Game Theory with Applications to Finance and Marketing

Lecture 1: Games with Complete Information, Part I

Instructor: Chyi-Mei Chen, R1102
(TEL) (02) 3366-1086
(email) cchen@ntu.edu.tw

1. **Definition 1:** A game can be described by (i) the set of players, (ii) the strategies of players, and (iii) the payoffs of players as functions of their strategies. Players are assumed to maximize their payoffs and payoffs are generally interpreted as von Neumann-Morgenstern utilities. Players are *rational* in the sense that they will choose strategies to maximize their (expected) payoffs. A game described in this way is called a game in *normal form*, or a *strategic game*. (A game can also be described in *extensive form*; see below.) An event is the players' *mutual knowledge* if all players know the event, and an event is called the players' *common knowledge* if all players know it, all players know that they all know it, all players know that they all know that they know it, and so on and so forth. If the norm form of a game (and the rationality of all players) is the players' common knowledge, then the game is one with *complete information*.

   Example 1: Consider the Cournot game in normal form. The set of players: two firms, $i = 1, 2$. The strategies: the two firms' output quantities: $q_i \in S_i \equiv [0, \infty),\ i = 1, 2$. The payoff of each player i:

   $$\pi_i(q_i, q_j) = q_i[P(q_i + q_j) - c] - F,$$

   where $P(\cdot)$ is the inverse demand function, c and F are respectively the variable and fixed costs.

   Example 2: A two-player game with a finite number of actions (strategies) is usually represented by a bimatrix:

player 1/player 2	L	R
U	0,1	-1,2
D	2,-1	-2,-2

where there are two players in the game, who simultaneously choose actions, and action profiles (U,L), (U,R), (D,L) and (D,R) result in respectively payoff profiles $(0,1),(-1,2),(2,-1)$ and $(-2,-2)$ for the two players (where by convention the first coordinate in a payoff profile stands for player 1's payoff).

2. Our purpose of learning the non-cooperative game theory is practical. In application, we shall first describe an economic or business problem as a game in norm form (or more often in *extensive form*), and then proceed to *solve* the game so as to generate useful *predictions* about what the players involved in the original economic or business problem may do. For this purpose, we need to adopt certain *equilibrium concepts* or *solution concepts*. In the remainder of this note we shall review the following solution concepts (and illustrate them using a series of examples):

    - Rational players will never adopt *strictly dominated strategies*;
    - Common knowledge about each player's rationality implies that rational players will never adopt *iterated strictly dominated strategies*;
    - Rational players will never adopt *weakly dominated strategies*;
    - Common knowledge about each player's rationality implies that rational players will never adopt *iterated weakly dominated strategies*;
    - Rational players will never adopt *strategies that are never best responses*, or equivalently, rational players will adopt only *rationalizable strategies*;
    - Rational players will adopt *Nash equilibrium strategies*;
    - Rational players will adopt *trembling-hand perfect equilibrium strategies*;
    - Rational players will adopt *subgame-perfect Nash equilibrium strategies*;
    - Rational players will adopt *proper equilibrium strategies*;
    - Rational players will adopt *strong equilibrium strategies*;
    - Rational players will adopt *coalition-proof strategies.*

In practice, the solution concepts of Nash equilibrium (NE) and subgame-perfect Nash equilibrium (SPNE) are most widely accepted. In rare cases only will we turn to other equilibrium concepts.

3. **Definition 2:** A pure strategy is one like U, R, D, or L in example 2. It has to be a complete description of a player's actions taken throughout the game. A mixed strategy is a probability distribution over the set of pure strategies. In a broad sense, a pure strategy is a mixed strategy.

4. **Definition 3:** Consider a game in normal form,

$$G = (\mathcal{I} \subset \Re; \{S_i; i \in \mathcal{I}\}; \{u_i : \Pi_{i\in\mathcal{I}} S_i \to \Re; i \in \mathcal{I}\}),$$

where $\Re$ is the set of real numbers, $\mathcal{I}$ is the set of players, S_i is the set of pure strategies feasible to player i (also known as player i's *pure strategy space*), $\Pi_{i\in\mathcal{I}} S_i$ is the Cartesian product of $\{S_i; i \in \mathcal{I}\}$, of which each element is referred to as a *strategy profile*, and $u_i(\cdot)$ is player i's payoff as a function of the strategy profile. We say that G is a *finite game* if $\mathcal{I}$ and S_i are finite for all $i \in \mathcal{I}$. If S_i has cardinality m, where $m \geq 2$ is a positive integer, then the set of feasible mixed strategies for player i, denoted Σ_i, is a simplex of dimension $m - 1$.[1] Each element of Σ_i is a probability distribution over the set S_i. Define $S \equiv \Pi_{i\in\mathcal{I}} S_i$ and $\Sigma \equiv \Pi_{i\in\mathcal{I}} \Sigma_i$. We denote a generic element of S_i, Σ_i, S, and Σ by respectively s_i, σ_i, s, and σ. When S_i is a countable set, we let $\sigma_i(s_i)$ denote the probability assigned by σ_i to the pure strategy s_i. Given a mixed strategy profile σ, player i's payoff is

$$u_i(\sigma) \equiv \sum_{s\in S} [\Pi_{j=1}^{I} \sigma_j(s_j)] u_i(s),$$

where we have abused the notation a little to let $u_i(\sigma)$ denote the expected value of $u_i(s)$ under the joint probability distribution σ. Take Example 2 for example, where we have

$$S = S_1 \times S_2 = \{(U, L), (U, R), (D, L), (D, R)\},$$

and if we define $p \equiv \sigma_1(U)$ and $q \equiv \sigma_2(L)$, then

$$u_1(\sigma_1, \sigma_2) = \sum_{s\in S} [\Pi_{j=1}^{I} \sigma_j(s_j)] u_i(s)$$

[1]The simplex in $\Re^n$ is the set $\{\mathbf{x} \in \Re^n_+ : \mathbf{x}'\mathbf{1} = \sum_{j=1}^{n} x_j = 1$. This set is convex, closed (see Homework M) and bounded, and hence compact.

$$= \sigma_1(U)\sigma_2(L)u_1(U,L) + \sigma_1(U)\sigma_2(R)u_1(U,R)$$
$$+\sigma_1(D)\sigma_2(L)u_1(D,L) + \sigma_1(D)\sigma_2(R)u_1(D,R)$$
$$= pqu_1(U,L) + p(1-q)u_1(U,R)$$
$$+(1-p)qu_1(D,L) + (1-p)(1-q)u_1(D,R)$$
$$= pq \cdot 0 + p(1-q) \cdot (-1) + (1-p)q \cdot 2 + (1-p)(1-q) \cdot (-2),$$

which is a polynomial of (p,q), and hence a continuous function of (p,q). Note that, consistent with the notion of non-cooperative games, the mixed strategies of players are assumed to be stochastically independent.

5. **Definition 4:** Let σ_{-i} be some element of

$$\Sigma_{-i} \equiv \Sigma_1 \times \Sigma_2 \times \cdots \Sigma_{i-1} \times \Sigma_{i+1} \times \cdots \Sigma_I,$$

where I, a positive integer, is the cardinality of $\mathcal{I}$. Let $r_i(\sigma_{-i}) \in \Sigma_i$ be the set of *best responses* of player i against σ_{-i}; i.e., $\sigma_i \in r_i(\sigma_{-i})$ if and only if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Sigma_i$. For any (mixed) strategies $\sigma_i, \sigma_i' \in \Sigma_i$, we say that σ_i is *weakly dominated* by σ_i' for player i if

$$u_i(\sigma_i, \sigma_{-i}) \leq u_i(\sigma_i', \sigma_{-i}),\ \forall \sigma_{-i} \in \Sigma_{-i},$$

with the inequality being strict for at least one σ_{-i}; and we say that σ_i is *strictly dominated* by σ_i' for player i if the above inequality is strict for all $\sigma_{-i} \in \Sigma_{-i}$.

Our first equilibrium concept is that *rational players will not use strictly dominated strategies*. Consider the following normal form game, known as the *prisoner's dilemma*:

player 1/player 2	Not Confess	Confess
Not Confess	0,0	-3,1
Confess	1,-3	-2,-2

In this game, "Not Confess" is strictly dominated by "Confess," and hence the unique undominated symmetric outcome is the one where both players confess the crime. Note that unlike Walrasian equilibrium, which by the first fundamental theorem of welfare economics is typically

Pareto efficient,[2] the current equilibrium is Pareto inefficient.[3] That equilibria in a game are in general Pareto inefficient is the first lesson to be learned here (and various economic theories start from here).

6. Based on the implicit assumption that the whole normal form game is the players' *common knowledge*, the above dominance argument can be extended further so that we shall be looking at outcomes that survive from the procedure of *iterative deletion of strictly dominated strategies*. The following is an example.

player 1/player 2	L	M	R
U	0,-1	0,0	1,1
M	2,3	3,1	$\frac{3}{2}$,-1
D	4,2	1,1	2, $\frac{3}{2}$

Note that M is not strictly dominated by L from player 2's perspective, but since U is strictly dominated by M from player 1's perspective, and in the absence of U, M is strictly dominated by L from player 2's perspective, we should not expect player 2 to use M. It follows that player 2 will use L and hence player 1 will use D.

Observe that we have repeatedly used the assumption that rationality of all players is the players' common knowledge. Player 1 knows that player 2 knows that player 1 is rational and would not use U, and hence player 1 can deduce that player 2 would never use M or R (since player 1 knows that player 2 is rational). Thus player 1 chooses to use D, which player 2 can deduce, and hence player 2 chooses to use L.

7. Mixed strategies that assign strictly positive probabilities to strictly dominated pure strategies are themselves strictly dominated. A mixed

[2]See https://en.wikipedia.org/wiki
/Fundamental_theorems_of_welfare_economics#Proof_of_the_first_fundamental_theorem

[3]An arrangement or outcome X for a set of players is Pareto efficient if, relative to X, there is no other (feasible) arrangement or outcome Y that can offer each player a weakly higher payoff and at least one player a strictly higher payoff. (If such Y does exist, then we shall say that Y *Pareto Dominates* X.) In the game *prisoner's dilemma*, for example, (Confess,Confess) is not Pareto efficient, because it is Pareto dominated by (Not Confess, Not Confess).

strategy that assigns strictly positive probabilities only to pure strategies which are not even weakly dominated may still be strictly dominated. Consider the following example:

player 1/player 2	L	R
U	1,3	-2,0
M	-2,0	1,3
D	0,1	0,1

The mixed strategy $(0.5, 0.5, 0)$ for player 1 is strictly dominated by D.

8. It can be shown that in a finite game iterated deletion of *strictly* dominated strategies will lead to a set of surviving outcomes which is independent of the order of deletion; see section 60 for a proof. The same is not true, however, for iterated deletion of *weakly* dominated strategies.

   Consider the following finite strategic game:
   Example 3:

player 1/player 2	L	R
U	1,1	0,0
M	1,1	2,1
D	0,0	2,1

   If we delete U and then L, then we conclude that the payoff profile would be $(2, 1)$. If we delete D and then R, then we conclude that the payoff profile would be $(1, 1)$.

9. **Definition 5:** A normal form game is *dominance solvable* if all players are indifferent between all outcomes that survive the iterative procedure where all the weakly dominated actions of each player are eliminated simultaneously at each stage. Is the game in example 3 dominance solvable?[4] (**Hint**: Player 1 is *not* indifferent about (M,R) and (M,L).)

[4] Definition 5 can be found in Osborne and Rubinstein's *Game Theory*. A related definition is the following: A game is solvable by iterated strict dominance, if the iterated deletion of strictly dominated strategies leads eventually to a unique undominated strategy profile; see Fudenberg and Tirole's *Game Theory*, Definition 2.2.

10. Consider the following two-player game. They each announce a natural number a_i not exceeding 100 at the same time. If $a_1 + a_2 \leq 100$, then player i gets a_i; if $a_1 + a_2 > 100$ with $a_i < a_j$, then players i and j get respectively a_i and $100 - a_i$; and if $a_1 + a_2 > 100$ with $a_i = a_j$, then each player gets 50. Determine if this game is dominance solvable.

11. Consider example 1 with the further specifications that $c = F = 0$, $P(q_1 + q_2) = 1 - q_1 - q_2$. Like the game of *prisoner's dilemma*, there is a unique outcome surviving from the iterated deletion of strictly dominated strategies. We say that this game has a *dominance equilibrium*. To see this, note that the best response for firm i as a function of q_j (called firm i's *reaction function*) is

$$q_i = r_i(q_j) = \frac{1 - q_j}{2}.$$

Observe that $q_1 > \frac{1-0}{2} = r_1(0)$ is strictly dominated for firm 1.[5] Since rationality of firm 1 is the two firms' common knowledge, firm 2 will realize that any $q_2 < \frac{1-\frac{1}{2}}{2} = r_2(r_1(0))$ is strictly dominated by $r_2(r_1(0))$.[6] But then, common knowledge about rationality again implies that any $q_1 > \frac{1-\frac{1}{4}}{2}$ is strictly dominated for firm 1, which in turn implies that $q_2 < \frac{1-\frac{3}{8}}{2}$ is strictly dominated for firm 2.

[5]Observe that (i) given any $q_j \in \Re_+$, $\Pi_i(\cdot, q_j)$ is a strictly decreasing function on $[r_i(q_j), +\infty)$; and (ii) $r_i(\cdot)$ is a strictly decreasing function on $\Re_+$. Thus if $q_i > \frac{1-0}{2} = r_i(0)$, then for all $q_j \in \Re_+$, we have

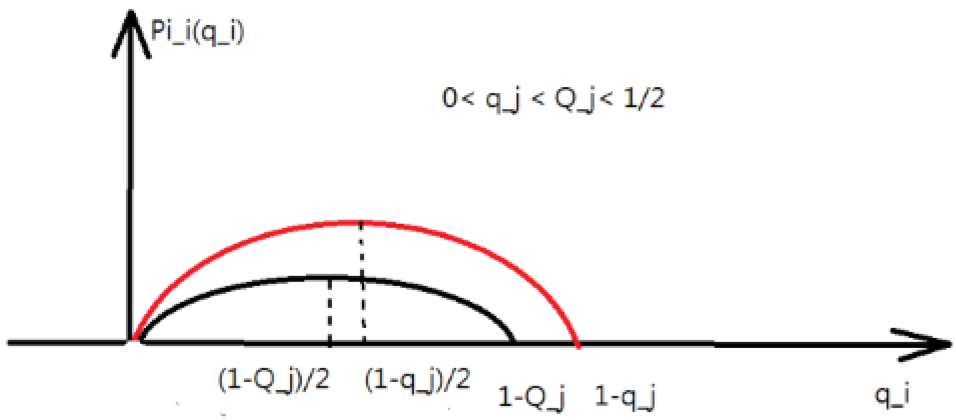
$$q_i > r_i(0) \geq r_i(q_j) \Rightarrow \Pi_i(r_i(0), q_j) > \Pi_i(q_i, q_j),$$

proving that any such q_i is strictly dominated by $r_i(0)$, regardless of firm j's choice q_j. That is, firm i will never choose a q_i that exceeds $\frac{1}{2}$.

[6]Note that $\Pi_2(\cdot, q_1)$ is a strictly increasing function on $[0, r_2(q_1)]$. Since by common knowledge about firm 1's rationality, firm 2 believes that any q_1 chosen by firm 1 must satisfy $q_1 \leq r_1(0)$, we must have

$$q_2 < r_2(r_1(0)) \leq r_2(q_1) \Rightarrow \Pi_2(q_2, q_1) < \Pi_2(r_2(r_1(0)), q_1),$$

proving that any such q_2 is strictly dominated by $r_2(r_1(0))$ from firm 2's perspective, no matter which $q_1 \leq r_1(0)$ chosen by the rational firm 1. That is, given that firm 1 will never choose a q_1 that exceeds $\frac{1}{2}$, firm 2 will never choose a q_2 below $\frac{1}{4}$.



By symmetry, the above argument can be repeated to show that for each firm $i \in \{1, 2\}$, and for all positive integers n, if $q_i > q_n^u$ is strictly dominated then so is $q_i > q_{n+1}^u \equiv \frac{1-\frac{1-q_n^u}{2}}{2}$; and if $q_i < q_n^d$ is strictly dominated then so is $q_i < q_{n+1}^d \equiv \frac{1-\frac{1-q_n^d}{2}}{2}$, where we define

$$q_1^d = 0, \ q_1^u = \frac{1}{2}.$$

Note that $\{q_n^u\}$ and $\{q_n^d\}$ are respectively decreasing and increasing sequences with $q_n^u > \frac{1}{3} > q_n^d$ for all n. It follows from the *nested intervals theorem* (section 28 of Lecture M) that for both firms there is exactly one output level surviving the iterated deletion of strictly dominated strategies, which is the unique element contained in

$$\bigcap_{n=1}^{\infty} [q_n^d, q_n^u] = \{\frac{1}{3}\};$$

so that $(q_1^*, q_2^*) = (\frac{1}{3}, \frac{1}{3})$ is the (unique) dominance equilibrium. (Recall that it is shown in Lecture 0 that $(q_1^*, q_2^*) = (\frac{1}{3}, \frac{1}{3})$ is the unique pure-strategy Nash equilibrium for this game.)

12. (**Market Share Competition**)

player 1/player 2	Don't Promote	Promote
Don't Promote	1,1	0,2-c
Promote	2-c,0	1-c,1-c

In the above, two firms can spend $c > 0$ on promotion. One firm would capture the entire market by spending c if the other firm does not do so. Show that when c is sufficiently small, this game has a dominance equilibrium.

13. Consider also the following moral-hazard-in-team problem: Two workers can either work (s=1) or shirk (s=0). They share the output $4(s_1 + s_2)$ equally. Working however incurs a private cost of 3. Show that this game has a dominance equilibrium.

14. **Definition 6:** A pure strategy Nash equilibrium (NE) for a game in normal form is a set of pure strategies (called a *pure strategy profile* from now on), one for each player, such that if other players play their specified strategies, a player also finds it optimal to play his specified strategy. In this case, these strategies are *equilibrium strategies.* For instance, (U,R) and (D,L) are two pure strategy NEs for the game in example 2. A mixed strategy NE is defined similarly, where the equilibrium strategies are mixed strategies.[7]

    Formally, a profile $\sigma \in \Sigma$ is a (mixed strategy) NE if and only if

    $$u_i(\sigma) \geq u_i(s_i, \sigma_{-i}),\ \forall i \in \mathcal{I},\ \forall s_i \in S_i.$$

    Observe that this definition asks us to check unilateral deviations in *pure strategy* only! Show that, however, it implies that

    $$u_i(\sigma) \geq u_i(\sigma_i', \sigma_{-i}),\ \forall i \in \mathcal{I},\ \forall \sigma_i' \in \Sigma_i.$$

    Show that the following is an equivalent definition for (mixed strategy) NE: a profile $\sigma \in \Sigma$ is an NE if and only if for all $i \in \mathcal{I}$, for all $s_i, s_i' \in S_i$,

    $$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \Rightarrow \sigma_i(s_i) = 0.$$

[7] The following interpretation may be helpful. In example 2, if player 2 expects player 1 to play U, then his best response is to play R, and if player 1 correctly expects that the above is player 2's belief, then he cannot gain by picking an action that proves that player 2's belief is incorrect. An NE is by definition a situation where, if the two players expect their strategy profile, they cannot gain by making *unilateral deviations.*

This new equivalence condition says that if a pure strategy s_i is assigned by σ_i with a strictly positive probability, then s_i has to be a best response against σ_{-i}.

15. A single auction consists of a seller facing more than one buyer (or a buyer facing more than one seller) with some object(s) to sell (respectively, to buy), where the multiple buyers (respectively, the multiple sellers) submit bids to compete for the object(s). The object is of private value if learning other bidders' valuations for the object does not change one's own valuation, and it is of common value if bidders have an (unknown) identical valuation. The four popular auction rules are English auction, Dutch auction, the first-price sealed-bid auction, and the second-price sealed-bid auction. With an English auction, the seller-auctioneer will start with a minimum acceptable price and raise the price gradually to attain a situation where there is exactly one bidder remains interested in the object, and in the latter situation the object is rendered to that bidder in exchange for the price at which the winning bidder is selected. With a Dutch auction, the seller-auctioneer will start with a very high price and then reduce the price gradually to attain a situation where at least one bidder shows interest in the object at that price, and in the latter situation the object is sold to the first bidder who reveals his interest. In the two sealed-bid auctions mentioned above, all bidders must submit secretly their bids at the same time to an auctioneer, and the object is delivered to the bidder submitting the highest bid. The difference between the two sealed-bid auctions is that in the first-price auction, the winning bidder must pay his own bid, but in the second-price auction, the winning bidder will pay the highest losing bid. See Milgrom and Weber (1982).

    Now suppose that an indivisible object will be sold to one of the N bidders whose private valuations for the object are respectively $v_1, v_2, \cdots, v_N$. Show that a second-price sealed-bid auction with private values for an indivisible object has an NE where all bidders bid their valuations for the object. (**Hint**: Consider bidder 1's problem. Let the random variable $\tilde{B}$ denote the maximum of the bids submitted by bidders $2, 3, \cdots, N$. Compare any bid b submitted by bidder 1 with the bid v_1 we suggest. Note that with the bidding strategy b, bidder

1's expected payoff is

$$E[(v_1 - \tilde{B})1_{[b>\tilde{B}]}(b)],$$

where (i) we have assumed that $\tilde{B}$ is a continuous random variable and so have ignored the probability of a tie, and (ii) $1_A(x)$ is an indicator function for event A, so that it equals one if $x \in A$ and zero if otherwise. Show that b and v_1 make a difference only when $v_1 > \tilde{B} > b$ or $v_1 < \tilde{B} < b$ (again ignoring the probability of a tie).[8] Deduce that b is weakly dominated by v_1.)

16. Consider N bidders competing for one indivisible object, for which they attach values $v_1 < v_2 < \cdots < v_N$ respectively. Show that if the seller adopts a first-price sealed-bid auction, then in all NE's for this game, player 1 gets the object; see Appendix for a discussion about the case $N = 2$.

17. A game is called a *zero-sum* game if $\sum_{i \in \mathcal{I}} u_i(\sigma)$ is independent of $\sigma \in \Sigma$.[9] A zero-sum game may have multiple NE's, but players obtain the same payoff profile in all the NE's. To see this, suppose that there are two players, and that σ and σ' are both NE's for the zero-sum game. It must be that

$$u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1', \sigma_2)$$

and

$$-u_1(\sigma_1', \sigma_2') = u_2(\sigma_1', \sigma_2') \geq u_2(\sigma_1', \sigma_2) = -u_1(\sigma_1', \sigma_2)$$

so that

$$u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1', \sigma_2').$$

The same argument shows that

$$u_1(\sigma_1, \sigma_2) \leq u_1(\sigma_1', \sigma_2')$$

[8] Consider the case where $\tilde{B}$ may be discrete. Submitting b or v_1 makes no difference in the event where $v_1 = \tilde{B}$. Note that v_1 dominates b in the events $b = \tilde{B} > v_1$ and $b = \tilde{B} < v_1$, because submitting b may win with a positive probability in the former event, and submitting b may lose with a positive probability in the latter event. Winning is good if $\tilde{B} < v_1$ but is bad if $\tilde{B} > v_1$.

[9] Thus we had better call it a *constant-sum* game. However, recall that a von Neumann-Morgenstern utility function is determined only up to a positive affine transform (see my note in Investments, Lecture 2), and hence deducting the payoffs by the same number really changes nothing relevant.

and hence we conclude that

$$u_1(\sigma_1, \sigma_2) = u_1(\sigma_1', \sigma_2')$$

implying that

$$u_2(\sigma_1, \sigma_2) = u_2(\sigma_1', \sigma_2')$$

as well.

18. Unlike dominance equilibrium, which may not exist for a game, a Nash equilibrium always exists for a finite normal form game. This is stated in Theorems 1 and 2 below.

Theorem 1 (Wilson, 1971; Harsanyi, 1973): Almost every finite strategic game has an odd number of NE's in mixed strategies; more precisely, with the set of players and their strategy spaces fixed, the set of payoff functions that result in a strategic game having an even number of NE's has zero Lebesgue measure.[10]

Following Theorem 1, we guess that there is at least one mixed strategy NE for the game in Example 2. Let us find them. The key observation here is that, if a mixed strategy assigns strictly positive probabilities to more than one pure strategy, then the player must feel indifferent about these pure strategies. Let x be the prob. that player 1 uses U and y the prob. that player 2 uses L. We must have

$$1 \cdot x + (-1) \cdot (1 - x) = 2 \cdot x + (-2) \cdot (1 - x) \Rightarrow x = \frac{1}{2};$$

$$0 \cdot y + (-1) \cdot (1 - y) = 2 \cdot y + (-2) \cdot (1 - y) \Rightarrow y = \frac{1}{3}.$$

These mixed strategies are said to be *totally mixed*, in the sense that they assign strictly positive prob.'s to *each and every* pure strategy. Here we have only one mixed strategy NE. If we have two totally

[10]Take Example 2 for example. There are 8 real numbers appearing in the bi-matrix, which we can draw randomly from $\Re^8$, according to, say, an eight-dimensional multivariate normal distribution. The probability that the 8 numbers are drawn from a set $A \subset \Re^8$ of zero Lebesgue measure is then zero. In this sense, the chance that the game in Example 2 has an even number of NEs when we alter the 8 payoff numbers is zero!

mixed NE's, then we naturally have a continuum of mixed strategy NE's (why?). From now on, we denote the set of totally mixed strategies of player i by Σ_i^0. Observe that Σ_i^0 is simply the *interior* of Σ_i (see Lecture M or Homework M), when Σ_i is endowed with the usual Euclidean topology.

19. (**Matching Pennies**) Note that some games do not have pure strategy Nash equilibrium:

player 1/player 2	H	T
H	1,-1	-1,1
T	-1,1	1,-1

20. **Theorem 2** (Nash, 1950): Every finite normal-form game has at least one mixed-strategy NE (which may or may not be a pure-strategy NE).

    Theorem 2 is actually a special version of the following more general theorem (see Fudenberg and Tirole's *Game Theory*, Theorem 1.2).

    Theorem 2′: (Debreu-Glicksberg-Fan) Consider a strategic game

    $$G = \{\mathcal{I}, S = (S_1, S_2, \cdots, S_I), (u_i : S \to \Re; i \in \mathcal{I})\}$$

    with $\mathcal{I}$ being a finite set. If for all $i \in \mathcal{I}$, S_i is a nonempty compact convex subset of some Euclidean space, and if for all $i \in \mathcal{I}$, u_i is continuous in s and quasi-concave in s_i when given s_{-i},[11] then G has a pure-strategy NE.

21. Suppose that $r : X \to Y$ is a correspondence (a multi-valued function), where X and Y are some subsets of $\Re^n$.

[11] A function $f : \Re^n \to \Re$ is quasi-concave if for all $r \in \Re$, the pre-image $f^{-1}([r, +\infty))$ is a convex subset of $\Re^n$. It is easy to see that f is quasi-concave if f is concave: let $x, y \in f^{-1}([r, +\infty))$, so that $f(x), f(y) \geq r$, implying that for all $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y) \geq r,$$

and hence $\lambda x + (1 - \lambda)y \in f^{-1}([r, +\infty))$, showing that $f^{-1}([r, +\infty))$ is a convex set. Verify that the function $f : \Re^1 \to \Re^1$ defined by $f(x) = e^x$ is quasi-concave but not concave.

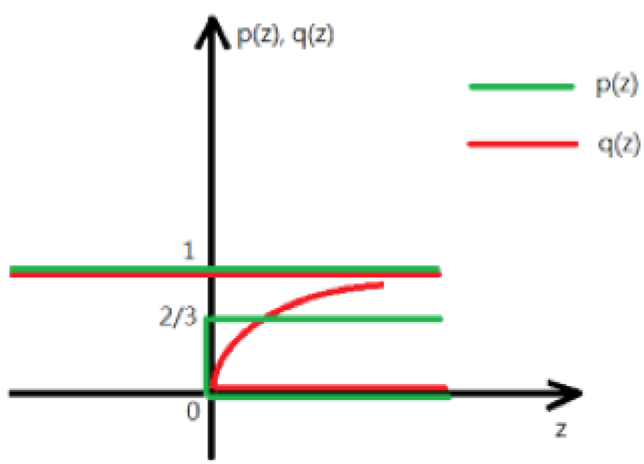
- If $r(x) \neq \emptyset$ for all $x \in X$, then we say that $r(\cdot)$ is non-empty.
- If $r(x)$ is a convex (compact, closed) subset of $\Re^n$ for all $x \in X$, then we say that $r(\cdot)$ is convex-valued (compact-valued, closed-valued), or simply convex.
- We say that $r(\cdot)$ is upper hemi-continuous (u.h.c.) at x if for every open set V containing $r(x)$, there exists an open set U containing x such that $a \in U \Rightarrow r(a) \subset V$.[12] We say that $r(\cdot)$ is u.h.c. if it is u.h.c. at each and every $x \in X$.
- Suppose that $r(\cdot)$ is closed. If $r(\cdot)$ is u.h.c. at x, then (Θ) whenever $\{x_n\}$ converges to x and $\{y_n \in r(x_n)\}$ converges to y, we have $y \in r(x)$. If Y is compact, then $r(\cdot)$ is u.h.c. at x if (Θ) holds.
- We say that $r(\cdot)$ is lower hemi-continuous (l.h.c.) at $x \in X$ if for all open set $V \subset Y$ such that $V \bigcap r(x) \neq \emptyset$, there exists some open set $U \in X$ such that $a \in U \Rightarrow r(a) \bigcap V \neq \emptyset$. We say that $r(\cdot)$ is l.h.c. if it is l.h.c. at each and every $x \in X$.
- It can be shown that $r(\cdot)$ is lower hemi-continuous (l.h.c.) at $x \in X$ if and only if whenever $\{x_n\}$ converges to x, for each $y \in r(x)$, there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and a sequence $\{y_{n(k)} \in r(x_{n(k)})\}$ such that the latter sequence $\{y_{n(k)}\}$ converges to y.
- If r is u.h.c. and compact-valued, then $r(A)$ is compact in Y if A is compact in X.
- The correspondence $r(\cdot)$ is said to be continuous if it is both upper and lower hemi-continuous.

22. Here we give some examples of upper hemi-continuous correspondences. First consider the following strategic game

player 1/player 2	L	R
U	1,1	0,0
D	0,0	z,2

[12] See: https://en.wikipedia.org/wiki/Hemicontinuity#Sequential_characterization.

Let us consider all mixed strategy NE's of this game. Any mixed strategy NE can be represented by (p, q), where p is the probability that player 1 adopts U and q the probability that player 2 adopts L. Simple calculations give the following results. For $z > 0$, the game has three NE's, $(p, q) = (1, 1)$, $(p, q) = (0, 0)$, and $(p, q) = (\frac{2}{3}, \frac{z}{1+z})$; for $z < 0$, iterated deletion of strictly dominated strategies implies that the game has a unique NE, which is $(p, q) = (1, 1)$; and for $z = 0$, the game has an (uncountably) infinite number of NE's: besides $(1, 1), (0, 0)$, and $(\frac{2}{3}, 0)$, which are obtained from the above three NE's by letting $z \downarrow 0$, any $(p, 0)$ with $0 \leq p \leq \frac{2}{3}$ are also NE's. The idea is that as long as player 2 is willing to play R with probability one, given $z = 0$, player 1 is genuinely indifferent about U and D, and hence can randomize with any probability p, and for $p \leq \frac{2}{3}$, player 2 really prefers R to L.

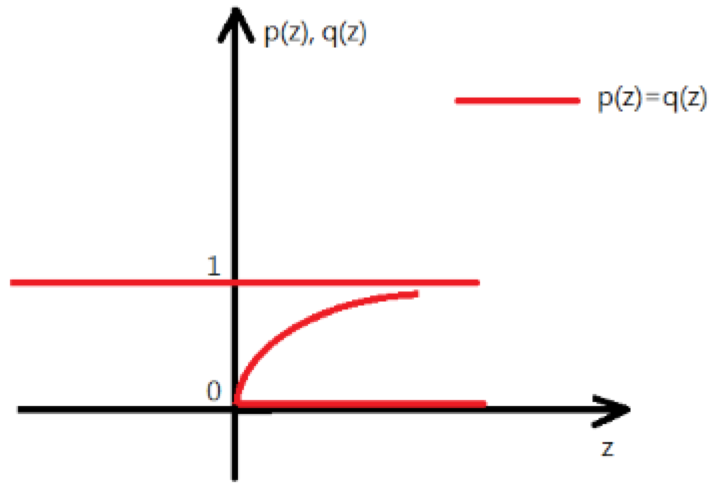


Apparently, player 1's equilibrium mixed strategy p depends on z (which is the sole parameter here to distinguish one strategic game from another). This dependence defines a correspondence $p(z)$, called the *Nash equilibrium correspondence*. This correspondence is easily seen to be upper hemi-continuous (and if you draw the graph of $p(\cdot)$, then you will realize why we also refer to upper hemi-continuity by the *closed graph* property), but it fails to be lower hemi-continuous. Observe that any $p \in (0, \frac{2}{3})$ is contained in $p(0)$, but it is not the limit of a sequence $p(z_n)$ that corresponds to a sequence of z_n that converges to 0. Note that except for the case where $z = 0$ (which is an event of zero Lebesgue measure on $\Re$), this game has an odd number of NE's, verifying Wilson's theorem.

Next, consider the following strategic game

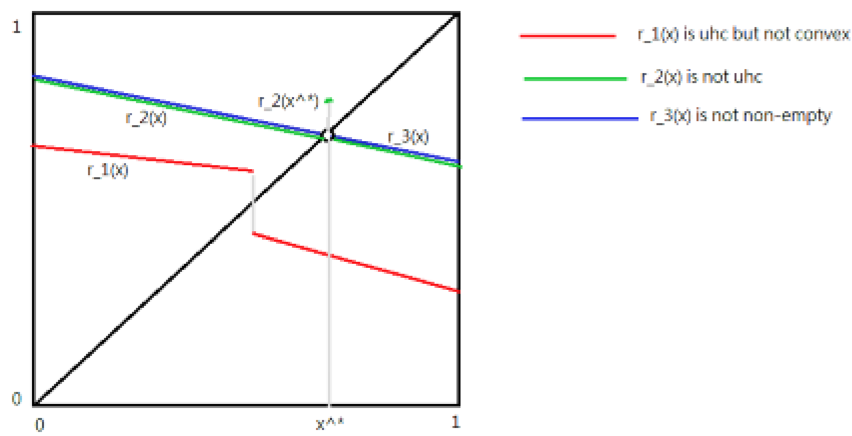
player 1/player 2	L	R
U	1,1	0,0
D	0,0	z,z

Again, any mixed strategy NE can be represented by (p, q), where p is the probability that player 1 adopts U and q the probability that player 2 adopts L. Simple calculations give the following results. For $z > 0$, the game has three NE's, $(p, q) = (1, 1)$, $(p, q) = (0, 0)$, and $(p, q) = (\frac{z}{1+z}, \frac{z}{1+z})$; for $z < 0$, iterated deletion of strictly dominated strategies implies that the game has a unique NE, which is $(p, q) = (1, 1)$; and for $z = 0$, the game has two NE's: $(1, 1)$ and $(0, 0)$. Again, consider the correspondence $p(z)$. It is still upper hemi-continuous, although the number of NE's drops in the limit as $z \downarrow 0$. Note also that this game has an odd number of NE's except when $z = 0$.



23. **Theorem 3** (Kakutani, 1941): Let $r : \Sigma \to \Sigma$ be a nonempty, convex, upper hemi-continuous correspondence where $\Sigma \subset \Re^n$ is nonempty, convex and compact. Then, there exists $\sigma^* \in \Sigma$ such that $\sigma^* \in r(\sigma^*)$, where we refer to σ^* as a fixed point of correspondence $r(\cdot)$.

24. The following figure, in which $\Sigma = [0, 1]$ and three correspondences are all single-valued, shows how a correspondence may fail to have a fixed point. Note that for a single-valued correspondence, upper hemi-continuity is equivalent to continuity. In the figure, $r_1(\cdot)$ is not convex-valued, $r_2(\cdot)$ is not u.h.c, and $r_3(\cdot)$ is not non-empty. The three correspondences do not intersect with the 45-degree line, and hence have no fixed points.



25. We now sketch the proof of Theorem 2 using Theorem 3. Let S_i and Σ_i be respectively player i's pure strategy space and mixed strategy space. Assume there are n players. Then, Σ_i is a simplex which is compact. Denote $\Sigma = \Pi_{i=1}^n \Sigma_i$. Let σ_i be a typical element of Σ_i, and let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ and $\sigma_{-i} = (\sigma_1, \sigma_2, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n)$. Let $r_i(\sigma_{-i})$ be player i's best response against other players' mixed strategies σ_{-i}, and define $r(\sigma)$ as the Cartesian product of $r_i(\sigma_{-i})$. Now $r : \Sigma \to \Sigma$ is a correspondence defined on Σ! If we can show the existence of a fixed point of r, then we are done.

    Since players' payoff functions are linear (hence continuous) in mixed strategies, and since the simplex Σ_i is compact, there exist solutions to players' best response problems (Weierstrass theorem). That is, r is nonempty. Moreover, if $\sigma_{i1}, \sigma_{i2} \in r_i(\sigma_{-i})$, then clearly for all $\lambda \in [0, 1]$,

$$\lambda\sigma_{i1} + (1 - \lambda)\sigma_{i2} \in r_i(\sigma_{-i}),$$

    proving that r_i is convex, which in turn implies that r is convex. Finally, we claim that r is u.h.c. To see this, suppose instead that r fails to be upper hemi-continuous at some profile $\sigma \in \Sigma$, then there must exist some player i, a sequence $\{\sigma_{-i,n}; n \in \mathbf{Z}_+\}$ converging to σ_{-i} in Σ_{-i}, and a sequence $\{\sigma'_{i,n}; n \in \mathbf{Z}_+\}$ in Σ_i converging to some $\sigma'_i \in \Sigma_i$ with $\sigma'_{i,n} \in r_i(\sigma_{-i,n})$ for all $n \in \mathbf{Z}_+$ such that σ'_i is not contained in $r_i(\sigma_{-i})$. This would mean that there exists some $\sigma''_i \in \Sigma_i$ such that $u_i(\sigma''_i, \sigma_{-i}) > u_i(\sigma'_i, \sigma_{-i})$; that is, σ''_i is a better response than σ'_i for player i against the other players' strategy profile σ_{-i}. Since $u_i(\cdot)$ is continuous, and since the sequence $\{\sigma'_n; n \in \mathbf{Z}_+\}$ converges to σ', this would mean that $u_i(\sigma''_i, \sigma_{-i,n}) > u_i(\sigma'_{i,n}, \sigma_{-i,n})$ for n sufficiently large, a contradiction to the fact that $\sigma'_{i,n} \in r_i(\sigma_{-i,n})$ for all $n \in \mathbf{Z}_+$.

Now since $r(\cdot)$ is non-empty, convex, and u.h.c., and since Σ is non-empty, convex, and compact in $\Re^n$, by theorem 3 there must exists some $\sigma^* \in \Sigma$ such that $\sigma^* \in r(\sigma^*)$.

26. From the proof of Theorem 2, we can infer that discontinuous payoff functions may result in non-existence of NE's.

When payoff functions fail to be continuous, it may happen that the best response correspondence is not non-empty, or fails to be upper hemi-continuous. For example, consider the single-player game where his payoff function $f : \Re \to \Re$ is defined as $f(x) = -|x| \cdot 1_{[x \neq 0]} - 1_{[x=0]}$, then the player has no best response. Even if all best responses are well defined, the best response correspondence my fail to be upper hemi-continuous, as shown by the following example, which is taken from section 1.3.3 of Fudenberg and Tirole (1991).[13]

Suppose we have a two-player strategic game where $S_1 = S_2 = [0, 1]$, $u_1(s) = -(s_1 - s_2)^2$, and

$$u_2(s) = -(s_1 - s_2 - \frac{1}{3})^2 \cdot 1_{[s_1 \geq \frac{1}{3}]} - (s_1 - s_2 + \frac{1}{3})^2 \cdot 1_{[s_1 < \frac{1}{3}]}.$$

Observe first that $r_1(s_2)$ is a (single-valued) function. Indeed, given player 2's mixed strategy $\tilde{s}_2$, which by the fact that $S_2 = [0, 1]$ must have a finite variance, player 1 would seek to

$$\min_{s_1 \in [0,1]} f_1(s_1) \equiv E[(s_1 - \tilde{s}_2)^2],$$

[13]Fudenberg, D., and J. Tirole, 1991, *Game Theory*, MIT Press.

so that the necessary and sufficient first-order condition gives

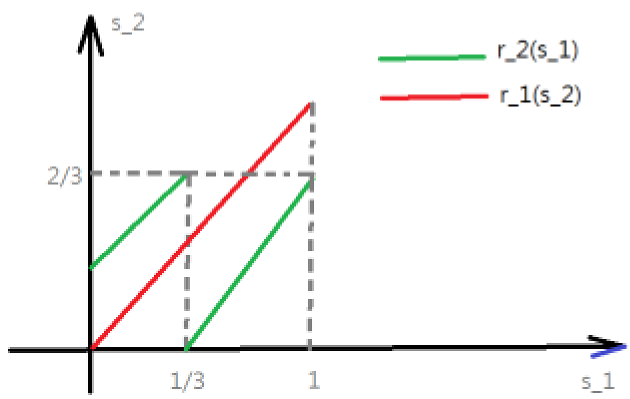
$$r_1(\tilde{s}_2) = E[\tilde{s}_2].$$

On the other hand, given player 1's pure strategy s_1, $r_2(s_1)$ is also a (single-valued) function. If $s_1 \geq \frac{1}{3}$, then player 2 would seek to

$$\min_{s_2 \in [0,1]} (s_1 - \frac{1}{3} - s_2)^2 \Rightarrow r_2(s_1) = s_1 - \frac{1}{3};$$

and if $s_1 < \frac{1}{3}$, then player 2 would seek to

$$\min_{s_2 \in [0,1]} (s_1 + \frac{1}{3} - s_2)^2 \Rightarrow r_2(s_1) = s_1 + \frac{1}{3}.$$

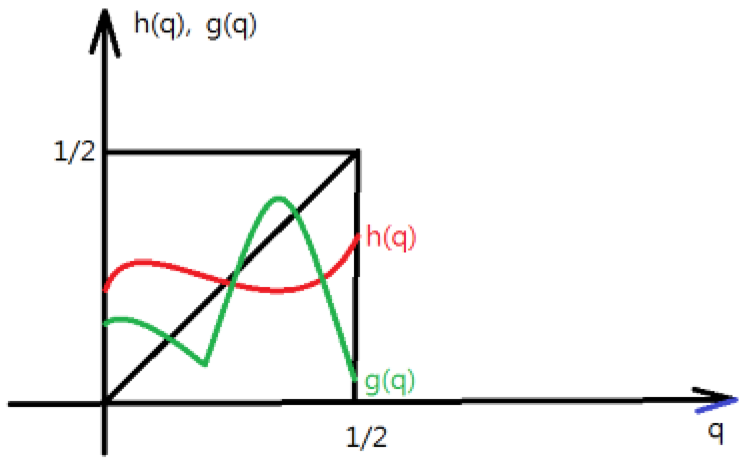


It is easy to see that $r_2(\cdot)$ is not upper hemi-continuous; upper hemi-continuity becomes the usual continuity of a single-valued function when the correspondence is actually single-valued. Here, evidently, $r_2(s_1)$ and $r_1(s_2)$ do not intersect, and hence this two-player game has no (pure-strategy or mixed-strategy) NE at all.

27. Example 1 (see the section on dominance equilibrium) can be used to understand the above Theorems 1 and 2. Define $h_i(\cdot) \equiv r_i(r_j(\cdot))$. Apparently, this function can be restricted to the domain of defintion $[0, \frac{1}{2}]$, which is a non-empty, convex, compact subset of $\Re$, and moreover, the functional value of h_i is also contained in $[0, \frac{1}{2}]$. If this function intersects with the 45-degree line, then the intersection defines a pure strategy NE. (This game has no mixed strategy NE because a firm's profit is a strictly concave function of his own output level given any output choice of its rival.) Now it follows from Brouwer's fixed point theorem (a special version of theorem 3) that if $h_i(\cdot)$ is continuous[14] then there exists an NE for the game, and moreover, as one can verify, generically a continuous $h_i(\cdot)$ will intersect the 45-degree line in an odd number of times; here genericity means that we disregard the following exceptional cases where $h_i(0) = 0$ or $h_i(\frac{1}{2}) = \frac{1}{2}$ or $h_i'(q) = 1$ at some point $q \in [0, \frac{1}{2}]$.[15]

[14]With the specification $P(q_1 + q_2) = 1 - q_1 - q_2$, $h_i(q_i) = \frac{1 - \frac{1-q_i}{2}}{2}$, which is indeed a continuous function of q_i.

[15]Brouwer's fixed point theorem says that if $f : A \to A$ is continuous, where $A \subset \Re^n$ is non-empty, compact and convex, then there exists $x \in A$ such that $f(x) = x$.



28. **Theorem 4**: Consider a two-player game where each player i simultaneously chooses a number s_i in the unit interval, and where player i's payoff function $u_i(s_1, s_2)$ is continuous in (s_1, s_2). This game has a mixed strategy NE.
Proof Call this game Γ. Consider a sequence of modified games $\{\Gamma_n\}$ in normal form where the set of players and the payoff functions are the same as in Γ, but the players' common pure strategy space is

$$S_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\}.$$

Theorem 2 implies that for all $n \in \mathbf{Z}_+$, Γ_n has an NE (denoted σ^n) in mixed strategy. Since the set of probability measures on $[0,1]$ is weakly compact (see my note on "distribution functions" in the course of stochastic processes), and the products of compact spaces are compact (see my note on "basic ideas in topology" in the course of stochastic processes), the sequence $\{\sigma^n\}$ has a convergent subsequence in weak topology, for which let σ be the limiting probability measure. Let the convergent subsequence be $\{\sigma^{n_k}; k \in \{Z\}_+\}$. We claim that σ is a mixed strategy NE for Γ. The proof proceeds in two steps. First, every prob. measure on $[0,1]$ is the limit (in weak topology) of a sequence of prob. measures, where the k-th term in the sequence is a prob. measure on S_{n_k} (recall Helly's convergence theorem). Second, let σ_i' be any prob. measure on $[0,1]$, then by step 1, there exists a sequence $\{[\sigma_i^{n_k}]'\}$ that converges weakly to it. Fix $i \in \{1,2\}$. For each $k \in \mathbf{Z}_+$, we have by definition,

$$u_i(\sigma_i^{n_k}, \sigma_j^{n_k}) - u_i([\sigma_i^{n_k}]', \sigma_j^{n_k}) \geq 0,$$

which, by the continuity of u_i in (s_1, s_2) and the definition of weak convergence of prob. measures on $[0,1]$, implies that

$$u_i(\sigma_i, \sigma_j) - u_i(\sigma_i', \sigma_j) \geq 0,$$

proving that σ is an NE in mixed strategy for Γ.

29. Theorem 4 is a special case of the following theorem:
Theorem 4′: (Glicksberg, 1952) Consider a strategic game

$$G = \{\mathcal{I}, S = (S_1, S_2, \cdots, S_I), (u_i : S \to \Re; i \in \mathcal{I})\}$$

with $\mathcal{I}$ being a finite set (the game has a finite number of players). If for all $i \in \mathcal{I}$, S_i is a nonempty compact subset of some (common) metric space M, and if for all $i \in \mathcal{I}$, u_i is continuous, then G has an NE in mixed strategy.

30. A strategic game is *symmetric* if $u_i = u_j$ and $S_i = S_j$ for all $i, j \in \mathcal{I}$.
Theorem 5: A finite symmetric game has a symmetric Nash equilibrium in mixed strategy.
Proof Suppose that in an I-person finite game, $S_i = S_1$ and $u_i(\cdot) = u_1(\cdot)$ for all $i = 1, 2, \cdots, I$. This implies that $\Sigma_i = \Sigma_1$ for all i. We show this game has a symmetric NE. For any profile σ, we write

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}).$$

For any $\sigma_1 \in \Sigma_1$, define

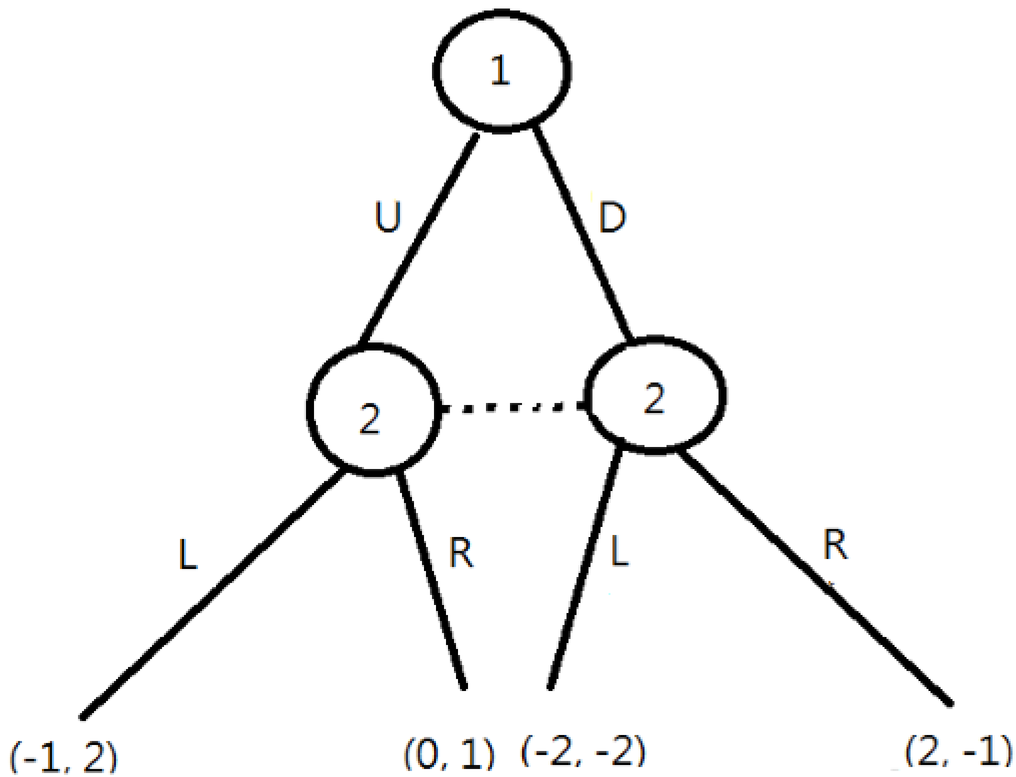
$$R_i(\sigma_1) = arg \max_{\sigma_i \in \Sigma_1} u_i(\sigma_i, \sigma_{-i})$$

subject to

$$\sigma_{-i} = \sigma_1^{I-1}.$$

It is clear that $R_i(\cdot)$ is independent of i. Since Σ_1 and $R_i : \Sigma_1 \to \Sigma_1$ satisfy all the requirements for Kakutani's theorem to apply, we conclude that for some $\sigma_1 \in \Sigma_1$, $\sigma_1 \in R_i(\sigma_1)$, where the profile σ_1^I is obviously an NE.

31. **Definition 7:** A game can be described in *extensive form*, which needs to specify the timing of players' moves and what they know when they move, in addition to the things specified in normal form. A game in extensive form is also known as an *extensive game*. Usually, a game in extensive form is depicted as a *game tree*:



Note that player II's information set is denoted by the two nodes connected by a dotted line, which says that player II, when determining her own moves, does not know whether player I has moved up or down. Formally, an *information set* is a set of decision nodes for a player, who cannot tell which node in the information set he is currently on. A game where at least one player has incomplete knowledge about the history of the game up to a point when he or she is ready to take a move is called *a game with imperfect information*. Such a game has at least one non-singleton information set in the game tree. In contrast, a game where all the information sets appearing in its game tree are singletons is a *game with perfect information.*

32. Because the extensive form specifies two more things than the normal form, there does not exist a one-to-one correspondence between the two. In fact, a normal form can correspond to more than one extensive form. Consider the following extensive game. Player 1 first chooses among pure strategies A, B, and C. If A is chosen, then player 1 gets 2 and player 2 gets 1. Otherwise, upon seeing player 1's choice of action, player 2 can react by choosing either L or R. The payoffs resulting from these strategies are summarized in the following bimatrix.

player 1/player 2	L	R
B	4,2	1,0
C	0,1	$\frac{3}{2}$,0

Draw the extensive form for this game.

33. So, how do we construct the strategic form from its extensive counterpart? The complication is that we need to define pure and mixed strategies for the normal form game, while all we know is what each player can do at each of his information sets in the extensive game. First, we define a *pure strategy* for player i as a complete description of which action player i will take (with probability one) at each of his information sets. Second, if two pure strategies of player i always ("always" means when put together with any profile $\sigma_{-i} \in \Sigma_{-i}$) generate the same payoff for player i, then they are said to be *equivalent*, and a *reduced normal form* game is obtained from the original extensive game if for each player, equivalent pure strategies are identified (all of them are removed but one).[16] A *mixed strategy* for player i is then defined as

[16]For example, consider an extensive game where player 1 first chooses between A and B, and if A is chosen then the game ends; or else, following B, players 1 and 2 simultaneously

a probability distribution over all feasible pure strategies in the reduced normal form game. To fix terminology, mixed strategies in the original extensive game are referred to as *behavior strategies*, and each behavior strategy of player i specifies a set of probability distributions for player i, where each probability distribution corresponds to one information set of player i, indicating how player i may randomly choose among the actions feasible at that particular information set.

34. What are the relations between mixed strategies in the strategic form and behavior strategies in the extensive form? The answer is that, each mixed strategy generates a unique behavior strategy, but a behavior strategy can be generated by more than one mixed strategy. Kuhn's theorem tells us that in any game of perfect recall, every mixed strategy is equivalent to the behavior strategy that it generates, and every behavior strategy is equivalent to all the mixed strategies that generate it; see Fudenberg and Tirole's *Game Theory* for details.[17]

35. Consider the Cournot game in example 1. Let

$$P(Q) = 1 - Q,\ Q = q_1 + q_2,\ c = F = 0.$$

This is a game with *simultaneous moves*, and hence a game with *imperfect information*. Let us solve for the pure strategy NE.[18] By definition, it is a pair (q_1^*, q_2^*), such that given q_2^*, q_1^* is profit maximizing for firm 1, and given q_1^*, q_2^* is profit maximizing for firm 2. The procedure is first to find the reaction function for firm i given any q_j firm j might choose:

$$\max_{q_i} f(q_1) = q_i(1 - q_i - q_j),$$

where $f(\cdot)$ is concave and hence the first-order condition is necessary and sufficient for optimal solution. This gives

$$q_i^* = \frac{1 - q_j}{2}.$$

choose between a and b. Player 1 has four pure strategies, and two of them, (A,a) and (A,b), are equivalent strategies; for another example, see the game *Battle of Sex* below.

[17] A game of perfect recall is an extensive game with special restrictions on its information sets: players never forget what they knew in the past. In application, almost all games we shall encounter are of perfect recall.

[18] For any mixed strategy σ of firm j, firm i's payoff function $\pi_i(q_i;\sigma) = q_i(1-q_i-E_\sigma[q_j])$ is strictly concave. This implies that the set of best responses must be a singleton, and hence this game has no mixed strategy equilibria.

An NE, by definition, is the intersection of the two firms' reaction functions. Solving simultaneously

$$q_1^* = \frac{1 - q_2^*}{2}$$

and

$$q_2^* = \frac{1 - q_1^*}{2}$$

we have $q_1^* = q_2^* = P^* = \frac{1}{3}$.

36. Let us reconsider example 1 by assuming that firm 1 moves first by setting q_1, and firm 2 observes q_1 before determining its q_2. Now this is a game with *sequential moves*, and hence a game with *perfect information*. (This game has a name called Stackelberg game.) The key difference here is that a pure strategy of firm 2 is not a quantity q_2; rather, it is a function $q_2(q_1)$ which means that for different firm 1's q_1 observed, q_2 may vary. Before we solve the pure strategy NE for this game, we need a definition.

37. **Definition 8:** A *subgame* is the remainder of a game tree starting from some singleton information set (an information set composed of only one node).

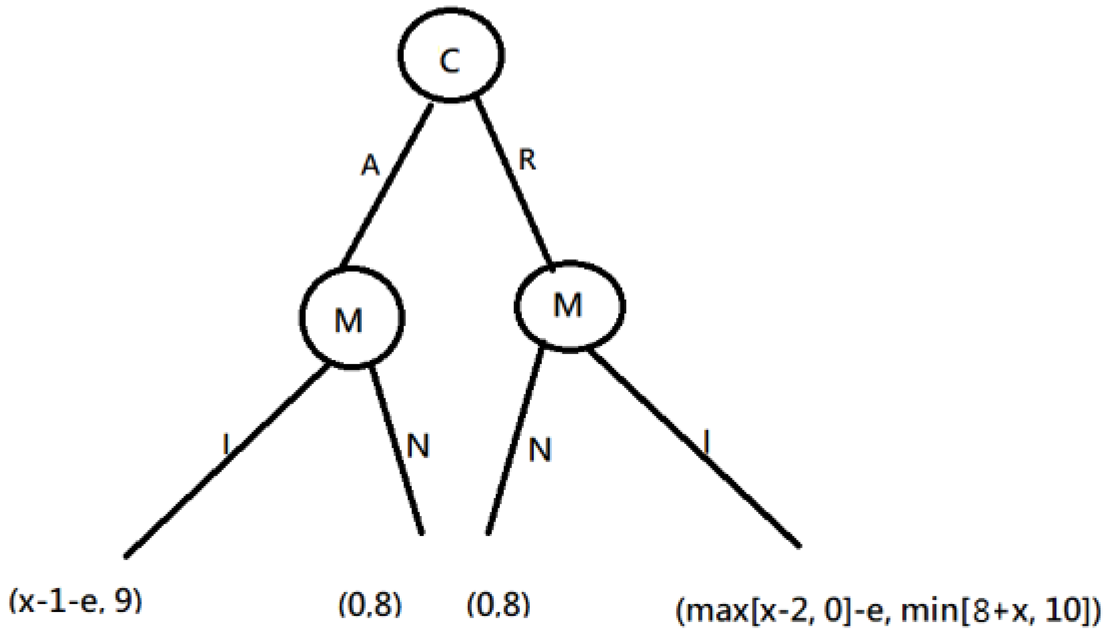
38. **Definition 9:** (Selten, 1965) A subgame perfect Nash equilibrium (SPNE) is an NE for a game in extensive form, which specifies NE strategies in each and every subgame. (This definition is further generalized into *sequential rationality* by Kreps and Wilson (1982).)

39. A game where all information sets are singletons is one of *perfect information*. Players must move sequentially in a game of perfect information. The way to solve the SPNE's of a finite game with perfect information is referred to as *backward induction*: we first consider the (last) subgames of which the immediate successors are the penultimate nodes, and since each such subgame has a single player facing a finite number of actions, there will be an optimal action for him in this last subgame; then, we can move backward on the game tree to the immediate predecessors of these last subgames, and in solving the NE's for the subgames starting from these predecessors, we should assume that

the players know which optimal actions will be taken if one of those last subgames is reached; and then we move backward on the game tree again, and so on and so forth. Because we are given a finite extensive game with perfect information, with the above procedure we will eventually reach the beginning of the game tree, thereby determining an equilibrium path, which by definition is a pure strategy SPNE. This procedure is usually referred to as *Kuhn-Zermelo algorithm*. Observe that if no two terminal nodes give any player the same payoff, the obtained SPNE is unique.

Example 4: Every NE in the extensive game depicted in section 30 is also an SPNE.

Example 5: M is the owner-manager of a firm which is protected by limited liability against its creditor(s). The debt due one year from now has a face value equal to \$10. There is a single debtholder, referred to as C. The total assets in place are worth only \$8 in one year. Just now, a new investment opportunity with NPV=$x > 1 + e \geq 1$ became available, which requires that M make an unobservable effort but no addition investment. Making the effort would incur a disutility $e \geq 0$ to M. M has told C that he will make the effort for the new investment project only if C agrees to reduce the face value of debt by \$1. The extensive game proceeds as follows. First C can accept (A) or reject (R) M's request. Then, M can choose to (I) or not to (N) make the effort. Both M and C are risk-neutral without time preferences.



(i) Suppose $x > 2 + e$. Show that there is an NE in which the creditor agrees to reduce the face value of debt and M makes the investment.

(ii) Show that the NE in (i) is not an SPNE. Find an SPNE.

(iii) How may your conclusion about (ii) change if $1 + e < x \leq 2 + e$?

(iv) Define bankruptcy as a state where the firm's equity value drops to zero. Explain why bankruptcy does not take place in (iii).

(v) Verify that the equilibrium firm value is increasing in x, but the equilibrium equity value may not.

Solution. Note that M can choose one action following A and another action following R. Hence C has 2 pure strategies, A and R, but M has

4 pure strategies

$$\begin{pmatrix} A \to I \\ R \to I \end{pmatrix},$$

$$\begin{pmatrix} A \to N \\ R \to N \end{pmatrix},$$

$$\begin{pmatrix} A \to I \\ R \to N \end{pmatrix},$$

and

$$\begin{pmatrix} A \to N \\ R \to I \end{pmatrix}.$$

The normal-form bimatrix is as follows.

M/C	A	R
$\begin{pmatrix} A \to I \\ R \to I \end{pmatrix}$	$(x-1-e, 9)$	$(\max(x-2, 0) - e, \min(8+x, 10))$
$\begin{pmatrix} A \to N \\ R \to N \end{pmatrix}$	$(0, 8)$	$(0, 8)$
$\begin{pmatrix} A \to I \\ R \to N \end{pmatrix}$	$(x-1-e, 9)$	$(0, 8)$
$\begin{pmatrix} A \to N \\ R \to I \end{pmatrix}$	$(0, 8)$	$(\max(x-2, 0) - e, \min(8+x, 10))$

In part (i), the strategy profile

$$(\begin{pmatrix} A \to I \\ R \to N \end{pmatrix}, A)$$

is indeed a pure strategy Nash equilibrium. However, it is not an SPNE: given that C has chosen R, M would be better off choosing I over N. Things are different in part (iii), where the above strategy profile becomes an SPNE.

For part (iv), as we explained in class, M has the control right before debt maturity, which is the reason that the firm has a positive equity value in the SPNE. For example, assume that $e = 0$ and $x = 1.95$. Even though $x + 8 < 10$, the equilibrium equity value equals $0.95 > 0$.

For part (v), assume that $e = 0$ and compare the case with $x = 2.01$ to the case with $x = 1.95$. Since $x > 0$, the new investment project will be undertaken in an SPNE (Coase Theorem!). Undoubtedly, the firm value increases in x. However, the equity value is 0.01 in the former case, but 0.95 in the latter case.

40. Now reconsider Example 5, assuming that Chen must first choose the amount y that the creditor is asked to reduce from the face value of debt, and then Chen and the creditor will play the game stated in Example 5. What is the optimal y chosen by Chen in the SPNE?[19]

41. Example 5 shows that in a dynamic game, only SPNEs are reasonable outcomes among NEs.

42. Now we apply the procedure of backward induction to solve the SPNE for the Stackelberg game. First, consider the subgame starting with firm 2's decision. A subgame is distinguished by firm 1' choice of q_1. Given q_1, firm 2 has infinite possible strategies $q_2(\cdot)$, but which one is the best? Of course, the profit maximizing strategy is the best, and hence

$$q_2 = \frac{1 - q_1}{2}.$$

Now go back to firm 1's problem. Firm 1 will

$$\max_{q_1} q_1(1 - q_1 - q_2(q_1)).$$

Check the concavity first and solve for the optimal q_1. We have $q_1^* = \frac{1}{2}$, $q_2^* = \frac{1}{4}$, $P^* = \frac{1}{4}$. Here some remarks are in order: (i) Committing to be the leader (the one who moves first) is beneficial in quantity-setting games; (ii) In Stackelberg game price is socially more efficient.

Note that with sequential moves firm 1 (the leader) can affect firm 2's choice of q_2 by committing to any q_1 it wants, and since $r_2(q_1)$ is decreasing in q_1 (when firm 1 expands output firm 2 will respond by cutting back its own output level in order to prevent the price from dropping too much; in this sense the firms' output choices are *strategic substitutes*), firm 1 has more incentives to expand output than in the

[19] Show that if $x > 2$, the choice of y no longer matters, for the creditor will always turn down Chen's offer. Show that if $x \leq 2$, the creditor will turn down any $y > 2$, and Chen will never offer any $y \leq x - 2$. Show that if $x \geq 2$, the optimal choice for Chen is $y = 2$.

simultaneous (Cournot) game.[20] This explains why the leader enjoys a higher equilibrium supply quantity, and why the equilibrium price is lower than that in the (simultaneous) Cournot game. More precisely, observe that in the Cournot game firm 1 believes that the product price it is faced with is $1-q_1-q_2$, meaning that one unit of output expansion will result in a dollar price reduction, while in the Stackelberg game, firm 1 believes that the price it is faced with is $1-q_1-r_2(q_1)$, meaning that the price will drop less when it increases its output by one unit.

43. Now consider the famous *Bertrand game*. Two firms produce a homogeneous good and sell it to homogeneous consumers. Producing one unit of the good costs c, where $0 < c < 1$. Each consumer is willing to pay as much as 1 dollar to buy 1 unit of the good. The total population of consumers is (normalized to) 1. The firms are competing in price. Consumers maximize consumer surplus and firms maximize profits. A pure strategy Nash equilibrium is (p_1, p_2), where p_i is the price chosen by firm i, $i = 1, 2$. Assume that firms get the same market share if they choose the same price. Show that there is a unique Nash equilibrium (called the *Bertrand outcome*) where $p_1 = p_2 = c$.

Proof Let $F(p)$ and $G(p)$ be the distribution functions for p_1 and p_2 chosen by firm 1 and firm 2 respectively.[21] Thus F and G define completely a mixed strategy profile for this game. Note that given G, firm 1's expected profit from using a pure strategy x is

$$(x-c)\{[1-G(x)] + \frac{1}{2}\Delta G(x)\}.$$

That is, if firm 2 prices strictly above x, which occurs with probability $1-G(x)$, then firm 1 may earn $x-c$ with probability 1; but if firm 2 prices exactly at x also, which occurs with probability $\Delta G(x)$, then firm 1 may earn $x-c$ only with probability $\frac{1}{2}$.

[20] Strategic substitutes and complements were first defined by Bulow, Geanakoplos, and Klemperer (1985).

[21] A weakly increasing function $F : \Re \to \Re$ satisfying (i) (**right-continuity**) $\lim_{x<y,y\to x} F(y) = F(x)$ for all $x \in \Re$; (ii) $\lim_{x\to-\infty} F(x) = 0$; and (iii) $\lim_{x\to+\infty} F(x) = 1$ is called a distribution function. It can have at most a countably infinite number of discontinuity points, and each such point is referred to as a point of jump. We denote by $\Delta F(x-) \equiv F(x) - \lim_{y<x,y\to x} F(y) \equiv F(x) = F(x-)$ the probability that $F(\cdot)$ assigns to the point of jump x.

We shall assume that (F, G) is an NE, and from here we shall derive a series of necessary conditions on F and G (to be summarized in the following steps). At first, let S_F and S_G be respectively the supports of F and G.

Step 1: $S_F \subset [c, 1]$, $S_G \subset [c, 1]$, $\inf S_F = \inf S_G = \underline{p}$, and $\sup S_F = \sup S_G = \overline{p}$.

Since pricing below c results in sure losses and pricing above 1 will sure push consumers away, we have $S_F \subset [c, 1]$, $S_G \subset [c, 1]$. We claim that the two sets S_F and S_G have the same infimum (greatest lower bound) and supremum (least upper bound). To see this, note that if $F(P) = 1$ at P, then any price $q > P$ is not a best response for G, because facing q, regardless of p_1, consumers would rather purchase from firm 1. Similarly, if $F(p) = 0$ at p then there is no reason that G will randomize over $q < p$: q is dominated by $\frac{q+p}{2}$, because both q and $\frac{q+p}{2}$ will attract all consumers for sure but apparently the price $\frac{q+p}{2}$ generates a higher profit. Thus for some $\underline{p}$ and $\overline{p}$, we have

$$\inf S_F = \inf S_G = \underline{p}, \ \sup S_F = \sup S_G = \overline{p}.$$

In particular, we have $S_F \subset [\underline{p}, \overline{p}] \subset [c, 1]$ and $S_G \subset [\underline{p}, \overline{p}] \subset [c, 1]$.
To ease notation, let $p = \underline{p}$ and $P = \overline{p}$.

Step 2: Fix any $x \in (p, P)$. Then neither F nor G can have a jump at x.

To see this, suppose instead that $x \in (p, P)$ is a point of jump of F. Thus firm 1 may randomize on the price x with a strictly positive probability $\delta > 0$. In this case, there exists $\epsilon > 0$ small enough such that all prices contained in the interval $[x, x + \epsilon)$ are dominated for firm 2 by some price $q < x$ with q sufficiently close to x.[22] Since (F, G) is by assumption an NE, it must be that G assigns zero probability to

[22] Write $q = x - e$ for a tiny $e > 0$. Since $F(\cdot)$ has only a countably infinite number of points of jump, we can find such a q at which $F(\cdot)$ does not jump. By choosing the

the interval $[x, x+\epsilon)$. Again, since F is the best response against G, and since $G(\cdot)$ is flat on the interval $[x, x+\epsilon)$, this must imply that x is not a best response for firm 1 (the price $x + \frac{\epsilon}{2}$, for example, is strictly better than x), and firm 1 should not have assigned a strictly positive probability to x, a contradiction. We conclude that F and G are continuous except possibly at p and P (when $p = c$ or $P = 1$).

Step 3: If $P > p$ then F and G are both continuous and strictly increasing on the interval (p, P).

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pure-strategy q, given firm 1's mixed strategy $F(\cdot)$, firm 2's expected payoff is

$$(x - e - c)[1 - F(x - e)],$$

which we claim is greater than

$$(x + d - c)[1 - F(x + d) + \frac{1}{2}\Delta F(x + d)], \ \forall d \in [0, \epsilon).$$

To see this, note that

$$1 - F(x - e) = \text{prob.}(\tilde{p}_1 > x - e),$$

and

$$1 - F(x + d) + \frac{1}{2}\Delta F(x + d) \leq 1 - F(x + d) + \Delta F(x + d) = \text{prob.}(\tilde{p}_1 \geq x + d).$$

Since $\Delta F(x) = \delta > 0$, we know that

$$\text{prob.}(\tilde{p}_1 > x - e) - \text{prob.}(\tilde{p}_1 \geq x + d) > \delta,$$

implying that

$$[1 - F(x - e)] - [1 - F(x + d) + \frac{1}{2}\Delta F(x + d)] > \delta.$$

Now, for very small $e > 0$ and $\epsilon > 0$,

$$(x + d - c) - (x - e - c) = e + d < e + \epsilon$$

is very small relative to the fixed δ. Hence we conclude that

$$(x - e - c)[1 - F(x - e)] > (x + d - c)[1 - F(x + d) + \frac{1}{2}\Delta F(x + d)], \ \forall d \in [0, \epsilon),$$

showing that no pure strategy lying in the interval $[x, x+\epsilon)$ can be a best response against $F(\cdot)$ from firm 2's perspective, which implies that $G(\cdot)$ must be flat on this interval.

That F and G are continuous has been proved in Step 2. Suppose instead that, say, F is flat on an interval $[a, b] \in (p, P)$. Since p is the infimum of S_F, $F(b) > 0$.[23] Moreover, since $b < P$, $F(b) < 1$.[24] There exists a smallest $x \in [p, P]$ such that $F(x) = F(b) = F(a)$ so that $0 < F(x) < 1$: By the right continuity of F, we have $x = \inf\{y \in (p, P) : F(y) = F(b)\}$. We claim that x is a best response for firm 1. Either $x = p$ so that $F(x) > 0$ implies that p is a point of jump, or $x > p$ but for all $\epsilon > 0$, $F(x - \epsilon) < F(x)$ (by definition of x)[25] so that there exists a best response y_n in each interval contained in the sequence of intervals $\{(x - \frac{1}{n}, x]; n \in \mathbf{Z}_+\}$ with $\lim_{n\to\infty} y_n = x$. In the former case, where x is a point of jump for $F(\cdot)$, $x = p$ is clearly a best response. We claim that the same conclusion holds in the latter case also. In the latter case, note that each best response y_n gives rise to the same expected profit Π_1 for firm 1. Since by Step 2 $G(\cdot)$ is continuous on the interval (p, P), and since the sequence of y_n converges to x, we have

$$(x - c)[1 - G(x)] = \lim_{n\to\infty} (y_n - c)[1 - G(y_n)] = \lim_{n\to\infty} \Pi_1 = \Pi_1,$$

implying that, given $G(\cdot)$, the pure strategy x also yields for firm 1 the expected profit Π_1, proving that x itself is a pure-strategy best response for firm 1.

On the other hand, given that $F(\cdot)$ is flat on (x, b), firm 2 would never randomize over (x, b), because each $p_2 \in (x, b)$ is dominated by, say, $\frac{p_2+b}{2}$ from firm 2's perspective. However, given that $G(\cdot)$ is also flat on (x, b), x itself is weakly dominated by, say, $\frac{b+x}{2}$, and hence x cannot be a best response for firm 1, which contradicts our earlier conclusion that x *must be* a best response for firm 1. Hence we conclude that $F(\cdot)$ cannot be flat on any sub-interval contained in (p, P). Together with Step 2, this says that F and G are strictly increasing and continuous on (p, P), implying that each point in (p, P) is a best response for both firms.

[23] If $F(b) = 0 = F(p)$, then b is a lower bound for S_F, and $b > p$, which is a contradiction to the fact that p is the greatest lower bound of S_F.

[24] If instead $F(b) = 1$ then b is an upper bound of S_F and yet $b < P$, a contradiction to the fact that P is the least upper bound of S_F.

[25] If there exists $\epsilon > 0$ such that $F(x - \epsilon) = F(x)$, then $F(x - \epsilon) = F(b)$ and $x - \epsilon < x$, which is a contradiction to the definition of x.

We have just reached the conclusion that $S_F = S_G = [p, P]$. We next show that $p = P$.

Step 4: $p = P$.

Suppose instead that $P > p$, and consider any $x \in (p, P)$ for firm 1. We claim that G cannot have a jump at P if $P > p$: If instead G does jump at P, then P must be a pure-strategy best response for firm 2, and we claim that in this case P cannot be a best response for firm 1 (pricing slightly below P is a better response than pricing at P from firm 1's perspective), implying by definition that firm 1 must price below P with probability one, which in turn implies that pricing at P will lose customers for sure (and hence yields zero profits) for firm 2, while picking any price in the interval (p, P) can yield a strictly positive expected profit for firm 2, a contradiction to the assumption that P is a best response for firm 2.

Thus $G(\cdot)$ is continuous at P. Now, since all $x \in (p, P)$ are pure-strategy best responses for firm 1 and they generate the same expected profit Π_1 for firm 1, we must have

$$(x - c)[1 - G(x)] = \Pi_1, \ \forall x \in (p, P).$$

Let $\{x_n\}$ be any increasing sequence contained in the interval (p, P) and converging to P. We have for all n,

$$(x_n - c)[1 - G(x_n)] = \Pi_1$$

implying that

$$\lim_{n \to \infty} (x_n - c)[1 - G(x_n)] = \lim_{n \to \infty} \Pi_1 = \Pi_1,$$

where note that

$$\lim_{n \to \infty} (x_n - c)[1 - G(x_n)] = (P - c)[1 - G(P)] = 0,$$

following the fact that, by definition, $G(P) = 1$. Thus $\Pi_1 = 0$. But then, for all $x \in (p, P)$,

$$(x - c)[1 - G(x)] = 0,$$

which implies that

$$G(x) = 1, \ \forall x \in (p, P).$$

By the fact that $G(\cdot)$ is right-continuous and weakly increasing, we have

$$G(p) = G(p+) = \inf\{G(x) : \ x \in (p, P)\} \Rightarrow G(p) = 1!$$

This implies that $p = P$, so that $S_F = S_G$ and they are a singleton set.

Thus the above Bertrand game has a unique symmetric pure strategy Nash equilibria.[26] It is now easy to show that such an equilibrium is unique, in which $p = P = c$. To sum up, the unique (mixed-strategy) NE for this game can be written as $F(x) = G(x) = 1_{[c,+\infty)}(x), \ \forall x \in \Re_+$.[27]

44. Continue with the Bertrand game discussed in the preceding section but with the following modifications:

    Suppose that the two firms can first announce a *best price in town* policy, which promises their customers that they will match the lowest price a consumer can find in this town. For example, if $p_1 = \frac{1}{3} > p_2 = \frac{1}{4}$, then everyone having bought the product from firm 1 will receive an amount $\frac{1}{3} - \frac{1}{4}$ from firm 1. Assume that both firms have announced the *best price in town* policy. Reconsider the price competition between the two firms. Show that in one NE the *best price in town* policy gives rise to the worst price for the consumers: In equilibrium the consumers are left with no surplus.

    Proof Let p_m be the optimal price chosen by a monopolistic firm (which is equal to 1 with the previous specifications). Then setting price at p_m by both firms forms an NE in the presence of the best-price-in-town policy: Apparently, no firm can gain from unilaterally raising the price (which implies that the transaction is still p_m!), but if one firm unilaterally lowers the price to below p_m, it does not increase

[26] See Appendix for more discussions about Bertrand game equilibria.

[27] What do you think may happen if the two firms must move sequentially? Show that the Bertrand outcome is again the unique SPNE.

sales volume because consumers purchasing from the other firm will automatically get compensated with the difference in the two firms' prices, and so those consumers have no incentives to switch to the firm charging the lower price. This proves that no firms can benefit from a unilaterally deviation from the price p_m, and hence p_m defines a symmetric NE.

This game has other pure strategy NE's as well. In fact, any price contained in the interval $[c, p_m]$ defines a symmetric NE. The NE where both pick the price p_m is the Pareto undominated equilibrium for the two firms. In that NE, consumers have the lowest possible consumer surplus (as if they were faced with a monopolistic firm, or a perfectly colluding cartel).

45. Continue with the above extensive game. Assume now that the two firms can first simultaneously choose to or not to announce the best-price-in-town policy, and then upon observing the two firms' announcements, the two firms engage in the price competition. Show that if the firms will reach only Pareto undominated equilibria in a subgame, then in equilibrium at least one firm will announce the best-price-in-town policy in the first stage, and then both firms price at p_m in the second stage.[28]

46. Suppose there are two firms, 1 and 2, with symmetric demand functions:

$$q_1 = 1 - p_1 + 0.5p_2,\ q_2 = 1 - p_2 + 0.5p_1.$$

Assume no costs for either firm. Suppose firms compete in price. Find the NE for the cases of (i) simultaneous moves; and (ii) sequential moves. The firms' prices are *strategic complements*, and hence with sequential moves the equilibrium output level will be less efficient; see the previous discussions about the relationship between the Cournot game and the Stackelberg game and argue analogously.

47. Two heterogeneous goods are produced by firm 1 and firm 2 respectively, and the demands for the two goods are

$$D_i(p_i, p_j) = \max(0, 1 - ap_i + bp_j),\ a, b > 0,\ i \neq j,\ i, j \in \{1, 2\}.$$

[28]Committing to the best-price-in-town policy essentially allows a firm to convince its rival that it will react to the latter's price cut *in no time*. This removes its rival's incentive to lower the price, and hence both firms can benefit from pricing high.

Find the pure strategy NE for this game assuming that the firms are competing in price. What restrictions do you need to impose on the parameters a, b so that firms will not get unbounded profits? Now we solve the prices from the above system of equations to get the inverse demands for the two firms:

$$P_i(q_i, q_j) = \max(0, \frac{1 + \frac{b}{a}}{a - \frac{b^2}{a}} - \frac{1}{a - \frac{b^2}{a}} q_i - \frac{\frac{b}{a}}{a - \frac{b^2}{a}} q_j), i \neq j, \ i, j \in \{1, 2\}.$$

Now suppose instead that the two firms are competing in quantity. In which case (competing in price or quantity) do the two firms obtain higher profits in an NE? Why?

Solution Suppose that the two firms are competing in quantity. In an NE of this quantity setting game, a firm considering raising its quantity always assumes that its rival will not react. Since the game is really one of price-setting, what the quantity-setting conjectural variation says is really that when a firm lowers its price (so that its quantity is expanded) it expects its rival to move the latter's price in such a manner that with the new prices the rival's quantity does not change. This implies that one's lowering its price is expected to be reacted right away by its rival by also lowering the latter's price. Thus the firms in a quantity-setting game (again, they are really playing the price-setting game) have less incentives of lowering prices and expanding outputs. As a consequence firms have higher profits in equilibrium.[29]

48. Having considered firms' imperfect competition in quantity and price, let us consider location (or spatial) competition. (Think of product

[29] We can provide a similar argument for a price-setting game in the context where the two firms are really playing the quantity setting game given $P_i(q_i, q_j)$. We claim that the two firms have more incentives to expand outputs in the price-setting game. To see this, note that in the quantity-setting game a firm i that considers expanding its output q_i believes that its rival j will not react by changing q_j, and consequently one unit of increase in q_i reduces the firm's own price P_i by $\frac{1}{a - \frac{b^2}{a}}$. In the price-setting game, on the other hand, a firm i that considers expanding its output q_i believes that its rival j will react immediately by *reducing* q_j in such a manner that P_j will stay unchanged, and this implies that one unit of increase in q_i results in a reduction in P_i by less than $\frac{1}{a - \frac{b^2}{a}}$. Consequently, the two firms have more incentives to expand outputs in the price-setting game, and therefore obtain lower equilibrium profits.

positioning.) Two firms must each choose (simultaneously) a location on the Hotelling main street (the $[0,1]$ interval). Thus $S_1 = S_2 = [0,1]$. For any pure strategy profile (x_l, x_r), where $0 \leq x_l \leq x_r \leq 1$, the firm choosing x_l has payoff

$$x_l + \frac{x_r - x_l}{2};$$

and the firm choosing x_r has payoff

$$\frac{x_r - x_l}{2} + (1 - x_r).$$

Show that the unique pure strategy NE of this game is $(\frac{1}{2}, \frac{1}{2})$. Show that the same is true if the two firms move sequentially. Now consider the same game but assume that there are three firms. A profile (x_l, x_m, x_r) generates $x_l + \frac{x_r - x_l}{2}$ for the firm choosing x_l, $\frac{x_m - x_l}{2} + \frac{x_r - x_m}{2}$ for the firm choosing x_m, and $\frac{x_r - x_m}{2} + (1 - x_r)$ for the firm choosing x_r. Show that this game has no pure strategy NE. What if the three firms move sequentially?

Let $L_i \in [0,1]$ be the location of firm i. An innocuous assumption is that $L_3 = \frac{1}{2}(L_1 + L_2)$ whenever firm 3 decides to locate at somewhere between L_1 and L_2. (This assumption does matter, because firms 2 and 3 when deciding their own locations must form expectations about how firm 3 will locate. It is innocuous because firm 3 should be indifferent about any locations in between L_1 and L_2 once staying in between L_1 and L_2 is its decision. This seems to be the natural choice of firm 3.) In the following, $L_R \equiv \max(L_1, L_2)$, $L_L \equiv \min(L_1, L_2)$.

By backward induction, we should first consider firm 3's reaction function $L_3^*(L_1, L_2)$. Firm 3's payoff is (nearly)

$$\max(1 - L_R, L_L, \frac{L_R - L_L}{2}),$$

and its best response is

$$L_3^* = \begin{cases} L_R + \epsilon, & \text{if} \quad \max(1 - L_R, L_L, \frac{L_R - L_L}{2}) = 1 - L_R; \\ \\ L_L - \epsilon, & \text{if} \quad \max(1 - L_R, L_L, \frac{L_R - L_L}{2}) = L_L; \\ \\ \frac{L_R + L_L}{2}, & \text{if} \quad \max(1 - L_R, L_L, \frac{L_R - L_L}{2}) = \frac{L_R - L_L}{2}. \end{cases}$$

Note that

$$\max(1 - L_R, L_L, \frac{L_R - L_L}{2}) = 1 - L_R$$

$$\Leftrightarrow 1 - L_R \geq \frac{L_R - L_L}{2}, \quad 1 - L_R \geq L_L \Leftrightarrow L_R \leq \min(\frac{2 + L_L}{3}, 1 - L_L).$$

Re-arranging the conditions $\max(1-L_R, L_L, \frac{L_R-L_L}{2}) = L_L$ and $\max(1-L_R, L_L, \frac{L_R-L_L}{2}) = \frac{L_R-L_L}{2}$ analogously, we obtain

$$L_3^*(L_1, L_2) = \begin{cases} L_R + \epsilon, & \text{if } L_R \leq \min(\frac{2+L_L}{3}, 1 - L_L) \ (3-1) \\ \frac{L_R+L_L}{2}, & \text{if } L_R \geq \max(3L_L, \frac{2+L_L}{3}) \ (3-2) \\ L_L - \epsilon, & \text{if } L_R \in [1 - L_L, 3L_L] \ (3-3). \end{cases}$$

Note that (3-3) says that firm 3 will never choose $L_3 = L_L - \epsilon$ unless $L_L \geq \frac{1}{4}$.

Moving backwards, we now derive $L_2^*(L_1)$ using firm 2's correct expectations regarding $L_3^*(L_1, L_2)$. Since possible L_1 and L_2 are $\in [0, 1]$, we must consider the following six regions:
(a) $L_2 \geq L_1, \ L_2 \leq \min(\frac{2+L_1}{3}, 1 - L_1)$ (so that $L_1 \in [0, \frac{1}{2}]$);
(b) $L_2 \geq L_1, \ L_2 \geq \max(\frac{2+L_1}{3}, 3L_1)$; (so that $L_1 \in [0, \frac{1}{3}]$)
(c) $L_2 \geq L_1, \ L_2 \in [1 - L_1, 3L_1]$ (so that $L_1 \in [\frac{1}{4}, 1)$);
(d) $L_2 \leq L_1, \ L_1 \leq \min(\frac{2+L_2}{3}, 1 - L_2)$ (so that $L_1 \in [0, \frac{3}{4}]$);
(e) $L_2 \leq L_1, \ L_1 \geq \max(\frac{2+L_2}{3}, 3L_2)$ (so that $L_1 \in [\frac{2}{3}, 1]$);
(f) $L_2 \leq L_1, \ L_1 \in [1 - L_2, 3L_2]$ (so that $L_1 \in [\frac{1}{2}, 1)$).
Note that from firm 3's perspective, in (a),(b), and (c) $L_1 = L_L$ and $L_2 = L_R$, and in (d), (e), and (f), $L_1 = L_R$ and $L_2 = L_L$.

We shall first compute regionally optimal $L_2^i(L_1)$ for each region $i \in \{$ a,b,c,d,e,f$\}$, and then obtain the globally optimal $L_2^*(L_1)$ from the six regionally optimal $L_2^i(L_1)$'s.

- First consider firm 2's optimal choice in region (a). Given L_1, if firm 2 chooses L_2 such that $L_2 \geq L_1$ and $L_2 \leq \min(\frac{2+L_1}{3}, 1 - L_1)$, then (3-1) holds with $L_2 = L_R$ and $L_1 = L_L$, and hence $L_3^* = L_2 + \epsilon$, which implies that $L_1 \leq L_2 < L_3^*$, and firm 2's payoff is

$$\Pi_2^a = \frac{L_2 - L_1}{2},$$

which is increasing in L_2, so that firm 2 should choose L_2 as large as possible. The largest L_2 that satisfies $L_2 \geq L_1$ and $L_2 \leq \min(\frac{2+L_1}{3}, 1-L_1)$ is exactly firm 2's regionally optimal choice

$$L_2^a(L_1) = \min(\frac{2+L_1}{3}, 1-L_1).$$

With this choice, firm 2's payoff becomes

$$\Pi_2^a = \frac{L_2^a(L_1) - L_1}{2} = \frac{\min(\frac{2+L_1}{3}, 1-L_1) - L_1}{2}.$$

- Next, consider firm 2's optimal choice in region (b). Given L_1, if firm 2 chooses L_2 such that $L_2 \geq L_1$ and $L_2 \geq \max(\frac{2+L_1}{3}, 3L_1)$, then (3-2) holds with $L_2 = L_R$ and $L_1 = L_L$, and hence $L_3^* = \frac{L_1+L_2}{2}$, which implies that $L_1 < L_3^* < L_2$, and firm 2's payoff is

$$\Pi_2^b = \frac{L_2 - L_1}{4} + (1 - L_2),$$

which is decreasing in L_2, so that for firm 2 the smaller L_2 is the better. The smallest L_2 that satisfies $L_2 \geq L_1$ and $L_2 \geq \max(\frac{2+L_1}{3}, 3L_1)$ is exactly firm 2's regionally optimal choice

$$L_2^b(L_1) = \max(\frac{2+L_1}{3}, 3L_1).$$

With this choice, firm 2's payoff becomes

$$\Pi_2^b = \frac{L_2^b(L_1) - L_1}{4} + 1 - L_2^b(L_1) = 1 - \frac{L_1}{4} - \frac{3}{4}\max(\frac{2+L_1}{3}, 3L_1).$$

- Next, consider firm 2's optimal choice in region (c). Given $L_1 \geq \frac{1}{4}$, if firm 2 chooses L_2 such that $L_2 \geq L_1$ and $L_2 \in [1-L_1, 3L_1]$, then (3-3) holds with $L_2 = L_R$ and $L_1 = L_L$, and hence $L_3^* = L_1 - \epsilon$, which implies that $L_3^* < L_1 \leq L_2$, and firm 2's payoff is

$$\Pi_2^c = \frac{L_2 - L_1}{2} + (1 - L_2),$$

which is decreasing in L_2, so that for firm 2 the smaller L_2 the better. The smallest L_2 that satisfies $L_2 \geq L_1 \geq \frac{1}{4}$ and $L_2 \in [1 - L_1, 3L_1]$ is exactly firm 2's regionally optimal choice

$$L_2^c(L_1) = \max(1 - L_1, L_1) \geq \frac{1}{2}.$$

With this choice, firm 2's payoff becomes

$$\Pi_2^c = \frac{L_2^c(L_1) - L_1}{2} + 1 - L_2^c(L_1) = 1 - \frac{L_1}{2} - \frac{1}{2}\max(1 - L_1, L_1).$$

- Following a similar procedure, we can obtain

$$L_2^d(L_1) = \min(1 - L_1, L_1) \le \frac{1}{2}, \quad \Pi_2^d = \frac{L_1 + \min(L_1, 1 - L_1)}{2};$$

$$L_2^e(L_1) = \min(3L_1 - 2, \frac{L_1}{3}), \quad \Pi_2^e = \frac{L_1 + 3\min(3L_1 - 2, \frac{L_1}{3})}{4};$$

and

$$L_1 \ge L_2^f(L_1) = \max(1-L_1, \frac{L_1}{3}) \ge \frac{1}{4}, \ L_1 \ge \frac{1}{2}, \ \Pi_2^f = \frac{L_1}{2} - \frac{1}{2}\max(1-L_1, \frac{L_1}{3}).$$

Now, to obtain firm 2's best response from the above six regional optima, we compare the six corresponding payoffs for firm 2, which are summarized in the table below (where an asterisk indicates the maximum payoff).

	$L_1 \in [0, \frac{1}{4}]$	$L_1 \in [\frac{1}{4}, \frac{1}{2}]$	$L_1 \in [\frac{1}{2}, \frac{2}{3}]$	$L_1 \in [\frac{2}{3}, \frac{3}{4}]$	$L_1 \in [\frac{3}{4}, 1]$
Π_2^a	$\frac{1-L_1}{3}$	$\frac{1}{2} - L_1$	—	—	—
Π_2^b	$\frac{1-L_1}{2}^*$	$1 - \frac{5}{2}L_1$	$1 - \frac{5}{2}L_1$	$1 - \frac{5}{2}L_1$	$1 - \frac{5}{2}L_1$
Π_2^c	—	$\frac{1}{2}^*$	$1 - L_1$	$1 - L_1$	$1 - L_1$
Π_2^d	L_1	L_1	$\frac{1}{2}^*$	$\frac{1}{2}^*$	—
Π_2^e	—	—	—	$\frac{5L_1-3}{2}$	$\frac{L_1}{2}^*$
Π_2^f	—	—	$L_1 - \frac{1}{2}$	$L_1 - \frac{1}{2}$	$\frac{1}{3}L_1$

From the table we obtain

$$L_2^*(L_1) = \begin{cases} \frac{2+L_1}{3}, & \text{if } L_1 \le \frac{1}{4}; \ (2-1) \\ 1 - L_1, & \text{if } \ L_1 \in [\frac{1}{4}, \frac{3}{4}]; \ (2-2) \\ \frac{L_1}{3}, & \text{if } \ L_1 \ge \frac{3}{4} \ (2-3). \end{cases}$$

Now, we consider firm 1's optimal decision, L_1^*. For L_1 such that $(2-1)$ holds, from the above analyses,

$$L_2^*(L_1) = \frac{2 + L_1}{3}, \ L_3^*(L_1, L_2^*(L_1)) = \frac{1 + 2L_1}{3},$$

which implies that firm 1's payoff is $L_1 + \frac{1}{2}(\frac{1+2L_1}{3} - L_1) = \frac{1+5L_1}{6}$, which is increasing in L_1, and hence the best L_1 in this region is $\frac{1}{4}$. On the other hand, if firm 1 chooses any L_1 such that $(2-2)$ holds, then

$$L_2^*(L_1) = 1 - L_1,\ L_3^*(L_1), L_2^*(L_1)) = L_1 - \epsilon,$$

which implies that firm 1's payoff is decreasing in L_1. Once again the regionally optimal L_1 is $\frac{1}{4}$. How about those L_1 that satisfy $(2-3)$? In this case, $(3-2)$ holds, and hence firm 1's payoff is $(1-L_1) + \frac{L_1 - \frac{L_1}{3}}{4}$, which is decreasing in L_1. Hence, the regionally optimal $L_1 = \frac{3}{4}$, which is obviously a strategic equivalent of $L_1 = \frac{1}{4}$. Thus, this game has two subgame perfect Nash equilibria (assuming that firm 3 always locates itself right in the middle between firms 1 and 2 whenever applicable):

$$(L_1^*, L_2^*, L_3^*) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{2}),$$

and

$$(L_1^*, L_2^*, L_3^*) = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}).$$

49. Find all mixed strategy NEs for the following strategic game:

Player 1/Player 2	L	M	R
U	$4, 0$	$0, 1$	$-1, -100$
D	$2, 2$	$1, 1$	$0, 3$

Solution It is easy to show that (D,R) is a pure strategy NE. Consider mixed strategy NE's. Let π be the equilibrium probability that player 1 chooses U, and p and q respectively the equilibrium probabilities that player 2 chooses L and M. It is easy to verify that there are two mixed strategy NE's of this game, which are

$$(\pi, p, q) = (\frac{1}{2}, \frac{1}{3}, \frac{2}{3}),\ (\pi, p, q) = (\frac{1}{101}, \frac{1}{3}, 0).$$

Note that in the first mixed strategy NE, given player 1's mixed strategy, R is not a best response for player 2, and that is why player 2 only randomizes over L and M. In the second mixed strategy NE, given player 1's mixed strategy, M is not a best response for player 2, and that is why player 2 randomizes only over L and R.

50. Mr. A and B are asked to simultaneously name a number in $\{1, 2, \cdots, 100\}$, and if the two numbers are identical they each get 1 dollar, or else they get zero. Find all the NE's. (**Hint**: Each non-empty subset $E \subset \{1, 2, \cdots, 100\}$ defines a mixed strategy NE where A and B both assign probability $\frac{1}{\#(E)}$ to each and every element in E, where $\#(E)$ is the number of elements in E.)

51. Consider the following strategic game:

Player 1/Player 2	L	M	R
U	$5, 5$	$-1, 6$	$-2, -2$
M	$6, -1$	$0, 0$	$-2, -2$
D	$-2, -2$	$-2, -2$	$-6, -6$

Now assume this game is played twice, and each player maximizes the sum of his payoffs in the two rounds of play. Each player observes everyone's first-period action before the second-period game starts. This is called a repeated game, where the above *stage game* is played twice.

(a) What are the highest symmetric payoffs in any NE?
(b) What are the highest symmetric payoffs in any SPNE?
(**Hint**: For part (b), the only SPNE of this game consists of the two players playing (M,M) in both rounds.[30] For part (a), note first that each player has 3×3^9 pure strategies, and in each pure strategy a player must specify which among (U,M,D) (or (L,M,R)) will be chosen at the first stage, and at the beginning of stage 2, in each of the 9 possible subgames which among (U,M,D) (or (L,M,R)) will be chosen. Now show that the following pure strategy profile constitutes an NE, which is not subgame-perfect: player 1 plays U at the first stage and will play M at the second stage if (U,L) is the outcome of the first stage and he

[30]The reader may get the wrong impression that an SPNE of a finitely repeated game is nothing but a repetition of the NE in the stage game. In fact, strategic games that have a unique NE are quite unusual. If the stage game has more than one NE, then a repeated game will have a lot of SPNE's which differ from any series of NE's obtained from the static game, as long as the number of repetitions is large enough. We shall have more to say on this in a subsequent Lecture.

will play D at the second stage if the first-stage outcome is not (U,L); player 2 plays L at the first stage and will play M at the second stage if (U,L) is the outcome of the first stage and he will play R at the second stage if the first-stage outcome is not (U,L).)

52. Consider two firms engaged in Cournot competition. The inverse demand is

$$p = 3 - q_1 - q_2.$$

Assume that both firms have marginal cost equal to 1, but only firm 1 has the chance to spend F and brings the marginal cost down to zero. Firm 1 moves first by deciding to invest F or not to, which is unobservable to firm 2 (imperfect information). The two firms then play the Cournot game by selecting their own outputs. Compute the set of pure strategy Nash equilibria for this imperfect information game for $F \in \mathcal{R}_+$. (**Hint**: for different ranges of F, the equilibria may differ.)
Solution Suppose that firm 2 believes (correctly) that in equilibrium firm 1 makes the investment with probability $1 - \pi$, where $0 \leq \pi \leq 1$. Let the equilibrium quantity of firm 1 be $q_1^*(c)$ if firm 1's marginal cost is $c \in \{0, 1\}$. Let q_2^* be firm 2's equilibrium quantity. Then, as part of the Nash equilibrium, we must have

$$q_1^*(1) = \arg\max_{q_1} q_1(3 - q_1 - q_2^* - 1),$$

$$q_1^*(0) = \arg\max_{q_1} q_1(3 - q_1 - q_2^* - 0),$$

$$q_2^* = \arg\max_{q_2} q_2(3 - E[q_1^*] - q_2 - 1),$$

where

$$E[q_1^*] = \pi q_1^*(1) + (1 - \pi) q_1^*(0).$$

The (necessary and sufficient) first-order conditions of these three maximizations give three equations with three unknowns. Solving, we have

$$q_1^*(1) = \frac{5 - \pi}{6}, \quad q_1^*(0) = \frac{8 - \pi}{6}, \quad q_2^* = \frac{1 + \pi}{3}.$$

Let $Y_i(c; \pi)$ be the equilibrium profit of firm i before subtracting the expenditure on F, given that firm 1's marginal cost is c and firm 2's beliefs are such that firm 1 spends F with probability $1-\pi$. Then, one can show that

$$Y_1(1;\pi) = [\frac{5-\pi}{6}]^2, \quad Y_1(0;\pi) = [\frac{8-\pi}{6}]^2.$$

Suppose that there exist pure strategy NE's. In a pure strategy NE, firm 1 either spends F with prob. 1 or does not spend F with prob. 1. Suppose first that there is a pure strategy NE in which firm 1 invests. Then this is expected correctly by firm 2, and hence $\pi = 0$. From the preceding discussion, this requires, for firm 1 to follow its equilibrium strategy,

$$Y_1(0;0) - F \geq Y_1(1;0),$$

or $F \leq \frac{39}{36}$.

Next, suppose instead that there is a pure strategy NE in which firm 1 does not invest. The earlier discussion implies that $\pi = 1$ and in equilibrium it must be that

$$Y_1(0;1) - F \leq Y_1(1;1),$$

or $F \geq \frac{33}{36}$.

Finally, consider mixed strategy NE's. For firm 1 to randomize between to and not to invest with respectively prob. $(1-\pi)$ and π,[31]

$$Y_1(0;\pi) - F = Y_1(1;\pi),$$

which, after recognizing $\pi \in (0,1)$, implies that $F \in (\frac{33}{36}, \frac{39}{36})$. Note that for all $F \in \mathcal{R}_+$, this game has an odd number of NE's.[32]

[31] Note again that neither firm 1 nor firm 2 can randomize in the quantity-setting stage (why?), and hence a mixed strategy NE can at most involve firm 1 randomly making investment decisions.

[32] Let us solve in detail the pure strategy NE where firm 2 chooses q_2^* and firm 1 spends F with probability one and then choose q_1^*. In this equilibrium, firm 2 correctly expects firm 1's equilibrium strategy, and hence firm 2 correctly expects that firm 1's marginal cost becomes 0 when firm 1 is ready to choose q_1^*. If firm 1 has chosen to spend F in the earlier stage, then its choice q_1^* should be optimal against firm 2's q_2^*; that is,

$$q_1^* = \arg\max_q q(3 - q - q_2^* - 0),$$

53. Two firms are competing in a declining industry in continuous time. If at time $t \in [0, +\infty)$ both stay, firm i gets profit density $\pi_d^i(t)$; if only firm i stays, it gets $\pi_m^i(t)$. (d and m stand for respectively 'duopoly' and 'monopoly'.) It is given that

$$\pi_d^1 = 2 - 2t,$$

$$\pi_d^2 = 1 - 2t,$$

$$\pi_m^1 = \frac{11}{4} - t,$$

$$\pi_m^2 = \frac{7}{4} - t.$$

---

which requires that

$$q_1^* = \frac{3 - q_2^*}{2}.$$

Similarly, firm 2's q_2^* must be a best response against firm 1's q_1^*, so that

$$q_2^* \in \arg\max_q q(3 - q_1^* - q - 1),$$

which implies that

$$q_2^* = \frac{2 - q_1^*}{2}.$$

Solving, we obtain

$$q_1^* = \frac{4}{3}, \quad q_2^* = \frac{1}{3}.$$

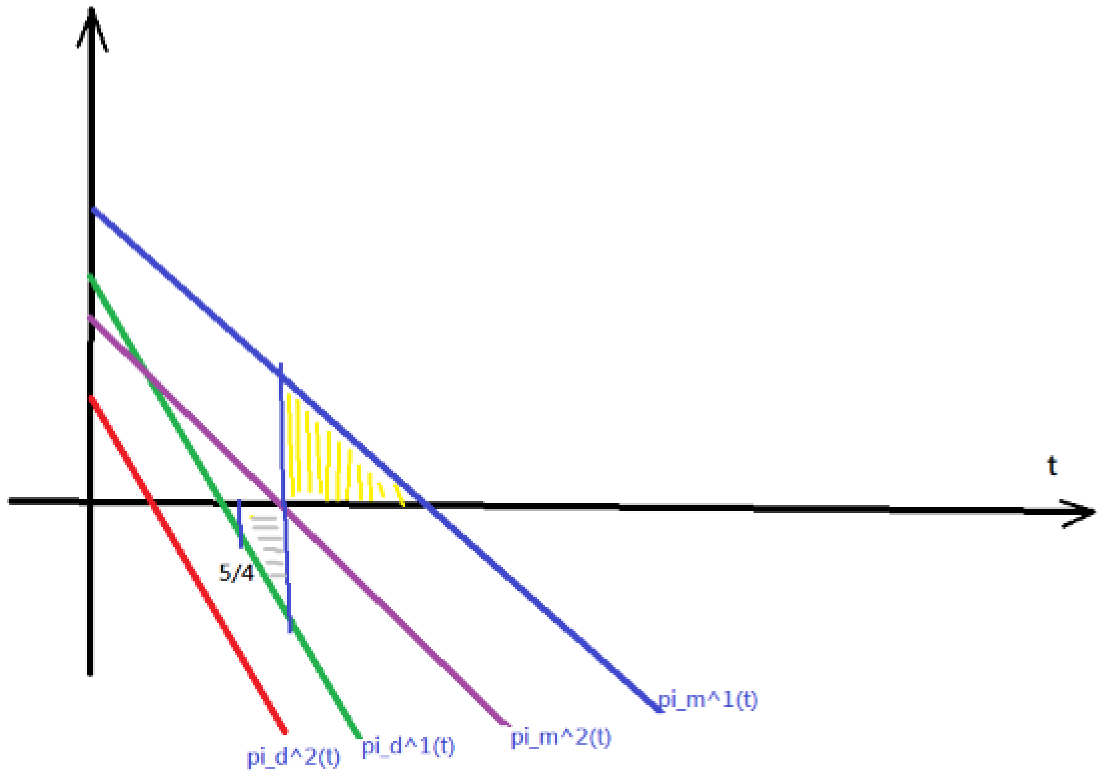
It follows that firm 1's equilibrium profit is $\frac{4}{3}(3 - \frac{4}{3} - \frac{1}{3} - 0) - F = \frac{16}{9} - F$, which must be greater than the profit that firm 1 would make by choosing not to spend F in the first place. The latter *deviation* would generate the following payoff for firm 1:

$$\max_q q(3 - q - \frac{1}{3} - 1) = \frac{25}{36};$$

that is, with unit cost being 1 instead of 0 following the deviation, firm 1 would reduce its sales volume from $\frac{8}{6}$ to $\frac{5}{6}$ and obtain the payoff of $\frac{25}{36}$. (Note that following firm 1's deviation, firm 2's output choice is still $q_2^* = \frac{1}{3}$, because firm 2 must choose its output without actually seeing firm 1's investment decision.) Hence such an equilibrium exists if and only if

$$\frac{16}{9} - F \geq \frac{25}{36} \Leftrightarrow \frac{39}{36} \geq F.$$

The other pure-strategy NE where firm 1 does not spend F in equilibrium can be derived analogously.



Show that there is a unique SPNE for this game. (**Hint**: Let T_i be the equilibrium point in time firm i chooses to exit. First observe that $T_1 \in [1, \frac{11}{4}]$ and $T_2 \in [\frac{1}{2}, \frac{7}{4}]$. Now repeatedly apply common knowledge about the two firms' rationality to make the following inferences: if at any $t \in [\frac{5}{4}, \frac{11}{4}]$ firm 1 is still operating, then firm 1 will stay till $\frac{11}{4}$, and knowing this, if at any $t \in [\frac{5}{4}, \frac{11}{4}]$ firm 2 is still operating, then firm 2 should leave immediately at t. Firm 1, knowing that firm 2 is rational and is able to make the above inference, will not leave until $\frac{11}{4}$ if it is still operating at any $t \in [1, \frac{5}{4}]$. Being able to make this last inference, firm 2 will leave immediately at $t = \frac{1}{2}$. Conclude thereby that $T_1 = \frac{11}{4}$ and $T_2 = \frac{1}{2}$ constitute the unique SPNE.)[33]

[33]This game does have other NE's which are not subgame perfect. Show that $T_1 = 1$ and $T_2 = \frac{7}{4}$ constitute one such NE. To see that this NE is not subgame perfect, note that in the subgame at time $t \geq \frac{5}{4}$ where neither firm has exited, $T_1 = \frac{11}{4}$ is a dominant

54. Three cowboys A, B, and C will take turn to shoot at one of their opponents (everyone can take one shot; A first, and then B, and then C, and then A (if still alive), and so on). The shooter's feasible strategies consist of deciding his targets. With probability 0.8, 0.2 and zero, A, B, and C may respectively miss their targets. Assume that if the target is hit, he dies. The game ends when there is exactly one cowboy remaining alive. The last survivor gets payoff 1, and others get zero. Find an SPNE.

Solution We shall look for a special kind of SPNE's, called a *stationary equilibrium*. In such an equilibrium, a player's strategy in a subgame depends only on the state at the current stage, but not on the history that reaches this subgame. In this game, the state of the current stage concerns how many opponents of the shooter are still alive. In such an equilibrium, in all the subgames where it is A's turn to shoot and both B and C are alive, A will adopt the same equilibrium strategy.

(a) In every subgame where there are two cowboys alive, the shooter's strategy is, trivially, to aim at his last opponent.

(b) Consider the subgame where a shooter is facing two opponents. If the shooter is C, then after C shoots the preceding (a) will apply, and hence in C's interest, C prefers taking out B rather than A.

(c) Consider the subgame where the shooter is B and B is facing two opponents. After B shoots, if the target is missed, then the above (b) will apply, and B will be dead regardless who B's current target is. On the other hand, if the target is not missed, then B is better off facing A. The conclusion is therefore B should aim at C.

(d) Consider the subgame where the shooter is A and A is facing two opponents. Again, we do not have to discuss the case where A misses his target. In case the target is hit, then A would rather let the last surviving opponent be B (so that A has the chance to shoot again). The conclusion is again that A should shoot at C.

Thus our conclusion is that both A and B should try to shoot at C, and C should try to shoot at B, until someone is hit and dead, and from then on the remaining two start shooting at each other.

strategy for firm 1, to which firm 2's best response is *not* $T_2 = \frac{7}{4}$. That is, the NE does not specify NE strategies for firm 2 in each and every subgame!

55. Consider the following voting game. An incumbent manager I is currently managing an all-equity firm. The firm's earnings will be Y_I under manager I's control, and manager I himself will receive a private benefit Z_I if he has the control. A raider C appeared just now to compete with I. The firm's earnings under C's control will be Y_C and manager C will receive a private benefit Z_C if he defeats I and becomes the new manager. There are two classes of stocks issued by the firm. Class i stockholders are entitled to a fraction S_i of the firm's earnings, and are given a fraction V_i of the votes in managerial election, where $i = A, B$. Assume that investors do not hold both stocks A and B at the same time. Investors are rational and can predict the outcome of control contest correctly. There are many small shareholders for each class of stock, so that no shareholder considers himself privotal. Suppose that the firm's charter states that the rival C gets control if he obtains a fraction α of the votes. For simplicity, assume that $\alpha = 1$.

The game proceeds as follows. The rival C first makes an offer to stockholders, and then the incumbent manager can make another offer given C's offer. Given the two offers, the stockholders then make tendering decisions.

(i) Suppose that $S_A = S_B = 50\%$, $V_A = 100\%$, $V_B = 0\%$. Suppose that it is common knowledge that $Y_I = 200, Y_C = 180$, and $Z_I = 0 < Z_C$. Suppose that offers must be in integers. Show that if $Z_C \geq 11$, C can purchase all class A stock at the price of 101, and the market value of the firm becomes 191.

(ii) Suppose that $S_A = 75\%$, $S_B = 25\%$, $V_A = 100\%$, $V_B = 0\%$. Moreover, Y_I, Y_C, Z_I, Z_C are as assumed in part (i). Show that the market value of the firm becomes 196 in equilibrium if $Z_C \geq 16$; or else the firm value is 200.

(iii) Suppose that $S_A = 100\%$, $S_B = 0\%$, $V_A = 100\%$, $V_B = 0\%$. Moreover, Y_I, Y_C, Z_I, Z_C are as assumed in part (i). Show that the market value of the firm becomes 201 in equilibrium if $Z_C \geq 21$; or else the firm value is 200.

(iv) From now on, assume $Z_I, Z_C > 0$. Suppose that $S_A = 100\%$, $S_B = 0\%$, $V_A = 100\%$, $V_B = 0\%$. Moreover, $Y_I = 10$, $Y_C = 100$, $Z_I = 0 < 1 < Z_C$. Show that the market value of the firm becomes 100 in equilibrium.

(v) Suppose that $S_B = 100\%$, $S_A = 0\%$, $V_A = 100\%$, $V_B = 0\%$. Moreover, $Y_I = 10$, $Y_C = 100$, $Z_I = 0 < 1 < Z_C$. Show that the market value of the firm becomes 101 in equilibrium.[34]

56. Players 1 and 2 are bargaining over 1 dollar. The game proceeds in $2N$ periods. In period i, where i is odd, if no concensus has been reached before, then player 1 can make an offer $(x_i, 1-x_i)$ to player 2, where x_i is player 1's share, and player 2 can either accept or reject that offer, and if the offer is accepted, then the dollar is so divided; or else, the game moves on to the $i+1$st period. In period j, where j is even, if no concensus has been reached before, then player 2 can make an offer $(x_j, 1-x_j)$ to player 1, where x_j is again player 1's share, and player 1 can either accept or reject that offer, and if the offer is accepted, then the dollar is so divided; or else, the game moves on to the $j+1$st period. Both players would have zero payoff if no concensus has been reached at the end of period $2N$.

    We assume that waiting is costly, because both players have a (common) discount factor $\delta \in (0, 1)$. Show that this game has a unique subgame perfect Nash equilibrium (SPNE) where player 1 proposes

[34] Thus one-share-one-vote scheme is optimal except in the case where one management team has higher values in both Y and Z! The idea here is that the managerial position typically gives a team private benefits which the investors of the firm cannot obtain, but via the design of a voting scheme the investors can force a management team to disgorge those benefits. When the voting scheme is not one-share-one-vote, as in parts (i) and (ii), the team with a lower earnings performance may still win (by paying a part of their private benefits to the vote-holding investors); this happens because the votes they purchase do not represent 100% of the cash-flow rights (and hence the differential performance between this team and a high-performance team is under-estimated by the vote-holding investors). The total firm value may not attain the maximal level because the securities without voting rights may receive less cash flows under the management of the winning team. When one team is better than the other team in both earnings performance and private benefits, forcing the former to buy purely voting securities can best extract that winning team's private benefits. Of course, ex-ante the firm in designing the voting scheme cannot tell whether there will be a team that dominates other teams in both Y and Z, and so the voting scheme must be chosen to be optimal against an "average" future situation.

$(\frac{1-\delta^{2N}}{1+\delta}, \frac{\delta+\delta^{2N}}{1+\delta})$ and player 2 accepts it in the first period.[35] [36]

57. Two firms are asked to deliver respectively $A > 0$ and $B > 0$ units of a homogeneous good at time 1. They do not have inventory at time 0. At each point in time, holding one unit of inventory incurs a cost $\alpha > 0$ (respectively $\beta > 0$) for firm A (respectively, firm B). Purchasing one unit of the good at time t incurs a unit cost

$$c(t) = 1 + a(t) + b(t),$$

where $a(t)$ and $b(t)$ are respectively quantities ordered by firms A and

[35]By backward induction, we have $x_{2N} = 0$, $x_{2N-1} = 1 - \delta(1 - x_{2N})$, $x_{2N-2} = \delta x_{2N-1}$, $x_{2N-3} = 1 - \delta(1 - x_{2N-2})$, and so on. Define $y(n) \equiv x_{2N-1-2(n-1)}$, for all $n = 1, 2, \cdots, N$. Then we have $y(1) = x_{2N-1}$, $y(2) = x_{2N-3}, \cdots, y(N) = x_1$. Observe that $y(\cdot)$ satisfies the following difference equation:

$$y(n+1) = 1 - \delta + \delta^2 y(n),$$

with the seed value $y(1) = 1 - \delta$. Note that the adjoint homogeneous equation $y(n+1) = \delta^2 y(n)$ has the general solution $b\delta^{2n}$ for some constant b, and that the inhomogeneous part is a constant, which induces us to guess a particular solution a for $y(n)$, where a is a constant independent of n. Solving $a = 1 - \delta + \delta^2 a$, we have $a = \frac{1}{1+\delta}$. Now since the general solution for the difference equation $y(n+1) = 1 - \delta + \delta^2 y(n)$ has to be the sum of the general solution for the adjoint homogeneous difference equation *and* the particular solution a for the original inhomogeneous difference equation (see http://en.wikipedia.org/wiki/Difference_equations), we have

$$y(n) = a + b\delta^{2n}.$$

Finally, using $y(1) = 1 - \delta$, we obtain $b = \frac{-1}{1+\delta}$, so that $x_1 = y(N) = \frac{1-\delta^{2N}}{1+\delta}$.

[36]Thus our theory predicts that if $N = 1$ then player 2, when he gets the chance to move, will offer player 1 with zero payoff. This prediction is at odds with existing experimental evidence. Usually, the player who makes the offer behaves a lot more generous in those experiments than our theory would describe. One possibility is that player 1 (in the experiment) is not an expected utility maximizer; rather, his behavior may be consistent with Kahneman and Tversky's (1979) prospect theory (see my note in Investments, Lecture 2, Part II). Say player 1 thinks that it is simply *fair* that he receives at least 0.3 dollars (called a *reference point*), and he is ready to reject any offer that gives him a payoff less than 0.3 dollars. If player 2 recognizes this, then player 2's optimal strategy is to offer player 1 0.3 dollars. This example does not necessarily imply any flaws of the game theory per se (I am not saying that the game theory is flawless though); rather, it points out the importance of specifying correctly the payoffs for the players. Mis-specified payoffs lead to incorrect predictions about the final outcome of the game. Since our purpose of learning the game theory is essentially to make correct predictions, the importance of specifying correctly the normal form for the game cannot be overstated.

B at time t. The inventory as a function of time must be continuously differentiable. Each firm seeks to minimize the total costs of making the delivery at time 1 (without discounting). Assume that

$$A > \frac{1}{6}(2\alpha - \beta),\ B > \frac{1}{6}(2\beta - \alpha),$$

and

$$2\alpha > \beta > \frac{\alpha}{2} > 0.$$

Find an (open-loop) NE for this game.[37] (**Hint**: Suppose that the two firms have inventory $x(t)$ and $y(t)$ at time t, where $x(0) = y(0) = 0$, and $x(1) = A$, $y(1) = B$. Then $x'(t) = a(t)$ and $y'(t) = b(t)$, so that the total cost facing the two firms are respectively

$$\int_0^1 (\alpha x(t) + x'(t)[1 + x'(t) + y'(t)])dt$$

and

$$\int_0^1 (\beta y(t) + y'(t)[1 + x'(t) + y'(t)])dt.$$

An open-loop pure strategy NE is a pair $((x(\cdot), y(\cdot))$ such that (i) given $y'(\cdot)$, $x(\cdot)$ minimizes $\int_0^1 (\alpha x(t) + x'(t)[1 + x'(t) + y'(t)]dt$ subject to $x(0) = 0$ and $x(1) = A$; and (ii) given $x'(\cdot)$, $y(\cdot)$ minimizes $\int_0^1 (\beta y(t) + y'(t)[1 + x'(t) + y'(t)]dt$ subject to $y(0) = 0$ and $y(1) = B$. Conjecture that both $x(\cdot)$ and $y(\cdot)$ are quadratic. See my note on "continuous-time calculus of variations" for the necessary and sufficient Euler equation.)

Solution Define

$$F(x, x', t; y') \equiv \alpha x(t) + x'(t)[1 + x'(t) + y'(t)],$$

and

$$G(y, y', t; x') \equiv \beta y(t) + y'(t)[1 + x'(t) + y'(t)],$$

and one can verify that F is convex in (x, x') and G is convex in (y, y'). Thus, from the Euler equation, we have

$$\alpha = \frac{d}{dt}[1 + 2x'(t) + y'(t)],\ \beta = \frac{d}{dt}[1 + 2y'(t) + x'(t)],$$

[37] In an open-loop NE, strategies are functions of time alone. By contrast, in a closed-loop or feedback control NE, strategies can be made contingent upon the history up to any time t. Which equilibrium concept is more proper for this problem?

implying that in equilibrium,

$$\alpha = 2x''(t) + y''(t),\ \ \beta = x''(t) + 2y''(t).$$

Integrating and using the boundary conditions, we have

$$x(t) = \frac{2\alpha - \beta}{6}t^2 + (A - \frac{2\alpha - \beta}{6})t,$$

and

$$y(t) = \frac{2\beta - \alpha}{6}t^2 + (B - \frac{2\beta - \alpha}{6})t.$$

Finally, conditions must be imposed so that the inventories are positive at all times.

58. Let us review the notion of *forward induction* using the following two-player game (Kohlberg and Mertens, 1986). Player 1 first chooses to read a book or to go to a concert. If he chooses to read a book, then the game ends with a payoff profile (2,2). If he chooses to go to a concert, then he and his girlfriend play the following subgame called *Battle of Sex* (BoS from now on). There are two concerts available for the night, on Bach (B) and on Stravinsky (S). Going to the same concert is perferred by the couple, but the boy and the girl prefer different music. Specifically, the normal form of this "BoS" subgame is as follows.

player 1/player 2	B	S
B	3,1	0,0
S	0,0	1,3

First, let us write down the normal form for the entire game.

player 1/player 2	B	S
Reading,B	2,2	2,2
Reading,S	2,2	2,2
Concert,B	3,1	0,0
Concert,S	0,0	1,3

Note that with reading, B is never reached by player 1, but the *definition* of a pure strategy requires that we write down what player 1 would do whenever he is called upon to make a move. Now, note that (Reading,B) and (Reading,S) are *equivalent strategies* for player 1, in the sense that the two always generate the same payoff for player 1 regardless of player 2's move. Removing one of them leads to the *reduced normal form* of the game as follows.

player 1/player 2	B	S
Reading	2,2	2,2
Concert,B	3,1	0,0
Concert,S	0,0	1,3

Now, we demonstrate the connection between weakly dominated strategy and the notion of *forward induction*. Note that giving up "Reading" and then choosing "Concert,S" is weakly dominated by "Reading" for player 1. What should player 2 think if nonetheless player 1 chooses to give up "Reading?" At this moment, the subgame has 2 pure strategy NE's and one mixed strategy NE. The payoffs of player 1 in these equilibria are respectively 3, 1 and $\frac{3}{4}$. Player 2 should conclude that player 1 is prepared to reach the NE ((Concert,B),B) if he gave up "Reading:" or else, player 1 could have done better by choosing "Reading!" In other words, a leader's move in a sequential game serves to influence the follower's beliefs regarding which NE will probably prevail.[38]

Consider the following game, known as *Burning Money* (Ben-Porath and Dekel, 1992). Suppose that player 1 can first choose to or not to give away 1 util before playing the BoS game (note that we have removed his option of staying home and doing some reading). The normal form of this new game is as follows.

[38]The literal interpretation of the game is not very sensible, however. The couple certainly can communicate with each other, without having to guess what is on the other's mind. They are lovers, aren't they?

player 1/player 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
DB	2,1	-1,0	2,1	-1,0
DS	-1,0	0,3	-1,0	0,3

In the above bimatrix, for example, (BS) stands for "go to Bach if player 1 did not discard the util but go to Stravinsky if player 1 did", and (0B) stands for "do not discard the util, and simply go to Bach." Note that allowing player 1 to have an opportunity of making a silly commitment actually is beneficial to player 1. Now he can be sure that (3,1) will be the outcome of the game. To see this, note that DS is (weakly) dominated by 0B; hence SS by SB; hence BS by BB; hence 0S by DB; hence SB by BB; hence DB by 0B. The single strategy profile that remains is (0B,BB)! The intuition is that, player 1 can always get a payoff equal to or greater than $\frac{3}{4}$ if he does not throw away the util, and hence his playing D (discarding utility) and then playing S makes no sense to player 2. Playing D should signal that player 1 is prepared to play B! This means that player 2's best response is to play B, yielding 2 for player 1. Realizing that player 2 would reason this way after seeing D, player 1 will not play 0 (playing 0 means not to throw away the util) unless he is prepared to play B afterwards. Knowing this, player 2 will then react by playing B, yielding 3 for player 1. Without this commitment, player 1 must run the risk that the NE may be (S,B), in which he receives only 1 instead of 3, or the mixed strategy NE, where his payoff is $\frac{3}{4}$.

At first glance, the above reasoning may seem odd: discarding the util together with any feasible strategy in the game of BoS is dominated by not discarding the util and then playing that feasible strategy. The crucial idea here is, however, that these decisions are made sequentially: discarding the util need not be a weakly dominated move if it successfully affects player 2's belief about what player 1 will play in the subsequent game of BoS. Without this seemingly redundant stage, all three NE's for BoS look likely to player 2, and player 1 may run the risk of getting a payoff less than or equal to 1; but with the *burning money* stage, player 2, upon seeing player 1 discard the util, must conclude that player 1 believes that he and player 2 will subsequently play the outcome (Bach, Bach): discarding the util and then reaching either the mixed strategy NE or the pure strategy NE (Stravinsky, Stravinsky) will generate a non-positive payoff for player 1, and hence is definitely worse than not discarding the util (in the latter case player 1 will get no less than $\frac{3}{4} > 0$). Thus in view of the role that *burning money* plays in helping player 2 rule out the mixed strategy NE and the pure strategy NE (Stravinsky, Stravinsky), *burning money* is not a weakly dominated strategy for player 1 (recall that it ensures a payoff of 2).

59. (**Value of Commitment**) Recall the prisoner's dilemma:

player 1/player 2	Don't Confess	Confess
Don't Confess	0,0	-3,1
Confess	1,-3	-2,-2

Suppose that the two players both commit to give away 2 utils to a third party if ex-post they are found to have played the strategy "Confess." Now neither can profitably deviate. A smaller strategy space leads to a higher welfare for both.

One problem pertaining to this arrangement is that the commitment, in order to be credible, must be *irreversible*. Suppose that player 1 can secretly come back and talk to the guy to whom he commits to transfer his utility. Suppose player 1 offers to give a tiny payoff $e > 0$ if the guy is willing to forget about the aforementioned commitment. It is easy to see that the guy should accept the offer: if he refuses, then the commitment is still binding for player 1, and since player 1 will never play "Confess" in this case, the guy will get nothing. The question is then, "Can player 1 benefit from sneaking back and renegotiating with the guy?" The answer is positive, for $e > 0$ small enough. If player 2 will not play "Confess" then player 1 gets 1 by playing "Confess," and since

$$1 - e > 0,$$

where 0 would be player 1's payoff if player 1 does not play "Confess," player 1 does benefit from the above renegotiation. Always remember that nobody is fool. If we can deduce all this, so can player 2. Rationally expecting (although not seeing directly) player 1's incentive of sneaking back, player 2 will not take player 1's "commitment" seriously in the first place. (The argument goes from player 1's perspective too.)

Thus the game remains the same as before even if the two players are allowed to make the announcements that they are willing to give away payoffs if they play the undesired strategies.[39]

Although making arrangements with a third party is not credible when a player can renegotiate those arrangements with the third party without being seen by the rival, committing with the rival directly is credible. (Think of deposits.) By committing with the rival, re-negotiation of the commitment will be detected directly, and there is actually no room for such re-negotiation. Since we have assumed that contracting is feasible (or else how did player 1 commit with the third party in the above scenario?), we can allow the two players to write a binding contract, saying that a player that is found to have played "Confess" ex-post will be liable to paying the rival 2 (units of utility). To implement the contract, one player can draw it, and write explicitly that the contract is valid if and only if it has both players' signatures on it. It is easy to see that with this explicit statement both players will sign it, and given it, both players are better off.[40]

In decision theory, a single decision maker's welfare always increases

[39] Recall the Stackelberg game where two firms compete in quantity to maximize revenues (no production costs) given the inverse demand $p = 1 - q_1 - q_2$. Recall that the leader (firm 1) has $q_1^* = \frac{1}{2}$ and the follower (firm 2) has $q_2^* = \frac{1}{4}$. Note that if firm 1 could secretly change q_1 after announcing $q_1^* = \frac{1}{2}$, then it would prefer $q_1 = \frac{3}{8}$ instead. Why? Simply because $\frac{3}{8}$ rather than $\frac{1}{2}$ is the best response for firm 1 against firm 2's quantity $\frac{1}{4}$. However, rationally expecting this, firm 2 will not take $q_1^* = \frac{1}{2}$ as "given" any more! In fact, if firm 2 believes that firm 1's announcement $q_1^* = \frac{1}{2}$ can be secretly reversed, then the game becomes simultaneous again, and the only possible NE is where $q_1 = q_2 = \frac{1}{3}$. This example shows that for $q_1^* = \frac{1}{2}$ to be taken seriously by firm 2, firm 1 must have *commitment power*; that is, it must be able to somehow convince firm 2 that q_1^*, once announced, can never be altered. Commitment power actually explains why firm 1 rather than firm 2 may get to be the leader in the Stackelberg game: if firm 1 has commitment power while firm 2 does not, then firm 1 gets to be the leader of the game.

[40] In reality, explicit binding contracts may not be feasible, since the two players may be considered collusive, which is illegal and detrimental to other people's welfare; imagine that two players are the two firms in the preceding game of *market share competition*. In a subsequent Lecture on repeated games, we shall demonstrate the possibility of sustaining a commitment between two self-interested players by allowing the latter to tacitly coordinate their intertemporal strategies. The basic idea is that *relationship has value*: with a long-term relationship, a short-term move against the two-players' common interest will be retaliated in the future, and this threat helps the players attain Pareto improved outcomes even if signing binding contracts is not feasible.

when the feasible set of choices is enlarged. The non-cooperative game theory essentially treats multiple decision makers' decision-making problems. The example we present here shows that, with multiple decision makers, enlarging their strategy spaces need not raise their welfare; they may become worse off actually.

With a single decision maker, he is always better off with more information. In a game-theoretic context with multiple agents, the same need not be true. This is the famous *Hirshleifer effect*; see the following example. Suppose that A and B will consume a single commodity at date 2, while their endowments in the commodity in the two equally probable states at date 2 are summarized in the following table:

endowments/ states	state 1	state 2
player A's endowments	2	0
player B's endowments	0	2

Suppose that A and B are endowed with an increasing, strictly concave von Neumann-Morgenstern utility function $u(\cdot)$ for date-2 consumption. The current time is date 0, and A and B can sign an insurance contract at date 1.

First suppose that no new information arrives at date 1. It is clear that A and B can benefit from the following insurance contract: if state 1 occurs at date 2, then A should give B one unit of the commodity; and if state 2 occurs at date 2, then B should give A one unit of the commodity. Jensen's inequality implies that

$$\frac{1}{2}u(1) + \frac{1}{2}u(1) = u(1) = u(\frac{0+2}{2}) > \frac{1}{2}u(0) + \frac{1}{2}u(2),$$

and hence both A and B are better off (have a higher expected utility) at date 0, in anticipation of the opportunity to sign an insurance contract at date 1.

Now suppose instead that some public information arrives at date 1, which reveals completely the state at date 2. If state 1 will prevail at date 2, A will refuse to trade with B; and if state 2 will prevail at date 2, then B will refuse to trade with A. This information destroys the possibility of insurance! With this information A and B are worse off at date 0.

60. In this last section, we shall prove that the order of deletion is irrelevant in leading to a set of strategy profiles that survive iterated deletion of strictly dominated strategies.

Consider two players, labeled A and B, in a simultaneous game with their *finite* strategy spaces being respectively

$$S^A = \{a_1, a_2, \cdots, a_J\},$$

and

$$S^B = \{b_1, b_2, \cdots, b_K\}.$$

Suppose that iterative deletion of strictly dominated strategies[41] in two different sequential orders V and T leads to two maximal reductions (V_N^A, V_N^B) and (T_M^A, T_M^B) for (S^A, S^B), where N (respectively, M) is the number of steps taken to reach the maximal reduction (V_N^A, V_N^B) (respectively, (T_M^A, T_M^B)). We shall prove by contraposition that $V_N^A = T_M^A$ and $V_N^B = T_M^B$. To this end, assume without loss of generality that $a_j \in V_N^A \bigcap [T_M^A]^c$, and we shall demonstrate a contradiction.

Since T_M^A does not contain a_j, a_j must be strictly dominated by some mixed strategy σ_A and deleted in some step $m \leq M-1$ under T. Can we have $V_N^B \subset T_m^B$? We shall show that the answer is negative.

Suppose instead that $V_N^B \subset T_m^B$. If σ_A remains feasible when A is restricted to using pure strategies in V_N^A, then since a_j is strictly dominated by σ_A, a_j cannot be contained in V_N^A, which is a contradiction. What if σ_A becomes infeasible when A is restricted to using pure strategies in V_N^A? In the latter case, there must exist $a_i \in T_m^A \bigcap [V_N^A]^c$ with $\sigma_A(a_i) > 0$, but since V_N^A does not contain a_i, a_i must be strictly dominated by some σ_A^i and deleted at some step $n_i < N$ under V. For all a_i as such, we can replace a_i by σ_A^i in σ_A, in the sense that we assign probability $\sigma_A(a_i)$ to σ_A^i rather than a_i, and call the resulting A's mixed strategy σ'_A. Clearly, σ_A (and hence a_j) is dominated strictly by

[41] Let $S_0^i \equiv S^i$ and $\Sigma_0^i \equiv \Sigma^i$. Define correspondingly S_0^{-i} and Σ_0^{-i}. Define for all $n \geq 1$, $\Sigma_n^i \equiv \{\sigma^i \in \Sigma^i : \sigma^i(s^i) > 0 \Rightarrow s^i \in S_{n-1}^i\}$ and $S_n^i = S_{n-1}^i \setminus \{s^i \in S_{n-1}^i : \exists \sigma^i \in \Sigma_n^i \ni u_i(\sigma^i, \sigma^{-i}) > u_i(s^i, \sigma^{-i}), \forall \sigma^{-i} \in \Sigma_n^{-i}\}$. Define $S_\infty^i \equiv \bigcap_{n=1}^\infty S_n^i$. Because S^A and S^B are finite sets, it takes a finite number of steps to reach S_∞^i, for $i = A, B$. We must show that S_∞^i is independent of the order of deletion of strictly dominated strategies.

σ'_A and hence a_j cannot be contained in V^A_N, as long as σ'_A is feasible when A is restricted to using pure strategies in V^A_N. Note that if σ'_A is again infeasible when A is restricted to using pure strategies in V^A_N, then by repeating the above argument we can again create some σ''_A that dominates strictly σ'_A. In the end, since S^A is a finite set, there must be some σ^*_A that remains feasible when A is restricted to using pure strategies in V^A_N,[42] and σ^*_A dominates strictly a_j, implying that V^A_N cannot contain a_j, which is a contradiction.

Thus we conclude that there must exist some $b_k \in V^B_N \bigcap [T^B_m]^c$ such that when player B uses b_k, player A feels that using a_j is no worse than using σ_A. As T^B_m does not contain b_k, there exists σ_B that dominates strictly b_k and makes b_k deleted at some step $l \le m-1$ under T. If $V^A_N \subset T^A_l$, then by mimicking the reasoning in the preceding paragraph, we can establish that V^B_N does not contain b_k, which would be a contradiction. Thus there must exist some $a_h \in V^A_N \bigcap [T^A_l]^c$, such that when player A uses a_h, player B feels that using b_k is no worse than using σ_B.

Note that a_j and a_h are distinct: a_j has not been deleted at step $l \le m-1$ under T, and if $a_j = a_h$, then b_k cannot be deleted at step l under T!

Now we can turn the spotlight away from $a_j \in V^A_N \bigcap [T^A_M]^c$ but on to $a_h \in V^A_N \bigcap [T^A_l]^c$ instead. Define $a_1 \equiv a_j$ and $a_2 \equiv a_h$, and repeat the above argument. We thus obtain an infinite sequence of distinct elements of S^A, which, given that S^A is finite, is a contradiction. We conclude that $V^A_N \subset T^A_M$, and by symmetry, that $T^A_M \subset V^A_N$, so that $V^A_N = T^A_M$. This conclusion applies to player B as well.

[42] First note the following fact: If σ_A strictly dominates a_j when B is restricted to using pure strategies in $S_B \subset S^B$, then for all $b_r \in S_B$, $\sum_{i=1}^m u_A(a_i, b_r)\sigma_A(a_i) - u_A(a_j, b_r) > 0$, implying that $\sigma^j_A \equiv \frac{1}{1-\sigma_A(a_j)}(\sigma_A(a_1), \sigma_A(a_2), \cdots, \sigma_A(a_{j-1}), \sigma_A(a_{j+1}), \cdots, \sigma_A(a_m))$ is also a mixed strategy that dominates strictly a_j. Note that σ^j_A assigns zero probability to a_j. Thus when a_j is strictly dominated, we can always assume that it is dominated by a mixed strategy that assigns positive probabilities only to A's pure strategies that differ from a_j. Now, define a^1 as a_i mentioned above, and if σ'_A is infeasible when A is restricted to using pure strategies in V^A_N, then there is $a^2 \in [V^A_N]^c$ with $\sigma'_A(a^2) > 0$ and a^2 differs from a^1. Repeating this argument, we see that if the afore-mentioned σ^*_A does not exist, then we would obtain an infinite sequence $\{a^n\}$ contained in $S^A \bigcap [V^A_N]^c$, which contradicts the fact that S^A is finite.

Appendix

Nash Equilibria for Bertrand Duopolists with Diverse Unit Costs

1. Consider the following extensive game. Firms 1 and 2 must compete for the patronage of a single buyer with unit demand. The firms produce the same product with different unit costs. Let $c_j > 0$ be firm j's unit cost of production. Let v be the buyer's valuation for the product. Assume that $v > c_2 > c_1 > 0$, and these parameter values are the firms' and the buyer's common knowledge. The game proceeds as follows. At first, the two firms must simultaneously choose prices $p_1 \in \Re$ and $p_2 \in \Re$ respectively, and then, upon seeing these prices, the buyer either purchase from one of the firms, or the buyer can choose to give up buying.

2. Let $F_j(\cdot)$ be the distribution function for p_j in a mixed-strategy Nash equilibrium.[43] Let Π_j be firm j's equilibirum payff (expected profit). The support S_j of p_j is the smallest closed set on which p_j may take values with probability one. Apparently, c_j is a lower bound for S_j (because pricing below c_j is never a best response for firm j) and S_j is non-empty (because elements contained in S_j must altogether occur with probability one), the set S_j has a greatest lower bound (or an infimum). Similarly, S_j is bounded above by v. This means that S_j has a lowest upper bound (or a supremum). Denote the greatest lower bound and lowest upper bound of S_j by respectively $\underline{p}_j$ and $\overline{p}_j$.[44]

[43] In the current context where all random variables are totally independent (because firms choose p_1 and p_2 simultaneously and independently), a single random variable p_j can be identified by its distribution function $F_j(\cdot)$. Recall that a real-valued function $F_j(\cdot) : \Re \to \Re$ is a distribution function if and only if it is right-continuous, weakly increasing, and it satisfies $\lim_{z\to+\infty} F_j(z) = 1$ and $\lim_{z\to-\infty} F_j(z) = 0$. A weakly increasing function is almost everywhere differentiable with respect to Lebesgue measure on $\Re$, and its has at most a countably infinite number of discontinuity points. A weakly increasing function $F_j(\cdot)$ is *regular* in the sense that at each and every $x \in \Re$, its left-hand limit $F_j(x-) \equiv \lim_{z<x,z\to x} F_j(z)$ and its right-hand limit $F_j(x+) \equiv \lim_{z>x,z\to x} F_j(z)$ both exist, so that all of its discontinuity points are of the first kind; that is, x is a discontinuity point of $F_j(\cdot)$ if and only if $F_j(x) = F_j(x+) > F_j(x-)$.

[44] In general, $\inf A$ and $\sup A$ denote respectively the greatest lower bound and the lowest upper bound for a subset A of $\Re$. Recall that $\inf A$ exists for any non-empty $A \subset \Re$ that

We shall focus on those equilibria in which firm 2 does not adopt a weakly dominated strategy. We shall show that there is a continuum of such equilibria, but all of them are equivalent in the sense that the firms receive the same pair of equilibrium payoffs in all of these equilibria. (Other equilibria will be briefly mentioned in section 3.)

- **Step 1.** In equilibrium, $\overline{p}_2 > \underline{p}_2$; that is, there does not exist an equilibrium where firm 2 adopts a pure strategy p_2^*.

  Suppose instead that such an equilibrium exists, with firm 2 pricing at p_2^* with probability one. We take cases.
  (1) $p_2^* > v$. In this case, firm 1's best response would be to price at v for sure, but firm 2 can deviate and price slightly below v and become better off, a contradiction.
  (2) $p_2^* \in (c_1, v]$. In this case, firm 1's best response would be to price below but as close to p_2^* as possible, implying the absence of a best response for firm 1, a contradiction.
  (3) $p_2^* \leq c_1$. If $p_2^* < c_1$, then firm 1 feels indifferent about any $p_1 > p_2^*$, implying that firm 2's equilibrium payoff is negative, a contradiction. If $p_2^* = c_1$, then firm 1 will never price below p_2^*, so that the buyer will buy from firm 2 with a positive probability, which again implies a negative equilibrium payoff for firm 2, a contradiction.

- **Step 2**. In equilibrium, $\underline{p}_2 \geq c_2$.

  Suppose instead that $\underline{p}_2 < c_2$. Then, then there exists a sequence $p_2^n \downarrow \underline{p}_2$ such that for all n, p_2^n is an equilibrium pure-strategy best response for firm 2. (See footnote 2.) For n sufficiently large, we have $p_2^n < c_2$, but firm 2 could have priced at c_2 and ensured a non-negative profit. This is a contradiction.

- **Step 3.** In any equilibrium, $\underline{p}_1 = \underline{p}_2$.

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has a lower bound, and that $\sup A$ exists for any non-empty $A \subset \Re$ that has an upper bound. Since $\inf A$ is the largest among all lower bounds of A, for any positive integer n, there exists some $a_n \in A$ such that $a_n < \inf A + \frac{1}{n}$; that is, $\inf A + \frac{1}{n}$ is not a lower bound of A. Thus there exists a sequence $\{a_n\}$ in A such that for all n, $\inf A + \frac{1}{n} > a_n \geq \inf A$, and hence this sequence $\{a_n\}$ must satisfy $\inf A = \lim_{n\to\infty}(\inf A + \frac{1}{n}) \geq \lim_{n\to\infty} a_n \geq \inf A$, or simply, $\{a_n\}$ must converge to $\inf A$. It is clear that we can choose $\{a_n\}$ to be a weakly decreasing sequence. By the same argument, there exists a weakly increasing sequence $\{b_n\}$ in A that converges to $\sup A$.

By pricing at any $p_1 < \underline{p}_2$ firm 1 will win the buyer's patronage for sure, but pricing at $\frac{1}{2}(p_1 + \underline{p}_2)$ is even better. Hence no $p_1 < \underline{p}_2$ can be an equilibrium best response for firm 1. We conclude that $\underline{p}_1 \geq \underline{p}_2$.
Now, if $\underline{p}_1 > \underline{p}_2$, then there exists some equilibrium best response $p_2 \in [c_2, \underline{p}_1)$ for firm 2. However, such a p_2 can never be a best response for firm 2, for it is obviously dominated by the price $\frac{1}{2}[p_2 + \underline{p}_1]$. Hence in equilibrium we must have $\underline{p}_1 = \underline{p}_2$.

- **Step 4**. In equilibrium, $F_1(\cdot)$ is continuous at any firm 2's best reponse $p_2 \in (\underline{p}_2, \overline{p}_2]$.

  Suppose not. Then

  $$\Delta F_1(p_2) \equiv F_1(p_2) - \lim_{z \uparrow p_2} F_1(z) > 0,$$

  at some $p_2 \in (\underline{p}_2, \overline{p}_2]$. Since $p_2 > \underline{p}_2$, we have

  $$0 < F_1(p_2-) \equiv \lim_{z \uparrow p_2} F_1(z),$$

  and since $F_1(p_2) \leq 1$, we have

  $$F_1(p_2-) \equiv \lim_{z \uparrow p_2} F_1(z) < 1.$$

  In this case, firm 2 would become better pricing at some $p_2 - \epsilon$ below but extremely close to p_2 than pricing at p_2, which contradicts the assumption that p_2 is one equilibrium best response for firm 2.

- **Step 5.** If firm j's equilibrium payoff $\Pi_j > 0$, then $\overline{p}_j \leq \overline{p}_i$, for all $i, j \in \{1, 2\}$, $i \neq j$.

  Indeed, if instead that $\overline{p}_j > \overline{p}_i$, then there exists some firm j's equilibrium best response $p_j > \overline{p}_i$, which implies that $\Pi_j = 0$, a contradiction.

- **Step 6.** There is no equilibrium in which $\underline{p}_2 > c_2$.

  Suppose instead that in equilibrium we have $\underline{p}_2 > c_2$. We claim that $\Delta F_1(\underline{p}_2) = 0$. Indeed, if $\Delta F_1(\underline{p}_2) > 0$, then some equilibrium best response $\underline{p}_2 + \epsilon$ which lies above but extremely close to $\underline{p}_2$ will become dominated by some other price $\underline{p}_2 - \delta$ which lies below but extremely close to $\underline{p}_2$, which would be a contradiction.

Now, we claim that if $\underline{p}_2 > c_2$ and $\Delta F_1(\underline{p}_2) = 0$, then $\underline{p}_2$ itself is an equilibrium best response for firm 2. Indeed, by the definition of $\underline{p}_2$, there exists a decreasing sequence $\{p_n\}$ in S_2 that converges to $\underline{p}_2$ such that for all n, p_n is one equilibrium best response for firm 2. By step 4, $F_1(\cdot)$ is continuous at p_n for all n, and hence for all n, by pricing at p_n, firm 2's payoff is

$$(p_n - c_2)[1 - F_1(p_n)] = \Pi_2,$$

where the constant Π_2 is firm 2's equilibrium expected profit. It follows that, by the right continuity of $F_1(\cdot)$ and the assumption that $\Delta F_1(\underline{p}_2) = 0$, firm 2 would get

$$(\underline{p}_2 - c_2)[1 - F_1(\underline{p}_2)] = \lim_{n\to\infty}(p_n - c_2)[1 - F_1(p_n)] = \lim_{n\to\infty}\Pi_2 = \Pi_2$$

by pricing at $\underline{p}_2$, and hence the latter is also one equilibrium best response for firm 2.

We have just reached the conclusion that $\Pi_2 = \underline{p}_2 - c_2 > 0$ if $\underline{p}_2 > c_2$. Note that $\Pi_1 > 0$ because firm 1 can always price at $\frac{c_1+c_2}{2}$ and obtain a strictly positive payoff. Thus if $\underline{p}_2 > c_2$, then by step 5, we have $\overline{p}_2 = \overline{p}_1$. Either $\Delta F_1(\overline{p}_2) > 0$ or $\Delta F_1(\overline{p}_2) = 0$. In the former case, $\overline{p}_2$ is one equilibrium best response for firm 1, which results in a zero payoff for firm 1, contradicting $\Pi_1 > 0$. In the latter case, since by steps 1 and 3, $\overline{p}_1 = \overline{p}_2 > \underline{p}_2 = \underline{p}_1$, there exists a sequence $\{p_n\}$ of equilibrium best responses for firm 2 such that for all n, $p_n \le \overline{p}_2$, and

$$\lim_{n\to\infty} p_n = \overline{p}_2,$$

we have

$$\Pi_2 = \lim_{n\to\infty}\Pi_2 = \lim_{n\to\infty}(p_n - c_2)[1 - F_1(p_n)] = (\overline{p}_2 - c_2)[1 - F_1(\overline{p}_2)] = 0,$$

where the last equality follows from the definition of $\overline{p}_2$. Hence we must have $\Pi_2 = 0$, which contradicts $\Pi_2 > 0$.

To sum up, we have shown that a contradiction will always arise if $\underline{p}_2 > c_2$.

- **Step 7.** In any equilibrium, $\Pi_2 = 0$.

  By pricing at $\underline{p}_2 = c_2$, firm 2's payoff is

$$
\begin{aligned}
0 = (c_2 - c_2)\{\frac{1}{2}\Delta F_1(c_2) + [1 - F_1(c_2)]\} \\
\geq (c_2 - c_2)[1 - F_1(c_2)] \\
= \lim_{p_n \downarrow c_2} (p_n - c_2)[1 - F_1(p_n)] = \lim_{n \to \infty} \Pi_2 = \Pi_2,
\end{aligned}
$$

  where $\{p_n\}$ is a decreasing sequence of firm 2's equilibrium best responses which converges to $\underline{p}_2 = c_2$.

- **Step 8.** In any equilibrium, $\underline{p}_1 = \overline{p}_1 = c_2$.

  Suppose instead that $\overline{p}_1 > p_1 > \underline{p}_1 = c_2$ for some p_1, then by pricing slightly below p_1 firm 2 can ensure itself a strictly positive payoff, a contradiction.

- **Step 9.** $\Delta F_2(c_2) = 0$, and hence $\Pi_1 = c_2 - c_1$; that is, in equilibrium firm 1 must win the buyer's patronage with probability one.

  By Steps 2 and 6, we know that $\underline{p}_2 = c_2$. If $\Delta F_2(c_2) > 0$, then for $e > 0$ small enough, pricing at $p_1 = c_2 - e$ is better than pricing at $p_1 = c_2$ from firm 1's perspective, which contradicts Step 8. It follows that $\Pi_1 = c_2 - c_1$.

- **Step 10.** There exists a continuum of payoff-equivalent equilibria with continuous $F_2(\cdot)$, where in a typical equilibrium firm 1 prices at $p_1 = c_2$ with probability one, and $F_2(\cdot)$ is such that $c_2 = \underline{p}_2 < \overline{p}_2 \leq v$ with $\Delta F_2(c_2) = 0$, and for all $x \in (c_2, \overline{p}_2]$,

$$(x - c_1)[1 - F_2(x)] \leq c_2 - c_1,$$

  or equivalently,

$$\frac{x - c_2}{x - c_1} \leq F_2(x), \ \forall x \in (c_2, \overline{p}_2].$$

  In words, firm 2's equilibrium mixed strategy must ensure that firm 1 does not wish to deviate from its equilibrium pure strategy, and this imposes a restriction on $F_2(\cdot)$—the latter must be first-order stochastically dominated by a benchmark distribution function.

3. What if $c_2 = c_1 = c < v$? In this case, the above analysis implies that both firms, just like the above firm 2, must have a zero equilibrium payoff (step 7), which implies that their rivals must adopt a pure strategy (step 8), showing that the Bertrand outcome where both firms price at c is indeed the unique Nash equilibrium outcome of this game.

4. The Bertrand game analyzed above has an exact counterpart in auction theory. Consider a first-price sealed-bid private-value auction with one seller and two bidders. The seller's reservation value for the (indivisible) auctioned object is c, and for $j = 1, 2$, bidder j's valuation for the object is v_j. Assume that

$$v_2 > v_1 > c \geq 0,$$

   and these parameter values are the seller's and the bidders' common knowledge.

   The game proceeds as follows. The two bidders simultaneously submit their bids $B_1 \in \Re$ and $B_2 \in \Re$ to the seller, and then the seller announces the bidder with the highest submitted bid as the winner, and then the winning bidder must pay his bid to the seller in exchange of the auctioned object. (In case of a tie, each bidder is announced as the winning bidder with probability $\frac{1}{2}$.)

   Mimicking the above analysis for the Bertrand game, one can obtain a continuum of payoff-equivalent equilibria. Let $G_j(\cdot)$ be the distribution function for bidder j's equilibrium bid B_j, and let U_j be bidder j's equilibrium payoff. Let $\overline{B}_j$ and $\underline{B}_j$ be respectively the supremum and the infimum of the support of B_j. One can proceed in the following steps.

   (a) There does not exist an equilibrium in which $\overline{B}_1 = \underline{B}_1$.

   (b) In each equilibrium , $\overline{B}_2 = \overline{B}_1 \leq v_1$.

   (c) There does not exist an equilibrium in which $\overline{B}_1 < v_1$.

   (d) In each equilibrium, $U_1 = 0$.

   (e) In each equilibrium, $\overline{B}_2 = \underline{B}_2 = v_1$, and hence $U_2 = v_2 - v_1$.

   (f) $G_1(\cdot)$ must first-order stochastically dominate a benchmark distribution function.