probability a, and B may verify with probability d following A's report $\hat{z} = 0$. It can be shown that in equilibrium A must feel indifferent about telling the truth or lying when $z = 3$, and B must feel indifferent about spending or not spending c after A reports that $\hat{z} = 0$. It follows that, given F'', we must have

$$\begin{cases} 3 - F'' = 3(1 - d); \\ 0 = -c + \frac{\frac{1}{2} \cdot a}{\frac{1}{2} \cdot a + \frac{1}{2} \cdot 1} \cdot 3, \end{cases}$$

but A's payoff would drop to zero if A chose to lie and to trigger state verification. That is, A's IC constraint is strictly satisfied under the contract in part (i). This is wasteful, because A would still be willing to tell the truth if the probability d following $\hat{z} = 0$ were reduced slightly. Note that we have assumed that this cannot be done in part (i), where A can only commit to a deterministic policy of state verification.

Now, since any $d \in [0, 1]$ is allowed in part (ii), A can do better by offering a date-0 contract (F', d) such that both IR_B and IC_A are binding. Solving the following system of equations simultaneously,

$$\begin{cases} \frac{1}{2}F' - \frac{1}{2}dc = I; \\ \\ 3 - F' = 3(1 - d), \end{cases}$$

we obtain

$$d = \frac{I}{\frac{3}{2} - \frac{1}{10}} = \frac{5}{7}, \quad F' = \frac{3}{\frac{3}{2} - \frac{1}{10}} = \frac{15}{7}.$$

In this case, A's equilibrium payoff is

$$\frac{1}{2}(3 - F') = \frac{3}{7}.$$

Finally, consider part (iii). Given F'', what would happen in the date-1 subgame where A has reported $\hat{z}$? It can be shown that

- B will never verify if A has reported $\hat{z} = 3$; and
- given that B may verify with probability d after A has reported $\hat{z} = 0$, A will always tell the truth when $z = 0$.

Thus suppose that A may lie about the true profit $z = 3$ with probability a, and B may verify with probability d following A's report $\hat{z} = 0$. It can then be shown that in equilibrium

$$0 < a, d < 1;$$

that is, A must feel indifferent about telling the truth or lying when $z = 3$, and B must feel indifferent about spending or not spending c upon seeing A's report that $\hat{z} = 0$.

It follows that, given F'', we must have

$$\begin{cases} 3 - F'' = 3(1-d); \\ \\ 0 = -c + \frac{\frac{1}{2}\cdot a}{\frac{1}{2}\cdot a + \frac{1}{2}\cdot 1} \cdot 3, \end{cases}$$

so that

$$d = \frac{F''}{3}, \quad a = \frac{\frac{1}{10}}{\frac{1}{2}(3 - \frac{1}{10})} = \frac{1}{14}.$$

Now, at date 0, F'' must be the solution to the following maximization problem:

$$\max_F \frac{1}{2}[3a(1-d) + (1-a)(3-F)]$$

subject to

$$d = \frac{F''}{3}, \quad a = \frac{\frac{1}{10}}{\frac{1}{2}(3 - \frac{1}{10})} = \frac{1}{14},$$

and the following IR_B:[2]

$$\frac{1}{2}(1-a)F'' - I \geq 0.$$

At optimum, IR_B must be binding, and hence we obtain

[2]B no longer expects to incur a state-verification cost! At the state verification date, upon seeing $\hat{z} = 0$, B either chooses to not verify the state (which costs nothing), or B may verify and yield a return which is expected to exactly cover B's verification cost. These two actions are equally good, and hence B will randomize over them upon seeing the earnings report $\hat{z} = 0$.

$$F'' = \frac{3 - \frac{1}{5}}{\frac{3}{2} - \frac{1}{5}} = \frac{28}{13}.$$

It follows that A's equilibrium payoff becomes

$$\frac{1}{2}(3 - F'') = \frac{11}{26}.$$

Remark. In this exercise, A's payoff in part (ii) is the highest, and his payoff in part (i) is the lowest. This is quite natural: B will break even under the optimal contract obtained in part (i) or part (ii) or part (iii), but the expected cost for state verification differs over the three scenarios.

Part (ii) allows A to directly minimize that expected cost, and hence it maximizes A's payoff. Via the mixed-strategy equilibrium of the date-1 subgame, part (iii) can implement a probability of state verification lower than $d = 1$, but since in equilibrium state verification may occur with probability $ad = \frac{1}{14} \times \frac{28}{39}$ even in the true state $z = 3$, A's optimal payoff in part (iii) is less than in part (ii). Indeed, since in part (iii) state verification may occur in the state $z = 3$, in order for B to break even in both part (ii) and part (iii), we must have $F'' > F'$, and hence A's payoff in part (iii), which is $\frac{1}{2}(3 - F'')$ is less than A's payoff in part (ii), which is $\frac{1}{2}(3 - F')$.

4. (**Part II.**) Now we review a few game-theoretic models of debt financing with imperfectly competitive firms.

(**Example 2.**) (**Risky Short-term Debt May Lead to More Aggressive Cournot Competition.**)
Consider two firms 1 and 2 engaging in Cournot competition at date 1. They produce the same product and face the following inverse demand:

$$P = k(1 - q_1 - q_2),$$

where $k > 0$ is a constant. Firm 2's unit production cost is kc, and firm 1's unit production cost is $k\tilde{c}_1$ which is equally likely to be $k(c+d)$ and $k(c-d)$. Firm 1 must choose q_1 before seeing the realization of $\tilde{c}_1$. We assume that

$$1 > c+d > c > c-d > 0.$$

Firms are risk-neutral without time preferences.
(i) Suppose that firms seek to maximize expected profits. Find the Nash equilibrium.
(ii) Suppose that firm 1 has borrowed some debt prior to date 1. Let the face value of firm 1's debt be F, with

$$(\Lambda)\quad \max(k[\frac{1-c-4d}{3}][\frac{1-c+2d}{3}], k[\frac{(1-c)^2}{9} - \frac{(1-c)d}{3}]) < F < k[\frac{1-c+2d}{3}]^2.$$

Assume that each firm seeks to maximize its equity value, and investors are all risk-neutral without time preferences. Find a Nash equilibrium in which firm 1 does not always default on its debt.

Solution. It is useful to consider a situation where firm i's unit cost is c_i. It is easy to show that firm i's reaction function is

$$R_i(q_j) = \frac{1-q_j-c_i}{2}, \quad i,j = 1,2,$$

so that in equilibrium

$$(\Gamma)\quad q_i^* = \frac{1+c_j-2c_i}{3}, \quad i,j = 1,2.$$

In part (i), note that $E[\tilde{c}_1] = c$, and since both firms are risk-neutral (so that only the expected unit cost matters), in equilibrium both firms produce the same output, which is

$$q_1^* = q_2^* = \frac{1-c}{3}.$$

Consider part (ii). Let $\tilde{\Pi}_j$ denote firm j's profit. Then by assumption firm 1 seeks to maximize

$$E[\max(\tilde{\Pi}_1 - F, 0)],$$

where

$$\tilde{\Pi}_1 = kq_1(1 - q_1 - q_2 - \tilde{c}_1).$$

Since firm 2's profit $kq_2(1 - q_1 - q_2 - c)$ is strictly concave in q_2, firm 2 will adopt a pure strategy q_2^* in equilibrium. In an equilibrium where firm 1 does not default on its debt in the low-cost state, there are two possibilites: either firm 1 never defaults on its debt, or it defaults on its debt only in the high-cost state. Observe that in either case, firm 1 must also adopt a pure strategy q_1^* in equilibrium. Hence there are two possibilities regarding the equilibrium (q_1^*, q_2^*): either

$$kq_1^*(1 - q_1^* - q_2^* - c - d)q_1^* \leq F < kq_1^*(1 - q_1^* - q_2^* - c + d)q_1^*$$

or

$$kq_1^*(1 - q_1^* - q_2^* - c - d)q_1^* > F$$

must be true. In the former case, given q_2^*, firm 1's equilibrium best response is

$$q_1^* = \frac{1 - q_2^* - c + d}{2},$$

which together with firm 2's reaction function

$$q_2^* = \frac{1 - q_1^* - c}{2}$$

implies that

$$(q_1^*, q_2^*, p^*) = (\frac{1 - 2(c - d) + c}{3}, \frac{1 - 2c + (c - d)}{3}, \frac{k[1 + c + (c - d)]}{3}),$$

and in this equilibrium, firm 1's profit is

$$kq_1^*(1 - q_1^* - q_2^* - c - d)q_1^* = k[\frac{1 - c - 4d}{3}][\frac{1 - c + 2d}{3}]$$

if its realized unit cost is $c + d$; and firm 1's profit is

$$kq_1^*(1 - q_1^* - q_2^* - c + d)q_1^* = k[\frac{1 - c + 2d}{3}]^2$$

if its realized unit cost is $c - d$. Condition (Λ) implies that this equilibrium does exist. In the latter case, firm 1 will never default on its debt in equilibrium, and hence the equilibrium is as stated in part (i). If the latter equilibrium exists, then firm 1's equilibrium profit would be

$$k[\frac{(1-c)^2}{9} - \frac{(1-c)d}{3}]$$

when its realized unit cost is $c + d$, but condition (Λ) implies that firm 1 must default on its debt when its realized unit cost is $c + d$, a contradiction to the conjecture that firm 1 never defaults on its debt in equilibrium. Hence the latter equilibrium does not exist.[3]

Our conclusion is thus that, under condition (Λ), in equilibrium firm 1's objective function becomes

$$\frac{1}{2}[kq_1(1 - q_1 - q_2^* - c + d) - F] + \frac{1}{2} \cdot 0.$$

Note carefully that, given q_2^*, firm 1 has essentially become a firm with unit cost $c - d$! In equilibrium, firm 2 must reduce its output to below $\frac{1-c}{3}$, because firm 2 realizes that firm 1 has become more aggressive than in part (i) in choosing its output, and, expecting this, firm 2's best response to cut back on its own output. Indeed, in the SPNE we have, using the formulae in (Γ),

$$q_1^{**} = \frac{1 + c - 2c + 2d}{3} = \frac{1 - c + 2d}{3} > \frac{1-c}{3} = q_1^*,$$

[3]Condition (Λ) is needed here because without such a condition, this game may not have a (mixed- or pure-strategy) Nash equilibrium. The following is a numerical example. Suppose that

$$k = 1, \ c = d = \frac{1}{10}, \ F = \frac{109}{1800}.$$

In equilibrium, either firm 1 defaults on its debt in the high-cost state, or it does not. But one can verify that neither is consistent with equilibrium. Indeed, if firm 2 expects firm 1 to default on its debt in the high-cost state, then firm 2 expects firm 1 to expand output, and hence firm 2 must cut back on its own output, resulting in firm 1 having a realized profit in the high-cost state exceeding F. On the other hand, if firm 2 expects firm 1 to never default on its debt, then the two firms will choose the same output level, leading to firm 1 having a realized profit in the high-cost state which is less than F. The non-existence of equilibrium is not surprising, because a game where players' strategy spaces are not finite sets may not have a Nash equilibrium. Existence of equilibrium requires certain convexity conditions.

and

$$q_2^{**} = \frac{1+c-d-2c}{3} = \frac{1-c-d}{3} < \frac{1-c}{3} = q_2^*.$$

To sum up, by issuing risky short-term debt, firm 1 gains a competitive advantage, since it behaves as if its unit cost were low with probability one, and expands output accordingly. By the fact that the two firms' output choices are strategic substitutes, this forces its rival to produce less and concede in market share. Note that to have this commitment value, firm 1's debt must be risky: firm 1 must get nothing when its unit cost is high and the profit that it makes cannot fully repay the debt. Note that var$[\tilde{\Pi}_1]$ rises because of firm 1's risky debt, which is the *asset substitution* effect pointed out in Jensen and Meckling (1976). That risky short-term borrowing may induce a firm to expand output and compete more aggressively was first pointed out by Brander and Lewis (1986, *AER*).

5. (**Example 3**.) (**Long-term Debt May Promote Collusion**.) Now, assume that the two firms in Example 2 also compete at date 0, and that firm 1 borrows debt at date 0 when it is penniless. Again, the debt with face value F will be due at date 2. It is now referred to as a long-term debt. At date 0, each firm can spend a cost h on promotion. If neither promotes, each firm gets v at date 0. If exactly one firm promotes, then that firm gets $2v-h$, leaving the other firm with zero profit. If both promote, then each gets $v-h>0$.

   To make things interesting, let us assume that

$$(\Delta)\quad \frac{kd(2-2c-d)}{9} > v-h,$$

   and that

$$(\Sigma)\ F-(2v-h) < \max(k[\frac{1-c-4d}{3}][\frac{1-c+2d}{3}], k[\frac{(1-c)^2}{9}-\frac{(1-c)d}{3}])$$

$$< F-(v-h) < k[\frac{1-c+2d}{3}]^2.$$

   Note that, were the date-1 competition not existent, the two firms would compete at date 0 with their only concerns being their date-0 profits. It is clear that they are playing a version of the game called the prisoner's dilemma, where the only Nash equilibrium is the one in which both promote.

Since firms are rational, and since they know that they have to compete again at date 1, they seek to maximize the sum of profits over dates 0 and 1 when taking date-0 actions. Condition (Σ) says that if firm 1 does not make enough profits at date 0, it will panic because the long-term debt will default at date 2 in the event that $\tilde{c}_1 = c + d$, and hence it will expand output, which will hurt firm 2. Realizing this, firm 2 must "make firm 1 look good" at date 0, so that at the beginning of date 1, firm 1 knows that its debt will never default at date 2, which will then induce firm 1 to choose a low output, thereby raising firm 2's output and date-1 profit. Condition (Δ) says that by conceding at date 0 and then gaining at date 1, firm 2 will become better off. Hence in the SPNE, firm 2 does not promote at date 0, leaving firm 1 with a profit $(2v - h)$, which results in a symmetric date-1 output choice $q_i^* = \frac{1-c}{3}$ as in part (i) of Example 5.

Remark. That long-term borrowing can mitigate competition before the debt maturity date gets close was first pointed out by Glazer (1994, *JET*). We have examined in Example 3 the case where only firm 1 is financially leveraged.

Now consider the case where both firms have issued long-term debt with identical face value, and where both firms have unit cost $k\tilde{c}_1$. If the two firms compete only at date 1, as in Example 2, then risky short-term debt would make both of them worse off. This happens because both firms would behave as if their unit cost were sure to be $c - d$! (This is like a prisoner's dilemma, where in equilibrium each firm has a lower value; recall that firm value is the sum of debt value and equity value.) On the other hand, If these firms compete at both dates 0 and 1, as in Example 3, then debt actually benefit both of them: each firm would want to make the other firm look good at date 0 in order to boost its own date-1 profit. Consequently these two firms would be able to avoid the inefficient date-0 outcome of the prisoner's dilemma. Moreover, the high profits they make at date 0 also allow them to avoid competing aggressively at date 1! Note that without long-term borrowing, the two firms would be trapped in an inefficient outcome at date 0, although their date-1 equilibrium output choices would be as efficient as in the presence of long-term borrowing.

6. (**Example 4.**) **The Bolton-Scharfstein (1990) CSV model**.

   Firm B needs F dollars to operate in the product market at respectively date 0 and date 1. Profits are generated at respectively date 1 and date 2. Firm B has no cash initially, and it has to borrow from an investor who has all bargaining power against firm B. Profits are only observable to firm B and verifying profits is prohibitively costly for the investor. The revelation principle implies that the contract-design problem between B and the investor can be modeled as a direct revelation game with no loss of generality. In the direct revelation game, the repayment of the financial contract only depends on the firm's report of profit. Assume that at each date t ($t = 1, 2$) the profit of B can be either π_1 (with probability θ) or π_2, with $\pi_2 > \pi_1$, $\overline{\pi} \equiv E(\pi) = \theta\pi_1 + (1-\theta)\pi_2 > F$, and $\pi_1 < F$. Also, assume all parties are risk neutral with no time preference.

   (i) Show that if firm B operates for only one period, the investor will refuse to lend F. (**Hint**: If the investor does, B will *always* report $\pi_1 < F$.)

   Because of (i), we now suppose that B operates for two periods and that $\pi_2 - \pi_1 < F$. The financing is assumed to proceed as follows. At date 0, the investor lends F to B. Then at date 1, B reports its date-1 profit $\pi_i \in \{\pi_1, \pi_2\}$. If B reports its date-1 profit to be π_i, then it has the obligation of paying the investor R_i at date 1. After this repayment is made, with probability β_i the debt is renewed. In case the debt is renewed at date 1, then the investor gives F to B at date 1, and at date 2, B reports its profit π_j. The second period repayment is denoted by R_{ij} if at date 1 firm B has reported π_i and at date 2 it reports π_j.

   The game proceeds as follows. The investor first decides to or not to lend at date 0. If lending is the decision, then the investor offers a financial contract $(R_1, R_2, \beta_1, \beta_2, R_{11}, R_{12}, R_{21}, R_{22})$ to B, and B can either accept or reject.[4] Such a contract specifies only variables that can be subsequently observed by both contracting parties and can be verified by the court of law (so that the latter can enforce it). When specifying these contract variables, the investor must make sure that B will accept (accepting generates for B a utility higher than otherwise), which is called B's individual rationality condition (IR condition). The investor must also make sure that B will truthfully report its profits (truthtelling is better than lying), which is called B's incentive compatibility condition (IC condition). Finally, the repayment R_i and R_{ij} must really be affordable by B when the true profts are respectively π_i and π_j at dates 1 and 2. This is called the limited liability condition (LL condition).

Any contract satisfying these three conditions is said to be *feasible*. The investor wants to find a feasible contract that maximizes her own expected utility. The solution is called an *optimal contract* (because such a contract is Pareto optimal within the set of feasible contracts). Thus, when deciding to lend at date 0, the investor's optimal contract problem is

$$\max_{\beta_i, R_i, R^i} -F + \theta[R_1 + \beta_1(R^1 - F)] + (1 - \theta)[R_2 + \beta_2(R^2 - F)],$$

subject to
(IC at date 1) $\pi_2 - R_2 + \beta_2(E(\pi) - R^2) \geq \pi_2 - R_1 + \beta_1(E(\pi) - R^1)$;
(LL at date 1) $\pi_i \geq R_i$,
(LL at date 2) $\pi_i - R_i + \pi_1 \geq R^i, i = 1, 2$;
(IR at date 0) $\theta[\pi_1 - R_1 + \beta_1(E(\pi) - R^1)] + (1 - \theta)[\pi_2 - R_2 + \beta_2(E(\pi) - R^2)] \geq 0$.
Note that in the above, we have used the fact that R_{ij} must be independent of π_j in order to satisfy B's second period IC condition, and we have written R_{ij} as R^i. This is also why we did not impose B's IC condition at date 2.

(ii) Show that under optimal contract, firm B always tells the truth

[4] This implies that the borrower may be a small firm, which lacks bargaining power when negotiating the loan contract with a large bank.

when reporting the second-period profit. (**Hint**: Like the reasoning in part (i), if the repayment were dependent on the second-period report, B would always report π_1 in the second period, violating B's IC constraint.)

(iii) Show that the optimal contract $(R_1^*, \beta_1^*, R_2^*, \beta_2^*)$ is $(\pi_1, 0, E(\pi), 1)$ if

$$\theta F + (1-\theta)E(\pi) > \pi_1,$$

and $(\pi_1, 1, \pi_1, 1)$ if otherwise. (**Hint**: Show that the above IC condition has to be binding at optimum. Thus,

$$\pi_2 - R_2 + \beta_2(E(\pi) - R^2) = \pi_2 - R_1 + \beta_1(E(\pi) - R^1).$$

Replace this equality into the objective function and note that the objective function becomes strictly increasing in β_2. This implies that the objective function does not depend on R_2 and R^2 separately; rather, it depends on $R_2 + R^2$ only (and similarly for the constraints.) Thus, there is no loss to set $R^{2*} = \pi_1$. Also, the objective function is increasing in both R_1 and R^1. Finally, note that the objective function is decreasing in β_1 if and only if

$$\theta F + (1-\theta)E(\pi) > \pi_1.$$

Depending on whether this inequality holds, the optimal contract can be fully solved.)

(iv) Show that the investor lends F to B at date 0 if and only if

$$F < \frac{(\pi_1 + (1-\theta)E(\pi))}{2-\theta}.$$

Up to now, we have assumed that $\pi_2 - \pi_1 < F$, and so refinancing at date 1 is necessary for firm B to continue its business in the second period.
(v) Show that, if instead,

$$\min(\pi_2 - F, F) \geq \pi_1,$$

then the investor refuses to lend at date 0 even if B can operate for two periods.

Solution. Consider part (i). Apparently, B will *always* report $\pi_1 < F$ in this one-period setting, and expecting this, investors never want to lend to B in the first place.

Consider part (ii). As in part (i), if the second-period repayment were made dependent on the second-period profit report in a non-trivial way, then B will always report π_1 in the second period, violating B's second-period IC constraint. Thus, the optimal contract requires that R_{ij} be independent of π_j.

Consider part (iii). First it can be proved that the above IC condition must be binding at optimum. Thus,

$$\pi_2 - R_2 + \beta_2(E(\pi) - R^2) = \pi_2 - R_1 + \beta_1(E(\pi) - R^1).$$

Replace this equality into the objective function and note that the objective function becomes strictly increasing in β_2. This means that $\beta_2^* = 1$, which in turn implies that the objective function does not depend on R_2 and R^2 separately; rather, it depends on $R_2 + R^2$ only (and the same is also true for the constraints). Thus, there is no loss to set $R^{2*} = \pi_1$. Also, the objective function is increasing in both R_1 and R^1. Thus, $R_1 = R^1 = \pi_1$, according to LL. Finally, note that the objective function is decreasing in β_1 if and only if

$$\theta F + (1 - \theta)E(\pi) > \pi_1,$$

and in this case, $\beta_1^* = 0$. On the other hand, when

$$\theta F + (1 - \theta)E(\pi) \leq \pi_1,$$

it is optimal to set $\beta_1^* = 1$.

Next, consider part (iv). We need to show that the investor lends F to B at date 0 if and only if

$$F < \frac{(\pi_1 + (1 - \theta)E(\pi))}{2 - \theta}.$$

This follows from the fact that the investor's expected profit is

$$\pi_1 - F + (1-\theta)(E(\pi) - F),$$

which cannot be negative.

Finally, assume in part (v) that

$$\min(\pi_2 - F, F) \geq \pi_1.$$

We need to show that the investor would refuse to lend to B at date 0 even if B can operate for two periods. To see that this is so, note that if the above inequality holds, B will always report π_1, with no concern about whether he will get refinancing. This happens because, by the above assumption, once B gets to operate for one period, B will collect enough money to cover the second-period F. Recognizing this fact, investors will not lend to B in the first place. This is one version of the *free cash flow* problem discussed in Jensen (1986, *American Economic Review*). There, Michael Jensen points out that there may be substantial benefits resulting from a voluntary reduction of a firm's internal funds (by buying shares back, paying dividends, or repaying existing debts).

7. (**Part III.**) In this part, we shall discuss several applications of game theory in stock trading.

   (a) (**Going Public Before Acquisition**) A start-up firm endowed with a highly innovative new product may find it difficult to obtain financing from traditional loan-making commercial banks, but it may get financed by venture capitalists or angel investors, who specialize in screening and fostering high-risk investment projects. Funds are provided in a stage-finance manner, where success in operations in previous stages is necessary for the start-up firm to obtain financing in the next stage. Most start-up firms end up failing in the first several stages, and the very few of them that survive may turn out to be highly profitable. A successful start-up firm tends to attract attention from large established firms,[5]

[5] Google was once considered a takeover target by Yahoo, when Google was still cheap. For some reasons Yahoo passed up the takeover; see for instance https://finance.yahoo.com/news/remember-yahoo-turned-down-1-132805083.html.

and may become a takeover target. Zingale (1995) discusses why a successful start-up firm may benefit from going public before contacting an acquiring firm, and how much ownership the initial entrepreneur should retain when going public.[6] Intuitively, by selling shares to a continuum of small investors, the start-up firm can utilize the free-rider problem to extract most monetary surplus from the acquiring firm, but small investors would always ignore the non-pecuniary private benefits that directing large shareholders possess, and to best extract such benefits from the acquiring firm, the initial entrepreneur must possess an adequate ownership. Consequently, a start-up firm would generally benefit from going public, and there generally exists an optimal percentage of ownership that the initial entrepreneur should continue to possess following the IPO.

(b) (**Coase Theorem and Trade Efficiency**) Before discussing Zingale's story, let us first review the famous Coase Theorem[7] and the free-rider problem that arises when an acquiring firm makes a tender offer to a continuum of small shareholders of a target firm.

Consider a bilateral trading model, where a seller is about to sell a product to a buyer. The seller's reservation value for the product is c, and the buyer's valuation for the product is v. The transaction price is p.

When v and c are the seller and the buyer's common knowledge, and when bargaining is costless, Coase theorem says that trade will happen for sure if $v > c$, with p lying somewhere between v and c. Underlying Coase theorem is the fact that both agents are rational, and they can share the surplus $v - c$ when trade occurs.

When trade does not occur, the seller gets c, and the buyer gets 0. We call these payoffs the two agents' status-quo payoffs. If the two agents have equal bargaining power, as assumed in the Nash bargaining solution, then they will split the transaction surplus (or efficiency gain) $v - c$ equally, so that after bargaining the seller's

[6] Zingales, L., 1995, Insider Ownership and the Decision to Go Public, *Review of Economic Studies*, 62, 425-448.

[7] Coase, R., 1937, The Nature of the Firm, *Economica*, 4, 386-405.

payoff becomes $c + \frac{1}{2}(v - c) = \frac{1}{2}(v + c)$, and the buyer's payoff becomes $0 + \frac{1}{2}(v - c) = \frac{1}{2}(v - c)$. The transaction price to fulfill the Nash bargaining solution is $p = \frac{1}{2}(v + c)$, so that the seller gets p and the buyer gets $v - p$.

The Nash bargaining solution has assumed that the seller and the buyer have the same bargaining power. If the seller has all bargaining power, then he will set $p = v$; and if instead the buyer has all bargaining power, then he will set $p = c$. In any case, under Coase theorem, trade will occur for sure, and so the expected surplus from trade is a fixed amount, and it is in this situation that maximizing one's own share of surplus is the same as minimizing the rival's share of surplus.

Now, what if $v < c$? Then trade efficiency dictates that the seller should keep the product. No trade can take place. When $v = c$, the two parties feel indifferent about to or not to trade, and we shall assume that trade would still take place.

Finally, we should emphasize that under information asymmetry, Coase theorem no longer applies, and there usually is a loss of trade in the above bilateral trading model. For example, suppose that the seller has all bargaining power, so that $\psi = 1$. Under symmetric information, trade would occur with $p = v$, so that the seller's payoff is v and the buyer's payoff is zero. Under asymmetric information, say $c = 0$ for sure and v can take on v_2 with probability π or v_1 with probability $1 - \pi$, where $v_2 > v_1 > c$. When $\pi > \frac{v_1 - c}{v_2 - c}$, it is in the seller's best interest to announce $p = v_2$, but then trade cannot occur when actually $v = v_1$. In fact, it is a well-known result in contract theory that under information asymmetry, attaining full trade efficiency is almost always impossible (Myerson-Satterthwaite Theorem).[8]

(c) (**Tender Offer and the Free-rider Problem.**) In Grossman and Hart (1980),[9] the share value of an all-equity target firm would

[8]Myerson, R., and M. Satterthwaite, 1983, Efficient Mechanisms for Bilateral Trading, *Journal of Economic Theory*, 29, 265-281.

[9]Grossman, S., and O. Hart, (1980), Takeover Bids, The Free-Rider Problem, and the Theory of the Corporation, *Bell Journal of Economics*, 11(1): 42-64.

be v if a takeover succeeds, and it is q in the absence of the takeover attempt, where $v > q$. The target firm has a continuum of small shareholders. In this case, since a single shareholder's decision regarding whether to tender his shares has no impact on the likelihood of success of the raider's takeover attempt, each shareholder is willing to tender his shares if and only if $p \geq v$, where p is the unconditional tender price made by the raider. Now, if the raider obtains no private benefit (that cannot be transferred to the target shareholders) from the takeover, then the raider cannot make a profit by submitting an acceptable bid $p \geq v$. In this case, the takeover cannot succeed even if $v > q$.

Note that the above *free-rider problem* occurs because (i) the target firm's shares are diffusely held; and (ii) the target firm's stockholders and the raider have the same valuation for the target firm.

Grossman and Hart (1980) and the subsequent literature have recommended several ways to get around (i) and (ii) stated above. Shleifer and Vishny (1986)[10] analyze the efficiency role of large shareholders. The presence of a large shareholder can get around (i). To see this, suppose that the target has a single stockholder S. If the raider has all bargaining power against S, then the tender offer with $p = q$ will be accepted by S: the takeover succeeds if and only if S is willing to tender his shares, and given p, tendering is optimal for S. In general, if the raider and S both have some bargaining power, and if v, q are their common knowledge, then efficient takeovers will always succeed. This is an application of a theorem due to Ronald Coase. For example, let S and the raider trade via a double auction, where they must simultaneously submit bids s and b respectively, and trade takes place at price $p = \frac{s+b}{2}$ if and only if $b > s$. In this case, for all $p \in [q, v]$, $b = s = p$ is one equilibrium, and the takeover succeeds in each and every such equilibrium.

Regarding (ii), Grossman and Hart emphasize that a takeover may still succeed in the absence of a large shareholder if the raider is

[10]Shleifer, A., and R. Vishny, (1986a), Large Shareholders and Corporate Control, *Journal of Political Economy*, 94, 461-488.

allowed to obtain some *private benefits* from a successful takeover. A private benefit is by definition a benefit that the raider cannot credibly share with other stockholders. One way to generate private benefits for the raider is to stipulate in the target firm's charter that following a successful takeover the raider is allowed to take some actions that benefit the raider at the expense of the target's remaining shareholders who have refused to tender their shares. This likely *dilution* of minority shareholders' post-takeover benefits reduces the latter's payoff from holding out, and encourages them to tender their shares. Alternatively, without the help from the target firm's charter, the raider may benefit from making a two-tier offer. One obvious drawback with these treatments is that they may also promote the likelihood of success of inefficient takeovers.

(d) (**Zingale's IPO Story.**) Consider a start-up firm run by an initial owner (i), who anticipates correctly that a raider (r) will subsequently show up to propose a takeover bid. The firm will generate cash earnings v^r if the raider gets control, and v^i if the initial owner continues to get control. The party $k \in \{r, i\}$ getting control would obtain a private benefit B^k, which other people can never share or try to take away.

If we regard the initial owner as a seller with reservation value $c = B^i + v^i$ and the raider as a buyer with valuation $v = B^r + v^r$ as in the preceding section, then trade would not take place if $B^i + v^i > B^r + v^r$. Hence we consider only the case where $B^i + v^i \leq B^r + v^r$.

There are three cases to consider.

(i) $B^r < B^i$. In this case, $B^i + v^i \leq B^r + v^r$ implies that there exists a solution $\phi^* \in (0, 1)$ to the following equation (with ϕ being the unknown)

$$B^i + \phi v^i = B^r + \phi v^r \Rightarrow \phi^* = \frac{B^i - B^r}{v^r - v^i},$$

so that the initial owner would optimally give a share $(1 - \phi^*) <$

100% of cash flow rights (but with no voting rights) to small shareholders.

The idea is as follows. Since Coase theorem applies and $B^i + v^i \leq B^r + v^r$, the bargaining game becomes a constant-sum game, and in such a game the initial owner's goal to maximize his own payoff is equivalent to minimizing the raider's payoff.

What is the raider's payoff? By bargaining with the initial owner (who has sold a fraction $(1-\phi)$ of cash flow rights to outside small investors), the raider would get a fixed fraction of the transaction surplus $v - c$ when $v \geq c$ (so that trade would take place) and he would get a zero payoff when $v < c$ (so that no trade happens), where $v \equiv B^r + \phi v^r$ and $c \equiv B^i + \phi v^i$.[11] Thus to minimize the raider's payoff, the initial owner should select ϕ to minimize $v - c$ while making sure that trade would take place (i.e., $v \geq c$). Since $B^i + v^i \leq B^r + v^r$ and $B^r < B^i$, it must be that $v^r > v^i$. Thus to minimize

$$v - c = -(B^i + \phi v^i) + (B^r + \phi v^r),$$

which is increasing in $\phi \in [0, 1]$, while making sure that trade would not lose (i.e., $v \geq c$), the initial owner should optimally choose $\phi^* = \frac{B^i - B^r}{v^r - v^i}$.

(ii) $v^r < v^i$. In this case, to minimize the raider's payoff, the initial owner would again try to reduce $v-c$ while making sure that trade would take place (i.e., $v \geq c$). Since $v^r < v^i$, to minimize

$$v - c = -(B^i + \phi v^i) + (B^r + \phi v^r),$$

which is now decreasing in $\phi \in [0, 1]$, while making sure that trade would not lose, the initial owner should optimally choose $\phi^* = 1$, so that the firm is maintained private in the first stage.

(iii) $v^r \geq v^i$, $B^r \geq B^i$. In this case, to minimize the raider's payoff, the initial owner would again try to reduce $v-c$ while making sure

[11] And because of the assumed free-rider problem, the raider would get zero additional payoffs if trying to buy more shares from small shareholders.

that trade would take place (i.e., $v \geq c$). To minimize

$$v - c = -(B^i + \phi v^i) + (B^r + \phi v^r),$$

which is increasing in $\phi \in [0,1]$, while making sure that trade would not lose (i.e., $v \geq c$), the initial owner should optimally choose $\phi^* = 0$, so that the initial owner would hold all voting rights but no cash flow rights.

Note that the equation

$$B^i + \phi v^i = B^r + \phi v^r$$

has a solution ϕ contained in $[0,1]$ only in case (i). Hence with $0 < \psi < 1$ the initial owner can fully extract the raider's surplus only in case (i).

(e) (**Optimal Design of a Tender Offer When Target Firm Has a Finite Number of Large Shareholders.**) We have shown above that when the target firm has a continuum of small shareholders the free-rider problem is so severe that the raider can only keep his private benefit at the end of a tender offer.

Holmström and Nalebuff (1992)[12] show that following a successful tender offer a raider typically possesses 5 to 15 percent ownership and does not always enjoy large private benefits. They show that when the target-firm shareholders are no longer infinitesimal a raider can typically keep 50% or more of the takeover gain.

Using Zingales' notation, let us suppose that $\phi = 0 = B^i = B^r = v^i = 0$, and $v^r = 1$. What would happen if the raider attempts a takeover with an un-conditional tender offer v, where $0 < v < 1$?

Suppose that the target firm has N shareholders, each holding $\frac{1}{N}$ of the firm's ownership (or eash holding one share). The raider must acquire K shares to get control. Suppose that the raider has

[12]Holmström, B., and B. Nalebuff, 1992, To the Raider Goes the Surplus? A Re-examination of the Free-rider Problem, *Journal of Economics & Management Strategy*, 1, 1, 37-62.

announced a bid $v \in (0, 1)$ (for the entire firm). What would be the outcome of this tender offer game?

When N is very large, as pointed out by the authors, asymmetric equilibria are very unconvincing: people hardly know one another, but in equilibrium they need to forecast correctly who would be those players doing one thing, and who would be those players doing another thing. In particular, when players are anonymous, it is natural to focus on symmetric equilibrium. In a symmetric equilibrium, every small shareholder chooses to tender his share with probability p. It is immediate that $0 < p < 1$ (why?), and hence we can only have a symmetric mixed-strategy equilibrium. In such an equilibrium, a shareholder gets $\frac{v}{N}$ by tendering his share, and hence by not tendering his share, he must expect a probability v of success of the takeover attempt (so that he would get an expected payoff of $\frac{1}{N} \times v$).
For example, suppose that $N = 3$, $K = 2$, and $v = \frac{1}{2}$. In this case, by not tendering his own share, he expects that the takeover attempt may succeed with probability p^2, so that

$$p^2 = v = \frac{1}{2} \Rightarrow p = \frac{\sqrt{2}}{2}.$$

The takeover attempt may succeed with probability

$$3p^2(1-p) + p^3 = \frac{3-\sqrt{2}}{2} \sim 0.8,$$

so that it generates a total surplus of

$$(3p^2(1-p) + p^3) \cdot (v^r - v^i) = \frac{3-\sqrt{2}}{2},$$

from which the three target-firm shareholders obtain $N \cdot \frac{v}{N} = v = \frac{1}{2}$, and hence the raider ends up with a surplus of

$$\frac{2-\sqrt{2}}{2}.$$

In general, in the symmetric mixed-strategy equilibrium, by not tendering his share(s), a single shareholder must believe that the other $N-1$ shareholders would altogether tender at least K shares with probability v. This allows the authors to show that, given v, or given the p induced by v, the raider would obtain a surplus of

$$\begin{pmatrix} N-1 \\ \\ K-1 \end{pmatrix} p^K(1-p)^{N-K}.$$

The authors then go on to show that it is in the raider's best interest to offer

$$v = \frac{K}{N}.$$

(Thus it is optimal for the raider to offer $v = \frac{2}{3}$ in the above numerical example.) With the optimal bidding policy, the raider would obtain a surplus of

$$\begin{pmatrix} N-1 \\ \\ K-1 \end{pmatrix} (\frac{K}{N})^K(\frac{N-K}{N})^{N-K}.$$

Now, if 50% majority is needed for control (i.e., $\frac{K}{N} = \frac{1}{2}$), then the authors show that the raider's surplus is about 4% when $N = 100$ and 0.4% when $N = 1000$. As $N \uparrow \infty$, it approached zero, as it becomes less and less likely that a single shareholder would become pivotal. This gives a rationale for Grossman and Hart's doctrine that with a continuum of small shareholders, the raider would obtain no surplus.

A surprising (but actually intuitive) result is that the raider would become better off if the target firm has imposed a super-majority rule (a seemingly anti-takeover action). In fact, it is in the raider's best interest if the target firm sets $K = N$: every share held by a target-firm shareholder now becomes pivotal, and hence all shareholders are willing to tender at any $v > 0$.

(f) (**A Numerical Example.**) An acquirer (Mr. A) is attempting to take over a target firm, T. Firm T is all-equity financed, and it has three shareholders, each holding 1 share of the firm's equity (so that the firm has 3 shares of common stock outstanding). The current value of firm T is zero (as a normalization). If Mr. A is able to obtain 2 or more than 2 shares, then the takeover will succeed, and the value of firm T will become 18 in that event.

Suppose that Mr. A has announced a share price $p = 3$ to the three shareholders of firm T, saying that Mr. A is willing to buy as many shares from them as possible at the price p. The three shareholders must simultaneously and independently decide whether to sell his share to Mr. A at the price p.

We look at an equilibrium where each and every target shareholder may sell his share to Mr. A (or tender his share) with probability π, and compute the likelihood that Mr. A's takeover attempt may succeed. Then we compute Mr. A's best choice of p.

Analysis. We claim that this game has no pure-strategy NE's. Indeed, if $\pi = 0$ in the symmetric NE, then the takeover attempt fails for sure, but then a target shareholder's equilibrium payoff would be zero, while he can deviate unilaterally by selling his share and obtain a payoff of $p = 3$, which is a contradiction.

Similarly, if $\pi = 1$, then the takeover would succeed for sure, and a target shareholder's equilibrium payoff would be $p = 3$, while he can deviate unilaterally and obtain a payoff of 6 by keeping his share till the takeover is completed, which is another contradiction!

We conclude that a symmetric NE must involve each target shareholder using a mixed strategy; i.e., we must have $0 < \pi < 1$.

In this symmetric NE, a target shareholder can get $p = 3$ by tendering his share for sure, and for him to feel indifferent about tendering and not tendering his share, he must believe that without tendering his own share, the takeover attempt may still succeed with probability $\frac{p}{\frac{18}{3}} = \frac{1}{2}$. Recall that for the takeover attempt to succeed, Mr. A must obtain at least 2 shares. Thus, without tendering his own share, a target shareholder must believe that the takeover attempt may still succeed with probability π^2. Thus we have

$$\frac{1}{2} = \pi^2 \Rightarrow \pi = \frac{\sqrt{2}}{2}.$$

It follows that the probability that Mr. A's takeover attempt may succeed is

$$\text{prob.(at least two target shareholders would tender shares)}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \pi^2(1-\pi)^1 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \pi^3(1-\pi)^0$$

$$= 3\pi^2(1-\pi) + \pi^3 = \frac{3-\sqrt{2}}{2},$$

so that Mr. A's takeover attempt generates a total surplus of

$$[3\pi^2(1-\pi) + \pi^3] \cdot (18-0) = 27 - 9\sqrt{2},$$

from which the three target shareholders together take away $3p = 9$,[13] and hence Mr. A ends up with an expected profit of

$$27 - 9\sqrt{2} - 9 = 18 - 9\sqrt{2}.$$

Now, let us derive the optimal p for Mr. A. Note that

$$p = 6\pi^2$$

and Mr. A's expected profit from offering the share price $p = 6\pi^2$ is

$$(18-0)[3\pi^2(1-\pi) + \pi^3] - 3p = 36f(\pi),$$

where

$$f(x) = x^2(1-x).$$

[13] Note that from a target shareholder's perspective, tendering and not tendering his own share are both equilibrium best responses. Thus his equilibrium payoff equals the payoff of tendering his share, which is $p = 3$.

It is easy to verify that

$$f''(x) \leq 0 \Leftrightarrow x \geq \frac{1}{3};$$

$$f'(x) > 0 \Leftrightarrow 0 < x < \frac{2}{3};$$

$$f'(x) = 0 \Leftrightarrow x \in \{0, \frac{2}{3}\};$$

and

$$f(0) = f(1) = 0, \ f(\frac{2}{3}) = \frac{4}{27}.$$

Thus $f(x)$ has a unique maximum at $\frac{2}{3}$ over the unit interval $[0,1]$. Mr. A would optimally offer the share price

$$p^* = 6 \times (\frac{2}{3})^2 = \frac{8}{3},$$

which generates for Mr. A the expected profit of $36f(\frac{2}{3}) = \frac{16}{3} > 18 - 9\sqrt{2}$. The likelihood that the takeover attempt may succeed is $\frac{20}{27}$.

(g) (**The Bid-Ask Spread for a Stock Traded in an Over-the-counter Market.**) The true value $\tilde{v}$ of a traded stock is uniformly distributed over the unit interval $[0,1]$. Two market makers first announce bid and ask prices for absorbing one-share sell and buy orders simultaneously, and then a public trader will be selected to trade with market makers. It is equally likely that the public trader may either be an informed speculator that alone has observed the realization of $\tilde{v}$, or a liquidity trader who must buy or sell one share with equal probability for personal reasons. In stock market equilibrium, a market maker must make zero expected profits when absorbing a sell or buy order. We shall compute the equilibrium ask and bid prices.

(h) **Analysis.** Let μ be the likelihood that the public trader may turn out to be an insider (rather than a liquidity trader). Observe that from market makers' perspective, the probability that the selected public trader is a liquidity trader wishing to sell at the bid price B is $(1-\mu)\cdot\frac{1}{2}$; and the probability that he is instead a informed speculator wishing to sell at the bid price B is $\mu\cdot$ Pro.$(\tilde{v} \leq B)$. In order that market makers earn zero expected profits by posting B, it is necessary that

$$B = E[v|\text{The public trader agrees to sell at } B].$$

Expanding, we have

$$B = \frac{B}{2} \cdot \frac{\mu B}{(1-\mu)\frac{1}{2} + \mu B} + \frac{1}{2}\frac{(1-\mu)\frac{1}{2}}{(1-\mu)\frac{1}{2} + \mu B}.$$

Following a similar procedure, we can obtain

$$A = \frac{-\sqrt{(1+\mu)^2 - 2\mu(1+\mu)} + (1+\mu)}{2\mu}.$$

Now, we can obtain A and B by plugging in $\mu = \frac{1}{2}$, and we have

$$A = \frac{3-\sqrt{3}}{2}, \quad B = \frac{\sqrt{3}-1}{2}.$$

The bid-ask spread is equal to $2-\sqrt{3}$, which is positive because the market makers must make profits when trading with a liquidity trader in order to break even—they always incur losses when trading with an informed trader!

(i) (**Comparing Transaction Costs of Stock Trading Over Three Trading Mechanisms.**) The following example is taken from Biais, Foucault, and Salanié (1998).[14]

There are two liquidity suppliers (whom we shall refer to as dealers for simplicity) and a buyer that want to trade a stock in this

[14] Biais, B., T. Foucault, and F. Salanié, 1998, Floors, dealer markets and limit order markets, *Journal of Financial Markets*, 1, 253-284.

example. The buyer submits to a trading platform a market order to buy 4 shares of the stock. The buyer's payoff is $-\infty$ if he fails to buy exactly 4 shares; and his payoff is $-\sum_{j=1}^{2} t_j$ if he succeeds in buying the 4 shares by paying t_j to dealer j. Dealer j has payoff $t_j - V(q_j)$ when he sells q_j shares and receives a payment t_j. It is assumed that

q	1	2	3	4
$V_1(q)$	10	21	33	56
$V_2(q)$	12	25	39	64

If the Walrasian trading mechanism is adopted by the trading platform, as we have assumed in the preceding lectures, then it ensures that markets clear in equilibrium with each dealer's marginal rate of substitution between the stock and cash being equal to the price ratio of the stock to cash, so that

$$\frac{V_1'(q_1)}{1} = \frac{p}{1} = \frac{V_2'(q_2)}{1},$$

and

$$q_1 + q_2 = 4,$$

where cash is taken to be the numeraire. Thus we have $p = 12$, $q_1 = 3$ and $q_2 = 1$.

In reality, trading platforms do not use the Walrasian mechanism. Compare the following three mechanisms:

- In a floor market, the platform first announces the buyer's market order, and then the dealers must simultaneously announce share prices p_1 and p_2, saying that dealer j is willing to accept *any* number of shares $q_j \in [0, 4]$ that the platform subsequently asks him to deliver to the buyer at the price p_j.
- In a limit-order market, or an electronic communication network, the platform first announces the buyer's market order,

and then the dealers must simultaneously submit individual supply curves, where a supply curve indicates that, after having sold n shares, dealer j is willing to sell one additional share at some price $p_j(n+1) \in \Re_+ \bigcup\{+\infty\}$, with $p_j(n+1) \geq p_j(n)$ for all $n = 0, 1, 2, 3$.

- In an over-the-counter dealer market, the platform first announces the buyer's market order, and then the dealers must simultaneously announce menus of options that the buyer can select from, where a dealer's menu may consist of several bundles like: $\{(T_1, q_1), (T_2, q_2), \cdots\}$, from which a buyer must pay the dealer a total of T_k dollars and purchase q_k shares if he picks the option (T_k, q_k). Here, unlike in the limit-order market, the dealer may commit to selling at a lower price for one additional share.

In all three mechanisms described above, the trading platform will try to minimize the buyer's expenditure (so that price priority is enforced), and when ties occur, the platform is assumed to adopt a prorata rationing rule, and try to maintain equal sales for the two dealers.

In this example, dealers have the greatest freedom in choosing trading strategies if the platform is an over-the-counter dealer market.[15] The floor market offers dealers the least freedom in choosing trading strategies. It turns out that these two mechanisms are both conducive to tacit collusion between dealers, making the buyer suffer from high transaction costs.

We shall explain in the lecture how the following equilibria are

[15] Observe the different assumptions made in the preceding section and here. In the preceding section, when we derive the equilibrium bid and ask prices, we assume that dealers are faced with an adverse selection problem; that is, the buyer (or seller) may possess privileged inside information. Here the buyer is sure to be an uninformed liquidity trader. Moreover, dealers in the preceding section are risk-neutral, and here they are risk averse, which makes $V_j(q)$ a convex function of q. Finally, here the buyer can split his order between the two dealers, but in the preceding section, exactly one dealer will be chosen to absorb the buyer's order. In reality, dealers are faced with information asymmetry (as in the preceding section), but they can offer a client complicated trading choices (as assumed here), and a buyer can indeed split orders among multiple dealers (also as assumed here).

sustained under the different trading mechanisms.

- In the floor market, there is one equilibrium[16] where both dealers announce the share price 14, and the platform asks each dealer to deliver 2 shares at this price. Comparing this equilibrium to the Walrasian efficient outcome, we see that the buyer must pay 56 rather than 48 for the 4 shares, and the allocation of the stock between the dealers does not attain efficient risk sharing.
- In the limit-order market, there is a unique[17] equilibrium where dealer 1 submits a single limit order $(p_1, q_1) = (12, 12)$ and dealer 2 submits a single limit order $(p_2, q_2) = (12, 4)$. In this equilibrium, the aggregate supply is 16 at the price 12, and the prorara rationing rule applies, so that dealer 1 gets to sell 3 shares and dealer 2 1 share. As we can see, this outcome coincides with the Walrasian equilibrium outcome.
- In the over-the-counter dealer market, there is one equilibrium where dealer 1 offers two trading options $\{(12, 1), (57, 4)\}$ and dealer 2 offers two trading options $\{(45, 3), (57, 4)\}$. Take dealer 1 for example. Given dealer 2's strategy, which says that the buyer either must give up trading with dealer 2, or buy 3 shares from dealer 2 at a total price of 45, or buy the entire 4 shares at 57, dealer 1 is left with 3 choices: either

[16] An equilibrium is a vector of trading strategies, one for each dealer, such that given the trading strategy that the vector prescribes for the other dealer j, dealer i finds the strategy that the vector prescribes for him already a best choice. This is defined as a Nash equilibrium in non-cooperative game theory.

[17] Let us briefly explain the equilibrium uniqueness. In equilibrium, any two executed limit orders must have expressed the same price. Indeed, if (p_i, q_i) and (p_j, q_j) are both executed, and yet $p_i < p_j$, then raising p_i to $p_i + \epsilon$ would still ensure execution, which raises the payoff for the dealer submitting (p_i, q_i). Now, if all executed limit orders are executed at the price p, then a dealer j can submit $(p - \epsilon, q_j(p - \epsilon))$ to gain price priority and get sure execution, where $q(z)$ is the unique solution to $z = U_j'(x)$ with x being the unknown. To sustain the equilibrium, no dealer should find this a profitable move. Thus we reach the conclusion that in equilibrium, it must be that all executed limit orders have expressed the same price p, and dealer i has already expressed that he is willing to sell $q_i(p)$ shares at the price p. This means that in any limit-order market equilibrium, where dealer j sells q_j shares, it must be that $U_1'(q_1) = p = U_2'(q_2)$ and $q_1 + q_2 = 4$, proving that (q_1, q_2, p) is nothing but the Walrasian equilibrium outcome!

he can sell 4 shares to the buyer (when the buyer chooses to give up dealer 2), or he can sell 1 share to the buyer (when the buyer chooses to buy 3 shares from dealer 2), or he can sell nothing (when the buyer chooses to buy 4 shares from dealer 2). Given dealer 2 promises to offer the entire 4 shares at a total price of 57, if dealer 1 wishes to sell 1 share to the buyer, his price p_1 must satisfy $p_1 + 45 \leq 57$, so that his best choice is 12, explaining why he offers $(12, 1)$ in his own menu. In equilibrium, the platform executes the two options $(12, 1)$ and $(45, 3)$, so that dealer 1 sells 1 share and dealer 2 sells 3 shares, and the buyer's expenditure is 57.

To conclude, we see that in the absence of information asymmetry, the limit-order market best guards the buyer's interest, and it ensures allocative efficiency for the two dealers. The other two trading mechanisms tend to raise the buyer's trading costs, and fail to attain allocative efficiency for the dealers.

8. (**Part IV.**) **A Reputation Game with Entre Deterrence**.

Consider the following reputation game about a Cournot-competitive industry that extends for n periods. In each period t, the price of the homogeneous product (referred to as product X) supplied by all the firms is $P_t = A - Q_t$, where $A > n+1$ is a positive constant, and Q_t is the sum of supply quantities chosen by the firms.

This industry has an incumbent firm, I, and n potential entrants, E_j, $j = 1, 2, \cdots, n$. There is no discounting, and each firm seeks to maximize (the sum of) expected profits.

At $t = i \in \{1, 2, \cdots, n\}$, E_i can decide whether to spend a one-time cost $k_i \equiv k(n+1-i)$ to enter the industry, where $k > 0$ is a constant. Once it enters, it can costless supply 1 unit of product X at each period $t = i, i+1, \cdots, n$. Let m_i denote the number of entrants among E_1, E_2, $\cdots$, E_i which are operating at $t = i$. The incumbent firm's unit cost is $\tilde{c}$, and at $t = i$, *all* entrants believe that $\tilde{c}$ may take on 1 with probability x_i or 0 with probability $1 - x_i$. (Bayesian updating is applied whenever possible.) The timing of the relevant events is as follows.

- At $t = i$, before E_i enters, E_i can observe whether entry has occurred at an earlier point in time, and the supply quantities chosen by all the firms operating at that point in time. However, $\tilde{c}$ and the incumbent firm's past profits remain unobservable to E_i.
- Then, E_i must decide whether to spend k_i and enter the industry or stay out and get zero payoffs.
- Then, given m_i all the firms operating at $t = i$ must make output decisions simultaneously, where $m_0 = 0$, and for $i \geq 1$, $m_i = m_{i-1}$ if E_i stays out and $m_i = m_{i-1} + 1$ if E_i enters.
- Then P_t is realized at $t = i$ and the date-t profits accrue to the firms. Then the game ends if $i = n$; or else the game moves on to $t = i + 1$.

We shall assume that $n = 3$, $A > 4$, and $A - 1 < 2k < A$. Notice that $k_1 = 3k$, $k_2 = 2k$, and $k_3 = k$. The outcome of $\tilde{c}$ will be referred to as the incumbent's *type*.

(i) First consider $t = 3$. Given $m_3 \in \{0, 1, 2, 3\}$, the date-t supply quantity chosen by the type-1 incumbent is ___A___, and given x_3 and m_3, the expected date-t product price is ___B___. Thus E_3 enters if and only if $m_3 =$ ___C___ and x_3 satisfies the *weak* inequality (write it down!) ___D___.[18]

(ii) Now, consider $t = 2$. Suppose first that E_1 has entered at $t = 1$. In this case we can get m_2, so that the type-0 incumbent's date-2 output quantity plus the type-1 incumbent's date-2 output quantity must be equal to ___E___.

(iii) Continue with $t = 2$. Now, suppose that E_1 did not enter at $t = 1$. In this latter case, we can get m_2 also, and show that E_2 would stay

[18] Thus we are making the tie-breaking assumption that E_i would enter when feeling indifferent about entering or staying out.

out if and only if x_2 satisfies a *strict* inquality, and when E_2 does stay out, the type-1 incumbent's profit at $t = 2$ is equal to ___F___.

(iv) Now, consider $t = 1$. If E_1 has entered, then there is a (answer 'pooling' or 'separating') ___G___ PBE, where the type-1 incumbent's sum of expected profits over the date-1-date-3 period is equal to ___H___. Thus E_1 would stay out if and only if x_1 satisfies the *strict* inquality (write it down!) ___I___, and following that, the type-1 incumbent's sum of expected profits over the date-1-date-3 period is equal to ___J___.

Solution. We shall solve the PBE using backward induction, and we shall record our findings as a series of lemmas along the way.

Since once entering the industry, an entrant can supply 1 unit without incurring any costs, and since $A > n + 1$ (which implies that the product price is never negative), the optimal choice of output quantity in any operating period for such an entrant is exactly 1 unit.

Lemma 0. *The sum of output quantities supplied by the entrants operating at date t is m_t.*

Now observe that the type-0 incumbent has no concerns for reputation.

Lemma 1. Given m_t, the type-0 incumbent's date-t output choice is $\frac{A-m_t}{2}$.

Now we solve the PBE of the above reputation game using backward induction.

First consider the date-3 subgame where E_3 has just made its entry decision. Since this is the last period of the game, the incumbent has no reputation concern any more. By **Lemma 0**, given m_3 and $\tilde{c}$, the incumbent would seek to

$$\max_q q(A - m_3 - q - \tilde{c})$$

so that the type-$\tilde{c}$ incumbent's date-3 output choice is

$$q(\tilde{c}) = \frac{A - m_3 - \tilde{c}}{2}.$$

It follows that in state $(m_3, \tilde{c})$, the realized date-3 product price is

$$P_3(\tilde{c}) = A - m_3 - q(\tilde{c}) = \frac{A - m_3 + \tilde{c}}{2},$$

and hence given (m_3, x_3), E_3 expects its post-entry expected profit to be

$$1 \cdot [x_3 \times P_3(1) + (1 - x_3) \times P_1(0)] = \frac{A - m_3 + x_3}{2}.$$

Thus E_3 will enter in equilibrium if and only if, by our tie-breaking assumption,

$$\frac{A - m_3 + x_3}{2} \geq k \Leftrightarrow x_3 \geq 2k + m_3 - A,$$

where note that with E_3's entry we have $m_3 \geq 1$. Note that if E_3 enters and yet $m_3 \geq 2$, then

$$1 \geq x_3 \geq 2 + 2k - A > 1,$$

which is a contradiction. Thus we conclude that E_3 would enter in equilibrium if and only if $m_3 = 1$ and x_3 satisfies

$$x_3 \geq 1 + 2k - A.$$

Lemma 2. *If $m_2 \geq 1$ so that either E_1 or E_2 has already entered prior to date 3, then we have $m_3 = m_2$; and in the opposite case, we have $1 \geq m_3 \geq m_2 = 0$, so that $m_3 = 1$ if and only if $x_3 \geq 1 + 2k - A$.*

Now, consider the date-2 subgame where E_2 has just entered the industry.

Since $1 \geq m_3 \geq m_2 \geq 1$, the incumbent knows that by **Lemma 2** $m_3 = m_2$ and E_3 would never enter, so that the type-$\tilde{c}$ incumbent's date-3 profit, according to part (i), will be

$$(\frac{A - m_2 - \tilde{c}}{2})^2,$$

which is independent of the incumbent's choice of date-2 output quantity. Thus there is a separating date-2 equilibrium, with the type-$\tilde{c}$ incumbent's date-2 output choice being

$$\frac{A - m_2 - \tilde{c}}{2}.$$

It follows that in state $(m_2, \tilde{c})$, the realized date-2 product price is

$$P_2(\tilde{c}) = \frac{A - m_2 + \tilde{c}}{2},$$

so that before making its entry decision, E_2 would expect its post-entry profit at date 2 (and at date 3 also, why?) to be

$$1 \cdot [x_2 \times P(1) + (1 - x_2) \times P(0)] = \frac{A - m_2 + x_2}{2}.$$

Note that E_2 would not deviate and stay out if and only if

$$2 \times \frac{A - m_2 + x_2}{2} \geq 2k \Leftrightarrow x_2 \geq 2k + m_2 - A,$$

implying that $m_2 = 1$. Thus we conclude that there exists a PBE at the date-2 subgame where E_2 enters for sure if and only if E_1 did not enter at date 1 *and* if $x_2 \geq 1 + 2k - A$.

Next, consider the date-2 subgame where E_2 has just decided to stay out. Then $m_2 = m_1 = 1$ if E_1 entered at date 1 and $m_2 = m_1 = 0$ if E_1 did not.

In the former case, by **Lemma 2** E_3 would never enter, so that the type-$\tilde{c}$ incumbent's date-3 profit, according to part (i), will be independent of the incumbent's choice of date-2 output quantity. Thus there is again a separating date-2 equilibrium where the expected date-2 product price is

$$\frac{A-1+x_2}{2},$$

and we must check that E_2 indeed would not deviate and make entry: following a deviation the expected date-2 product price would become

$$\frac{A-2+x_2}{2},$$

and we must require that

$$2\times\frac{A-2+x_2}{2}<2k\Leftrightarrow x_2<2+2k-A,$$

but the last inequality holds always! Thus if E_1 has entered at date 1, there is a separating PBE at date 2 where E_2 does not enter, and following that the type-$\tilde{c}$ incumbent would choose the output quantity

$$\frac{A-1-\tilde{c}}{2}$$

at both date 2 and date 3.

Now, consider the latter case, where E_1 and E_2 have both chosen to stay out. Can there be a separating PBE at this point, where the two types of the incumbent choose different output quantities? In such an equilibrium, the type-1 incumbent would expect E_3 to enter at date 3 after seeing its date-2 output choice, which differs from the type-0 incumbent's output choice $\frac{A}{2}$; recall **Lemma 2**. Thus in this supposed separating PBE, the type-1 incumbent would choose the output quantity $\frac{A-1}{2}$, yielding for the type-1 incumbent the continuation payoff $\frac{(A-1)^2+(A-2)^2}{4}$. If the type-1 incumbent deviates and chooses $\frac{A}{2}$ instead, then it would get the date-2 payoff

$$\frac{A}{2}\times(A-\frac{A}{2}-1)=\frac{A(A-2)}{4},$$

but this would lead to $x_3=0$ and $m_3=0$, so that the type-1 incumbent would get

$$\frac{A-1}{2} \times (A - \frac{A-1}{2} - 1) = \frac{(A-1)^2}{4}.$$

Thus the type-1 incumbent would surely want to deviate! This proves that there cannot be a separating PBE.

Can there be a pooling PBE for the latter case, where both types of the incumbent produce $\frac{A}{2}$ units at date 2? Note that a deviation will be taken as evidence that the deviator is the type-1 incumbent, so that the optimal deviating output choice for the type-1 incumbent is $\frac{A-1}{2}$. In this PBE, we have $x_3 = x_2$, and if $x_2 \geq 1 + 2k - A$, then by **Lemma 2** E_3 would enter even though no deviation at date 2 is detected, which would then induce the type-1 incumbent to strictly prefer producing $\frac{A-1}{2}$ units instead of $\frac{A}{2}$ units. Thus for such a PBE to prevail at date 2, it is necessary that $x_2 < 1 - 2k + A$. When this inequality does hold, the type-1 incumbent would get $\frac{A(A-2)}{4}$ at date 2 and $\frac{(A-1)^2}{4}$ at date 3 in equilibrium, and he would get $\frac{(A-1)^2}{4}$ at date 2 and $\frac{(A-2)^2}{4}$ at date 3 after a deviation. Thus this pooling PBE does exist given that $x_2 < 1 - 2k + A$.

Lemma 4. The date-2 equilibrium given (m_1, x_2) is as follows.

- If $m_1 = 0$ and $x_2 \geq 1 + 2k - A$, then E_2 would enter for sure, leading to $m_2 = 1$, and following that there is a *separating* date-2 equilibrium, with the type-$\tilde{c}$ incumbent's date-2 and date-3 common output choice being

$$\frac{A - 1 - \tilde{c}}{2}.$$

- If $m_1 = 0$ and $x_2 < 1 - 2k + A$, then E_2 would stay out for sure, leading to $m_2 = 0$, and following that there is a *pooling* date-2 equilibrium, with $\frac{A}{2}$ being the equilibrium date-2 output choice for both types of the incumbent, and upon seeing this date-2 output choice E_3 would stay out for sure. The type-$\tilde{c}$ incumbent would then produce $\frac{A-\tilde{c}}{2}$ units at date 3.

- If $m_1 = 1$, then regardless of x_2, E_2 would stay out for sure, leading to $m_2 = 1$, and following that there is a *separating* date-2 equilibrium, with the type-$\tilde{c}$ incumbent's date-2 and date-3 common output choice being
$$\frac{A-1-\tilde{c}}{2}.$$

Now, consider the date-1 subgame where E_1 has just entered, so that $m_1 = 1$, and by **Lemma 4** and **Lemma 2**, E_2 and E_3 would both stay out for sure. We claim that following entry by E_1, there is a separating PBE, where the type-1 incumbent gets

$$3 \times \frac{(A-2)^2}{4}.$$

To see this, note that if the type-1 incumbent deviates and produces $\frac{A-1}{2}$ at date 1, then it would choose exactly the same output quantity at dates 2 and 3, just like deviation never occurs; recall the last statement in **Lemma 4**. Thus following E_1's entry, this separating PBE exists always! It follows that there is a date-1 equilibrium where E_1 would enter for sure if and only if $x_1 \geq 1 + 2k - A$.

Finally, consider the date-1 subgame where E_1 has just chosen to stay out, so that $m_1 = 0$. We claim that there is no separating equilibrium at date 1. If there were, then $x_2 = 1$ after the type-1 incumbent makes the equilibriu date-1 output choice, and by **Lemma 4** E_2 and E_3 would enter at date 2 and stay out at date 3 respectively. The type-1 incumbent's payoff in this supposed equilibrium would be

$$\frac{A(A-2)}{2} + 2 \times \frac{(A-2)^2}{4}.$$

By deviating and choosing the output $\frac{A}{2}$ at date 1, the type-1 incumbent can ensure that $x_2 = 0$, so that by **Lemma 4** E_2 would stay out for sure, and following that the type-1 incumbent can again choose $\frac{A}{2}$ as its date-2 output to ensure that E_3 would stay out for sure; the type-1 incumbent would then produce $\frac{A-1}{2}$ units at date 3. Thus with a series of deviations, the type-1 incumbent can get the payoff

$$2 \times \frac{A(A-2)}{2} + \frac{(A-1)^2}{4},$$

showing that the deviation payoff is higher!

Now, can there be a pooling equilibrium following E_1's staying out? Note that if $x_1 \geq 1 + 2k - A$, then upon seeing the incumbent's date-1 outpout choice $\frac{A}{2}$ in the pooling equilibrium, by **Lemma 4** E_1 would enter, and following that there would be a date-2 separating outcome. It is clear that the type-1 incumbent had better deviate at date 1 in this case!

Thus we focus on the case where $x_1 < 1 + 2k - A$. By **Lemma 4**, following the date-1 pooling choice of output, E_2 would stay out, and following that there is again a date-2 pooling equilibrium that induces E_3 to also stay out. Thus in this pooling PBE the type-1 incumbent gets the equilibrium payoff

$$2 \times \frac{A(A-2)}{4} + \frac{(A-1)^2}{4},$$

whereas after choosing the date-1 output $\frac{A-1}{2}$ during a deviation, by **Lemma 4**, the type-1 incumbent would expect both E_2 and E_3 to enter, so that its deviation payoff is

$$\frac{(A-1)^2}{4} + 2 \times \frac{(A-2)^2}{4}.$$

Clearly, no deviation would occur.

Lemma 5. The date-1 equilibrium depends on x_1.

- If $x_1 < 1+2k-A$, then there is pooling at date 1 and date 2, and all three entrants would stay out.
- If $x_1 \geq 1 + 2k - A$, then there is separating at date 1, and only E_1 enters in equilibrium.

The type-1 incumbent would always pool with the type-0 incumbent as long as no entrants have ever entered before. The type-1 incumbent would instead distinguish itself from the type-0 incumbent following the first occurrence of entry. Note that the condition

$$0 < 1 + 2k - A < 1$$

says that under full information E_1's decision is to enter if and only if the incumbent is of type 1. Given x_1, the incumbent's expected output quantity following entry of E_1 is at least x_1, and it is exactly equal to x_1 if following entry of E_1 the type-1 incumbent would rather distinguish itself from the type-0 incumbent, which is exactly what would happen given $x_1 \geq 1 + 2k - A$. Thus E_1 would stay out if and only if $x_1 < 1 + 2k - A$.

9. (**Part V.**) **Nash Implementation**.

10. A set of I players are facing uncertain states of nature contained in the sample space Θ, and they must make a collective choice for each state $\theta \in \Theta$. Let A denote the set of feasible social choices, @ and a a typical element of A. Suppose that each player $i \in \{1, 2, \cdots, I\}$ can separately see the realized θ, but θ is not verifiable in the court of law, and hence is not contractible. Moreover, for each θ, there exists a subset $f(\theta) \subset A$ which consists of all the social choices acceptable to a central planner. The central planner, who is not one of the I players, would like to design a game form for the I players, such that the set of Nash equilibrium outcomes of that game form in state θ coincides with $f(\theta)$. (Here we assume that participating in the game form is mandatory for each player, so that we do not need to impose an IR condition.) We shall refer to $f(\cdot)$ as a social choice rule (SCR) or a social choice correspondence (SCC).

    A game form is a pair (g, S), where S is the common strategy space for each and every player, and $g : S^I \Rightarrow A$ maps each strategy profile $s \equiv (s_1, s_2, \cdots, s_I)$ into a social choice. Note that player i's payoff function is θ-dependent, and hence the game form (g, S), failing to fully describe the payoff function for each player, is not a normal form game.

However, for each $\theta \in \Theta$, (g, S, θ) is indeed a normal-form game. Let $s^*(\theta)$ be one pure-strategy Nash equilibrium of the game (g, S, θ). Let $E_g(\theta)$ be the set of all pure-strategy Nash equilibria in state θ. Then, define $g(E_g(\theta)) \equiv \{g(s^*) : s^* \in E_g(\theta)\}$ as the set of equilibrium social choices. We say that (g, S) *fully implements* f in Nash equilibrium if and only if $g(E_g(\theta)) = f(\theta)$ for all $\theta \in \Theta$. We say that f is Nash implementable if we can find at least one game form (g, S) that fully implements f.

11. (**Example 5.**) Two women, Amy and Beth, carry one baby to the king, and each of them claims to be the mother of the baby. There are two possible states: the mother is either Amy (state α) or Beth (β). Thus let $\Theta = \{\alpha, \beta\}$. For the king, there are 4 feasible actions: to give the baby to Amy (a); to give it to Beth (b); to cut the baby in half and let each woman take one half (c); or to let both women and the baby die (d). Can you find a game form to fully implement the social choice rule f satisfying $f(\alpha) = a$ and $f(\beta) = b$?

12. **Theorem N1.** (Maskin 1977; Maskin 1999) f is Nash implementable only if f is monotonic.[19]

Proof. Define the lower contour set at a for agent i in state θ by

$$L^i(a, \theta) \equiv \{b \in A : aR^i(\theta)b\}.$$

We shall prove Theorem N1 by contraposition. Recall from section 17 that f is not monotonic if and only if there exist $\theta, \phi \in \Theta$ and $a \in A$ such that for all $i = 1, 2, \cdots, I$,

$$L^i(a, \theta) \subset L^i(a, \phi),$$

and yet $a \in f(\theta) \setminus f(\phi)$.[20] We show that in this case no game forms (g, S) can fully implement f in Nash strategy.

[19] Maskin, E., 1977, Nash Equilibrium and Welfare Optimality, MIT working paper. Maskin, E., 1999, Nash Equilibrium and Welfare Optimality, *Review of Economic Studies*, 66, 23-38.

[20] An equivalent definition for f being monotonic is this: for any $\theta, \phi \in \Theta$ and $a \in A$ with $a \in f(\theta) \setminus f(\phi)$, there must exist agent i and some $b \in A$ such that $b \in L^i(a, \theta) \setminus L^i(a, \phi)$, or such that $aR^i(\theta)b$ but $bP^i(\phi)a$.

Suppose instead that there were such a game form (g, S). Then there exists some Nash equilibrium s^* for the game (g, S, θ) such that $g(s^*) = a \in f(\theta)$. That is, for agent i, given his rival agents would play s^*_{-i} in state θ,

$$a = g(s^*_i, s^*_{-i})R^i(\theta)g(s_i, s^*_{-i}),\ \forall s_i \in S^i,$$
$$\Rightarrow g(s_i, s^*_{-i}) \in L^i(a, \theta) \subset L^i(a, \phi),\ \forall s_i \in S^i,$$

but then s^*_i continues to be agent i's best response against s^*_{-i} in state ϕ! As this is true for all agents i, we conclude that s^* would also arise as a pure-strategy Nash equilibrium in state ϕ. But then $a \in g(E_g(\phi)) \setminus f(\phi)$, showing that (g, S) does not fully implement f in Nash equilibrium.

13. To state our next result, we introduce the notion of no veto power. An SCC f satisfies (weak) no veto power if for all $i \in \{1, 2, \cdots, I\}$, for all $\theta \in \Theta$, and for all $a \in A$,

$$L^j(a, \theta) = A, \forall j \neq i \Rightarrow a \in f(\theta).$$

In words, if a is top ranked by all agents $j \neq i$ in state θ, then $a \in f(\theta)$ whether agent i likes a or not. (Agent i has no veto power!)

14. **Theorem N2.** (Maskin 1977; Repullo 1987[21]) Suppose that $I \geq 3$, and that f is monotonic and satisfies no veto power. Then f is Nash implementable.

Proof. The proof is by construction of a canonical game form (g, S) which fully implements f. Define for all i, $S^i = \Theta \times A \times \mathbf{Z}_+$, where $\mathbf{Z}_+$ denotes the set of positive integers, and define $g : S \to A$ as follows:

(a) If s is such that there exists $i \in \{1, 2, \cdots, I\}$ such that $s_i = (\eta, a_i, k_i)$ and for all $j \neq i$, $s_j = (\theta, a, k)$ with $a \in f(\theta)$, then

$$g(s) = \begin{cases} a_i, & \text{if } a_i \in L^i(a, \theta); \\ a, & \text{otherwise.} \end{cases}$$

[21] Repullo, R., 1987, A Simple Proof of Maskin's Theorem on Nash Implementation, *Social Choice and Welfare*, 4, 39-41.

(b) If s is such that (a) does not apply, then $g(s) = a_i$ where i is an agent announcing the highest k_i, with ties being broken by selecting among the agents announcing the highest k_i the person with the smallest i.

We shall show first that $f(\theta) \subset g(E_g(\theta))$ for all $\theta \in \Theta$, and then that $g(E_g(\theta)) \subset f(\theta)$ for all $\theta \in \Theta$.

- $f(\theta) \subset g(E_g(\theta))$ for all $\theta \in \Theta$.

  Given any $a \in f(\theta)$, define for all i, $s_i = (\theta, a, 1)$. Then s is such that (a) holds, and if agent i alone would like to deviate and to implement another a_i, he must choose some $a_i \in L^i(a, \theta)$, and hence he has no incentive to make unilateral deviations. Thus $a \in g(E_g(\theta))$, and this being true for all $\theta \in \Theta$ and for all $a \in f(\theta)$, we conclude that $f(\theta) \subset g(E_g(\theta))$ for all $\theta \in \Theta$.
- $g(E_g(\theta)) \subset f(\theta)$ for all $\theta \in \Theta$

  Let $s \in E_g(\theta)$, and we shall show that $g(s) \in f(\theta)$. Suppose θ is the true state. We take cases.
  - Suppose that s is such that $s_i = (\eta, a, k)$ $\forall i \in \{1, 2, \cdots, I\}$, with $a \in f(\eta)$, so that $g(s) = a$.

    For all $i \in \{1, 2, \cdots, I\}$, if agent i wishes to deviate unilaterally from s, then according to (a) above, agent i must announce some $s'_i = (\phi, a_i, k_i)$ with $a_i \in L^i(a, \eta)$. Since s is a Nash equilibrium in the true state θ, agent i weakly prefers the equilibrium outcome $a = g(s)$ to a_i in the true state θ, and this implies that

    $$a_i \in L^i(a, \eta) \Rightarrow a_i \in L^i(a, \theta),$$

    and this being true for all $i \in \{1, 2, \cdots, I\}$, we conclude that $a \in f(\theta)$ since f is monotonic.
  - Suppose that s is such that $s_i = (\eta, a, k)$ $\forall i \in \{1, 2, \cdots, I\}$, with $a \notin f(\eta)$.

In this case, by (b), any agent i can deviate and announce $s_i' = (\phi, a_i, k')$, where $k' > k$, so that the outcome a_i rather than $g(s)$ would be implemented. Since s is a Nash equilibrium in the true state θ, it must be that

$$g(s)R^i(\theta)a_i, \ \forall a_i \in A,$$

or equivalently,

$$L^i(g(s), \theta) = A,$$

and with this being true for each single agent i, we conclude that $g(s) \in f(\theta)$ by the fact that f satisfies no veto power.

– Suppose that s is such that there exist $i \neq j$, $s_i \neq s_j$.

In this case, thanks to the fact that $I \geq 3$, some agent $h \notin \{i, j\}$ can implement any $a_h \in A$ by announcing an integer k_h exceeding k_n for all $n \neq h$. Since s is a Nash equilibrium in the true state θ, it must be that

$$L^h(g(s), \theta) = A, \ \forall h \notin \{i, j\}.$$

Moreover, it is impossible that $s_h = s_i$ and $s_h = s_j$, simply because $s_i \neq s_j$. Suppose that $s_h \neq s_i$. Then we can repeat the above argument and conclude that

$$L^j(g(s), \theta) = A.$$

It follows that $g(s)$ is top ranked in state θ by all agents $n \neq i$, so that $g(s) \in f(\theta)$ by the fact that f satisfies no veto power.

References

1. Bolton, P., and D. Scharfstein, 1990, A Theory of Predation Based on Agency Problems in Financial Contracting, *American Economic Review*, 80, 93-106.

2. Brander, J., and T. Lewis, 1986, Oligopoly and financial structure: the limited liability effect, *American Economic Review*, 76, 956-970.

3. Gale, D., and M. Hellwig, 1985, Incentive-compatible debt contracts: the one-period problem, *Review of Economic Studies*, 52, 647-663.

4. Glazer, J., 1994, The strategic effects of long-term debt in imperfect competition, *Journal of Economic Theory*, 62, 428-443.

5. Jensen, M., and W. Meckling, 1976, Theory of the firm: managerial behavior, agency cost and ownership structure, *Journal of Financial Economics*, 3, 305-360.