Game Theory with Applications to Finance and Marketing, I

Homework 1, due at noon, September 25, 2024

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Problems 1-4 are required. Problem 5 is optional. One study group should submit exactly *one* set of solutions.

1. (**SPNE and Value of Commitment**.) Consider the following version of Cournot game, where firm 1 and firm 2 are both endowed with the technology of producing 1 unit of product Y using 1 unit of input X (and nothing else), where the price of input X is $c \in (0,1)$ at all times. The two firms are risk-neutral without time preferences.

   At date 0, firm 1 alone can purchase an amount $\overline{q} \geq 0$ of input X, which firm 1 can keep intact till date 1 without incurring any inventory costs. However, the input X purchased at date 0 cannot be returned to its seller subsequently, and it's date-1 resale value is zero.

   At date 1, upon seeing firm 1's date-0 choice $\overline{q}$, firm 1 and firm 2 simultaneously choose output levels q_1 and q_2 for product Y, where the date-1 inverse demand function for product Y (in the relevant region) is

$$P = 1 - q_1 - q_2.$$

   Note that each choice $\overline{q}$ made by firm 1 at date 0 defines a distinct date-1 subgame, where the associated pure-strategy Nash equilibrium will be denoted by (q_1^*, q_2^*). Given $\overline{q}$, there are three possible cases: (A) $q_1^* > \overline{q}$; (B) $q_1^* < \overline{q}$; and (C) $q_1^* = \overline{q}$. We shall denote the pure-strategy equilibria in these cases by respectively (q_1^A, q_2^A), (q_1^B, q_2^B), and (q_1^C, q_2^C).

   (i) Show that if $\overline{q} < \frac{1-c}{3}$ then there exists a pure-strategy equilibrium $(q_1^*, q_2^*) = (q_1^A, q_2^A)$ for the date-1 Cournot game, where firm 1's equilibrium payoff is

$$\Pi_1^A(\overline{q}) = \frac{(1-c)^2}{9}.$$

(ii) Show that if $\overline{q} > \frac{1+c}{3}$ then there exists a pure-strategy equilibrium $(q_1^*, q_2^*) = (q_1^B, q_2^B)$ for the date-1 Cournot game, where firm 1's equilibrium payoff has a least upper bound equal to

$$\Pi_1^B(\overline{q}) = \frac{(1+c)^2}{9} - c(\frac{1+c}{3}).$$

(iii) Show that if $\frac{1+c}{3} \geq \overline{q} \geq \frac{1-c}{3}$, then there exists a pure-strategy equilibrium $(q_1^*, q_2^*) = (q_1^C, q_2^C)$ for the date-1 Cournot game, where, when $c > \frac{1}{5}$, firm 1's equilibrium payoff has a least upper bound equal to

$$\Pi_1^C(\overline{q}) = \frac{(1-c)^2}{8},$$

which is attained at $\overline{q} = \frac{1-c}{2}$.

(iv) Show that

$$\Pi_1^C(\overline{q}) - \Pi_1^A(\overline{q}) = \frac{(1-c)^2}{72} > 0, \quad \Pi_1^C(\overline{q}) - \Pi_1^B(\overline{q}) = \frac{(1-5c)^2}{72} \geq 0.$$

Conclude that when $c > \frac{1}{5}$, in equilibrium firm 1 obtains the leading firm's payoff in a Stackelberg duopoly.

2. (**Pure- and Mixed-strategy NE.**) Consider the following two-player normal-form game:

Player 1/Player 2	L	M	R
U	$40, 0$	$0, 20$	$-10, -800$
D	$20, 10$	$10, 0$	$0, 30$

Let π be the probability that player 1 may choose U in equilibrium, and p and q respectively the probabilities that player 2 may choose L and M in equilibrium. This game has three (pure- and mixed-strategy) Nash equilibria. Find them all.

3. (**Forward Induction.**) Consider the following strategic game:

player 1/player 2	L	R
U	1,1	0,0
D	0,0	3,2

Any NE for the above strategic game can be represented by (p, q), where p is the probability that player 1 adopts U and q the probability that player 2 adopts L.

(i) Show that the above strategic game has 3 NE's, where (p, q) equals respectively $(1, 1)$, $(0, 0)$, and $(\frac{2}{3}, \frac{3}{4})$.

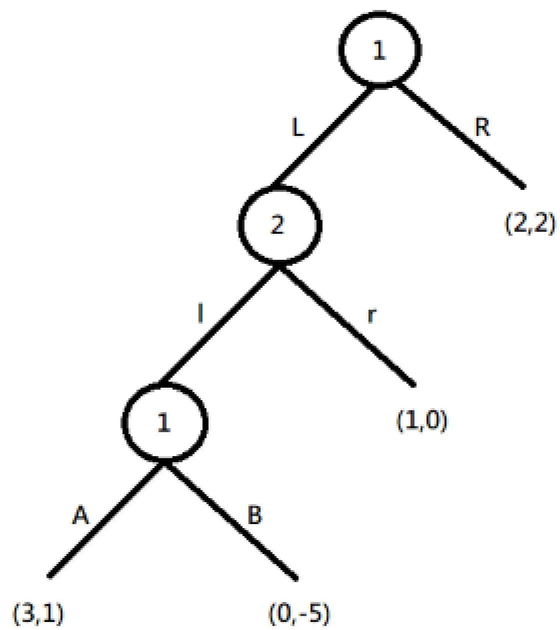
(ii) Now, consider the following new version of the above strategic game.

- At the first stage, player 1 can invite either A or B to become player 2 for the above strategic game, and here we assume that A does not know B's presence, and B does not know A's presence.
- At the second stage, player 1 and the selected player 2 then play the above strategic game. A (or B) would get player 2's payoffs described in the above strategic game, if he accepts the invitation to play the game. Without playing the game, A can get a (reservation) payoff of $\frac{1}{200}$ on his own, and B can get a (reservation) payoff of $\frac{3}{2}$ on his own.
- The game proceeds as follows. First, player 1 can invite either A or B, and if the invitation is accepted, then the game moves on to the second stage; and if the invitation gets turned down, then player 1 can invite the other candidate. If both A and B turn down player 1's invitations, then the game ends with A getting $\frac{1}{200}$, B getting $\frac{3}{2}$, and player 1 getting 0.

Which one between A and B should player 1 invite first? Compute player 1's equilibrium payoff.

(iii) Re-consider part (ii), but assume instead that it is common knowledge that A and B both exist and have the assumed reservation payoffs. Which one between A and B should player 1 invite first? Compute player 1's equilibrium payoff.

4. (**Trembling-hand Perfect Equilibrium and SPNE.**) Consider the following sequential game, which appears in section 3 of Lecture 1, Part II.



(i) Verify that the *reduced normal form* of the extensive game, where *equivalent strategies* are identified, can be represented by the following bi-matrix:

player 1/player 2	l	r
R	2,2	2,2
(L,A)	3,1	1,0
(L,B)	0,-5	1,0

(ii) Verify that given $\epsilon > 0$ small, if player 1 adopts the totally mixed strategy

$$\begin{bmatrix} \text{R} & \rightarrow & 1-\epsilon-\epsilon^2 \\ \\ \text{(L,A)} & \rightarrow & \epsilon^2 \\ \\ \text{(L,B)} & \rightarrow & \epsilon \end{bmatrix},$$

then r is player 2's best response, but l is not. Hence if player 2's restricted to assigning l a positive probability in an ϵ-perfect equilibrium, then she will assign l with a positive probability less than ϵ (and hence r must be assigned with a probability of at least $1 - \epsilon$).

Verify that given player 2's totally mixed strategy

$$\begin{bmatrix} \text{l} & \rightarrow & \epsilon \\ \\ \text{r} & \rightarrow & 1-\epsilon \end{bmatrix},$$

R is player 1's best response, but (L,A) and (L,B) are not. Conclude that player 1's totally mixed strategy

$$\begin{bmatrix} \text{R} & \rightarrow & 1-\epsilon-\epsilon^2 \\ \\ \text{(L,A)} & \rightarrow & \epsilon^2 \\ \\ \text{(L,B)} & \rightarrow & \epsilon \end{bmatrix}$$

and player 2's totally mixed strategy

$$\begin{bmatrix} \text{l} & \rightarrow & \epsilon \\ \\ \text{r} & \rightarrow & 1-\epsilon \end{bmatrix}$$

indeed form an ϵ-perfect equilibrium given the specified $\epsilon > 0$. Since this is true for all small $\epsilon > 0$, by letting $\epsilon \downarrow 0$, verify that indeed, (R,r) is a trembling-hand perfect equilibrium for the strategic game

player 1/player 2	l	r
R	2,2	2,2
(L,A)	3,1	1,0
(L,B)	0,-5	1,0

(iii) Show that (R,r) is neither a subgame perfect Nash equilibrium in the original extensive game, nor a proper equilibrium in the corresponding reduced normal form of the game.

(iv) Now, let us consider a new game similar to the game above, but with one difference: "the player 1" that gets to choose between A and B after player 2 chooses l in the original game is now replaced by a new player, called player 3. In this new game, player 3 and player 1 have the same payoff function, which is the payoff function of the player 1 in the original game. To represent this three-player normal-form game, we draw two bi-matrices as follows (where we identify player 3's payoff with player 1's payoff):

player 3/player 2	l	r	
A	3,1	1,0	L
B	0,-5	1,0	

player 3/player 2	l	r	
A	2,2	2,2	R
B	2,2	2,2	

In this new normal-form game, player 1 first chooses a bi-matrix for players 2 and 3, knowing that he will get what player 3 will get in equilibrium, and then players 2 and 3 must play the bi-matrix selected by player 1. Note that the first normal-form game corresponds to player 1 choosing L, and the second normal-form game corresponds to player 1 choosing R. This modified game is referred to as the *agent normal form* of the original extensive game.

Show that in this modified game, the only trembling-hand perfect equilibrium is the unique SPNE in the original game, where player 1 chooses L, and then player 2 chooses l, and then player 3 chooses A.

5. **Location Game.** Consider the *simultaneous* location game discussed in section 48 of Lecture 1, Part I. We shall characterize the pure-strategy equilibria.

Assume that there are $N \geq 2$ firms choosing locations at the same time, and fix a pure-strategy Nash equilibrium. Let y be the location of a firm which has no rival firms on its left, and let x be the location of a firm which has no rival firms on its right. Suppose that there are m_y firms located at y and m_x firms located at x.

(i) Show that $m_y \geq 2$ and $m_x \geq 2$. This shows that, if $N = 2$ then the two firms must choose the same location in a pure-strategy equilibrium.

(ii) Suppose that $N \geq 3$. Let z be the location of those firms that are closest to y among the firms located on the right of y. Show that the equilibrium payoff for a firm located at y is $\frac{z+y}{2m_y}$, and hence, necessarily,[1] $m_y \leq 2$. Conclude therefore that

$$m_y = m_x = 2.$$

This shows that there is no pure-strategy equilibrium when $N = 3$, and when $N = 4$, in a pure-strategy equilibrium two firms must choose location y and the other two firms must choose location x. Moreover, when $N = 5$, in a pure-strategy equilibrium two firms must choose location y and two firms must choose location x, with the remaining one firm choosing some location w, where $y < w < x$.

(iii) Suppose that $N \geq 6$. By part (ii) there must exist at least one firm located at some w, with $y < w < x$. Let r be the location of those firms that are closest to w among the firms located on the right of w, and l be the location of those firms that are closest to w among the firms located on the left of w. Show that the equilibrium payoff for a firm located at w is

$$\frac{r-l}{2m_w},$$

and hence, necessarily,[2] $m_w \leq 2$.

[1] A firm located at y can obtain essentially the payoff y by unilaterally deviating to $y - \epsilon$ or obtain essentially the payoff $\frac{z-y}{2}$ by unilaterally deviating to $y + \epsilon$.

[2] A firm located at w can obtain essentially $\frac{w-l}{2}$ by unilaterally deviating to $w - \epsilon$ or obtain essentially $\frac{r-w}{2}$ by unilaterally deviating to $w + \epsilon$.

(iv) Suppose that there are $N = 2n$ firms choosing locations at the same time, with $n \geq 3$. Show that the following strategy profile constitutes one pure-strategy Nash equilibrium: for $j \in \{1, 2, \cdots, n\}$, there are exactly two firms located at x_j, where $x_1 = \frac{1}{2n}$, $x_n = 1 - \frac{1}{2n}$, and $x_j = x_1 + \frac{j-1}{n}$, for all $j \in \{2, 3, \cdots, n-1\}$.