Game Theory with Applications to Finance and Marketing, I

Lecture 6: Network Formation and Competitive Platforms

1. This note consists of two parts. Part I gives a brief introduction to the theory of economic networks, and Part II online trading platforms.

2. **Part I**.

3. We proceed as follows. We shall first give a formal description for a network consists of a finite number of economic agents and define *strong efficiency*. Then we define an equilibrium concept referred to as *pairwise stability*. In many cases a pairwise stable equilibrium fails to be (strongly) efficient as a single agent does not fully internalize the externality that his decision to establish a new link or sever an existing link may create to the other agents. Then we discuss the dynamic evolution of networks in a discrete-time model where at each stage a pair of agents will be randomly selected and they can decide whether to sever an existing link or establish a new link, and to possibly sever other existing links in the latter case. It is shown that the dynamic process will lead to either a pairwise stable network or a cycle that consists of a set of networks such that in the end each network contained in the cycle will be repeatedly and indefinitely reached, and networks lying outside of the cycle are no longer reached.

4. Let $\mathcal{N} \equiv \{1, 2, \cdots, N\}$ be a finite set of players (or agents). Let ij denote the subset of $\mathcal{N}$ containing exactly i and j, and is referred to as *the link* ij. A network relation among these N agents is represented by a graph g, which is a collection of *links* ij. The interpretation is that if $ij \in g$, then players i and j are *directly connected*, and we say that i and j are *adjacent*. If $ij \notin g$, then players i and j are not *directly connected*. An example is the *complete graph* g^N, which represents the network in which everyone is directly connected to someone else. An empty graph then represents a network with no direct links. If g is non-empty and there exists player i (the *star center*) such that $jk \in g$ implies either $j = i$ or $k = i$, then g is referred to as a *star*.

Define $g+ij \equiv g \bigcup \{ij\}$ and $g-ij \equiv g \setminus \{ij\}$. Given a network g, define $N(g)$ as the set of players that are directly connected to someone else in g; i.e., $N(g) \equiv \{i | ij \in g \text{ for some } j\}$. Let $n(g)$ denote the cardinality of $N(g)$. A *chain* in g connecting i_1 and i_n is a set of *distinct* elements $\{i_1, i_2, \cdots, i_n\} \subset N(g)$ such that $\{i_1 i_2, i_2 i_3, \cdots, i_{n-1} i_n\} \subset g$.[1] We say that $g' \subset g$ is a *component* of g if there exists a chain connecting i and j for any distinct $i, j \in N(g')$, and if for any $i \in N(g')$, $j \in N(g)$, we have $ij \in g'$ whenever $ij \in g$. Thus two distinct components of g are disjoint.

We assume that forming a link ij requires the consent of both i and j, but severing a link can be done unilaterally. Moreover, forming a direct link ij will incur a cost $c > 0$ to both i and j. To describe a player i's payoff $u_i(g)$ from joining a network g, let t_{ij} denote the number of links in the *shortest* chain[2] between i and j (and define $t_{ij} = +\infty$ if no such chain exists), and let $\delta \in (0,1)$ be a constant. The network model is *symmetric* if $u_i(g)$ is independent of i, and it is a *symmetric connections model* if

$$u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - \sum_{j: t_{ij}=1} c.$$

In the symmetric connections model, agent i can benefit δ from an adjacent neighbor, and δ^2 from a player who is adjacent to one of agent i's adjacent neighbors, and so on, and so forth. The discussions to follow will mainly focus on the symmetric connections model.

5. (**Value of Network and Strong Efficiency**) Given g, define the *value* of the network g by $V(g) \equiv \sum_i u_i(g)$. We say that g is a *strongly efficient* network for $\mathcal{N}$ if $V(g) \geq V(g')$ for all other networks g'. First suppose that $c < \delta - \delta^2$.

[1] A chain is referred to as a *path* in Jackson and Wolinsky (1996). Jackson and Watts (2002) reserve the term a *path* to mean a sequence of networks, and a sequence of links is then referred to as a chain. The chain $g \equiv \{i_1 i_2, i_2 i_3, \cdots, i_{n-1} i_n\}$ is also referred to as a *line* passing through $i_1, i_2, \cdots, i_n$, and in this case the network $g + i_n i_1$ is referred to as a *circle* or a *ring*.

[2] This shortest chain is also referred to as the *geodesic* between i and j, and t_{ij} is referred to as the *geodesic distance*.

Lemma 1 *The complete network g^N is the unique strongly efficient network if $c < \delta - \delta^2$.*

To see the intuition, note that for any $i \neq j$ with $t_{ij} \geq 2$, we have

$$0 < \delta^{t_{ij}} \leq \delta^2 < \delta - c,$$

and hence it is in i and j's common interest to form a direct connection, which would also weakly raise $u_k(g)$ for all $k \neq i, j$.

6. Now suppose that $c > \delta - \delta^2$. There are three possibilities about a network g: either g is empty, or it is a single component, or it consists of several disjoint components.

Lemma 2 *Suppose that $g' \subset g$ and g' is a non-empty component containing m players and k links. Then $k \geq m-1$. Moreover, if g' has the highest value among all m-person components, then g' is a star with $k = m-1$.*

To see the intuition, note that it takes at least $m-1$ links to create a m-person component.[3] Now, observe that for a component having $m-1$ links, the component's value increases with the sum of payoffs generated by indirect connections.[4] The value generated by an indirect connection is no greater than $2\delta^2$. Thus a m-person star dominates any other m-person networks with $m-1$ links.

[3]Suppose that the m-person component involves the set $M \equiv \{1, 2, \cdots, m\}$ of agents. Let A be any non-empty proper subset of M and $\{A, A^c\}$ a partition of M. By the definition of a component, some agent i in A must be directly linked to some agent j in A^c. Thus agent 1 must linked to some agent $(2) \in M \setminus \{1\}$, and $\{1, (2)\}$ must be linked to some agent $(3) \in M \setminus \{1, (2)\}$. More generally, $\{1, (2), (3), \cdots, (k)\}$ must be linked to some agent $(k+1) \in M \setminus \{1, (2), \cdots, (k)\}$, for all $k \leq m-1$. Hence it takes at least $m-1$ links to form a m-person component.

[4]Each direct connection generates $2(\delta - c)$, and hence direct connections generate $2(m-1)(\delta - c)$ in total, regardless of the shape of the network.

Lemma 3 *Suppose that g is a m-person non-empty component. Then g is a single star if g is strongly efficient.*

If g is strongly efficient and non-empty, then it is either a single component, which by the preceding lemma must be a single star, or it consists of a finite number of disjoint components. In the latter case, strong efficiency requires that each of the disjoint components, by the preceding lemma, must be a single star. Suppose that g is composed of two disjoint stars, with $m-1$ and $n-1$ links respectively, and star-center agents i and j respectively. (Thus g is a $(m+n)$-person network, where $n+m \leq N$.) To be strongly efficient, it must be that both the m-person star and the n-person star have positive values. Note that the value of a m-person star is

$$V_m = (m-1)(2\delta - 2c) + (m-1)(m-2)\delta^2,$$

and we have $V_m > 0 < V_n$ if and only if

$$(2\delta - 2c) + (m-2)\delta^2 > 0 < (2\delta - 2c) + (n-2)\delta^2.$$

The latter two inequalities imply that

$$2(\delta - c) + (\frac{m+n}{2} - 2)\delta^2 > 0.$$

Now, imagine that the $m-1$ non-star-center agents in the former m-person star all sever their links to agent i and establish links to agent j at the same time, and being left alone, agent i also connects directly to agent j. How does this new arrangement affect the value of the network? It apparently benefits all non-star-center agents of both the n-person star and the m-person star: without additional linking costs, they are able to reach more non-adjacent agents and obtain a payoff δ^2 from each of the latter agents. For the two star-center agents, the sum of their payoffs before the change is

$$(m-1)(\delta - c) + (n-1)(\delta - c) = (m+n-2)(\delta - c),$$

which, after the change, becomes

$$(m+n-1)(\delta-c)+[(\delta-c)+(m+n-2)\delta^2] = (m+n)(\delta-c)+(m+n-2)\delta^2.$$

Thus the change benefits the two star-center agents as a whole if and only if

$$0 < 2(\delta - c) + (m + n - 2)\delta^2,$$

which is true given that

$$2(\delta - c) + (\frac{m+n}{2} - 2)\delta^2 > 0.$$

We conclude that a strongly efficient network cannot have disjoint components. Thus g must be a single star.

Lemma 4 *Suppose that $c > \delta - \delta^2$. Then a network g is strongly efficient if it is either empty or it is a N-person star.*

- *A N-person star is strongly efficient if $\delta - \delta^2 < c < \delta + \frac{(N-2)\delta^2}{2}$.*
- *The empty network is strongly efficient if $c > \delta + \frac{(N-2)\delta^2}{2}$.*

The earlier lemmas show that if g is non-empty and strongly efficient, then it is a single m-person star. The current lemma shows that a non-empty strongly efficient network must be a single star with $m = N$. To see what happens, note that the value of a N-person star is

$$V_N \equiv (N-1)(2\delta - 2c) + (N-1)(N-2)\delta^2,$$

which is positive (and hence dominates the empty network) if and only if $c < \delta + \frac{(N-2)\delta^2}{2}$. More generally, for $m \in \{2, 3, \cdots, N\}$, the value of a m-person star is

$$V_m = (m-1)(2\delta - 2c) + (m-1)(m-2)\delta^2,$$

which is positive if and only if

$$(2\delta - 2c) + (m-2)\delta^2 > 0 \Leftrightarrow m > m^* \equiv 2(\delta^2 - \delta + c),$$

and when $m > m^*$, we have

$$V_m - V_{m-1} = (2\delta - 2c) + 2(m-2)\delta^2 = 2m - m^* > 0.$$

Hence whenever $V_N > 0$, the single star with $m = N$ is strongly efficient. When $V_N < 0$, there is no m-person single star that can have a positive value, so that the empty network is efficient.

Proposition 1 *(***Jackson and Wolinky 1996***.)*[5] *The unique strongly efficient network in the symmetric connections model is*

- *the complete network* g^N *if* $0 < c < \delta - \delta^2$*;*
- *a* N*-person star if* $\delta - \delta^2 < c < \delta + (\frac{N-2}{2})\delta^2$*; and*
- *the empty network if* $\delta + (\frac{N-2}{2})\delta^2 < c$*.*

7. (**Pairwise Stable Equilibrium Networks.**) Consider a game where every pair of players can simultaneously meet and discuss whether to create a direct connection or to sever an existing connection. An equilibrium outcome for this game is referred to as a *pairwise stable network* g, which satisfies the following conditions:
(a) for all $ij \in g$, we have $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$; and
(b) for all $ij \notin g$, we must have $u_j(g) < u_j(g + ij)$ if $u_i(g) > u_i(g + ij)$.

In plain words, g is pairwise stable, if no single agent i wants to sever an existing link $ij \in g$, and no pair of agents i and j want to create a new link $ij \notin g$. Behind this definition is the assumption that severing an existing link can be done unilaterally, but adding a new connection must have approval from both agents that are involved in the new link.

Lemma 5 *Suppose that* g *is pairwise stable. Then* g *contains at most one non-empty component.*

To see the intuition, suppose instead that g', g'' are both components of g, with $ij \in g'$ and $kl \in g''$. Recall that g' and g'' must be disjoint. Since agent i is either directly or indirectly connected to every agent $h \in g'$, $h \neq i, j$, and agent k is totally unconnected to such agent h, agent k would get more than agent i does by spending c and connecting to agent j.[6] Thus we conclude that $u_k(g''+kj)-u_k(g'') > u_i(g')-u_i(g'-ij) \geq 0$. Similarly, we have $u_j(g' + kl) - u_j(g') > u_l(g'') - u_l(g'' - kl) \geq 0$. It

[5]Jackson, M., and A. Wolinsky, 1996, A Strategic Model of Social and Economic Networks, *Journal of Economic Theory*, 71, 44-74.

[6]Note that by spending c to build the link ij, agent i gets to weakly reduce the geodesic distance to each agent $h \in g'$, $h \neq i$. To see this, for each such h, let t_{ih} and t'_{ih} denote

follows that g, being pairwise stable, should have contained the link kj, a contradiction. Hence g cannot contain two disjoint non-empty components.

Lemma 6 *Suppose that $c < \delta - \delta^2$. Then the unique pairwise stable network is g^N.*

The intuition is obvious. Any two agents i and j that are not directly connected would get at most δ^2 from their un-connected or indirectly connected relationships. They would each obtain $\delta - c > \delta^2$ if they both agree to create a new link ij. Thus the unique pairwise stable network is g^N.

Lemma 7 *Suppose that $\delta - \delta^2 < c < \delta$. Then a star encompassing all players is pairwise stable, but there can be other pairwise stable equilibria.*

It is easy to check that neither the center agent nor the non-center agents in the N-person star would want to sever a link when $\delta - \delta^2 < c < \delta$. For the last assertion, consider the case with $N = 4$:

- The N-person circle is pairwise stable if $\delta - \delta^2 < c < \delta - \delta^3$; and
- The line $\{12, 23, 34\}$ is pairwise stable if $\delta - \delta^3 < c < \delta$.

Lemma 8 *Suppose that $\delta < c$.*

- *Then, the empty network is pairwise stable.*
- *In a non-empty pairwise stable network each player i must have at least two direct connections.*

respectively the geodesic distance from agent i to agent h in respectively the absence and the presence of link ij. Agent i's benefit from adding ij is $-c + \sum_{h \in g', h \neq i} [\delta^{t'_{ih}} - \delta^{t_{ih}}]$. Correspondingly, agent k's net benefit from adding kj is is $-c + \delta^2 + \sum_{h \in g', h \neq i} [\delta^{t'_{ih}} - 0]$, where the second term δ^2 is agent k's payoff from indirectly connecting to agent i via agent j.

- *A m-person star is never stable.*

Lemma 9 *Suppose that $\delta < c$ and $N = 5$. The N-person circle is pairwise stable if*

$$(\Theta) \quad \delta + (\delta^2 - \delta^4) > c > \delta.$$

In the 5-person circle $\{12, 23, 34, 45, 51\}$, each agent gets $\Pi^* \equiv 2(-c + \delta + \delta^2)$. Unilaterally severing one link will result in the deviating agent getting the new payoff $-c+\delta+\delta^2+\delta^3+\delta^4 < \Pi^*$, when (Θ) holds. How about adding a link, say 13? The net change in a deviating agent's payoff would be $(\delta - c) - \delta^2 < 0$, when (Θ) holds. Thus the 5-person circle $\{12, 23, 34, 45, 51\}$ is indeed pairwise stable.

Proposition 2 *(***Jackson and Wolinky 1996***.) For the symmetric connections model, the following assertions are true.*

- *Suppose that $0 < c < \delta - \delta^2$. Then the unique pairwise stable network is g^N, which is also the unique strongly efficient network.*
- *Suppose that $\delta - \delta^2 < c < \delta$. Then the unique strongly efficient network is a N-person star, but there can be multiple pairwise stable networks. A N-person star is one, but a line or a circle may also be pairwise stable.*
- *Suppose that $\delta < c < \delta + (\frac{N-2}{2})\delta^2$. Again, the unique strongly efficient network is a N-person star, but a N-person star is never pairwise stable! There can be multiple pairwise stable networks. The empty network is one, and actually the most inefficient one, showing that the equilibrium network can be under-connected. The N-person circle is pairwise stable when $N = 5$ and c is sufficiently small, showing that a pairwise stable equilibrium may exhibit too many links and the equilibrium network is overly connected.*[7]

[7] Note that given $N = 5$, the N-person star has 4 links, and yet the N-person circle has 5 links.

- *Suppose that $\delta + (\frac{N-2}{2})\delta^2 < c$. In this case, the unique strongly efficient network is the empty network, which is also pairwise stable. There can be no other pairwise stable networks in this case.*[8]

Summing up, we conclude that a pairwise stable network can be under- or over-connected, because when a single agent decides whether to sever a link or when a pair of agents decide whether to create a link, they do not take into account the impact of their decisions on other agents joining the network.

There are many variations of the above connections model, for example, one can replace $\delta^{t_{ij}}$ by a decreasing function $f(t_{ij})$, and in particular, one can assume that truncation occurs when $t_{ij} > n$, where n is some positive integer: let $f(t) = \delta^t$ if $t \leq n$ and $f(t) = 0$ otherwise. This has a natural interpretation when $n = 2$: a relationship between i and j is valuable if and only if either i and j are friends or i and j have a mutual friend k. Interested reader should refer to Jackson and Wolinsky (1996).

8. (**Dynamic Evolution of Networks.**) We have defined and characterized pairwise stable networks for the connections model in the preceding section, but we did not mention how likely a pairwise stable network may emerge from a dynamic process in which the agents are free to form new links and sever existing links. In this section, we switch attention to the dynamic evolution of networks, assuming that agent i's payoff function is $u_i(g)$.

   At date 1, the N players are endowed with a network g_1. Then, at date $t \geq 2$, a new network g_t will form and replace g_{t-1}, where either $g_t = g_{t-1}$ or $g_t = g_{t-1} + ij$ or $g_t = g_{t-1} - ij$ for some randomly (and uniformly) chosen link ij (which g_{t-1} may or may not contain). An *improving path* from a network g to a network g' is a finite sequence of

[8] Suppose instead that there is a non-empty pairwise stable network g. Since $c > \delta - \delta^2$, Lemma 2 implies that the value of any non-empty pairwise stable network g must be less than or equal to the value of a N-person star, which is now negative! Recall that the value of g is the sum of the N agents' payoffs in g. This implies that some agent in g would become better off severing all his links to other agents in g, proving that g cannot be pairwise stable.

adjacent networks $g_1, g_2, \cdots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, 2, \cdots, K-1\}$, either
(a) $g_{k+1} = g_k - ij$ for some ij such that $u_i(g_k - ij) > u_i(g_k)$; or
(b) $g_{k+1} = g_k + ij$ for some ij such that $u_i(g_k + ij) > u_i(g_k)$ and $u_j(g_k + ij) \geq u_j(g_k)$.
In words, an improving path is a sequence of adjacent networks that might be observed in a dynamic process where players are adding and deleting links, one at a time. Note that there can be multiple improving paths emanating from a given network g, but if there is no improving path leaving g, then g must be pairwise stable.

A set of networks, denoted by C, is said to form a *cycle* if for any $g, g' \in C$, there exist both an improving path connecting g to g' and an improving path connecting g' to g. A cycle C is *maximal* if it is not a proper subset of another cycle. A cycle C is *closed* if for all $g \in C$, any improving path emanating from g is entirely contained in C.

Lemma 10 *(***Closed Cycles versus Maximal Cycles.***)*

- *A closed cycle C is necessarily maximal.*
- *An improving path emanating from $g \in C$, where C is a (non-closed) maximal cycle, visits only networks not contained in C once it leaves C.*

Suppose that C is closed but C is a proper subset of another cycle C' and $g \in C$ but $g' \in C' \setminus C$. Then there is an improving path from g to g', as C' is a cycle, but this improving path is not entirely contained in C, so that C cannot be closed. Thus the above first assertion holds. The second assertion says that if there is an improving path leaving $g \in C$ and visiting some $g' \notin C$, then after visiting g' this improving path can never visit any networks contained in C. This is obvious, because if the improving path leaves g' and visits some $g'' \in C$, then C cannot be a maximal cycle: the union of C and the set of networks visited by the improving path $g \to g' \to g''$ is a strictly bigger cycle than C!

Lemma 11 *Suppose that g is contained in some cycle C. Then there is a maximal cycle that contains g.*

Let C^* be the union of all cycles containing g, so that $g \in C^*$. Let g' and g'' be distinct networks. We claim that there exists an improving path leaving g'' and visiting g' and another improving path leaving g' and visiting g'' if $g'', g' \in C^*$, and hence C^* is a cycle. The assertion is obvious if either $g' = g$ or $g'' = g$, for then by the definition of C^* there is already some cycle C containing both g' and g'', and hence the assertion follows. If instead $g' \neq g \neq g''$, then since $g, g' \in C^*$ there must exist cycles C' and C'' such that $g, g' \in C'$ and $g, g'' \in C''$. Then there exists an improving path ($g' \to g$) leaving g' and visiting g (as C' is a cycle), and an improving path ($g \to g''$) leaving g and visiting g'' (as C'' is a cycle), but then $g' \to g \to g''$ is an improving path leaving g' and visiting g''. Similarly, there must exist an improving path leaving g'' and visiting g'. Hence C^* is a cycle. Now, if C^* were a proper subset of another cycle C''' containing g, then C''' is also subset of C^*, which is a contradiction. Hence C^* is exactly the maximal cycle that contains g.

Lemma 12 *In a symmetric network model, no cycles can contain the empty network.*

The lemma follows from the following observations:
(i) For the empty network to be contained in a cycle, there must be an improving path reaching it, and another improving path leaving it;
(ii) an improving path leaving the empty network must replace the empty network by a singleton network $\{ij\}$, implying that an agent prefers forming a link to having no links at all; and
(iii) if an improving path leaving some non-empty network visits the empty network, then it must visit a singleton network $\{kl\}$ right before visiting the empty network, implying that an agent having one link prefers having no links to keeping that single link.

Lemma 13 *In a symmetric network model where forming a link with a totally un-connected agent is no better than forming a link with an agent already having some links, no (non-empty) cycles exist if $N \leq 3$.*

At first, if a non-empty cycle exists, it must contain two different non-empty networks (so that we can focus on $N = 3$), and there are improving paths travelling between those two networks back and forth. We claim that the cycle cannot contain a singleton network. Note that an improving path leaving a singleton network must first visit either the empty network or a star network (because of $N \leq 3$). The former can be ruled out because an improving path leaving the singleton network and reaching the empty network cannot reach any other non-empty network. Thus for a cycle to contain a singleton network, the star network must improve on the singleton network; that is, both agents involved in the singleton network would prefer having 2 links than having 1 link. However, an improving path reaching the singleton network must reach the star network first. Since the star agent should prefer having the two links than severing 1 link, both of the agents having 1 link in the star network would rather having no links at all. However, those two agents apparently get more than the center agent from paying the cost for maintaining 1 link, if having indirect connections is a benefit (as assumed in the lemma). It follows that the center agent should prefer having 1 link than keeping the 2 links, a contradiction. Thus a cycle, if it exists, cannot contain singleton networks.

Hence when $N = 3$, a non-empty cycle must contain a star and the complete network. An improving path reaching the complete cycle must first visit a star network, whereas the star network must improve on the complete cycle. It follows that every agent in the complete cycle would rather have 1 link than keep 2 links, contradicting the requirement that the complete cycle also improves on the star network. We conclude that there can be no cycles when $N = 3$.

Proposition 3 *(***Jackson and Watts (2002).***)*[9] *For the N agents, there must exist either a pairwise stable network g or a closed cycle C (or both).*

Pick any network g. If g happens to be pairwise stable, then we are done; or else, there is an improving path leaving g, which may either

[9]Jackson, M., and A. Watts, 2002, The Evolution of Social and Economic Networks, *Journal of Economic Theory*, 106, 265-295.

end at some pairwise stable network g' (and we are done), or continue endlessly. Since N is finite, there are only $2^{C_2^N}$ possible networks (including g^N and the empty network) for the N agents, where

$$C_2^N = \begin{pmatrix} N \\ 2 \end{pmatrix}.$$

If an improving path leaving g can continue endlessly, then it must visit some network g' more than once. Thus the improving path contains a sequence $\{g' = g_1, g_2, \cdots, g_{m-1}, g_m = g'\}$ such that $C \equiv \{g_1, g_2, \cdots, g_{m-1}\}$ is a cycle. Thus there is either a pairwise stable network or a cycle for the N agents.

It remains to show that there exists a *closed* cycle if no pairwise stable networks exist. Thus assume that pairwise stable networks do not exist. Pick any network g. Since g is not pairwise stable, there must exist an improving path leaving g, which contains some maximal cycle C_1. If C_1 is closed, then we are done; or else there must exist an improving path leaving C_1, which contains some maximal cycle C_2. We can repeat the above argument to find an infinite sequence of cycles $\{C_k\}$ if no closed cycles exist. Since each C_k is a maximal cycle, an improving path leaving C_k cannot visit a network contained in $\bigcup_{j=1}^{k-1} C_j$; recall Lemma 10. Thus we have an infinite sequence of *disjoint* cycles, but given that there only exist a finite number of networks, and hence a finite number of disjoint maximal cycles, this is a contradiction. Hence for some k, there cannot exist any improving path leaving C_k, implying that C_k must be a closed cycle.

Exercise 1 *Recall the symmetric connections model. Show that if* $c < \delta - \delta^2$*, then there do not exist closed cycles, but instead,* g^N *is the unique pairwise stable network.*

Exercise 2 *Consider the symmetric connections model with* $N = 4$ *and* $\delta^3 < \delta - c < \delta^2$*. Let* $g_1 = \{12, 23, 31, 14\}$*. Show that any improving path leaving* g_1 *must have* $g_2 = g_1 - ij$*, where* $ij \in \{12, 23, 31\}$*, so that the improving path ends at a circle passing through each and every player, which, as we recall, is pairwise stable.*

Exercise 3 *Suppose that $N = 4$, and an agent's payoff is his expected utility from trading consumption goods with all other agents belonging to the same connected component. An agent's ex-post utility is xy if he ends up consuming x units of X and y units of Y. In each period, an agent may be endowed with 1 unit of X, or 1 unit of Y, with equal probability, and the 4 agents endowments are totally independent over time and cross-sectionally. In each period, one link can be added to or subtracted from the network from the preceding period, as assumed at the beginning of this section, with the link cost being $c = \frac{7}{96}$, and following that the agents' endowments in the current period are realized and publicly observed, and then the agents within the same connected component trade X and Y via a Walrasian mechanism, and, finally, the process moves on to the next period.*

(i) Show that this network model has no pairwise stable networks.[10]

[10]**Hint**: Consider an agent endowed with d units of X and $1-d$ units of Y, where $d = 0, 1$. Given that $p_x = 1$ and $p_y = p$, his demands for X and for Y are respectively $\frac{d+(1-d)p}{2}$ and $\frac{d+(1-d)p}{2p}$. If there are N_x agents having $d = 1$, then the Walrasian equilibrium price is $p = \frac{N_x}{N-N_x} = \frac{N_x}{4-N_x}$. It follows that an agent having $d = 1$ has equilibrium utility $\frac{4-N_x}{4N_x}$ and an agent having $d = 0$ has equilibrium utility $\frac{N_x}{4(4-N_x)}$. Before taking into account the link costs, the expected utility from joining a component having m agents, denoted u_m, is such that $u_1 = 0$, $u_2 = \frac{1}{8}$, $u_3 = \frac{3}{16}$, $u_4 = \frac{7}{32}$. For example, when $m = 3$, there are 8 equally likely endowment profiles, where with probability $\frac{2}{8}$ an agent's endowment coincides with the other two agents' endowments so that each agent's equilibrium utility is zero, with probability $\frac{2}{8}$ an agent's endowment differs from the other two agents' endowments so that the agent's equilibrium utility is $\frac{1}{2}$, and with probability $\frac{4}{8}$ an agent's endowment coincides with exactly one other agent's endowment so that the agent's equilibrium utility is $\frac{1}{8}$. This leads to $u_3 = \frac{2}{8}\cdot 0+\frac{2}{8}\cdot\frac{1}{2}+\frac{4}{8}\cdot\frac{1}{8} = \frac{3}{16}$. Similarly, when $m = 4$, there are 16 equally likely endowment profiles, where with probability $\frac{2}{16}$ an agent's endowment coincides with the other three agents' endowments so that each agent's equilibrium utility is zero, with probability $\frac{2}{16}$ an agent's endowment differs from the other three agents' endowments so that the agent's equilibrium utility is $\frac{3}{4}$, with probability $\frac{6}{16}$ an agent's endowment coincides with exactly two other agents' endowments so that the agent's equilibrium utility is $\frac{1}{12}$, and with probability $\frac{6}{16}$ an agent's endowment coincides with one other agent's endowment so that each agent's equilibrium utility is $\frac{1}{4}$. This leads to $u_4 = \frac{2}{16}\cdot 0 + \frac{2}{16}\cdot\frac{3}{4} + \frac{6}{16}\cdot\frac{1}{12} + \frac{6}{16}\cdot\frac{1}{4} = \frac{7}{32}$. Now, observe that if a component having m agents is pairwise stable, then it has exactly $m-1$ links: it takes $k-1$ links to directly or indirectly connect m agents, and adding more links is a waste because agents trade the two goods on equal positions whether they are directly or indirectly connected. It follows that pairwise

(ii) Now, suppose that in period 1, the 4 agents are endowed with the network $g_1 = \{12, 34\}$. Show that the following is a cycle (although it may not be a closed cycle):

$$C = \{g_1, g_2, g_3, g_4, g_1\},$$

where

$$g_2 = \{12, 23, 34\}, \quad g_3 = \{12, 23\}, \quad g_4 = \{12\}.$$

9. Watts (2001)[11] studies how networks may evolve from the empty graph in a discrete-time symmetric connections model, assuming that at each time a pair of agents will be randomly selected and allowed to form a new link or to sever an existing link (and simultaneously sever links that are no longer needed).

   Proposition 4 *If $\delta - c > \delta^2 > 0$, where $1 > \delta > 0$, then every link forms (as soon as possible) and remains (no links are ever broken), so that the process converges to the complete network. If $\delta - c < 0$, then no links ever form, and the process converges to the empty network.*

   When the dynamic process converges to a network, the limiting network must be pairwise stable, but a pairwise stable network may not be strongly efficient. In case $\delta - c > \delta^2 > 0$, the dynamic process converges to the efficient complete network, but things differ in the case $\delta - c < 0$, where the empty network is pairwise stable, but it is strongly efficient only if $c > \delta + (\frac{N-2}{2})\delta^2$. When $\delta < c < \delta + (\frac{N-2}{2})\delta^2$, the dynamic process fails to converge to the efficient network, which is the N-person star.

   Proposition 5 *(***Watts (2001)***.) Suppose that $N \geq 4$ and $0 < \delta - c < \delta^2$. Then the dynamic process may converge to the N-person star with a positive probability p_N, where p_N decreases with N, and $\lim_{N \uparrow +\infty} p_N = 0$.*

stability requires that a component have exactly one link, and with 4 agents in the model, the only possible pairwise stable network is the one that consists of two disjoint links. Finally, it is easy to show that the latter network is still unstable.

[11]Watts, A., 2001, A Dynamic Model of Network Formation, *Games and Economic Behavior*, 34, 331-341.

The basic idea is that, if the dynamic process converges to the N-person star, then before it reaches the N-person star, it must reach a network where $N-1$ agents have joined the star. If the remaining agent (referred to as agent N) has already formed some links with the non-center agents, then the center agent will refuse to form a link with the remaining agent, given that $\delta^2 > \delta - c$. Thus agent N must not have any links when considering joining the star. Since $\delta - c > 0$, and since agents are myopic in forming or severing links, we claim that agent N must have not met anyone before: if otherwise, a link would have been formed, which would be severed only if agent N subsequently formed a new link and found the old link redundant, but then agent N would still remain connected to some non-center agents. Now, the same argument can be applied the second last agent joining the star: he must have not linked to any non-center agents, or else the center agent will refuse to form a link with him. Also, he must not have a link with agent N, because we have reasoned above that agent N must be totally un-connected when considering joining the star. It follows that this second last agent has not met anyone before, again because $\delta^2 > \delta - c$. Repeating the same argument, we conclude that the process can converge to the N-person star only in the event that each and every non-center agent meets the center agent before meeting anyone else. In particular, each time a pair of agents is selected, the center agent is one of them. This event occurs with a positive probability, but the probability monotonically converges to zero as N tends to infinity.

Take for example the case $N = 4$. There are 6 possible pairs of agents to select for meeting at a time. If the selected sequence is 12, 13, 14, 23, 24, 34, say, then the process does converge to the 4-person star. If the selected sequence is 12, 34, 13, 14, 23, 24, then the process converges to a circle if $\delta - c > \delta^3$; and it converges to a line if $\delta - c < \delta^3$. In particular, given that $\delta - c < \delta^3$, after the third meetings a network $\{12, 34, 13\}$ has been formed, and at this point, agents 1 and 4 are considering whether to form a new link. Agent 4 does want to form the link 14 and simultaneously sever the link 34, but agent 1 will refuse to do so, given that $\delta - c < \delta^3$. Similarly, when the process moves on to the next point in time when agents 2 and 3 are considering forming a new link, agent 2 does want to form the new link 23 and sever the link 12, but agent 3 will refuse to do so. Finally, when agents 2 and 4 meet, they do not form a link given that $\delta - c < \delta^3$. Consequently, the process converges to the line $\{12, 34, 13\}$.

The bottom line here is that in a dynamic process of network formation, agents do not take into account the impact of their current decisions on other agents, which may in turn impact their own payoffs subsequently, and in presence of externality and myopia, the process does not always converge to a strongly efficient network.

10. **Part II.**

11. (**Competing Platforms, Part II-A.**) Consider the following extensive game with two competing platforms, I and E, and two segments of platform users, referred to as side 1 and side 2 respectively. Segment i has population equal to one, and is represented by the interval $(2i, 2i+1)$. There exists a one-to-one measure-preserving correspondence $f : (2,3) \to (4,5)$ such that when user $x \in (2,3)$ meets with user $f(x) \in (4,5)$ on either platform I or platform E, the two users receive payoffs u_1 and u_2 respectively.[12] We say that a match occurs for x and his right partner $f(x)$ in the latter event. Before a match occurs, a side-1 user x thinks that each point $f(x) \in (4,5)$ is equally likely to be his right side-2 partner, and each side-2 user $f(y)$ thinks that each point $y \in (2,3)$ is equally likely to be his right side-1 partner. The two agents receive zero payoffs if they fail to meet with each other. We assume that $u_2 > u_1 > 0$. The platforms can operate without costs, and the timing of relevant events is as follows:

    - Platform I (the incumbent) must first decide whether to remain inactive (so that it gets a zero payoff by doing nothing) or to announce a pair of two-part tariffs $(p_1^I, t_1^I, p_2^I, t_2^I)$, saying that if a side-$i$ user wishes to use platform I then he needs to pay an upfront *registration fee* (or *access fee*, or *subscription fee*) $p_i^I \in \Re$, and in

[12] The one-to-one correspondence is measure-preserving if the Lebesgue measure of any set $A \subset (2,3)$ equals the Lebesgue measure of the image set $f(A) \subset (4,5)$. The requirements for f are satisfied when f is piece-wise affine with slope equal to plus or minus one.

case a match occurs subsequently, then he needs to pay another *transaction fee* $t_i^I \geq 0$.[13]

- Upon seeing platform I's announcements, platform E (the entrant) can decide whether to remain inactive (so that it gets a zero payoff by doing nothing) or to announce a pair of two-part tariffs $(p_1^E, t_1^E, p_2^E, t_2^E)$.
- Then users of both sides arrive, and they must decide simultaneously whether to register for both platforms (multi-homing), or just one of the platforms (single-homing) and which one, or to just leave. If a side-i user has registered for one platform k, then he is restricted to searching for his right partner from side j on platform k; but if the side-i user has registered for both platforms, then he can search for his right partner from side j on both platforms. We assume that whenever x and $f(x)$ have both registered for platform k and have initiated the search, platform k can locate them and make the match occur without incurring any costs.

(i) First suppose that platform E is absent, so that platform I is a monopolist.

(i-a) Suppose that platform I cannot verify whether a match really occurs, and hence it has to set $t_1^I = t_2^I = 0$. Show that there are multiple subgame-perfect Nash equilibria: in one equilibrium platform I makes the maximum profits $u_1 + u_2$ by announcing $(p_1^I, p_2^I) = (u_1, u_2)$, but in another equilibrium platform makes profits u_2 by offering $p_1^I = 0$ and $p_2^I = u_2$, and the latter equilibrium is supported by the off-the-equilibrium beliefs that no users would register for platform I if both p_1^I and p_2^I are strictly positive, and that all side-i users would register for platform I if $p_i^I \leq 0$.

(i-b) Suppose instead that platform I can set positive transaction fees. Show that platform I can essentially obtain the maximum profits u_1+u_2 in equilibrium.

[13]If $t_i^I < 0$, then side-i users would have an incentive to forge a phony match in order to collect money from the platform.

(ii) Now, suppose that both platforms are present, but users can use at most one platform. (That is, users must do "single-homing.")
(ii-a) Suppose that the platforms cannot verify whether a match really occurs, and hence they must set zero transaction fees. Show that all users choose to use platform I, with platform I's equilibrium pricing decisions being $p_1^I = -\max(u_1, u_2 - u_1)$ and $p_2^I = u_2$. Show that in equilibrium, platform I's profits are $\min(u_1, u_2 - u_1)$.

(ii-b) Now, suppose instead that platforms can set positive transaction fees. Show that there exists an equilibrium where all users choose to use platform I, with platform I's equilibrium pricing decisions being $p_1^I = -u_2 - u_1$, $t_1^I = u_1$, $p_2^I = 0$, and $t_2^I = u_2$. Show that in this equilibrium, platform I has zero profits.

(iii) Now, suppose that both platforms are present, and users can use both platforms. (That is, users are allowed to "multi-home.") Suppose that the platforms cannot verify whether a match really occurs, and hence they must set zero transaction fees. Show that there exists an equilibrium where all users choose to use platform I, with platform I's equilibrium pricing decisions being $p_1^I = p_2^I = 0$. Show that in this equilibrium, platform I has zero profits.

Solution. Consider (i-a). We claim that it is an equilibrium where platform I announces $p_1^I = u_1$ and $p_2^I = u_2$ and all users then choose to register for platform I. Indeed, expecting all side-i users to accept $p_i^I = u_i$ and register for platform I, a side-j user is confident that he will have a perfect match and obtain a payoff of $u_j - p_j^I = 0$ after registering for platform I himself, and hence that all users accept $p_1^I = u_1$ and $p_2^I = u_2$ and register for platform I constitutes a subgame equilibrium after platform I announces $p_1^I = u_1$ and $p_2^I = u_2$. Note that platform I attains its maximum possible profits when announcing $p_1^I = u_1$ and $p_2^I = u_2$, and hence it has no incentives to deviate from these equilibrium pricing decisions.

Next, we claim that it is also an equilibrium where platform I offers $p_1^I = 0$ and $p_2^I = u_2$, under the beliefs that no users would register for platform I if both p_1^I and p_2^I are strictly positive, and that all side-i users would register for platform I if $p_i^I \leq 0$.

To see that the claim is true, observe first that under the stated beliefs, it is indeed a subgame equilibrium that no users would register for platform I after platform I announces $p_1^I > 0$ and $p_2^I > 0$: expecting no side-i users would accept $p_i^I > 0$ and register for platform I, registering for platform I would result in zero chance of having a match and hence a payoff $-p_j^I < 0$ to each and every side-j user. Observe also that registering for platform I is a weakly dominant strategy for side-i users given that $p_i^I \leq 0$. Now, under the above beliefs, platform I can get side-i users on board by announcing $p_i^I = 0$, which, because rationality is users' common knowledge, convinces side-j users that they would obtain payoff $u_j - p_j^I$ if they accept p_j^I and register for platform I. Thus platform I can obtain payoffs u_1 by offering $(p_1^I, p_2^I) = (u_1, 0)$ and payoffs u_2 by offering $(p_1^I, p_2^I) = (0, u_2)$. Since $u_2 > u_1$, platform I's equilibrium best response is to offer $(p_1^I, p_2^I) = (0, u_2)$.

If we restrict p_1^I and p_2^I to be non-negative, then there always exists an equilibrium where users never register for platform I and platform I gets zero equilibrium payoffs by announcing $p_1^I = p_2^I = 0$. The latter zero-payoff equilibrium, referred to as a *market breakdown* equilibrium, is Pareto inefficient, and its driving force is the indirect network externality pertaining to a two-sided market. This inefficient equilibrium would vanish when platform I can announce a negative registration fee: by setting $p_1^I = -\epsilon$ and $p_2^I = u_2 - \epsilon$, platform I can get all side-1 users on board, which convinces side-2 users that they will obtain the payoff $u_2 - p_2^I = \epsilon > 0$ if they accept p_2^I and register for platform. Thus platform I can essentially attain a payoff of $u_2 > 0$ when allowed to offer a negative registration fee, proving that platform I would never announce $p_1^I = p_2^I = 0$ in equilibrium.

Consider (i-b). Now platform I can resolve the issue of multiple equilibria in (i-a) by announcing $p_i^I = -\epsilon$ for one segment of users, say, side i, and in response, all side-i users will be willing to register for platform I even if $t_i^I = u_i$. Thus by offering $(p_i^I, t_i^I) = (-\epsilon, u_i)$, platform I can get all side-i users on board, which convinces side-j users that they would each obtain the payoff $u_j - t_j^I - p_j^I$ if they are willing to accept (p_j^I, t_j^I)

and register for platform I. Thus platform I can choose (p_j^I, t_j^I) such that $p_j^I + t_j^I = u_j - \epsilon$ to go along with $(p_i^I, t_i^I) = (-\epsilon, u_i)$ and obtain a payoff $u_1 + u_2 - 2\epsilon$, and as $\epsilon > 0$ can be chosen to be arbitrarily small, platform I can obtain essentially the payoff of $u_1 + u_2$.

Consider (ii-a). We shall refer to an equilibrium where all users register for one platform and no users register for the other platform as a *tipping equilibrium*. Here we shall focus on the tipping equilibrium where all users choose to register for platform I. We use backward induction, starting with the subgame where platform E is about to make pricing decisions given (p_1^I, p_2^I). In this tipping equilibrium, E can induce side-i users to switch and join platform E instead if and only if

$$0 \cdot u_i - p_i^E > 1 \cdot u_i - p_i^I \Leftrightarrow p_i^E < p_i^I - u_i,$$

where we have emphasized that everyone (including platform E itself) believes that side-i users' probability of having a match is one if they stick to platform I, and their probability of having a match becomes zero if they switch to platform E. Now, given that E has announced $p_i^E < p_i^I - u_i$ to get side-i users on board, a side-j user's payoff from staying with platform I would become $0 \cdot u_j - p_j^I$,[14] so that platform E can induce side-j users to also switch and join platform E by announcing p_j^E that satisfies

$$1 \cdot u_j - p_j^E > \max(0, 0 \cdot u_j - p_j^I) = -\min(0, p_j^I) \Leftrightarrow p_j^E < u_j + \min(0, p_j^I).$$

Thus, given (p_1^I, p_2^I), platform E can attain the payoff

$$\max_{(i,j)\in\{(1,2),(2,1)\}} p_i^I - u_i + u_j + \min(0, p_j^I).$$

We shall assume that platform E would remain inactive (i.e., it would announce no price offers to users) if, given (p_1^I, p_2^I), the above maximum payoff is less than or equal to zero.

Now, return to the stage where platform I is about to set (p_1^I, p_2^I). In the supposed tipping equilibrium, platform I seeks to

$$\max p_1^I + p_2^I$$

[14]These users are said to be *stranded* in the language of Farrell and Saloner (1986); see Farrell, J. and G. Saloner, 1986, Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation, *American Economic Review*, 76, 5, 940-955.

subject to

$$p_1^I - u_1 + u_2 + \min(0, p_2^I) \leq 0;$$
$$p_2^I - u_2 + u_1 + \min(0, p_1^I) \leq 0;$$
$$p_1^I \leq u_1;\ p_2^I \leq u_2.$$

First we claim that platform I would announce $p_2^I \geq 0$. Indeed, if instead $p_2^I < 0$, then the above first constraint requires that

$$p_1^I + p_2^I \leq u_1 - u_2 < 0,$$

but platform I can at least remain inactive! Now, if $p_1^I \geq 0$ also, then by summing up the first two constraints we obtain $p_1^I + p_2^I \leq 0$, implying that platform I's payoff is zero, but we shall show that platform I can actually obtain a strictly positive payoff by announcing *some* negative p_1^I.

Thus assume that $p_1^I < 0 \leq p_2^I$, and it follows that the above third constraint can be ignored, and we can re-state the remaining constraints as

$$p_1^I \leq -(u_2 - u_1);$$
$$p_2^I \leq u_2 - u_1 - p_1^I;$$
$$p_2^I \leq u_2.$$

Since the objective function is strictly increasing in p_2^I given p_1^I, the optimal p_2^I should make either the second or the last constraint binding; i.e., we have at optimum either

$$p_2^I = u_2 \text{ and } \ p_1^I \leq -u_1 \Rightarrow p_1^I = \min(-u_1, u_1 - u_2)$$

$$\Rightarrow p_1^I + p_2^I = u_2 + \min(-u_1, u_1 - u_2) = \min(u_1, u_2 - u_1).$$

or

$$u_2 \geq p_2^I = u_2 - u_1 - p_1^I \geq 2(u_2 - u_1)$$
$$\Rightarrow p_1^I + p_2^I = u_2 - u_1 \text{ when } \ u_1 \geq u_2 - u_1.$$

Summing up the above discussions, we conclude that if $u_2 - u_1 > u_1$, then $2(u_2 - u_1) > u_2$, so that at optimum $p_1^I = -(u_2 - u_1)$ and $p_2^I = u_2$; and if instead $u_1 \geq u_2 - u_1$, then $u_2 \geq 2(u_2 - u_1)$, so that at optimum

$p_1^I = -u_1$ and $p_2^I = u_2$. Thus as we have asserted earlier, $p_1^I < 0$ at optimum, and platform I's equilibrium payoff is $\min(u_1, u_2 - u_1) > 0$.

Now, consider (ii-b). Again, we start with the subgame where platform E must make price decisions given $(p_1^I, t_1^I, p_2^I, t_2^I)$. To induce side-$i$ users to deviate and join platform E instead, platform E must offer (p_i^E, t_i^E) such that

$$-p_i^E > u_i - p_i^I - t_i^I,$$

but the key difference here is that platform E can set $t_i^E = u_i$. This makes platform E's effort of stealing side-i users away from platform I less costly than in scenario (ii-a). Having offered

$$p_i^E = -u_i + p_i^I + t_i^I - \epsilon,\ t_i^E = u_i,$$

platform E can also steal side-j users away from platform I by offering (p_j^E, t_j^E) that satisfies

$$u_j - p_j^E - t_j^E > \max(-p_j^I, 0) \Leftrightarrow p_j^E + t_j^E < u_j + \min(p_j^I, 0).$$

Thus, given $(p_1^I, t_1^I, p_2^I, t_2^I)$, platform E can obtain essentially the following payoff by remaining active:

$$\max_{(i,j)\in\{(1,2),(2,1)\}} p_i^I + t_i^I + u_j + \min(0, p_j^I).$$

Now, return to platform I's pricing decisions. Platform I seeks to

$$\max p_1^I + t_1^I + p_2^I + t_2^I$$

subject to

$$p_1^I + t_1^I + u_2 + \min(0, p_2^I) \leq 0;$$
$$p_2^I + t_2^I + u_1 + \min(0, p_1^I) \leq 0;$$
$$p_1^I + t_1^I \leq u_1;\ 0 \leq t_1 \leq u_1;$$
$$p_2^I + t_2^I \leq u_2;\ 0 \leq t_2 \leq u_2.$$

For $(p_1^I, t_1^I, p_2^I, t_2^I)$ satisfying $p_1^I \geq 0$ and $p_2^I \geq 0$, we would have

$$p_1^I + t_1^I \leq -u_2,\ p_2^I + t_2^I \leq -u_1,$$

implying that platform I's payoff is strictly negative. Thus suppose that $p_i^I < 0$, which implies that

$$p_j^I + t_j^I + p_i^I + t_i^I \leq p_j^I + t_j^I + \min(p_i^I, 0) + u_i \leq 0.$$

Thus platform I cannot attain a strictly positive payoff. Note that one way for platform I to attain zero profits is to set $t_i^I = u_i$, $p_2^I = 0$ and $p_1^I = -u_1 - u_2$.

Consider part (iii). Again, we start with the subgame where platform E must make price decisions given (p_1^I, p_2^I). Note that users are allowed to register at both platforms, and hence any negative p_i^E can induce side-i users to adopt platform E as an additional platform. However, to ensure that users would like to meet at platform E after registering at both platforms, platform E must ensure that side-j users strictly prefer platform E to platform I after knowing that side-i users have chosen to register at both platforms; that is, platform E must offer (p_i^E, p_j^E) such that $p_i^E < 0$ (so that side-i users will adopt platform E as their second choice) and, given this fact, side-j users would rather have a match on platform E than on platform I; that is,

$$1 \cdot u_j - p_j^E > 1 \cdot u_j - p_j^I \Leftrightarrow p_j^E < p_j^I,$$

where note that, given side-i users' multi-homing decisions, everyone (including platform E) realizes that side-j users' chance of having a match is one no matter which platform side-j users choose to use to meet with side-i users. Thus platform E's optimal payoff given (p_1^I, p_2^I) is

$$\sup_{\epsilon \in \Re_{++}} (0, -\epsilon + p_1^I - \epsilon, -\epsilon + p_2^I - \epsilon) = \max(0, p_1^I, p_2^I).$$

Now, return to platform I's pricing decisions. Platform I seeks to

$$\max p_1^I + p_2^I$$

subject to

$$\max(0, p_1^I, p_2^I) \leq 0 \Rightarrow p_1^I, p_2^I \leq 0,$$

implying that platform I must choose $p_1^I = p_2^I = 0$!

Remark 1. We have focused on the so-called tipping equilibrium in the above analysis, where all users choose to join one platform. This type of equilibrium can prevail even if we assume that the two platforms act simultaneously, but with simultaneous moves, there may be other equilibria with symmetric equilibrium allocations. In general, when the dominant platform gets a zero payoff in a tipping equilibrium, the two platforms would also get zero payoffs in a symmetric equilibrium. Here we emphasize an unusual symmetric equilibrium with positive payoffs for the two platforms.

Suppose now that in part (iii) the two platforms must announce prices *simultaneously* before users arrive. Then it is an equilibrium where for platform $k \in \{I, E\}$, $p_1^k = 0$, $p_2^k = \frac{u_2}{2}$.[15] [16] In this equilibrium, side-1

[15]To check that no platforms would want to deviate unilaterally from the equilibrium pricing decisions, first note that lowering the registration fee for side-2 users is worthless: lowering the registration fee may in general encourage registration but side-2 users have already chosen multi-homing. Second, lowering the registration fee for side-1 users would make them choose multi-homing, but it has no influence on their (and side-2 users') choices regarding the platform on which they would like the match to take place.

[16]This equilibrium would not break down even if platforms must move sequentially. We first argue that, given that platform I announces $p_1^I = 0$ and $p_2^I = \frac{u_2}{2}$, platform E has no incentive to deviate. If instead platform E announces subsequently $p_1^E = -\epsilon$ and $p_2^E = \frac{u_2}{2} - \epsilon$, then side-1 users will do multi-homing, and side-2 users will do single-homing and drop platform I, and thus all matches will occur on platform E. However, since $t_i^E = 0$, platform E actually becomes worse off: platform E's deviation payoff becomes $1 \cdot p_1^E + 1 \cdot p_2^E < \frac{u_2}{2}$! Next, we argue that platform I has no incentive to deviate either if certain off-the-equilibrium beliefs are held by users and the two platforms. Specifically, if platform I deviates and aims at obtaining a payoff higher than $\frac{u_2}{2}$, then following the deviation platform I *must* announce $p_i^I > 0$ for some i. Thus there are two possible deviation announcements for platform I.

- If following the deviation, platform I announces $p_1^I, p_2^I > 0$, then we assume that side-i users will choose single-homing and join the platform k offering the lower p_i^k. In this case platform E can offer $p_i^E = p_i^I - \epsilon$ and make platform I's deviation payoff equal to zero. This removes platform I's incentive to deviate and offer those positive registration fees in the first place.
- Now, if following the deviation platform I offers $p_i^I \leq 0$ and $p_j^I > 0$, then we assume that whenever $p_i^E \leq 0$ all users believe that side-i users will choose multi-homing but side-j users will choose single-homing and join the platform k offering the lower p_j^k. In this case platform E can offer $p_i^E = 0$ and $p_j^E = p_j^I - \epsilon$, and this removes

users do single-homing and they split equally between the two platforms, but side-2 users do multi-homing.[17] A side-1 user's equilibrium payoff from joining platform k is $u_1 - p_1^k = u_1 > 0$, and a side-2 user's equilibrium payoff from joining both platforms is $u_2 - p_2^I - p_2^E = 0$. A platform k's equilibrium payoff is $\frac{1}{2} \cdot p_1^k + 1 \cdot p_2^k = \frac{u_2}{2}$.

Remark 2. The platforms studied above can be cybermediaries providing online dating services (like eharmony.com or match.com), or online search engines, or they can be e-commerce firms (e.g. Amazon.com) performing informational intermediation between, say, readers and books. Typically the value of an intermediary for a user on one side relates positively to the number of users on the other side, a phenomenon generally referred to as an *indirect network externality.* Platforms exhibiting this property is referred to as a two-sided market. This exercise is adapted from Caillaud and Jullien (2001).[18]

12. (**Competing Platforms, Part II-B.**) Here we shall modify Problem 1 as follows:

    - For some $\rho \in (0, 1)$, $u_2 = 1 + \rho$ and $u_1 = 1 - \rho$.
    - The platforms must set zero transaction fees and non-negative registration fees.
    - The timing of relevant events is modified as follows:
        - The two platforms simultaneously announce p_1^I and p_1^E.
        - Then side-1 users arrive and they simultaneously choose which platform(s) to use.
        - Then it becomes public information that there are x side-1 users having registered for platform I, y side-1 users having

platform I's incentive to deviate and announce $p_i^I \leq 0$ and $p_j^I > 0$ in the first place.

[17]Note that there is no *need* for side-1 users to do multi-homing, once they are sure that side-2 users are doing multi-homing. To have users from both sides doing multi-homing, for *each* side i, some platform k must be offering $p_i^k \leq 0$.

[18]Caillaud, B., and B. Jullien, 2001, Competing Cybermediaries, *European Economic Review (Papers and Proceedings)*, 45, 797-808.

registered for platform E, and z side-1 users having registered for both platforms, where $0 \leq z \leq x, y \leq 1$.

- Then the two platforms simultaneously announce p_2^I and p_2^E.
- Then side-2 users arrive and they simultaneously choose which platform(s) to use.

(i) Show that if all users must adopt single-homing then there are multiple equilibria, where in one equilibrium all side-1 users choose to use platform $k \in \{I, E\}$ with platform k pricing at $p_1^k = (1 - \rho)$ and then $p_2^k = (1 + \rho)$, and in another equilibrium the two platforms announce $p_1^I = p_1^E = 0$ with side-1 users randomly choosing a platform.

(ii) Suppose instead that side-1 users must adopt single-homing but side-2 users are allowed to multi-home. Show that given x, y, z, the two platforms will charge side-2 users respectively $p_2^I = (1 + \rho)(x - z)$ and $p_2^E = (1 + \rho)(y - z)$, so that, by backward induction, the two platforms announce $p_1^I = p_1^E = 0$ when serving side-1 users.

Solution. Consider part (i), which assumes that $z = 0$. Consider first the subgame where the two platforms must make price offers to side-2 users, given x, y. Note that a platform k will never offer $p_2^k < 0$ at this stage even if negative registration fees are allowed: if it did, it loses money from getting side-2 users on board, without helping attract side-1 users to come join the party (simply because side-1 users' registration decisions have already been made and the platform cannot charge a positive transaction fee). When $x > y$, platform I and platform E will offer the prices $p_2^E = 0$ and $p_2^I = (1 + \rho)(x - y)$ and all side-2 users will join platform I; and when $x < y$, platform I and platform E will offer the prices $p_2^I = 0$ and $p_2^E = (1 + \rho)(x - y)$ and all side-2 users will join platform E.[19] In case $x = y$, both platforms offer zero registration fees,

[19] In these two cases, the two platforms are like two Bertrand-competitive firms offering heterogeneous goods at zero costs, where buyers' common valuation for firm k's good is v_k, with $0 < v_1 < v_2$, say. In equilibrium, firm 1 will price at zero, and firm 2 will price at $v_2 - v_1$. The idea is that firm 1 can at best offer buyers a surplus of v_1 at zero price, which firm 2 can match by pricing at (or slightly below) $v_2 - v_1$. Thus firm 1 makes no profits, and firm 2's payoff is $v_2 - v_1$.

and side-2 users randomly pick a platform to join.

Now, return to the stage where the two platforms are about to make offers to side-1 users. According to the above analysis, essentially, a side-1 user will have a match only if he joins the platform that the majority of his side-1 fellow users choose to join. Thus if side-1 users believe that they will all join platform k, then this is a self-fulfilling equilibrium as long as $p_1^k \leq u_1$. This leads to a tipping equilibrium where some platform k offers $p_1^k = (1 - \rho)$ and the (non-negative) registration fee chosen by its rival is irrelevant. On the other hand, if side-1 users believe that they will all choose the platform k offering the lower p_1^k, and they will randomly choose a platform to join when $p_1^I = p_1^E$, then it is an equilibrium where the two platforms offer $p_1^I = p_1^E = 0$ and side-1 users randomly join the two platforms. Apparently, the former two tipping equilibria are efficient (despite having higher prices and platform profits than the symmetric equilibrium), as every user has a match in equilibrium. The latter symmetric equilibrium is not, where for each user a match may occur only with a probability less than one.

Consider part (ii). Define $x_I = x$ and $x_E = y$. The case where $xy = 0 < x^2 + y^2$ is easy; the platform k having $x_k > 0$ will announce $p_2^k = x_k(1 + \rho)$ and get all side-2 users on board. Now, suppose that $x, y > 0$, and note that by assumption side-1 users must single-home, so that we have $z = 0$ once again. Given that $x, y > 0 = z$, can it be an equilibrium where side-2 users single-home? The answer is negative: if side-2 users do not register for platform k, then platform k can always announce $0 < p_2^k < x_k(1 + \rho)$ to get side-2 users on board and raise its own payoff. Now, observe that side-2 users' valuation for platform k, given that they will also join the other platform, is $(x_k - z)(1 + \rho)$, and hence platform k will price at $p_2^k = (x_k - z)(1 + \rho)$ if expecting side-2 users to also join the other platform. We conclude that the equilibrium prices are $p_2^I = x(1 + \rho)$ and $p_2^E = y(1 + \rho)$ and side-2 users will all multi-home whenever $x, y > 0$.

Now, return to the stage where the two platforms are about to make offers to side-1 users. Rationally expecting side-2 users to multi-home whenever $x, y > 0$, side-1 users are confident that a match will occur with probability one no matter which platform they register for. Thus a side-1 user will join the platform k offering the lower p_1^k, regardless of what his side-1 fellow users will do. That is, $x_I = 0$ if $p_1^I > p_1^E$ and $x_E = 0$ if $p_1^I < p_1^E$. For $k, h \in \{I, E\}$, $k \neq h$, platform k's payoff from serving users from both sides is therefore $x_k(p_1^k + 1 + \rho)$ if $p_1^k < p_1^h$ and zero if $p_1^k > p_1^h$. This leads to the equilibrium outcome of $p_1^I = p_1^E = 0$, and facing these registration fees, side-1 users simply joint a platform at random. To sum up, in equilibrium the multi-homing side is facing high registration fees and left with no surplus, while the single-homing side is offered with free access. A platform's expected equilibrium payoff is $\frac{1}{2}(1 + \rho)$.[20]

Remark. When one side of users single-home, competition for those users will result in a winning platform, which exclusively gains the positive externality associated with those single-homing users. This fight for the positive externality intensifies competition, and can result in zero prices. When one side of users multi-home, multiple platforms can gain the positive externality associated with those multi-homing users at the same time, and this tends to mute the competition for those multi-homing users. These observations were first made in Armstrong (2006),[21] where the author wrote (pp. 669-670):

··· platforms have monopoly power over providing access to their single-homing customers for the multi-homing side. This monopoly power naturally leads to high prices being charged to the multi-homing side. ··· By contrast, platforms do have to compete for the single-homing users, and high profits generated from the multi-homing side are to a large extent passed on to the single-homing side in the form of low (or even zero) prices.

13. (**Competing Platforms, Part II-C.**) This section is adapted from a working paper of mine, Chen and Chou (2022).[22]

[20]This exercise is adapted from King, S., 2013, Two-sided Markets, *Australia Economic Review*, 46, 2, 247-258.

[21]Armstrong, M., 2006, Competition in Two-sided Markets, *Rand Journal of Economics*, 37, 3, 668-691.

[22]Chen, C.-M., and S.-Y. Chou, 2022, An Equilibrium Analysis for Physical Retailers Battling Online Retail Platforms under Financial Constraints: The Strategic Roles of

There are n bricks-and-mortar sellers, each being a local monopolist selling an identical product to $\frac{\delta}{n}$ local consumers (so that the total population of consumers is δ). Each consumer will buy either zero or one unit of the product, with willingness to pay equal to $\theta > 0$. In the absence of the online platforms, each seller will serve $\frac{\delta}{n}$ consumers and make a payoff $\frac{\theta\delta}{n}$. We shall assume that $n = 2$.

The timing of relevant events is as follows.

- The two platforms I and E simultaneously announce non-negative access fees ϕ^I and ϕ^E for the sellers and non-negative access fees κ^I and κ^E for the consumers (also referred to as online shoppers).
- Each seller must announce an offline price, and if the seller choose to also sell online, then the seller must also post an online price for each platform that the seller chooses to join.
- A consumer would automatically know his local seller's offline price. By paying access fees to one or both platforms and then observing the sellers' prices posted online, he may be able to find a better offer than his local store's offline price.
- Other than the access fees, a seller has zero operating expenses.

An online retail platform is a two-sided market exhibiting an indirect network externality in the sense that increasing the number of users on one side benefits users on the other side. This gives rise to a chicken-and-egg problem: the platform must have enough registered sellers to attract buyers, but sellers are willing to register only if they expect many buyers to come. Hence there are typically multiple equilibria for competing platforms, and here we shall focus on the tipping equilibrium suggested by Baye and Morgan (2001),[23] where only platform I remains active in equilibrium. We can think of this dominant platform as Amazon Marketplace.

Since the platforms must by assumption charge non-negative access fees ϕ^j and κ^j to sellers and consumers, the inactive platform cannot carry

Click-and-Collect Services.

[23] Caillaud and Jullien (2001) refer to the tipping equilibrium as a *dominant firm equilibrium.*

out a divide-and-conquer strategy so as to steal away the dominant platform's clients; see Caillaud and Jullien (2003) for details. The supporting beliefs for the tipping equilibrium are assumed to be consistent with the following criterion.[24]

Definition 1 *(***Beliefs** Γ.*) No consumers will purchase from platform E if* $\kappa^I > 0$ *and* $\kappa^I \geq \kappa^E \geq 0$*, and they will all purchase from platform I if* $\kappa^I = 0$.

The following lemma has been established in Baye and Morgan (2001), under the additional assumption that a seller would post the same price online and offline.[25] From now on, we let ϕ and κ to denote respectively ϕ^I and κ^I.

Lemma 14 *Suppose that consumers hold beliefs* Γ. *Then in order to make positive profits platform I must announce* $\kappa = 0$ *and choose some* $\phi \in [0, \frac{\theta}{2}]$*, and given* ϕ*, the two sellers would announce* θ *as their offline price, and with probability* $\alpha = 1 - \frac{2\phi}{\theta}$ *each seller may pay* ϕ *and then announce its online price p according to the following distribution function*

$$F(p) = \begin{cases} 0, & p \leq 2\phi; \\ 1 - (\frac{2\phi}{\theta - 2\phi})(\frac{\theta - p}{p}), & p \in [2\phi, \theta]; \\ 1, & p \geq \theta. \end{cases}$$

Consumers will all join platform I, and they will trade online with the seller that posted the lowest online price. Consumers will return to their local stores only if they find no posted prices online. Given ϕ*, each seller's equilibrium payoff is* ϕ*, and platform I's payoff is* $2(\phi - \frac{2\phi^2}{\theta})$*. In equilibrium, platform I would choose* $\phi = \frac{\delta\theta}{4}$*, so that each seller may join platform I with probability* $\frac{1}{2}$*, and in equilibrium* $\frac{\delta\theta}{4}$ *is the common payoff for the two sellers and platform I.*

[24] Caillaud and Jullien (2001) show that similar "bad expectation" or "pessimistic" beliefs against a platform is necessary to sustain a tipping equilibrium.

[25] Or equivalently, a seller is advertising its offline price at the online platform; all transactions are made offline. In their scenario, platforms I and E are Groupon-like platforms that provide consumers with price deals information.

Baye and Morgan (2001) assume that a seller's online price has to be the same as its offline price, and they show that the seller would price at θ if it chooses to not register for platform I. Since a consumer will trade with his un-registered local seller only after he finds no posted prices online, assuming that the seller's offline price is θ regardless of the online price that the seller posts will not alter the equilibrium.[26] This explains why our equilibrium coincides with the equilibrium derived in Baye and Morgan (2001).

Note that in equilibrium a consumer's payoff from joining the online market is

$$U^{\text{on}} \equiv \theta - E[\tilde{t}] - g(\delta),$$

in which $g(\delta)$ measures the waiting cost resulting from increasing difficulty in shipping and handling when a population δ of consumers choose to trade online at the same time, and the random variable $\tilde{t}$ is such that

$$\tilde{t} = \begin{cases} \theta, & \text{with probability } \frac{1}{4}; \\ \tilde{p}_1, & \text{with probability } \frac{1}{4}; \\ \tilde{p}_2, & \text{with probability } \frac{1}{4}; \\ \min(\tilde{p}_1, \tilde{p}_2) & \text{with probability } \frac{1}{4}, \end{cases}$$

where $\tilde{p}_i$ is the online price posted by seller i, with $\tilde{p}_1$ and $\tilde{p}_2$ being independent draws from the distribution function

$$F(p) = \begin{cases} 0, & p \leq \frac{\theta}{2}; \\ 2 - \frac{\theta}{p}, & p \in [\frac{\theta}{2}, \theta]; \\ 1, & p \geq \theta. \end{cases}$$

It is easy to verify that

$$E[\tilde{p}_i] = \theta \ln(2).$$

[26] Or put another way, no offline transactions can take place at an online price $p \neq \theta$ posted by the seller.

Note also that the distribution function of $\tilde{y} \equiv \min(\tilde{p}_1, \tilde{p}_2)$ is

$$G(y) = \begin{cases} 0, & y \leq \frac{\theta}{2}; \\ \frac{2\theta}{y} - \frac{\theta^2}{y^2}, & y \in [\frac{\theta}{2}, \theta]; \\ 1, & y \geq \theta, \end{cases}$$

so that

$$E[\tilde{y}] = \int_{\frac{\theta}{2}}^{\theta} y dG(y) = 2\theta[1 - \ln(2)].$$

Consequently, we have

$$E[\tilde{t}] = \frac{1}{4}\{\theta + 2 \cdot \theta \ln(2) + 2\theta[1 - \ln(2)]\} = \frac{3\theta}{4},$$

implying that

$$U^{\mathrm{on}} = \frac{\theta}{4} - g(\delta).$$

We can now ask what would happen if the sellers are not all-equity financed? Would debt financing lead to more aggressive competition between the sellers? Would it benefit or hurt the platforms?

Lemma 15 *Suppose that the sellers have all issued the same amount of debt, with face value D satisfying $\frac{\delta\theta}{2} \geq D \geq 0$. Then in equilibrium $\phi = \frac{\delta\theta}{4}$ and $\kappa = 0$, and all consumers join platform I for sure. The two sellers may each join platform I with probability $\frac{1}{2}$. A seller would always post θ as its offline price, and when it joins platform I, its online price p is a random draw from the distribution function*

$$F_D(p) = \begin{cases} 0, & p \leq \frac{\delta\theta + D}{2\delta}; \\ 2 - \frac{\delta\theta - D}{\delta p - D}, & p \in [\frac{\delta\theta + D}{2\delta}, \theta]; \\ 1, & p \geq \theta. \end{cases}$$

In equilibrium, each seller's payoff is $\frac{\delta\theta}{4} - \frac{D}{2}$ and platform I's payoff is $\frac{\delta\theta}{4}$.

Note that we can re-write $F_D(\cdot)$ as

$$F_D(p) = \begin{cases} 0, & p \le \frac{\theta+d}{2}; \\ 2 - \frac{\theta-d}{p-d}, & p \in [\frac{\theta+d}{2}, \theta]; \\ 1, & p \ge \theta, \end{cases}$$

where the only relevant parameter is $d \equiv \frac{D}{\delta}$.

Lemma 16 *An online shopper's equilibrium payoff is*

$$U^{on} \equiv \theta - E[\tilde{t_D}] - g(\delta),$$

in which the random transaction price $\tilde{t_D}$ is such that

$$\tilde{t_D} = \begin{cases} \theta, & \textit{with probability } \frac{1}{4}; \\ \tilde{p}_1, & \textit{with probability } \frac{1}{4}; \\ \tilde{p}_2, & \textit{with probability } \frac{1}{4}; \\ \min(\tilde{p}_1, \tilde{p}_2) & \textit{with probability } \frac{1}{4}, \end{cases}$$

so that we have

$$E[\tilde{t_D}] = \frac{\theta+d}{2} + \int_{\frac{\theta+d}{2}}^{\theta} [\frac{\theta-d}{x-d}]^2 dx.$$

Indirect network externality affects the third-party sellers' equilibrium pricing behavior when, and only when, $D > 0$.

Proof. An online shopper's equilibrium payoff is

$$U^{\text{on}} \equiv \theta - E[\tilde{t_D}] - g(\delta),$$

in which the random transaction price $\tilde{t_D}$ is such that

$$\tilde{t_D} = \begin{cases} \theta, & \text{with probability } \frac{1}{4}; \\ \tilde{p}_1, & \text{with probability } \frac{1}{4}; \\ \tilde{p}_2, & \text{with probability } \frac{1}{4}; \\ \min(\tilde{p}_1, \tilde{p}_2) & \text{with probability } \frac{1}{4}, \end{cases}$$

where $\tilde{p}_i$ is the online price posted by seller i, with $\tilde{p}_1$ and $\tilde{p}_2$ being independent draws from the distribution function $F_D(\cdot)$. Thus we have

$$E[\tilde{t_D}] = \frac{\theta}{4} + \frac{1}{4}\int_0^{+\infty}[3 - G_D(x) - 2F_D(x)]dx$$

$$= \frac{\theta}{4} + \frac{1}{4}[3(\frac{\theta+d}{2}) + \int_{\frac{\theta+d}{2}}^{\theta}\{[\frac{\theta-x}{x-d}]^2 + 2[\frac{\theta-x}{x-d}]\}dx]$$

$$= \frac{\theta}{4} + \frac{1}{4}[3(\frac{\theta+d}{2}) + \int_{\frac{\theta+d}{2}}^{\theta}\{([\frac{\theta-x}{x-d}]^2 + 2[\frac{\theta-x}{x-d}] + 1) - 1\}dx]$$

$$= \frac{\theta}{4} + \frac{1}{4}[3(\frac{\theta+d}{2}) + \int_{\frac{\theta+d}{2}}^{\theta}\{[\frac{\theta-d}{x-d}]^2 - 1\}dx]$$

$$= \frac{\theta+d}{2} + \int_{\frac{\theta+d}{2}}^{\theta}[\frac{\theta-d}{x-d}]^2 dx,$$

where $G_D(x) = 1-[1-F_D(x)]^2$ is the distribution function for $\min(\tilde{p}_1, \tilde{p}_2)$. Since $F_D(\cdot)$ and $G_D(\cdot)$ both shift downwards as D rises, it is clear that U^{on} increases with δ and decreases with D.

Summing up the above discussions, we obtain the following results:

(a) The presence of online retail platforms hurts a (third-party) seller as it spoils the seller's local monopoly power.

(b) The presence of online retail platforms can turn a seller's risk-free debt into risky debt.

(c) An increase in D results in the sellers offering fewer low prices and more high prices, thereby raising each seller's expected profits and reducing consumer surplus.

(d) The population of online shoppers, δ, affects sellers' online pricing behavior only when $D > 0$; when $D > 0$, an increase in δ benefits the platform and the sellers but hurt online shoppers.