Game Theory with Applications to Finance and Marketing

Some Examples

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1. Two decision-makers (or players) make decisions simultaneously. Let X be the set of decisions (or pure strategies) available for player 1, and likewise Y for player 2. Let $u_i(x,y)$ be player i's payoff when player 1's decision is x and player 2's decision is y. The above descriptions—the players, the players' feasible strategies, and the players' payoffs as functions of their choices of strategies—define a game in normal form. A pure-strategy Nash equilibrium (NE) for is game is a pair (x^*, y^*) such that $x^* \in X$, $y^* \in Y$, and

$$u_1(x^*, y^*) \geq u_1(x, y^*), \quad \forall x \in X, \tag{1}$$

$$u_2(x^*, y^*) \geq u_2(x^*, y), \quad \forall y \in Y. \tag{2}$$

   In plain words, an equilibrium is a pair of strategies (x^*, y^*), such that given player 2's strategy is y^*, x^* is player 1's best choice in X; and given player 1's strategy is x^*, y^* is player 2's best choice in Y.

2. A *mixed strategy* for player 1 in the above two-player normal-form game is a probability distribution f over X, and a mixed strategy for player 2 is a probability distribution g over Y. A mixed-strategy Nash equilibrium is a pair (f, g) such that f is player 1's best response against g, and g is player 2's best response against f.

3. (**Multiple Equilibria.**) Consider a two-player game, where the two must pick an integer from the set $\{1, 2, \cdots, 100\}$ at the same time. If they pick the same number, then they each get 1; or else, they each get zero. Find all the pure strategy NE's. Find all the mixed strategy NE's.

Solution. If (n_1, n_2) is a pure-strategy equilibrium, then necessarily $n_1 = n_2$, which is also a sufficient condition. Thus we have 100 distinct pure-strategy equilibria.

To characterize the mixed-strategy equilibria, we proceed in 3 steps.

(1) There can never be an equilibrium where one player is using a pure strategy and the other is using a genuine mixed strategy.

(ii) In a mixed-strategy equilibrium both players must be randomizing over the same set of integers.

(iii) All integers in the set that both players are randomizing over must be assigned with equal probability.

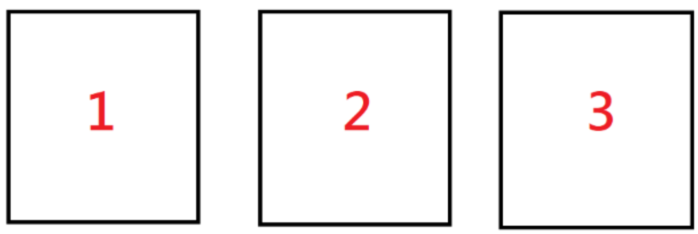
For example, it is a mixed-strategy equilibrium where both players choose 1,2, or 3 with probability $\frac{1}{3}$; and it is another mixed-strategy equilibrium where both players choose each feasible integer with probability $\frac{1}{100}$.

4. (**Guessing Hair Color.**) A village has three residents, and they all know that a resident's hair can be either red (R) or black (B). A resident can see the color of each neighbor's hair, but does not know the color of his own hair. The three residents are not allowed to communicate in any way. They must meet (quietly) for 1 hour at 9am each day, trying to figure out the color of his own hair. Suppose that a resident that figured out the color of his own hair by the evening of date t would be allowed (by Trump, unlikely?) to immigrate to the USA in the evening of date t. The three residents all wish to immigrate to the USA as early as they can. Suppose that exactly one of the three residents has black hair.
(A) Suppose that the three residents' first meeting is at date 1. When would a red head get to immigrate to the USA?
(B) Suppose that at date $n \geq 1$, an honest person passed through the village at 9:20am (whose honesty is well known to the residents), and he told the three residents during their daily meeting that *at least one of them has red hair*.

(i) At which date would a red head get to immigrate to the USA?
(ii) At which date would the black head get to immigrate to the USA?

Solution. For (A), the answer is never. For (B), the answer is, for part (i), date $n+1$; and for part (ii), date $n+2$.

5. (**The Three-box Quiz.**) Consider the following TV game show, where three boxes are presented to a guest (G) by the host (H). G understands that H knows which of the three boxes contains a prize even before the show begins. The show proceeds in 3 steps as follows. (Step 1.) G would have to choose one box. (Step 2.) Then H would open another box for G, and if the opened box contains the prize, then the prize is given to G; or else, (Step 3.) G can choose to or not to swap the box that G chose in Step 1 with the box that neither G nor H has touched.



Now, suppose that the show has finished Step 2, and H did not open a box containing the prize. Should G make the swap in Step 3?

(i) First assume that H would like to give the prize to G whenever possible (type a).
(ii) Next assume instead that H would prevent G from getting the prize whenever possible (type b).

(iii) Now, suppose that G believes that H may be type a for probability α. Then G should make the swap if and only if $\alpha < \alpha^*$, where $\alpha^* =$?

Solution. Note that G is faced with two sources of exogenous uncertainty: G does not know which box contains the prize, and G does not know H's type. These two random events are stochastically independent. Let the box picked by G be labeled box 1, and the box opened by H box 2. The remaining box is labeled box 3. Given that the prize is not in box 2, G holds the following beliefs:

- With probability $\frac{\alpha}{2}$ it may happen that box 3 contains the prize and H is of type a, and in that event H would have opened box 3 directly, so that the probability that box 2 was opened is zero;
- With probability $\frac{1-\alpha}{2}$ it may happen that box 3 contains the prize and H is of type b, and in that event H has no choice but to open box 2, so that the probability that box 2 was opened is one;
- With probability $\frac{1}{2}$ it may happen that box 1 contains the prize, and in that event H would have opened box 3 or box 2 with equal probability, regardless of H's type.

Thus from G's perspective, given that the prize is not in box 2, the probability that H may open box 2 is equal to

$$\frac{\alpha}{2} \cdot 0 + \frac{1-\alpha}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2},$$

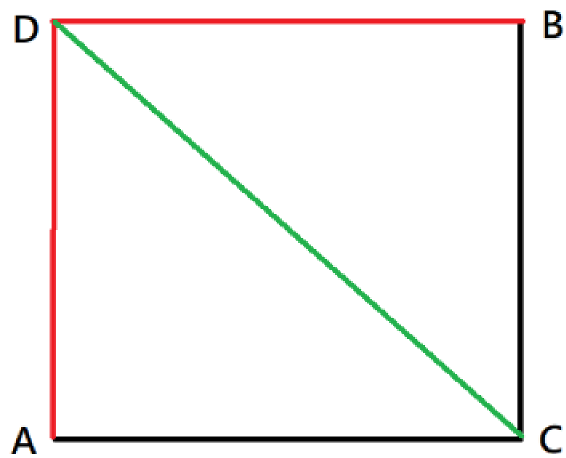
so that by Bayes Law, G believes that box 1 may contain the prize with probability

$$\frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{\alpha}{2} \cdot 0 + \frac{1-\alpha}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{3-2\alpha},$$

and G should make the swap if and only if this probability is less than $\frac{1}{2}$, or α is less than $\alpha^* = \frac{1}{2}$.

6. (**Equilibrium Inefficiency from Seemingly Efficient Network Expansion.**) On $\Re^2$ space, define 4 points as follows: $A \equiv (0,0)$, $C \equiv (0,1)$, $D \equiv (1,0)$, $B \equiv (1,1)$. There are 4000 cars trying to get from A to B at the same time, and a car driver's payoff is $-y$ if he spends y minutes to get from A to B. They simultaneously choose between the two routes below:

$$A \to C \to B \quad \text{or} \quad A \to D \to B.$$



- If there are x cars on the road $A \to C$, then each car must spend $\frac{x}{100}$ minutes to get from A to C.
- If there are x cars on the road $D \to B$, then each car must spend $\frac{x}{100}$ minutes to get from D to B.
- Regardless of the number of cars on the road $C \to B$, each car must spend 45 minutes to get from C to B.
- Regardless of the number of cars on the road $A \to D$, each car must spend 45 minutes to get from A to D.

(i) In equilibrium each driver's payoff is ___a____, and there are ___b____ cars taking the route $A \to C \to B$.

(ii) Now, suppose a new highway has been built to connect C and D. Regardless of the number of cars on this new road between C and D, each car can get from C to D or D to C in *zero* minutes. All drivers must simultaneously choose between the 4 routes. (Which 4?) In equilibrium each driver's payoff is ___c____, and there are ___d____ cars travelling through the road $A \to D$.

Solution. Consider part (i). Suppose first that there are x car drivers taking the route $A \to C \to B$, where $x > 2000$. Thus there are $4000-x$ cars taking the route $A \to D \to B$. But then each among the x car drivers taking the route $A \to C \to B$ would have an equilibrium payoff of $-(\frac{x}{100}+45)$, where $\frac{x}{100}+45 > 65 \geq \frac{4000-x+1}{100}+45$, so that he would become better off by taking the route $A \to D \to B$ instead. The case where $x < 2000$ is similar. Thus in the unique NE, we have $x = 2000$, so that each car driver's equilibrium payoff is $-(\frac{2000}{100}+45) = -65$.

Now, consider part (ii). There are 4 routes for each driver to choose from:

$$A \to C \to B,$$

or

$$A \to D \to B,$$

or

$$A \to C \to D \to B,$$

or

$$A \to D \to C \to B.$$

Recall that in the presence of the new highway connecting C and D, a driver can freely switch from C to D or from D to C. The following observations are useful:

(a) For a car driver that has already reached C, the route $C \to B$ is *strictly dominated* by the route $C \to D \to B$: even if all car drivers are taking the route $D \to B$, each of them would spend only 40 minutes, whereas it would take 45 minutes to finish the route $C \to B$.

(b) Similarly, for a car driver that has already reached D, the route $D \to C \to B$ is *strictly dominated* by the route $D \to B$.

(c) Given that all car drivers reaching C or D would subsequently take the route $D \to B$, the route $A \to D$ is *strictly dominated* by the route $A \to C$!

Thus we conclude that all car drivers would take the same route $A \rightarrow C \rightarrow D \rightarrow B$, so that each car driver's equilibrium payoff is $-\frac{4000}{100} - \frac{4000}{100} = -80$.

Remark. This example demonstrates the Braess's Paradox, which says that an extension of a road network may cause a redistribution of the traffic that results in longer individual running times; see Braess, D., A. Nagurney, and T. Wakolbinger, 2005, On a Paradox of Traffic Planning, *Transportation Science*, 39, 4, 446-450.

What happens here is that without the new highway connecting C and D, a car driver must choose between the two bundles below,

$$A \rightarrow C \text{ plus } \ C \rightarrow B$$

and

$$A \rightarrow D \text{ plus } \ D \rightarrow B,$$

and self-interested behavior by all car drivers causes them to balance perfectly between these two choices in equilibrium. In the presence of the new highway connecting C and D, the above two bundles are both broken up. A car driver is free to create his own bundle. In the latter case, each of them prefers the component $D \rightarrow B$ to the component $C \rightarrow B$, and for this reason, each of them also prefers the component $A \rightarrow C$ to the component $A \rightarrow D$. With all drivers taking the same route, each and every one of them becomes worse off.

For more on network economics, see for example Easley, D., and J. Kleinberg, 2010, *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*, Cambridge University Press.

7. (**Omni-channel Marketing and Dual-channel Strategy.**) A bricks-and-mortar store F can produce and sell product X costlessly to 2 consumers A and B, where A knows everything about internet, while B has no knowledge about it. F cannot distinguish A from B, but F knows that A (respectively, B) is willing to pay 2 (respectively, 5) dollars for 1 unit of X, and they both have unit demand for X.
(i) First suppose that e-commerce is unavailable. F first announces price p, and given p, A and B decide whether to buy X from F. What is p in equilibrium? What is F's profit?

(ii) Now suppose instead that F can first spend $t > 0$ and set up an online outlet, and if t is spent, then F would announce the online price q and the offline price p to A and B. A can then decide where (at the online or offline outlet) to buy X, but B can only buy from F's offline store. Should F spend t, and what is F's equilibrium prices?
(iii) Re-consider (ii) but assuming that B's reservation value is 3.9.
(iv) Would a higher demand would make it more likely that t is spent?

Solution. For part (i), F should set p at 5 dollars, which is also F's profit. For part (ii), if t is spent, then F should announce $p = 5$ and $q = 2$, so that F would gain $q - t$ by spending t. Thus F should spend t if and only if $t < 2$.[1] For part (iii), F's profit is 4 dollars if giving up the online outlet, while selling through both online and offline outlets would yield a profit of $3.9+2-t$, and hence F should sell through both channels if $t < 1.9$.[2]

Note that the assumption that B has a higher reservation value than A does is crucial in the above analysis. If the reservation values are reversed, F cannot price discriminate between A and B.

Finally, for part (iv), note that F's incentive to spend t may be reduced when A's reservation value increases: if A is already served without the internet, then the benefit from spending t is equal to the difference in A's and B's reservation values, and this difference *decreases* with A's reservation value.

For example, let $t = 1.85$ and B's valuation for X be 3.9. Suppose that A's valuation for X now increases from $v_A = 2$ to $v_A = 2.1$. If $v_A = 2$, then F would get $3.9 - 2 - t > 0$ by spending t; but F would get $3.9 - 2.1 - t < 0$ instead, if $v_A = 2.1$. Thus an increase in demand

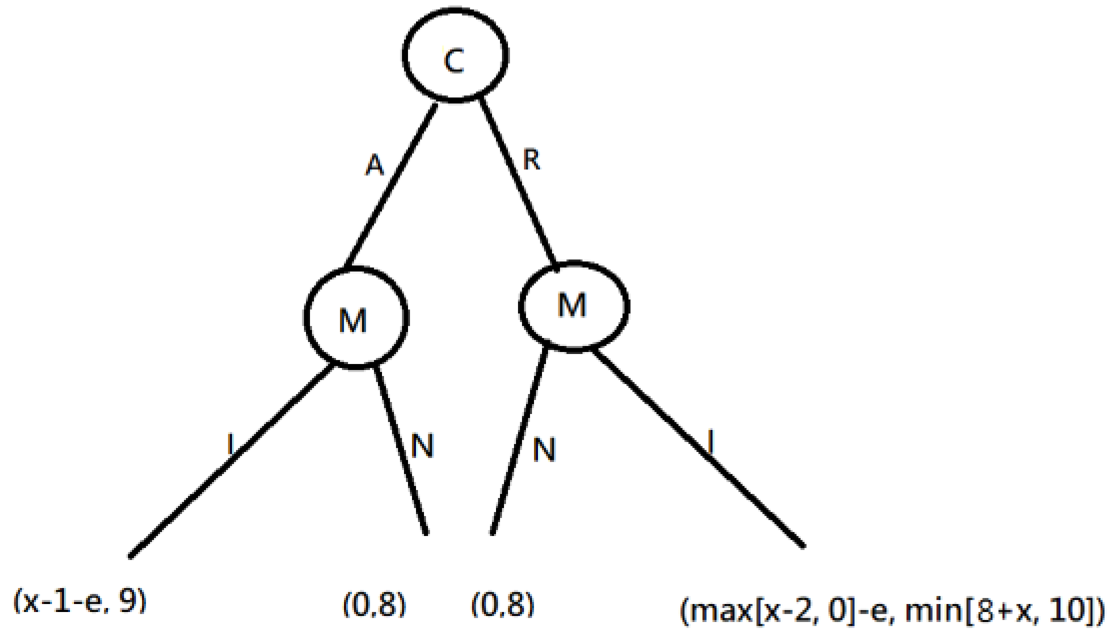
[1] Note that A is not served without on-line markets, and t is the cost that F incurs in order to extract the 2 dollars from A in the presence of on-line markets.

[2] Note that in (iii) selling to A online would create a loss: F would charge A slightly less than 2 dollars, but F has to spend $t > 0$. Spending t can still be beneficial because F can then charge B 3.9 instead of 2 dollars. Why? With A being directed to the on-line markets, B is identified as the only segment left to be served at the original store, and hence F can fully extract the consumer surplus from B. The extra 1.9=3.9-2 dollars that F can make from serving B must be greater than the loss incurred when F moves A from the offline outlet to the online outlet. Thus F should take this *dual channels* strategy if and only if $t < 1.9$.

(on A's part), other things equal, may *reduce* F's incentive to create an online outlet.

By contrast, an increase in B's demand does weakly encourage F to spend t: it does not affect F's preference about spending t if F would choose not to serve A in the absence of the internet, but it weakly increases F's benefit from spending t if F would choose to serve A in the absence of the internet.

8. (**Debt Renegotiation and Control Right.**) M is the owner-manager of a firm which is protected by limited liability against its creditor(s). The debt due one year from now has a face value equal to \$10. There is a single debtholder, referred to as C. The total assets in place are worth only \$8 in one year. Just now, a new investment opportunity with NPV=$x > 1+e \geq 1$ became available, which requires that M make an unobservable effort but no addition investment. Making the effort would incur a disutility $e \geq 0$ to M. M has told C that he will make the effort for the new investment project only if C agrees to reduce the face value of debt by \$1. The extensive game proceeds as follows. First C can accept (A) or reject (R) M's request. Then, M can choose to (I) or not to (N) make the effort. Both M and C are risk-neutral without time preferences.



(i) Suppose $x > 2 + e$. Show that there is an NE in which the creditor agrees to reduce the face value of debt and M makes the investment.

(ii) Show that the NE in (i) is not an SPNE. Find an SPNE.

(iii) How may your conclusion about (ii) change if $1 + e < x \leq 2 + e$?

(iv) Define bankruptcy as a state where the firm's equity value drops to zero. Explain why bankruptcy does not take place in (iii).

(v) Verify that the equilibrium firm value is increasing in x, but the equilibrium equity value may not.

Solution. Note that M can choose one action following A and another action following R. Hence C has 2 pure strategies, A and R, but M has

4 pure strategies

$$\begin{pmatrix} A \to I \\ R \to I \end{pmatrix},$$

$$\begin{pmatrix} A \to N \\ R \to N \end{pmatrix},$$

$$\begin{pmatrix} A \to I \\ R \to N \end{pmatrix},$$

and

$$\begin{pmatrix} A \to N \\ R \to I \end{pmatrix}.$$

The normal-form bimatrix is as follows.

M/C	A	R
$\begin{pmatrix} A \to I \\ R \to I \end{pmatrix}$	$(x-1-e, 9)$	$(\max(x-2,0)-e, \min(8+x,10))$
$\begin{pmatrix} A \to N \\ R \to N \end{pmatrix}$	$(0, 8)$	$(0, 8)$
$\begin{pmatrix} A \to I \\ R \to N \end{pmatrix}$	$(x-1-e, 9)$	$(0, 8)$
$\begin{pmatrix} A \to N \\ R \to I \end{pmatrix}$	$(0, 8)$	$(\max(x-2,0)-e, \min(8+x,10))$

In part (i), the strategy profile

$$(\begin{pmatrix} A \to I \\ R \to N \end{pmatrix}, A)$$

is indeed a pure strategy Nash equilibrium. However, it is not an SPNE: given that C has chosen R, M would be better off choosing I over N. Things are different in part (iii), where the above strategy profile becomes an SPNE.

For part (iv), as we explained in class, M has the control right before debt maturity, which is the reason that the firm has a positive equity value in the SPNE. For example, assume that $e = 0$ and $x = 1.95$. Even though $x + 8 < 10$, the equilibrium equity value equals $0.95 > 0$.

For part (v), assume that $e = 0$ and compare the case with $x = 2.01$ to the case with $x = 1.95$. Since $x > 0$, the new investment project will be undertaken in an SPNE (Coase Theorem!). Undoubtedly, the firm value increases in x. However, the equity value is 0.01 in the former case, but 0.95 in the latter case.

9. (**Tender Offer in a Hostile Takeover.**) A raider (Mr. A) attempts to take over a target firm, T. Firm T is all-equity financed, and it has three shareholders, each holding 1 share of the firm's equity (so that the firm has 3 shares of common stock outstanding). The current value of firm T is zero. If Mr. A is able to obtain 2 or 3 shares, then the takeover will succeed, and the value of firm T will become 18.

   Suppose that Mr. A has announced that he is willing to buy at price $p \in (0, 6)$ as many shares as possible from the three shareholders of firm T. Now, the three shareholders must simultaneously and independently decide whether to sell his share to Mr. A at the price p.

   We shall focus on symmetric equilibria where each and every target shareholder may sell his share to Mr. A (or tender his share) with the same probability π.

   (i) Characterize the symmetric NEs when $p = 3$.

   (ii) What is the best choice of p from the raider's perspective?

   Solution. Consider part (i). We claim that this game has no symmetric pure-strategy NEs. Indeed, if $\pi = 0$ in a symmetric NE, then the takeover attempt fails for sure, but then a target shareholder's equilibrium payoff would be zero, while he can deviate unilaterally by selling his share and obtain a payoff of $p = 3$, which is a contradiction.

   Similarly, if $\pi = 1$ in a symmetric NE, then the takeover would succeed for sure, and a target shareholder's equilibrium payoff would be $p = 3$, while he can deviate unilaterally and obtain a payoff of 6 by keeping his share till the takeover is completed, which is another contradiction!

We conclude that a symmetric NE must involve each target shareholder using a mixed strategy; i.e., we must have $0 < \pi < 1$.

In this symmetric NE, a target shareholder can get $p = 3$ by tendering his share for sure, and for him to feel indifferent about tendering and not tendering his share, he must believe that without tendering his own share, the takeover attempt may still succeed with probability $\frac{p}{\frac{18}{3}} = \frac{1}{2}$. Recall that for the takeover attempt to succeed, Mr. A must obtain at least 2 shares. Thus, without tendering his own share, a target shareholder must believe that the takeover attempt may still succeed with probability π^2. Thus we have

$$\frac{1}{2} = \pi^2 \Rightarrow \pi = \frac{\sqrt{2}}{2}.$$

It follows that the probability that Mr. A's takeover attempt may succeed is

$$\begin{aligned}
&\text{prob.(at least two target shareholders would tender shares)} \\
&= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \pi^2(1-\pi)^1 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \pi^3(1-\pi)^0 \\
&= 3\pi^2(1-\pi) + \pi^3 = \frac{3-\sqrt{2}}{2},
\end{aligned}$$

so that Mr. A's takeover attempt generates a total surplus of

$$[3\pi^2(1-\pi) + \pi^3] \cdot (18 - 0) = 27 - 9\sqrt{2},$$

from which the three target shareholders together take away $3p = 9$,[3] and hence Mr. A ends up with an expected profit of

$$27 - 9\sqrt{2} - 9 = 18 - 9\sqrt{2}.$$

[3]Note that from a target shareholder's perspective, tendering and not tendering his own share are both equilibrium best responses. Thus his equilibrium payoff equals the payoff of tendering his share, which is $p = 3$.

Now, consider part (ii). Note that given p, π must satisfy

$$p = 6\pi^2$$

and Mr. A's expected profit from offering the share price $p = 6\pi^2$ is

$$(18 - 0)[3\pi^2(1-\pi) + \pi^3] - 3p = 36f(\pi),$$

where

$$f(x) = x^2(1-x).$$

It is easy to verify that

$$f''(x) \leq 0 \Leftrightarrow x \geq \frac{1}{3};$$

$$f'(x) > 0 \Leftrightarrow 0 < x < \frac{2}{3};$$

$$f'(x) = 0 \Leftrightarrow x \in \{0, \frac{2}{3}\};$$

and

$$f(0) = f(1) = 0, \ f(\frac{2}{3}) = \frac{4}{27}.$$

Thus $f(\pi)$ has a unique maximum at $\pi = \frac{2}{3}$ over the unit interval $[0, 1]$. It follows that Mr. A should optimally offer the share price

$$p^* = 6 \times (\frac{2}{3})^2 = \frac{8}{3},$$

which generates for Mr. A the expected profit of $36f(\frac{2}{3}) = \frac{16}{3} > 18 - 9\sqrt{2}$. The likelihood that the takeover attempt may succeed with $p = \frac{8}{3}$ is $\frac{20}{27}$.

10. (**Debt Financing and Bargaining Power.**) Here is a bargaining example: One seller and one buyer meet to trade an object for which buyer's willingness to pay (WTP) is equally likely to be 4 or 7. The seller's reservation value (RV) is 0 and the seller can make a take-it-or-leave-it offer p to the buyer.

    The seller is in debt. After trading with buyer, the seller must repay D to his creditors; in default the seller would lose everything, including the object currently on sale if the seller fails to complete a trade with the buyer. The seller is risk-neutral and protected by limited liability.

The seller prefers the low-risk offer $p = 4$ if he is all-equity financed, but he would offer $p = 7$ and run the risk of being rejected by the buyer if D lies between 4 and 7. This example shows that debt financing reduces trading volume and raises the transaction price when the seller has all bargaining power against the buyer.

What if it is the buyer that has all bargaining power?

Suppose that the buyer's WTP is 7 but the seller's RV is equally likely to be 4 or 5. Then, facing an all-equity seller, the buyer would offer p=5.

When D lies between 4 and 5, the buyer would make an offer slightly greater than D! Intuitively, the seller with RV= 5 would reject this offer when $D = 0$, but he has no choice but to accept the offer to avoid default when he is in debt.

This example shows that when the buyer has all bargaining power, debt financing may reduce the transaction price without affecting trading volume.

11. (**Insider Trading and the Bid-Ask Spread.**) Let us modify the example in the preceding section by assuming that $\tilde{v}$ may take on 0 or 1 with equal probability, and that the public investor may be an insider with probability $\frac{1}{3}$ or a liquidity trader with probability $\frac{2}{3}$. Moreover, if the public investor is a liquidity trader, then he may want to buy or sell with equal probability, and conditional on him wanting to buy or sell, he may have to trade 1 share with probability $\frac{1}{4}$ or to trade 2 shares with probability $\frac{3}{4}$.

    The trade proceeds as follows. The two market makers must simultaneously announce their bid and ask prices $\{B_1, B_2, A_1, A_2\}$, where subscripts denote the different numbers of shares that the public investor may want to trade. Then the public investor learns whether he is an insider or he is a liquidity trader. If he is an insider, then he knows whether the stock is worth 1 dollar or nothing; and if he is a liquidity trader, then he knows whether he must buy or sell, and for how many shares.

Bertrand competition between the market makers ensures that they must both make zero expected profits in equilibrium. That is, if trade occurs at B_1, for example, then B_1 must equal the expected value of the stock conditional on trade occurring at B_1. Moreover, since an uninformed trader may want to trade either 1 share or 2 shares, the market makers had better refuse to absorb orders of other quantities. This implies that the insider can only submit 1-share or 2-share market orders also. It is clear that the insider should submit a buy order if he finds out that the stock is worth 1 dollar and a sell order if otherwise.

Let us now compute the equilibrium bid and ask prices for 1-share and 2-share orders.

First let us conjecture that the insider only wants to trade 2 shares in equilibrium. Given this conjecture, a 1-share sell order or buy order can only be submitted by the uninformed investor. Now, B_1 must equal the expected value of the stock, conditional on the arrival of a 1-share sell order, and A_1 must equal the expected value of the stock, conditional on the arrival of a 1-share buy order. Hence we have

$$B_1 = A_1 = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

On the other hand, a 2-share sell order may arrive at the stock market with probability

$$\frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4} = \frac{5}{12}.$$

Conditional on a 2-share sell order arriving at the stock market, it may have been submitted by the insider with probability

$$\frac{\frac{1}{3} \times \frac{1}{2}}{\frac{5}{12}} = \frac{2}{5},$$

and it may have been submitted by the uninformed investor with probability

$$\frac{\frac{2}{3} \times \frac{1}{2} \times \frac{3}{4}}{\frac{5}{12}} = \frac{3}{5}.$$

Now, B_2 must equal the expected value of the stock, conditional on the arrival of a 2-share sell order. Thus we have

$$B_2 = \frac{2}{5} \times 0 + \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}.$$

Similarly, we can compute A_2, which must equal

$$\frac{\frac{1}{3} \times \frac{1}{2} \times 1 + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4} \times \frac{1}{2}}{\frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4}} = \frac{7}{10}.$$

Now, if we conjectured the insider's behavior correctly, then facing these bid and ask prices the insider must indeed prefer 2-share orders to 1-share orders, and hence we must have

$$\frac{6}{10} = 2 \times [1 - A_2] > 1 \times [1 - A_1] = \frac{5}{10},$$

and

$$\frac{6}{10} = 2 \times [B_2 - 0] > 1 \times [B_1 - 0] = \frac{5}{10}.$$

Thus our conjecture is correct indeed!

The above example demonstrates how the bid-ask spreads are determined under a quote-driven mechanism like the one adopted by Nasdaq or LSE dealers (who must also compete with ECNs now). Note that in equilibrium we have

$$A_2 - B_2 = \frac{4}{10} > 0 = A_1 - B_1;$$

that is, *the equilibrium bid-ask spread increases with the trading volume.*

The above model can be extended into a model that allows a sequence of stock trading. For example, suppose that the market opens twice, and in each period the market makers face a public trader that may be insider with probability $\frac{1}{3}$ or an uninformed liquidity trader with probability $\frac{2}{3}$. The insiders that may appear in the two periods behave identically and the liquidity traders that may appear in the two periods have identically and independently trading needs.

Now, suppose that in period 1 trade has taken place at $B_2(1) = \frac{3}{10}$. At the beginning of period 2, the market makers must post new bid and ask prices for 1-share and 2-share orders. Using the same reasoning as above, we can deduce that

$$B_1(2; B_2(1)) = A_1(2; B_2(1)) = B_2(1) = \frac{3}{10},$$

$$B_2(2; B_2(1)) = \frac{\frac{1}{3} \times [1 - B_2(1)] \times 0 + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4} \times B_2(1)}{\frac{1}{3} \times [1 - B_2(1)] + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4}} = \frac{9}{58},$$

and

$$A_2(2; B_2(1)) = \frac{\frac{1}{3} \times B_2(1) \times 1 + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4} \times B_2(1)}{\frac{1}{3} \times B_2(1) + \frac{2}{3} \times \frac{1}{2} \times \frac{3}{4}} = \frac{7}{14} = \frac{1}{2},$$

if we conjecture that the period-2 insider will also submit only 2-share orders. Here note that $B_2(2; B_2(1)) = \frac{9}{58} < \frac{3}{10} = B_2(1)$; that is, *the market becomes even more pessimistic when facing a sequence of sell orders than just one sell order*!

Let us verify that the period-2 insider would indeed trade 2 shares when facing the above bid and ask prices. By buying 1 share given good news, the period-2 insider would get

$$1 \cdot [1 - \frac{3}{10}] = \frac{7}{10};$$

and the insider would get

$$2 \cdot [1 - \frac{7}{14}] = 1$$

if buying 2 shares. So, it indeed is more profitable to buy 2 shares. On the other hand, by selling 1 share given bad news, the period-2 insider would get

$$1 \cdot [\frac{3}{10} - 0] = \frac{3}{10} = \frac{9}{30};$$

and by selling 2 shares the insider would get

$$2 \cdot [\frac{9}{58} - 0] = \frac{9}{29} > \frac{9}{30}.$$

Thus our conjecture is correct; the insider would submit only 2-share orders.

12. (**Comparing Transaction Costs of Stock Trading Over Three Trading Mechanisms.**) The following example is taken from Biais, Foucault, and Salanié (1998),[4] in which we compare the equilibrium efficiency of a floor market (like the NYSE trading floor), a limit-order market (like an electronic communications network adopted by Tokyo Stock Exchange), and an over-the-counter dealer market (like the Nasdaq).

We assume that, independent of the market trading mechanism, there are only 3 traders in the stock market; two of them are liquidity suppliers (whom we shall refer to as dealers for simplicity) and the last one is a buyer that submits a market order and wishes to buy exactly 4 shares of the stock.[5] Dealer $j \in \{1, 2\}$ has payoff $t_j - V(q_j)$ when he sells q_j shares and receives a payment t_j. It is assumed that

q	1	2	3	4
$V_1(q)$	10	21	33	56
$V_2(q)$	12	25	39	64

First suppose that the textbook-described Walrasian trading mechanism is adopted by the trading platform, as we have assumed in the preceding lectures. If the stock is infinitely divisible, then at the equilibrium stock price markets clear and the two dealers' marginal rates of substitution between the stock and cash must equal the price ratio of the stock to cash; that is, with cash being the numeraire, we have at the market clearing price p,

$$\frac{V_1'(q_1)}{1} = \frac{p}{1} = \frac{V_2'(q_2)}{1},$$

and

$$q_1 + q_2 = 4,$$

[4] Biais, B., T. Foucault, and F. Salanié, 1998, Floors, dealer markets and limit order markets, *Journal of Financial Markets*, 1, 253-284.

[5] The buyer's payoff is $-\infty$ if he fails to buy exactly 4 shares; and his payoff is $-\sum_{j=1}^{2} t_j$ if he succeeds in buying the 4 shares by paying t_j to dealer j.

implying that $p = 12$, $q_1 = 3$ and $q_2 = 1$. In equilibrium, dealer 1 is indifferent about selling 2 or selling 3 shares, and dealer 2 is indifferent about selling 1 share or selling nothing.

In the following, we shall assume that the stock is not infinitely divisible; trade size is required to be a non-negative integer. Note that this new restriction does not alter the share price and allocation in the Walrasian equilibrium.

Now, let us ask what would happen if instead of the textbook Walrasian mechanism, the platform actually adopts one of the three aforementioned trading mechanisms. We shall compare the Nash equilibrium properties under the three trading mechanisms. Given the trading mechanism chosen by the platform, a Nash equilibrium for the two dealers is a pair of strategies (s_1, s_2), such that given dealer i's strategy is s_i, s_j is dealer j's best strategic choice.

- In a *floor market*, the platform first announces the buyer's market order, and then the dealers must simultaneously announce share prices p_1 and p_2, saying that dealer j is willing to accept *any* number of shares $q_j \in [0, 4]$ that the platform subsequently asks him to deliver to the buyer at the price p_j.
- In a *limit-order market*, or an electronic communication network, the platform first announces the buyer's market order, and then the dealers must simultaneously submit individual supply curves, where a supply curve indicates that, after having sold n shares, dealer j is willing to sell one additional share at some price $p_j(n+1) \in \Re_+ \bigcup \{+\infty\}$, with $p_j(n+1) \geq p_j(n)$ for all $n = 0, 1, 2, 3$.
- In an *over-the-counter dealer market*, the platform first announces the buyer's market order, and then the dealers must simultaneously announce menus of options that the buyer can select from, where a dealer's menu may consist of several bundles like: $\{(T_1, q_1), (T_2, q_2), \cdots\}$, from which a buyer must pay the dealer a total of T_k dollars and purchase q_k shares if he picks the option (T_k, q_k). Here, unlike in the limit-order market, the dealer may commit to selling at a lower price for one additional share.

In all three mechanisms described above, the trading platform is assumed to minimize the buyer's expenditure (so that price priority will be enforced), and when ties occur, the platform is assumed to adopt a pro rata rationing rule, and try to maintain equal sales for the two dealers.

In this example, dealers have the greatest freedom in choosing trading strategies if the platform adopts an over-the-counter dealer market.[6] The floor market offers dealers the least freedom in choosing trading strategies. It turns out that these two mechanisms are both conducive to tacit collusion between dealers, making the buyer suffer from high transaction costs.

- In the **floor market**, there is one equilibrium[7] where both dealers announce the share price 14, and the platform asks each dealer to deliver 2 shares at this price. To see that this is an equilibrium, note that if dealer i alone raises his offer to beyond 14, then dealer j would sell 4 shares and dealer i would sell nothing, because the platform would enforce price priority; and if dealer i alone reduces his offer to below 14, then he alone would have to sell 4 shares, implying that his payoff is less than $4 \times 14 - V_i(4) < 0$. Hence given that dealer j would announce the share price 14, dealer i's best choice is to also announce the share price 14.

  Comparing this equilibrium to the Walrasian efficient outcome, we see that the buyer must pay 56 rather than 48 for the 4 shares, and the equilibrium allocation does not allow the two dealers to attain efficient risk sharing.

[6]Observe the different assumptions made in the preceding section and here. In the preceding section, when we derive the equilibrium bid and ask prices, we assume that dealers are faced with an adverse selection problem; that is, the buyer (or seller) may possess privileged inside information. Here the buyer is sure to be an uninformed liquidity trader. Moreover, dealers in the preceding section are risk-neutral, and here they are risk averse, which makes $V_j(q)$ a convex function of q. Finally, here the buyer can split his order between the two dealers, but in the preceding section, exactly one dealer will be chosen to absorb the buyer's order. In reality, dealers are faced with information asymmetry (as in the preceding section), but they can offer a client complicated trading choices (as assumed here), and a buyer can indeed split orders among multiple dealers (also as assumed here).

[7]An equilibrium is a vector of trading strategies, one for each dealer, such that given the trading strategy that the vector prescribes for the other dealer j, dealer i finds the strategy that the vector prescribes for him already a best choice. This is exactly a Nash equilibrium that we defined in non-cooperative game theory.

- In the **limit-order market**, there is a unique equilibrium outcome, which coincides with the Walrasian equilibrium outcome; that is, dealer 1 would sell 3 shares and dealer 2 would sell 1 share in equilibrium. We prove this claim in three steps.

  Step 1. In equilibrium, any two *executed* limit orders must have expressed the same price p.

  Indeed, if (p_i, q_i) and (p_j, q_j) are both executed, and yet $p_i < p_j$, then raising p_i to $p_i + \epsilon$ would still ensure execution, which raises the payoff for the dealer submitting (p_i, q_i), proving that submitting (p_i, q_i) is not the latter dealer's best choice, a contradiction.

  Step 2. Given that all submitted limit orders express the same equilibrium price p, a single dealer j must be prepared to sell $q_j(p)$ shares in equilibrium, where $q_j(\cdot)$ is dealer j's supply curve under the Walrasian mechanism.[8]

  Indeed, in equilibrium a single dealer j alone can always deviate and submit $(p-\epsilon, q_j(p-\epsilon))$ to gain price priority and sell $q_j(p-\epsilon)$ shares for sure, which yields for dealer j the payoff

  $$(p-\epsilon) \times q_j(p-\epsilon) - U_j(q_j(p-\epsilon)).$$

  This deviation payoff must be less than or equal to

  $$p \times Q_j - U_j(Q_j),$$

  where Q_j denotes dealer j's equilibrium sales volume. Letting $\epsilon \to 0$, we conclude that

  $$p \times q_j(p) - U_j(q_j(p)) \geq p \times Q_j - U_j(Q_j) \geq p \times q_j(p) - U_j(q_j(p)),$$

  showing that $Q_j = q_j(p)$, as claimed.

[8] That is, $q_j(z)$ uniquely solves the following maximization problem:

$$\max_{q \geq 0} zq - U_j(q),$$

or equivalently $z = U_j'(q)$ with x being the unknown.

Step 3. In equilibrium dealer j must submit an order (p, q'_j) such that p is the equilibrium price and $q'_j \geq 4$.

Indeed, if $q'_j \leq 3$, then it is optimal for dealer i to submit (p_i, q'_i) with $q'_i = 4 - q'_j$ and p_i equal to a huge number strictly greater than p, a contradiction to Step 1.

Now by Steps 1-3, we must have, by the pro rata rationing rule,

$$Q_j = 4 \times \frac{q'_j}{q'_1 + q'_2} = q_j(p),$$

implying that

$$q_1(p) + q_2(p) = 4 \Rightarrow p = 12 \Rightarrow q_1(p) = 3,\ q_2(p) = 1.$$

Thus in *one* limit-order market equilibrium, for example, dealer 1 submits a single limit order $(p_1, q_1) = (12, 12)$ and dealer 2 submits a single limit order $(p_2, q_2) = (12, 4)$. In this equilibrium, at the price 12, the aggregate supply is 16 and the demand quantity is 4, so that the pro rata rationing rule applies. Consequently, dealer 1 gets to sell 3 shares and dealer 2 gets to sell 1 share. The limit-order market attains the same efficiency as the Walrasian market.

- In the **over-the-counter dealer market**, there is one equilibrium where dealer 1 offers two trading options $\{(12, 1), (57, 4)\}$ and dealer 2 offers two trading options $\{(45, 3), (57, 4)\}$. Take dealer 1 for example. Given dealer 2's strategy, which says that the buyer either must give up trading with dealer 2, or buy 3 shares from dealer 2 at a total price of 45, or buy the entire 4 shares at 57, dealer 1 is left with 3 choices: either he can sell 4 shares to the buyer (when the buyer chooses to give up dealer 2), or he can sell 1 share to the buyer (when the buyer chooses to buy 3 shares from dealer 2), or he can sell nothing (when the buyer chooses to buy 4 shares from dealer 2).
  - Dealer 1 would obtain zero payoff if he sells nothing.
  - If dealer 1 offers to sell 4 shares at a total price less than 57, dealer 1 would get a payoff less than $57 - V_1(4) = 1$.
  - Given dealer 2 promises to offer the entire 4 shares at a total price of 57, if dealer 1 wishes to sell 1 share to the buyer, his

price p_1 must satisfy $p_1+45 \leq 57$, so that his best choice is 12, explaining why he offers $(12, 1)$ in his own menu. By selling 1 share, dealer 1's payoff is thus $12 - V_1(1) = 2$. Apparently, selling 1 share is dealer 1's best choice!

The same reasoning would show that offering to sell 3 shares at a total price 45 is dealer 2's best choice given dealer 1's two offers $\{(12, 1), (57, 4)\}$. In equilibrium, ties occur because the platform can ensure a total expenditure of 57 for the buyer by asking either dealer to sell 4 shares, or by asking dealer 1 to sell 1 share and dealer to sell 3 shares. Recall that when ties occur, the platform would try to treat the two dealers as equitable as possible, and hence the platform picks the latter allocation. Thus in the over-the-counter market equilibrium, dealer 1 sells 1 share and dealer 2 sells 3 shares, and the buyer's expenditure is 57. Finally, observe that the sole role of $(57, 4)$ in dealer 2's offer is to discipline dealer 1 and to sustain the equilibrium. The platform never asks one dealer to sell 4 shares in equilibrium, and if it does and chooses dealer 2, then dealer 2 would realize a loss by selling 4 shares at a total price of 57. (Dealer 2 is using a weakly dominated strategy when offering $(57, 4)$!)

To conclude, we see that in the absence of information asymmetry, the limit-order market best guards the buyer's interest, and it ensures allocative efficiency for the two dealers. The other two trading mechanisms tend to raise the buyer's trading costs, and fail to attain allocative efficiency for the dealers. The OTC market results in dealer 1 selling fewer shares than dealer 2, which is highly inefficient.

13. Consider an auction website, where there are m buyers and n sellers for an identical indivisible good. Each buyer demands one unit, and each seller has one unit for sale. Buyers and sellers are risk-neutral and buyers have private values, which are independently drawn from a common distribution function $F(\cdot)$ with density $f(\cdot)$ on some interval $[0, \omega]$. Let v_i denote buyer i's valuation for the good.

A pair of buyer and seller can trade only if they are connected, or they have a "link." The configuration of links among buyers and sellers is graphically presented as a network, as shown in the following example, where $m = 5$ and $n = 3$, and where seller 1 is linked to buyers 1, 2, and 3; seller 2 is linked to buyers 3 and 4; and seller 3 is linked to buyers 3 and 5.

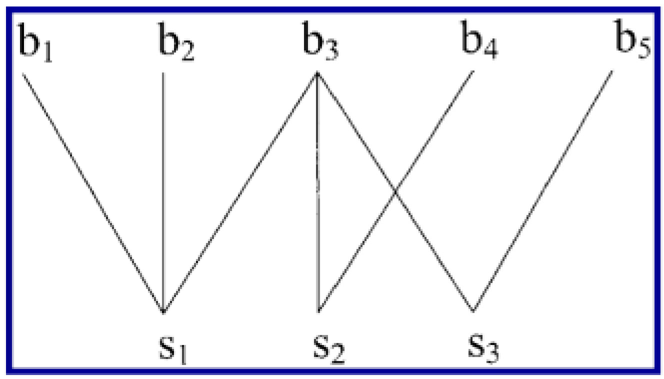


FIGURE 1. EXAMPLE OF A BUYER-SELLER NETWORK

The website implements the following auction rules:

- Sellers simultaneously hold ascending-bid auctions, where the going price is the same across all sellers at each moment. As this common price rises from zero, each buyer decides whether to drop out of the bidding of each of his linked sellers' auctions. It can be proven that it is an equilibrium (following elimination of weakly dominated strategies) for each buyer to remain in the bidding in each of his linked sellers' auctions up to his valuation of a good.
- The price rises until enough buyers have dropped out so that there is *a subset of sellers* for whom *demand equals supply.* We call such a subset a *clearable set of sellers.* The auctions of these sellers then "clear" at the current price. The set of buyers and sellers finishing trades at this market-clearing price is then removed from the original network, giving rise to a new, smaller network.
- As long as there are remaining sellers in the new network, the price continues to rise until another clearable set of sellers emerges, and another around of market-clearing then ensues.
- This process goes on and on until all sellers have sold their goods.

Now, suppose that the original network is as shown in the above example. Assume that sellers have zero valuation for the good, and that $v_1 > v_2 > v_3 > v_4 > v_5 > 0$. Let p_i denote the equilibrium price at which buyer i obtains one unit of the good, with $p_i = +\infty$ in case buyer i never obtains the good.

(i) In equilibrium, we have $p_1 =$ ______, $p_2 =$ ______, and $p_3 =$ ______.

(ii) Now, imagine that before the auctions get started, buyer 3 decides to eliminate his link with seller 3.

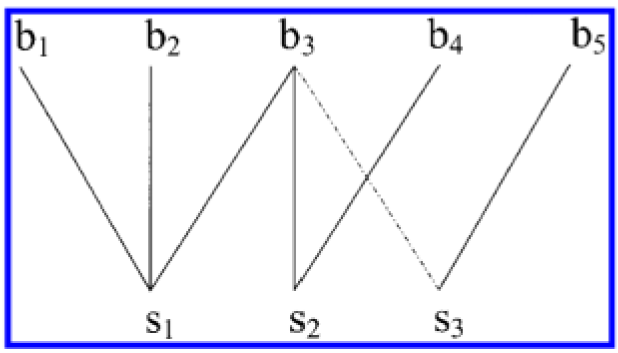


FIGURE 3. REMOVING A LINK

Following the removal of the link between buyer 3 and seller 3, the change in the social surplus (the sum of buyers' and sellers' payoffs) is equal to ______, and the change in buyer 3's personal payoff is equal to ______.

Solution. Consider part (i). By assumption, we have $v_1 > v_2 > v_3 > v_4 > v_5 > 0$, so that no clearable sets of sellers will arise before the price reaches v_5. When $p = v_5$, buyer 5 drops out, and sellers 2 and 3 form a clearable set of sellers: there are two units demanded by buyers 3 and 4, and sellers 2 and 3 together can supply two units. This is the unique clearable set of sellers at price $p = v_5$. Thus buyer 3 pays $p_3 = v_5$ and purchases 1 unit from seller 3, and buyer 4 pays $p_4 = v_5$ and purchases 1 unit from seller 2.

Upon removing buyers 3, 4, and 5 and sellers 2 and 3 from the original network, we obtain a smaller network in which seller 1 is connected with buyers 1 and 2. The price then rises and reaches v_2, and a new clearable set of sellers emerges after buyer 2 drops out. This clearable set contains only seller 1, who is facing 1 unit demand expressed by buyer 1. Thus buyer 1 pays $p_1 = v_2$ and purchases 1 unit from seller 1. We also conclude that $p_2 = p_5 = +\infty$.

Consider part (ii). This time, at $p = 0$ there is a clearable set of sellers, which consists of seller 3, who is facing 1 unit of demand expressed by buyer 5. Thus buyer 5 pays $p_5 = 0$ and purchases 1 unit from seller 3. Upon removing buyer 5 and seller 3, the price starts to rise above zero, and then reaches v_4. After buyer 4 drops out at $p = v_4$, a new clearable set of sellers emerges, where seller 2 is facing 1 unit of demand expressed by buyer 3. Thus buyer 3 pays $p_3 = v_4$ and purchases 1 unit from seller 2. After that, what happens is the same as in part (i). We conclude that $p_1 = v_2$, $p_2 = p_4 = +\infty$, $p_3 = v_4$, and $p_5 = 0$.

Comparing part (i) to part (ii), we conclude that the change in the social surplus is $v_5 - v_4 < 0$, showing that, the costs of maintaining a link aside, removing the link between buyer 3 and seller 3 is inefficient. As to buyer 3, his payoff also changes by $v_5 - v_4$, and this indicates that a single buyer's incentive to maintain or remove a link with a seller is consistent with social efficiency. This exercise is taken from Kranton and Minehart (2001).[9]

14. There are two competing night clubs, A and B, located at respectively the left and the right endpoints of the Hotelling main street denoted by $[0, 1]$. Girls and boys uniformly reside along the Hotelling street, and the population of boys is one, so is the population of girls. A boy or girl can visit at most one club. A boy located at x would obtain consumer surplus

$$V - p_b^A - t_b x + \beta N_g^A$$

if paying the entrance fee p_b^A charged by club A and visiting club A; and he would obtain

$$V - p_b^B - t_b(1 - x) + \beta N_g^B$$

[9] Kranton, R., and D. Minehart, 2001, A Theory of Buyer-Seller Networks, *American Economic Review*, 91, 485-508.

if paying the entrance fee p_b^B charged by club B and visiting club B instead, where N_g^j is the population of girls visiting club j, and $\beta > 0$ reflects that a boy would like to visit a night club where he can meet with more girls. Similarly, a girl located at y would obtain consumer surplus

$$V - p_g^A - t_g y + \gamma N_b^A$$

if paying the entrance fee p_g^A charged by club A and visiting club A; and she would obtain

$$V - p_g^B - t_g(1-y) + \gamma N_b^B$$

if paying the entrance fee p_g^B charged by club B and visiting club B instead, where N_b^j is the population of boys visiting club j, and $\gamma > 0$ reflects that a girl would like to visit a night club where she can meet with more boys.

The night clubs seek to maximize expected profits and boys and girls seek to maximize consumer surplus. For simplicity, assume that the two night clubs can operate without costs, and that V is sufficiently large so that boys and girls would visit exactly one club in equilibrium. Moreover, we assume that the night clubs must charge *non-negative* entrance fees.

The game proceeds as follows.

- The two clubs simultaneously announce entrance fees (p_b^A, p_g^A) and (p_b^B, p_g^B).
- Then given the announced entrance fees, boys and girls rationally conjecture that there is a boy x^* and a girl y^* such that

$$N_b^A = x^* = 1 - N_b^B,$$

$$N_g^A = y^* = 1 - N_g^B,$$

  and moreover,

$$V - p_b^A - t_b x^* + \beta N_g^A = V - p_b^B - t_b(1-x^*) + \beta N_g^B,$$

  and

$$V - p_g^A - t_g y^* + \gamma N_b^A = V - p_g^B - t_g(1-y^*) + \gamma N_b^B.$$

- Given the above conjecture, boys and girls simultaneously decide which night club they would go to.

We shall look for a subgame perfect Nash equilibrium, where the two night clubs rationally recognize the boys and girls' conjecture stated above when choosing entrance fees.

From now on, assume that

$$t_b = 2,\ \beta = 1.$$

(i) Suppose that $t_g = t_b$ and $\gamma = \beta$. In the symmetric equilibrium where $(p_b^A, p_g^A) = (p_b^B, p_g^B) = (p_b, p_g)$, we have $p_b + p_g =$ No. 7 and each night club's equilibrium profit is No. 8 .[10]

(ii) Ignore part (i). Suppose instead that $t_g = 4$ and $\gamma = 3$. In the symmetric equilibrium where $(p_b^A, p_g^A) = (p_b^B, p_g^B) = (p_b, p_g)$, we have $p_g + [p_g]^{p_b} =$ ___ and each night club's equilibrium profit is ___.

(iii) Ignore part (i) and part (ii). Suppose instead that $t_g = \frac{3}{4}$ and $\gamma = \frac{2}{3}$. In the symmetric equilibrium where $(p_b^A, p_g^A) = (p_b^B, p_g^B) = (p_b, p_g)$, we have $p_g - p_b =$ ___ and each night club's equilibrium profit is ___.

Solution. Note that under the boys and girls' rational beliefs, the

[10]**Hint**: Solve for N_b^A and N_g^A as functions of (p_b^A, p_g^A) and (p_b^B, p_g^B) from the four equations representing boys and girls' rational beliefs. (Note that $N_b^B = 1 - N_b^A$ and $N_g^B = 1 - N_g^A$.) Then, note that for $j, k \in \{A, B\}$, $j \neq k$, night club j seeks to maximize $\pi_j(p_b^j, p_g^j; p_b^k, p_g^k) = p_b^j N_b^j + p_g^j N_g^j$, which is concave in (p_b^j, p_g^j) given the parameter values assumed for parts (i)-(iii). Thus the Nash equilibrium can be solved from $\frac{\partial \pi_j}{\partial p_b^j} = 0$ and $\frac{\partial \pi_j}{\partial p_g^j} = 0$, for $j \in \{A, B\}$, if these four equations give rise to non-negative entrance fees. (Actually, you only need to look at the first-order conditions for $j = A$, because we are looking for a "symmetric" equilibrium. In equilibrium, $p_b^A = p_b^B$ and $p_g^A = p_g^B$ must satisfy the first-order conditions for $j = A$!) In the unfortunate event that the first-order conditions give rise to a negative entrance fee for, say the boys, then by the concavity of $\pi_A(\cdot, \cdot)$, you should assume that in equilibrium $p_b^A = p_b^B = 0$ and then solve for the equilibrium $p_g^A = p_g^B$. In this case, once again, you only need to solve the equation $\frac{\partial \pi_A}{\partial p_g^A} = 0$, by plugging in $p_b^A = p_b^B = 0$ and $p_g^A = p_g^B$. This exercise is actually easier than you might have imagined.

population of girls visiting night club A is

$$N_g^A = \frac{1}{2} + \frac{\gamma(p_b^B - p_b^A) + t_b(p_g^B - p_g^A)}{2(t_b t_g - \beta\gamma)} = \frac{(t_b t_g - \beta\gamma) + \gamma(p_b^B - p_b^A) + t_b(p_g^B - p_g^A)}{2(t_b t_g - \beta\gamma)},$$

and the population of boys visiting night club A is

$$N_b^A = \frac{1}{2} + \frac{\beta(p_g^B - p_g^A) + t_g(p_b^B - p_b^A)}{2(t_b t_g - \beta\gamma)} = \frac{(t_b t_g - \beta\gamma) + \beta(p_g^B - p_g^A) + t_g(p_b^B - p_b^A)}{2(t_b t_g - \beta\gamma)}.$$

Club A seeks to maximize

$$\pi_A(p_b^A, p_g^A) \equiv p_g^A N_g^A + p_b^A N_b^A$$

given the conjectured (p_b^B, p_g^B) and the above two equations; and club B seeks to maximize

$$\pi_B(p_b^B, p_g^B) \equiv p_g^B N_g^B + p_b^B N_b^B,$$

given the conjectured (p_b^A, p_g^A) and the above two equations.

We can then write

$$\pi_A(p_b^A, p_g^A) \equiv p_g^A N_g^A + p_b^A N_b^A$$

$$= \frac{1}{2(t_b t_g - \beta\gamma)}\{p_g^A[(t_b t_g - \beta\gamma) + \gamma(p_b^B - p_b^A) + t_b(p_g^B - p_g^A)]$$
$$+ p_b^A[(t_b t_g - \beta\gamma) + \beta(p_g^B - p_g^A) + t_g(p_b^B - p_b^A)]\},$$

It is easy to verify that π_A is a concave function of (p_b^A, p_g^A) if $4t_b t_g > (\gamma + \beta)^2$, which holds true in part (i), part (ii), and part (iii). Note that

$$\frac{\partial \pi_A}{\partial p_g^A} = (t_b t_g - \beta\gamma) + \gamma p_b^B + t_b p_g^B - \gamma p_b^A - 2t_b p_g^A - \beta p_b^A,$$

and

$$\frac{\partial \pi_A}{\partial p_b^A} = (t_b t_g - \beta\gamma) + \beta p_g^B + t_g p_b^B - \beta p_g^A - 2t_g p_b^A - \gamma p_g^A.$$

In a symmetric equilibrium, we must have

$$p_g^A = p_g^B \geq 0,\ p_b^A = p_b^B \geq 0,$$

and if the system of equations $\frac{\partial \pi_A}{\partial p_g^A} = 0$ and $\frac{\partial \pi_A}{\partial p_g^A} = 0$ with unknowns $p_b^A = p_b^B = X$ and $p_b^A = p_b^B = Y$ have a non-negative solution (X, Y), then (X, Y) must be the equilibrium entrance fees that we have been looking for. Thus we must solve

$$(t_b t_g - \beta\gamma) + \gamma Y + t_b X - \gamma Y - 2t_b X - \beta Y = 0,$$

and

$$(t_b t_g - \beta\gamma) + \beta X + t_g Y - \beta X - 2t_g Y - \gamma X = 0.$$

The solution is

$$Y = t_b - \gamma,\ X = t_g - \beta.$$

Now, in part (i), we indeed have $X, Y \geq 0$, and hence we have

$$p_b^A = p_b^B = Y = 1,\ p_g^A = p_g^B = X = 1,\ \pi_B(p_b^B, p_g^B) = \pi_A(p_b^A, p_g^A) = 1.$$

In part (ii), we have $Y < 0$, and hence in equilibrium we have $p_b^A = p_b^B = 0$. Now, assuming that $p_g^A = p_g^B = X$ satisfies $\frac{\partial \pi_A}{\partial p_g^A} = 0$ with $p_b^A = p_b^B = 0$, we have

$$(t_b t_g - \beta\gamma) + t_b X - 2t_b X = 0 \Rightarrow X = t_g - \frac{\beta\gamma}{t_b} = \frac{5}{2},$$

implying that

$$\pi_B(p_b^B, p_g^B) = \pi_A(p_b^A, p_g^A) = \frac{5}{4}.$$

Let us verify that the solution obtained here indeed defines a symmetric equilibrium. Suppose that club A expects B to price at $p_b^B = 0$ and $p_g^B = \frac{5}{2}$. Then by letting $p_g^A = x$, $p_b^A = y$, we can write[11]

$$\Pi_A(x, y) = \frac{1}{10}[-2x^2 - 4xy - 4y^2 + \frac{15}{2}y + 10x],$$

[11] A function $f : \Re^2 \to \Re$ defined by $f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ is strictly concave if $A < 0$ and $B^2 - 4AC < 0$.

with

$$\frac{\partial \Pi_A}{\partial x} = \frac{1}{10}[10 - 4x - 4y], \ \frac{\partial \Pi_A}{\partial y} = \frac{1}{10}[\frac{15}{2} - 4x - 8y].$$

$\Pi_A(x, y)$ has an un-constrained maximum at $(x^*, y^*) = (\frac{25}{8}, \frac{-5}{8})$, but for any $x \geq 0$, we have

$$\frac{\partial \Pi_A}{\partial x} = \frac{1}{10}[10 - 4x - 4y] > \frac{1}{10}[\frac{15}{2} - 4x - 8y] = \frac{\partial \Pi_A}{\partial y},$$

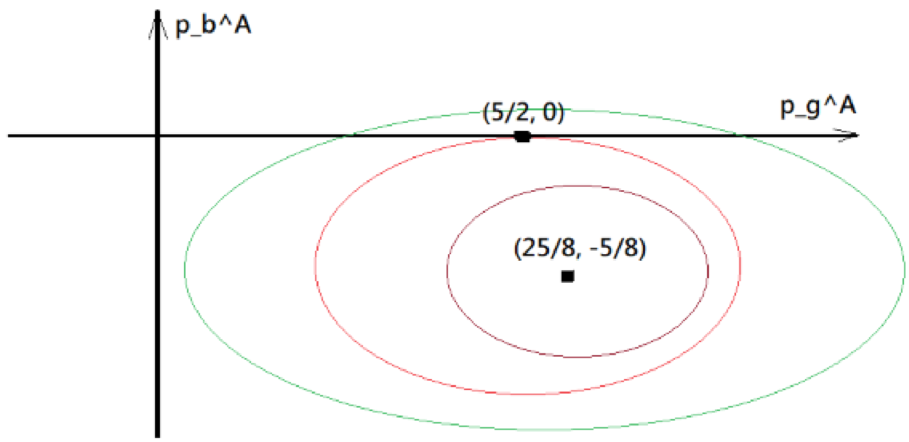
showing that club A's best response cannot be an interior solution satisfying

$$\frac{\partial \Pi_A}{\partial x} = 0 = \frac{\partial \Pi_A}{\partial y}.$$

We can at best have

$$\frac{\partial \Pi_A}{\partial x} = 0 > \frac{\partial \Pi_A}{\partial y},$$

which can only arise when $p_b^A = 0$! It can be verified that each isoquant of $\Pi_A(x, y)$ is an ellipse, and points located inside of a smaller ellipse generate higher values for Π_A; see the figure below. Club A's best response thus appears on the isoquant for which the horizontal axis is the tangent line.



Finally, in part (iii), we have $X < 0$, and hence in equilibrium we have $p_g^A = p_g^B = 0$. Now, assuming that $p_b^A = p_b^B = Y$ satisfies $\frac{\partial \pi_A}{\partial p_b^A} = 0$ with $p_g^A = p_g^B = 0$, we have

$$(t_b t_g - \beta\gamma) + t_g Y - 2t_g Y = 0 \Rightarrow Y = t_b - \frac{\beta\gamma}{t_g} = \frac{10}{9},$$

implying that

$$\pi_B(p_b^B, p_g^B) = \pi_A(p_b^A, p_g^A) = \frac{5}{9}.$$

Remark. In this exercise, each club sells two products—the chance for girls to meet with boys, and the chance for boys to meet with girls. In part (ii), because of the indirect network effect, reducing p_b to zero is better than maintaining a positive p_b, because it helps raise the club's sales volume to boys, thereby allowing the club to charge a higher p_g and make more profits from serving the girls.