Option Prices and Risk-neutral Densities for Currency Cross-rates

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Motivation

- Risk-neutral densities (RNDs) are important for
  - companies to manage risk exposures,
  - policy-makers to observe market expectations,
  - investors to price options.

- However, estimation of currency cross-rate RNDs is difficult because the option markets either don’t exist or are illiquid.
Objectives

- Without full price information for cross-rate options, but instead using only the information of two dollar-denominated exchange-rate options, to construct the theoretical formulae for the RND and the option prices of cross-rates.

- To select appropriate copula functions, utilized to define the bivariate RND of two dollar rates in the theoretical formulae.
Theoretical Framework

Bivariate RNDs $/£, $/€

Variable transformation with the correct numeraire

Univariate RND €/£

All cross-rate option prices
RND Formula for Cross-rates

- Three-currency framework: $, £, and €
  \[ S_t^{\$/$£} : \text{Spot rate of £1 in $} \quad S_t^{\$/$€} : \text{Spot rate of €1 in $} \]
  \[ F_t^{\$/$£} : \text{Forward rate of £1 in $} \quad F_t^{\$/$€} : \text{Forward rate of €1 in $} \]
  \[ f_s(y,z) : \text{Bivariate RND of } S_t^{\$/$£} \text{ and } S_t^{\$/$€} \]

- Call option to buy £1 for €\(X\)
  \[ = \text{Call option to exchange } S_T^{\$/$€} X \text{ for } S_T^{\$/$£} \]
  \[ C_\$ (X) = e^{-r_T S_T^{\$/$£}} E^Q [\max( S_T^{\$/$£} - X S_T^{\$/$€}, 0)] \]

- One-price law:
  \[ C_€ (X) = C_\$ (X) / S_0^{\$/$€} \]
RND Formula for Cross-rates

- Theory of Breeden & Litzenberger (1978)

\[
f_\epsilon(x) = e^{r_\epsilon T} \frac{\partial^2 C_\epsilon(X)}{\partial x^2} = \frac{1}{F_0^{\$/\epsilon}} \int_0^\infty z^2 f_\$(xz, z)dz
\]

\[
C_\epsilon(X) = e^{-r_\epsilon T} \frac{1}{F_0^{\$/\epsilon}} \int_X^\infty \int_0^\infty (x - X)z^2 f_\$(xz, z)dzdx
\]

- Risk-neutral for any \(f(y, z)\) as
  
  (1) \(f_\epsilon(x) \geq 0\)  
  (2) \(\int_0^\infty f_\epsilon(x)dx = 1\)  
  (3) \(\int_0^\infty xf_\epsilon(x)dx = F_0^{\$/\£}\)
Empirical Research Design

- $/£ options
- $/£ RND
- Dependence parameter
- Historical data
- ATM cross-rate option price
- $/€ RND
- Copulas
- $/€ options
- Bivariate RNDs $/£, $/€
- All cross-rate option prices
- Step 1
- Step 2
- $/€ RND
- Step 3
- Univariate RND €/£
- Variable transformation

Step 1: $/£ options
Step 2: Dependence parameter
Step 3: All cross-rate option prices
Constructions of Marginal & Bivariate RNDs

- **Marginal RNDs:** two popular methods
  - Lognormal Mixtures
  - General Beta distribution of the 2\textsuperscript{nd} kind (GB2)

- **Bivariate RNDs:** Use copula functions $c$ joining univariate marginal distributions $f$ and $g$ to a multivariate distribution $h$, using the univariate CDFs, $F$ and $G$, and one dependence parameter.

  $$h(y, z) = c(F(y), G(z)) \times f(y) \times g(z)$$

- Selective copula functions:
  - Gaussian, Frank, Plackett and Clayton copulas (different symmetry and tail dependence properties)
Contours and Copula Densities

Gaussian Copula

Plackett Copula

Frank Copula

Clayton Copula
(1) If the market price of one cross-rate option is available, we use numerical methods to obtain the implied estimate by equating the theoretical price with the market price.

(2) Otherwise, we can use the historical intraday record of two dollar-denominated exchange rates to calculate realized correlation, using the method of Andersen et al (2001), and then require the same Spearman’s rho across copulas.
Questions for Empirical Research

Q1: Which copula function(s) can model the dependence of foreign exchange rates relatively satisfactorily?

Q2: Is information about cross-rate density expectations integrated across different markets (OTC, CME)?
Data

- Settlement prices of options: USD/GBP, USD/EUR, EUR/GBP
  - Two sources: OTC for all exchange rates (7 deltas) and CME for USD rates only (21/24 strikes)
  - Period: from May to December 2000
  - Samples: 12 for 1-month (the 1st and 3rd Tuesday) and 3 for 3-month (12/6, 12/9 and 13/12)

- Spot foreign exchange rates and risk-free interest rates (Euro-currencies): Datastream

- Thirty-minute prices of exchange rate futures of USD/GBP and USD/EUR
Empirical Results (I)

- *When a cross-rate option price is available:*
  - Whether at OTC or CME, Gaussian and Frank copulas can model the dependence structure relatively satisfactorily. Average absolute error of implied volatility is usually below 0.4%, which is within the spread of bid and ask.
  - Market cross-rate volatility smile is between Gaussian and Frank smiles.
  - The implied volatility smiles from OTC and CME prices are similar.
    - Information is efficiently integrated across OTC and CME.
Cross-rate Implied Volatility
(With a known ATM cross-rate option price)
Marginal: Lognormal Mixtures

Average 1-month Implied Volatility

- Vanilla
- Gaussian
- Frank
- Plackett
- Clayton
OTC Implieds V.S. CME Implieds
(With a known ATM cross-rate option price)
Marginal: Lognormal Mixtures

Average 1-month Implied Volatility with Gaussian and Frank Copulas

Delta(%)  Implied

0.110
0.115
0.120
0.125
0.130

0 20 40 60 80 100

OTC-G  CME-G  OTC-F  CME-F
Empirical Results (II)

- *When a cross-rate option price is not available:*
  - Realized correlation coefficients are on average lower than implied correlation coefficients.
    - Biased historical estimates?
    - Mispriced ATM cross-rate options?
    - Premia for risks that can not be hedged such as jumps in volatility?
  - Due to the deviation of implied from realized correlation coefficients, the differences between theoretical implieds and market implieds increase.
    - 1% for one-month options; 2% for three-month options
Realized V.S. Implied Correlations

Implied and Realized Correlation Coefficients

- Implied
- Realized

Graph showing changes in implied and realized correlation coefficients from 05/01/00 to 12/25/00.
Cross-rate Implied Volatility
(With a known realized correlation coefficient)
Marginal: Lognormal Mixtures

Average 1-month Implied Volatility in OTC

- Vanilla
- Gaussian
- Frank
- Plackett
- Clayton
Summary & Conclusions

- **Theoretical formulae**
  - Considering the importance of numeraire, we derive the RND formula for cross-rates which is guaranteed risk-neutral for any bivariate density.
  - With copula functions, we successfully calibrate the formula to obtain RNDs of €/£ without using full information about options on €/£.
Empirical implications

- When ATM market prices of cross-rates are available, Gaussian and Frank copulas can model the dependence structure of $/£ and $/€ satisfactorily.

- When information about cross-rate option prices is not used, using realized correlation coefficients to estimate dependence parameters can still capture volatility smiles of cross-rates. However, the difference between theoretical implied volatility and market implied volatility increases.