Option prices and risk-neutral densities for currency cross-rates

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Abstract

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The dollar-rate option prices provide marginal RNDs for the two dollar rates. These two univariate RNDs and a copula function define the bivariate RND. Four one-parameter copula functions are compared empirically and the most satisfactory cross-rate RNDs are provided by the Gaussian and Frank copulas. Empirical comparisons are first made between cross-rate RNDs estimated from (a) one cross-rate option price and many dollar-rate option prices and (b) seven cross-rate option prices quoted by a bank. Further comparisons are made using cross-rate RNDs that are estimated solely from futures and options prices for dollar-rates.

Keywords: Option pricing, density estimation, exchange rates, cross-rate, copulas
JEL Classifications: F31, G13, G15

First version: March 9, 2004
This version: September 10, 2004

Acknowledgements:
This research has benefited from numerous conversations with Söhnke Bartram. We are indebted to seminar participants at the European Finance Association 2004 annual meeting for helpful comments and suggestions. The assistance of Aris Bikos with data collection is gratefully acknowledged.
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1. Introduction

Derivative markets are a rich source of information for gauging market sentiment. In particular, option prices reveal market expectations about the risk-neutral densities of future asset prices. Many numerical methods can be used to convert a set of option prices into a risk-neutral density\(^1\) (RND) and these are routinely applied to dollar exchange rates by traders and policymakers\(^2\). The estimation and analysis of risk-neutral densities for exchange cross-rates is, in contrast, a difficult problem because cross-rate options have low liquidity and are traded over-the-counter, at prices that are not usually available to researchers.

The primary contribution of this paper is a complete methodology for estimating cross-rate risk-neutral densities, from which cross-rate options can be priced. Our methods are illustrated for options on the euro/pound rate and they rely on prices from the highly liquid option markets for the dollar/pound and the dollar/euro rates. We use the triangular relationship between three currencies to infer the cross-rate RND from the bivariate density of the two dollar exchange rates. This bivariate density is given by the product of three terms: the two marginal densities for the dollar rates and a copula function that quantifies their dependence.

The second contribution of this paper is advice about selecting the copula function, that is based upon comparing our derived cross-rate RNDs with those implied by over-the-counter RNDs. The cross-rate RNDs that we derive can be used by banks, international businesses and central bankers to assess market expectations, to measure risks and to value options, without relying on over-the-counter markets that may be either non-existent or illiquid.

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\(^1\) See, for example, Jackwerth and Rubinstein (1996), Melick and Thomas (1997), Jackwerth (1999), Jondeau and Rockinger (2000) and Bliss and Panigirtzoglou (2002).

Bikos (2000), Rosenberg (2003) and Bennett and Kennedy (2003) also investigate cross-rate densities, their option prices and the implied dependence between dollar exchange rates. Bikos (2000) uses option prices for all three exchange rates to estimate the dependence between dollar rates. Rosenberg (2003) shows how the bivariate density can be estimated non-parametrically, using a copula function that is estimated from historical exchange rates. As we explain in Section 2, his derived cross-rate density is not risk-neutral because a required change of numeraire from dollars (for the bivariate density) to a foreign currency (for the cross-rate density) is overlooked. Bennett and Kennedy (2003) provide theoretical results for quanto options, that have dollar payoffs that are contingent upon cross-rates.

Our empirical results are all for the dollar, pound and euro currencies. We compare the option prices for cross-rates quoted by a bank with those that we derive from dollar option prices. Our derived prices depend on a source, a copula function and its dependence parameter. We compare the results for dollar option prices obtained from two sources: the OTC market and the Chicago Mercantile Exchange (CME). We compare results for four widely used copula functions, namely the Gaussian, Frank, Plackett and Clayton copulas. For each copula function, we fix the dependence parameter by using either the at-the-money, OTC cross-rate price or the recent historical record of exchange rates. We find first that the publicly available CME source gives essentially the same results as the OTC source, second that the Gaussian and Frank copulas provide the most satisfactory prices and third that historical correlations are on average lower than those implied by cross-rate, OTC option prices.

Sections 2 to 4 of this paper are theoretical, followed by empirical analysis in Sections 5 to 7. Section 2 derives the relationship between the RND for the cross-rate and the bivariate RND for two dollar exchange rates, using risk-neutral measures determined by the numeraires of payoffs. Section 3 introduces copula functions, which are used to characterize the relationship between the bivariate RND and the marginal RNDs for the dollar rates. Section 4
summarizes two standard methods for estimating marginal RNDs. The empirical framework
and the data used are described in Sections 5 and 6 respectively. Section 7 presents the
empirical results and comparisons. Finally, conclusions are stated in Section 8.

2. The risk-neutral density formula for cross-rates

Our first objective is to produce formulae for the cross-rate risk-neutral density and the prices
of cross-rate options by using the prices of dollar-denominated assets. Option payoffs depend
on the rates of exchange between dollars (\$, USD), pounds (\£, GBP) and euros (€, EUR) in
our three-currency framework. We denote the dollar price of one pound at time \( t \) by \( S_t^{\$/£} \) and
likewise the dollar price of one euro at the same time is denoted by \( S_t^{\$/€} \). The cross-rate
price of one pound in euros is then given by \( S_t^{€/£} = \frac{S_t^{\$/£}}{S_t^{\$/€}} \).

At time zero we assume there is a complete market for S/£ European call options,
priced in dollars, that expire at time \( T \). This implies the existence of a unique risk-neutral
density (RND) for \( S_T^{\$/£} \) that we denote by \( f_S(y) \); the dollar subscript emphasizes that the
numeraire of asset payoffs is dollars. Likewise, there is a RND for \( S_T^{\$/€} \) that we denote by
\( f_S(z) \). We also assume that it is possible to define a bivariate risk-neutral density for
\((S_t^{\$/£}, S_t^{\$/€})\), denoted by \( f_S(y, z) \), that can be used to price dollar payoffs that are
contingent on these two exchange rates; we defer discussion of the construction of \( f_S(y, z) \)
until Section 3.

Now consider an European option to buy \£1 for €\( X \) at time \( T \). This is identical to an
option to exchange \( XS_T^{\$/€} \) dollars for \( S_T^{\$/£} \) dollars at time \( T \). Hence its dollar payoff equals
\[ \max(S_T^{\$/£} - XS_T^{\$/€}, 0) \] and its fair price in dollars at time zero is:
Here $Q_S$ is the risk-neutral measure for the $\$\numeraire and $r_S$ is the dollar risk-free rate. The fair price of the same option in euros is therefore:

$$
C_S(X) = e^{-r_S T} E^{Q_S} \left[ \max(S_T^{S/\$} - X S_T^{S/\$}, 0) \right] \\
= e^{-r_S T} \int_0^\infty \int_0^\infty \max(y - Xz, 0) f_S(y,z) dydz.
$$

(1)

This must equal the following discounted expected payoff, that employs the risk-neutral measure $Q_e$ for the euro numeraire:

$$
C_e(X) = C_S(X) \bigg/ S_0^{S/\$} \\
= \left( e^{-r_S T} / S_0^{S/\$} \right) \int_0^\infty \int_0^\infty \max(y - Xz, 0) f_S(y,z) dydz.
$$

(2)

Here $r_e$ is the euro risk-free rate and $f_e(x)$ is the RND for one pound priced in euros.

A specific formula for the cross-rate RND defined by equation 3 is given by using the well-known result of Breeden and Litzenberger (1978),

$$
f_e(x) = e^{r_e T} \frac{\partial^2 C_e(x)}{\partial x^2},
$$

in conjunction with equation (2), to obtain:

$$
f_e(x) = \frac{1}{F_0^{S/\$}} \frac{\partial^2}{\partial x^2} \left[ \int_0^\infty \int_0^\infty \max(y - xz, 0) f_S(y,z) dydz \right]
$$

(4)

where $F_0^{S/\$} = S_0^{S/\$} e^{(r_S - r_e) T}$ is the forward price, at time zero, to exchange one euro for dollars at time $T$. Equation 4 simplifies, as is shown in the Appendix, to give the RND of the cross-rate as the following single integral:

$$
f_e(x) = \frac{1}{F_0^{S/\$}} \int_0^\infty z^2 f_S(xz,z) dz.
$$

(5)
Cross-rate option prices can be calculated either from the two single integrals defined by equations 3 and 5, or from the double integral in equation 2 which is the same as

\[ C_e(X) = e^{-r_e T} \frac{1}{F_0^S / \epsilon} \int_{X}^\infty \int_{0}^{\infty} (x - X) z^2 f_S(xz, z) \, dz \, dx. \]  

(6)

It is easy to check the three necessary conditions for the function in equation 5 to be a risk-neutral density. First, the function is obviously non-negative. Second, the substitution \( y = xz \) (with \( dy = zdx \)) can be used to show the function integrates to one and is therefore a density:

\[ \int_{0}^{\infty} f_e(x) \, dx = \frac{1}{F_0^S / \epsilon} \int_{0}^{\infty} \int_{0}^{\infty} z^2 \left[ \int_{0}^{\infty} f_S(xz, z) \, dz \right] \, dx \]

\[ = \frac{1}{F_0^S / \epsilon} \int_{0}^{\infty} \int_{0}^{\infty} zf_S(y, z) \, dy \, dz = E^{Q_S} \left[ S_T^S / \epsilon \right] / F_0^S / \epsilon = 1. \]  

(7)

Third, the same substitution establishes that the expectation of the spot cross-rate at time \( T \) is the forward cross-rate, and hence the density is risk-neutral:

\[ \int_{0}^{\infty} xf_e(x) \, dx = \frac{1}{F_0^S / \epsilon} \int_{0}^{\infty} \int_{0}^{\infty} xz^2 f_S(xz, z) \, dx \, dz \]

\[ = \frac{1}{F_0^S / \epsilon} \int_{0}^{\infty} \int_{0}^{\infty} yf_S(y, z) \, dy \, dz \]

\[ = E^{Q_S} \left[ S_T^S / \epsilon \right] / F_0^S / \epsilon = F_0^S / \epsilon / F_0^S / \epsilon = F_0^S / \epsilon. \]  

(8)

We provide the above checks because our RND for the cross-rate is not the same as the density derived in Rosenberg (2003). His density, in our notation, is

\[ f(x) = \int_{0}^{\infty} zf_S(xz, z) \, dz. \]  

(9)

This is the density of the cross-rate with respect to the risk-neutral measure for the dollar numeraire, i.e. \( Q_S \). Consequently, the Rosenberg density is not in general a cross-rate risk-neutral density and hence it should not be used to price cross-rate options. Indeed, a random variable whose density is given by equation (9) has expectation
\[ \left[ \int xz f_S(xz, z) dz dx \right] = \int \frac{y}{z} f_S(y, z) dy dz = E \frac{Q_S [S_T^S]}{S_T^S} \]

whenever \( \text{cov} Q_S (S_T^S, S_T^S) \neq 0 \).

3. The construction of bivariate RNDs

To price cross-rate options, using the cross-rate RND given by equation (5), we first need to generate the bivariate RND of the two dollar-denominated exchange rates, denoted \( f_S(y, z) \).

We use copula functions in this paper to convert the two marginal densities, \( f_S(y) \) and \( f_S(z) \), into the bivariate density \( f_S(y, z) \).

Copula methods are covered in the textbooks by Joe (1997), Nelsen (1999) and Cherubini et al (2004). There are many recent applications in finance research, to subjects such as credit risk (Li, 2000), portfolio allocations (Hennessy and Lapan, 2002) and the pricing of multivariate contingent claims (Rosenberg, 2003).

3.1 Definitions

Copula functions, denoted \( C(u, v) \), are the bivariate cumulative distribution functions of random variables \( U \) and \( V \) whose marginal distributions are uniform on the interval from zero to one. Employing a copula function permits modeling of the dependence between random variables by using marginal densities to construct bivariate densities that are consistent with the univariate marginals.

Let \( H \) be the bivariate cumulative distribution function for random variables \( Y \) and \( Z \),
with respective marginal cumulative functions $F$ and $G$ that are both continuous. By Sklar’s Theorem, there exists a unique copula function $C$ such that

$$H(y, z) = C(F(y), G(z)) \text{ for all } y \text{ and } z.$$  

Then $H(y, \infty) = C(F(y), 1) = F(y)$ and $H(\infty, z) = C(1, G(z)) = G(z)$ so that the joint c.d.f. has the correct marginal distributions. Assuming $F$ and $G$ have well-defined inverse functions, $C(u, v) = H(F^{-1}(u), G^{-1}(v))$.

The bivariate density for $(Y, Z)$ is given by

$$h(y, z) = c(F(y), G(z)) \times f(y) \times g(z)$$

where $c(u, v) = \partial^2 C(u, v)/\partial u \partial v$ is the density corresponding to the c.d.f. $C(u, v)$ and $f$ and $g$ are the marginal densities of $Y$ and $Z$.

### 3.2 A measure of dependence

The correlation between the uniformly distributed random variables $U = F(Y)$ and $V = G(Z)$ is a useful parameter for measuring the dependence between $Y$ and $Z$. This correlation defines Spearman’s “rho” for the variables $Y$ and $Z$, denoted by $\rho_S$:

$$\rho_S(Y, Z) = \text{cor}(U, V) = (E[UV] - \frac{1}{4})/\frac{1}{12} = 12 \int_0^1 \int_0^1 uv \, c(u, v) \, dudv - 3.$$  

An alternative formula, that is sometimes more convenient for calculations, is

$$\text{cor}(U, V) = 12 \int_0^1 \int_0^1 C(u, v) \, dudv - 3.$$  

Spearman’s rho is obviously invariant to increasing transformations of $Y$ and $Z$. In particular,

$$\rho_S(Y, Z) = \rho_S(U, V).$$

### 3.3 Specific copula functions
Many parametric copula functions have been applied in statistical literature. We focus on four functions that permit a wide range of possible dependence between two uniform variables. Each of the copula functions has a single parameter that characterizes the dependence. We make empirical comparisons between the Gaussian, Plackett, Frank, and Clayton copulas. All of these copulas can display either positive or negative dependence. The contours of the copula densities are shown on Figure 1, when Spearman’s rho equals 0.5 and the marginal distributions are standard normal.

The dependence structure is symmetric\(^3\) and tail-independent\(^4\) for the Gaussian, Frank, and Plackett copulas, but the Clayton copula has asymmetry and lower-tail dependence that has been used to model the dependence of equity prices (e.g. Cherubini and Luciano, 2002).

The Gaussian copula is defined by:

\[
C(u, v | \rho) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} f(y, z | \rho) \, dz \, dy
\]

where \(f(\cdot | \rho)\) denotes the standard bivariate normal density function with correlation \(\rho\) and the function \(\Phi\) is the univariate standard normal cumulative function. The density is:

\[
c(u, v | \rho) = \frac{1}{\sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \left[ s^2 + t^2 - 2 \rho st + \frac{1}{2} [s^2 + t^2] \right] \right\}
\]

with \(s = \Phi^{-1}(u)\) and \(t = \Phi^{-1}(v)\). Spearman’s rho for this density equals

\[
\rho_S = (6/\pi) \arcsin(\rho/2),
\]

see, for example, Kepner et al (1989). Note that the Gaussian copula will only generate a normal bivariate density when the marginal densities are normal.

The Plackett copula is defined by a constant cross-product ratio:

\[
\frac{P(U \leq u, V \leq v)}{P(U \leq u, V > v)} \frac{P(U > u, V > v)}{P(U > u, V \leq v)} = \frac{C(u, v)[1 - u - v + C(u, v)]}{[u - C(u, v)][v - C(u, v)]} = \theta
\]

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\(^3\) A copula \(C\) is symmetric if \((U, V)\) and \((1-U, 1-V)\) have the same c.d.f.

\(^4\) A copula is tail-independent if \(P(U > a | V > a)\) and \(P(U < b | V < b)\) both converge to zero as \(a \to 1\) and \(b \to 0\).
with $\theta$ a positive parameter. Consequently, with $\eta = \theta - 1$,

$$C(u, v|\theta) = \frac{1}{2} \eta^{-1} \{1 + \eta(u + v) - [(1 + \eta(u + v))^2 - 4\theta \eta uv]^{1/2}\}$$

and

$$c(u, v|\theta) = [(1 + \eta(u + v))^2 - 4\theta \eta uv]^{-3/2} \theta[1 + \eta(u + v - 2uv)]$$

The Plackett copula is stochastically increasing in the tails$^5$.

The Frank and Clayton copulas are both Archimedean copulas. These are constructed from some continuous, strictly decreasing function $\varphi$ from $(0, 1]$ to $[0, \infty)$ that has $\varphi(1) = 0$.

When $\varphi^{-1}$ is defined, $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$.

For the Frank copula,

$$\varphi(t) = -\ln[(\exp(-\theta t) - 1)/(\exp(-\theta) - 1)], \quad \theta \neq 0,$$

$$C(u, v|\theta) = -\frac{1}{\theta} \ln[1 + \left(\frac{e^{-\theta u} - 1}{e^{-\theta} - 1}\right)\left(\frac{e^{-\theta v} - 1}{e^{-\theta} - 1}\right)]$$

and

$$c(u, v|\theta) = -\theta(e^{-\theta u} - 1)(e^{-\theta v} - 1) \frac{1}{[e^{-\theta u} - 1 + (e^{-\theta u} - 1)(e^{-\theta v} - 1)]^2}.$$

For the Clayton copula, with $\theta$ positive$^6$,

$$\varphi(t) = (t^{-\theta} - 1)/\theta$$

$$C(u, v|\theta) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

and

$$c(u, v|\theta) = (1 + \theta)(uv)^{-\theta-1} (u^{-\theta} + v^{-\theta} - 1)^{-2-(1/\theta)}.$$

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$^5$ $Y$ is stochastically increasing in $X$ if $P(Y > y|X = x)$ is a non-decreasing function of $x$, for all $y$.

$^6$ The Clayton copula can also be defined, by different equations, when $-1 < \theta < 0$, but none of our estimates are within this range.
3.4 Estimation of the dependence parameter

If the market price of one cross-rate option is available, typically the at-the-money option, then we can use numerical methods to obtain the implied estimate of the dependence parameter by equating the market price with the theoretical price given by equation (6).

Otherwise, we can use a historical record of intraday dollar-denominated currency returns for two exchange rates to calculate their realized correlation, using the method of Andersen et al (2001a, 2001b). By assuming the historical correlation can approximate the risk-neutral correlation in the future, we can estimate Spearman’s rho for the Gaussian copula from equation (13). The dependence parameters for the other three copulas can then be obtained by constraining these copulas to have the same value for Spearman’s rho. We discuss the empirical relationship between the historical and risk-neutral correlations in Section 7. Following Andersen et al, the realized correlation coefficient between two sets of thirty-minute returns, \( \{r_{1,j}\} \) and \( \{r_{2,j}\} \), whose latest return is at the end of day \( t \), is given by the following formulae:

\[
\rho_t = \frac{\text{Cov}_t}{\sigma_{1,t} \sigma_{2,t}}, \quad \text{Cov}_t = \sum_{j=1}^{nm} n_{1,j} r_{2,j} \quad \text{and} \quad \sigma_{t,t}^2 = \sum_{j=1}^{nm} r_{t,j}^2,
\]

where \( n \) is number of thirty-minute intervals in one day and \( m \) is the number of days used for estimation.\(^7\)

4. Methodology for the marginal RNDs

Marginal risk-neutral densities for the dollar prices of one pound and one euro are required when we construct their bivariate density using equation (11) and an appropriate copula density. Many types of univariate RND have been proposed, including lognormal mixtures

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\(^7\) When using intraday returns to estimate realized volatility or correlations, the choice of frequency is a tradeoff between minimizing the bias and avoiding market microstructure effects. In our data, we find that the first order autocorrelation coefficients of five-minute return series are significantly negative, which signals that market microstructure effects may exist. Therefore, we use thirty-minute returns instead to calculate realized correlations since their first order autocorrelations are much lower.
(Ritchey, 1990, Melick and Thomas, 1997), generalized beta densities (Anagnou et al, 2002), multi-parameter discrete distributions (Jackwerth and Rubinstein, 1996) and densities derived from fitting spline functions to implied volatilities (Bliss and Panigirtzoglou, 2002). Providing options are traded for a range of exercise prices that encompass almost all of the risk-neutral distribution, several flexible density families will provide similar empirical estimates.

In this paper, we use lognormal mixtures and the generalized beta density of the second kind (GB2) to estimate univariate RNDs. These densities have few parameters and many desirable properties: general levels of skewness and kurtosis are allowed, the shapes of the tails are fat relative to the lognormal density and there are analytic formulae for the density, its moments and the prices of options. Furthermore, parameter estimation is easy and does not involve any subjective choices, the estimated densities are never negative and risk-neutral densities can be transformed analytically into real-word densities. Equations for risk-neutral and real-world densities, moments and call prices are provided by Liu et al (2003).

The univariate densities are now defined for an exchange rate $T$ years into the future when the spot rate for one unit of foreign currency is $S_T$ dollars. The dollar risk-free interest rate is $r$ and the forward rate at time zero for exchange at time $T$ is denoted by $F = E^Q[S_T]$.

Interest rates are assumed to be non-stochastic, so that $F$ is also a futures price. A general exercise price is represented by $X$, for a European call option that is written either on spot currency or on a futures contract that delivers at time $T$.

4.1 The lognormal mixture density

When asset prices follow geometric Brownian motion, with volatility $\sigma$, the risk-neutral density for $S_T$ is the following lognormal density:
\[
\psi(x|F, \sigma, T) = \frac{1}{x \sigma \sqrt{2\pi T}} \exp \left[ -\frac{1}{2} \left( \frac{\ln(x/F) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right)^2 \right].
\]

The price of a call option is then given by Black’s formula,

\[
C_B(F, T, X, r, \sigma) = F e^{-rT} \Phi(d_1) - X e^{-rT} \Phi(d_2)
\]

with

\[
d_1 = \frac{\ln(F/X) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T}.
\]

A mixture of two lognormal densities for \( S_T \) is defined as:

\[
f_{\text{mix}}(x) = p \psi(x|F_1, \sigma_1, T) + (1-p) \psi(F_2, \sigma_2, T).
\]

There are five non-negative parameters, \( \theta = (F_1, F_2, \sigma_1, \sigma_2, p) \), that must satisfy the constraints \( 0 \leq p \leq 1 \) and \( F = p F_1 + (1-p) F_2 \). Call prices are simply a mixture of Black prices:

\[
C(X|\theta, r, T) = p C_B(F_1, T, X, r, \sigma_1) + (1-p) C_B(F_2, T, X, r, \sigma_2).
\]

4.2 The generalized beta density

Bookstaber and McDonald (1987) proposed the GB2 density for asset prices, with four positive parameters that define a parameter vector \( \theta = (a, b, p, q) \). The risk-neutral density function for \( S_T \) is defined as

\[
f_{\text{GB2}}(x|a, b, p, q) = \frac{a}{b^{ap} B(p, q)} x^{ap-1} \left[ 1 + (x/b)^a \right]^{p+q}, \quad x > 0,
\]

where the function \( B \) is defined in terms of the Gamma function by \( B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q) \). Call prices can then be derived, with the following result:
where \( F_\beta \) is the incomplete beta function given by

\[
F_\beta(u|p,q) = \frac{1}{B(p,q)} \int_0^u t^{p-1}(1-t)^{q-1} dt
\]

and

\[
u(X,a,b) = \frac{(X/b)^a}{1 + (X/b)^a}.
\]

Risk-neutrality constrains the mean of the density to be

\[
F = bB[p+a^{-1},q-a^{-1}]/B(p,q),
\]

which also requires the constraint \( aq > 1 \).

4.3 Parameter estimation

Several loss functions can be considered when estimating the parameter vector \( \theta \) of either density family. As is common in the RND literature, we minimize the sum of squared pricing errors for a set of market call prices denoted by \( c_m(X_i) \). When there are market prices available for \( N \) distinct exercise prices, having a common expiry time \( T \), we estimate \( \theta \) by minimizing

\[
G(\theta) = \sum_{i=1}^{N} (c_m(X_i) - C(X_i|\theta))^2.
\]

5. Empirical framework

To obtain empirical estimates of the cross-rate risk-neutral density and then the prices of cross-rate options, we follow the steps shown in Figure 2. In the first step, we use market option prices for two dollar-denominated exchange rates, namely the dollar prices of one
pound and one euro, to estimate their univariate RNDs with respect to the risk-neutral measure for the dollar numeraire. The two univariate RNDs and a one-parameter copula function are employed in the second step to obtain their bivariate RND, again for the dollar numeraire. The third step produces the cross-rate RND for the euro price of one pound, with respect to the risk-neutral measure for the euro numeraire. It also provides the prices of cross-rate options.

The cross-rate RND depends both on the choice of the copula function and the value of its dependence parameter. The parameter is estimated either from historical data or by matching the theoretical price of the at-the-money cross-rate option with its market price. The latter approach requires a numerical method that repeats the second and third steps until the parameter value is estimated accurately.

To assess our cross-rate RND and option pricing methodology, we consider one-month and three-month maturities. In addition, we also investigate the following two questions:

1. Which copula function(s) can model the dependence structure of exchange rates most satisfactorily?

2. Is the information used to price cross-rate options efficiently shared across different markets for options on dollar exchange rates?

The cross-rate option prices generated by our copula formulae are compared with those for OTC cross-rate option prices. The differences between two sets of prices for the same options are summarized by three statistics: the Kolmogrov-Smirnov statistic, that equals the maximum difference between two cumulative risk-neutral distribution functions (K-S), the average of the absolute differences between call prices as a ratio of the OTC prices (G-call), and the average magnitude of the difference between implied volatilities (G-implied).

6. Data
The primary data are options prices for the $/£, $/€ and €/£ exchange rates. It is not possible to obtain useful exchange-traded option prices for the €/£ cross-rate. Some cross-rate settlement prices are available for the Chicago Mercantile Exchange (CME), but they correspond to almost no trading volume. Consequently, we have to rely on over-the-counter option prices for the cross-rate. Such prices are not in the public domain to the best of our knowledge. We make use of a confidential file of OTC option price midquotes, supplied by the trading desk of an investment bank, that covers the period from May to December 2000. The OTC quotes are for all three exchange rates, recorded at the end of the day in London. There are typically prices for seven exercise prices, based upon “deltas” equal to 0.1, 0.25, 0.37, 0.5, 0.63, 0.75 and 0.9.

A second source of $/£ and $/€ option prices is provided by CME settlement prices for options on currency futures contracts. These options are American. Their early exercise premia are estimated from the pricing approximation of Barone-Adesi and Whaley (1987) which allows us to deduce appropriate prices for European options. The average number of exercise prices for which there are option prices is 21 for the $/£ rate and 24 for the $/€ rate.

We generate both one-month and three-month RNDs. We obtain densities from the one-month OTC option prices for the first and third Tuesday of every month from May to October 2000. Thus our one-month results are based upon twelve cases. Because options traded at the CME have fixed expiry dates, on the Fridays preceding the third Wednesdays of the contract months, it is impossible to match them with OTC options whose expiry dates are two business day before the forward dates. Therefore, we define matching CME option prices from the prices of the two nearest-to-expiry contracts that are more than one week from expiry. These matching prices are obtained by assuming implied volatilities have a linear term structure. There is less usable OTC data for the generation of three-month densities. We use

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8 The term structure of the implied volatility of FX options is not flat and its slope changes frequently (Xu and Taylor, 1994). For a selected exercise price, we approximate the CME implied volatility for an option that
the three-month options prices on 12 June, 12 September, and 13 December 2000, from both
the OTC source and the CME.

The OTC data and our processing of the CME data provides us with implied
volatilities. The required one and three-month forward rates, when the CME closes a few
hours after the OTC quotes are recorded, are provided by the standard no-arbitrage equation
that relates forward rates to spot prices and domestic and foreign interest rates. These forward
rates are the expectations of the risk-neutral densities.

We also use the prices of $/£ and $/€ recorded every thirty minutes in the OTC
markets, to calculate one-month and three-month realized correlation coefficients for their
returns from equation (14). Spot exchanges rates and risk-free Eurocurrency interest rates are
obtained from Datastream.

7. Empirical results

Our empirical implementation is conducted using two types of information to estimate the
dependence parameter of the copula function. First we use the prices of at-the-money cross-
rate options, then we assess the alternative strategy of relying on historical data to provide
correlation estimates.

7.1 When the prices of ATM cross-rate options are used

The properties of €/£ cross-rate risk-neutral densities derived from OTC cross-rate option
prices are shown in the first column of Table 1, and also in the first column of Table 2. We
refer to these as “vanilla” densities as they merely depend upon cross-rate market prices. Each
vanilla density is derived from seven option prices. The remaining columns of Table 1 provide

expires after $T$ days by

$v = v_1 + (v_2 - v_1)(T - T_1)/(T_2 - T_1)$

where $v_1$ and $v_2$ are the implied volatilities of the
two near-to-expiry contracts, that have times $T_1$ and $T_2$ until expiry.
summary statistics for further cross-rate RNDs, each of which is obtained from seven OTC option prices for both the $/£ and the $/€ rates, a copula function and the OTC price of the at-the-money €/£ option\(^9\). Likewise, the remaining columns of Table 2 summarize cross-rate RNDs obtained from CME option prices for the $/£ and the $/€ rates, a copula function and the OTC price of the at-the-money €/£ option.

The summary statistics are averages across densities. For example, each “Variance” statistic for a set of one-month densities is the average value of twelve variances, one for each risk-neutral density. The statistics in Table 1 are generally similar to those in Table 2. There are also no important differences between the copula densities derived from the two families of marginal densities. Consequently, we discuss the results for the lognormal mixture marginals and the GB2 marginals together.

The means of all sets of twelve one-month densities are identical, because all the densities are risk-neutral, and likewise for all sets of three three-month densities. The average variances of the RNDs created from the various copula functions are very similar to those of the vanilla densities. There is more positive skewness in the vanilla densities than in the copula densities. The average skewness of the RNDs is positive, but small, for the Gaussian, Frank, and Plackett copulas while the averages are negative for the Clayton copula. The kurtosis averages for all sets of densities exceed three and the averages are fairly similar for the vanilla densities and those derived using copula functions. Generally, the cross-rate densities on a particular date are similar. This can be seen on Figure 3, which shows all the one-month densities formed on 5th September 2000; the cross-rate RNDs generated by the Gaussian, Frank and Plackett copulas are slightly nearer to the vanilla density than the densities obtained from the Clayton copula.

Next consider how near the densities derived from copulas are to the vanilla densities.

\(^9\) At-the-money refers to options whose delta equals 0.5.
The Kolmogorov-Smirnov (KS) statistic is an overall measure of the similarity of two densities. The averages of the KS statistics are always least for the Gaussian and Frank copulas and are then always less than 0.02. Similarity can also be assessed by comparing the option prices generated by different risk-neutral densities, for the options that are not at-the-money. The average absolute difference between a vanilla option price and a Frank option price is about 3% of the vanilla option price, with slightly higher average differences for the Gaussian and Plackett option prices. When implied volatilities are compared, the average absolute difference is less than the bid-ask spread (typically 0.4% to 0.5%) for all but the Clayton copula. Our comparisons show that the Gaussian and Frank copulas are satisfactory, with average absolute differences as low as 0.2%, the Plackett copula is fairly satisfactory but the Clayton copula systematically performs less well.

Figures 4 and 5 show averages of the implied volatility functions for the €/£ cross-rate, across density formation dates, respectively for OTC and CME dollar-rate data. These functions are plotted against the OTC option delta. All the copula functions generate volatility smiles. The vanilla densities and the Gaussian, Frank, and Plackett copulas all produce symmetric smiles. However, the Clayton smiles are asymmetric and hence are unsatisfactory. The curvature (or depth) of the smiles is highest for the Plackett smiles, and the Frank smiles are deeper than the Gaussian smiles. Generally the market volatility smile, defined by the vanilla densities, lies above the Gaussian smile and below the Frank smile.

Figure 6 compares the average OTC and CME volatility smiles generated by the Frank and Gaussian copulas from mixtures of lognormal densities. The differences between the OTC and the CME average smiles are seen to be small, particularly when delta is between 25% and 75%. The differences are all less than the bid-ask spread, even for the extreme values of delta. We conclude that the cross-rate densities obtained from the OTC and the CME option prices of dollar exchange rates are similar, so that price information is efficiently
shared by these two markets. Figure 6 also emphasizes that the Frank smiles are approximately twice as deep as the Gaussian smiles.

Our results show that the Gaussian and Frank copula functions are relatively satisfactory when modeling the dependence between the risk-neutral distributions of the $/£ and the $/€ exchange rates, for both the one-month and the three-month densities. The deeper smiles produced by the Plackett copula show that it is less satisfactory. Because the implied volatility pattern of foreign exchange rates is a smile, rather than a smirk, the asymmetric Clayton copula appears to be unsatisfactory, although it is often used to model the dependence structure of some equity prices satisfactorily (e.g. Cherubini and Luciano, 2002).

7.2 When the prices of ATM cross-rate options are not used

The previous strategy of estimating the dependence parameter by matching the theoretical ATM call price with the market price is like using an implied risk-neutral correlation to estimate the dependence parameter. When the ATM market price is either unavailable or considered to be an unfair price, we can instead use the realized historical correlation to estimate the dependence parameter.

Time series of implied and realized correlation coefficients through our sample period are shown on Figure 7. These are correlations between changes in the logarithms of the $/£ and the $/€ rates. The implied correlations are derived directly from the ATM implied volatilities of $/£, $/€ and €/£ OTC options one month from expiry, while the realized correlations are calculated from the previous month of thirty-minute returns for the $/£ and the $/€ rates.

It is seen very clearly that the realized correlation is on average lower than the implied correlation. There are many possible explanations, that include biased historical estimates\(^\text{10}\),

\(^{10}\) When using historical information to estimate future correlations, we assume the historical pattern of correlations can be applied to the future. Therefore, historical correlations may be biased estimates of future
mispriced ATM cross-rate options\textsuperscript{11} and premia for risks that cannot be hedged such as jumps in volatility\textsuperscript{12}. The inevitable consequence of the differences between the implied and realized correlations is that the differences between the vanilla and the copula results will increase, for densities, option prices and implied volatilities.

The properties of €/£ cross-rate risk-neutral densities derived from option prices for the $/£ and the $/€ rates, a copula function and the historical correlation are summarized in Tables 3 and 4, respectively for densities derived from OTC and CME prices. These summary statistics can be compared with those presented in Tables 1 and 2. The densities are guaranteed to remain risk-neutral. The most obvious difference, that occurs when historical correlations are used, is that the variances of the cross-rate RNDs increase significantly due to the lower dependence between the marginal dollar-rate densities. It is now hard to tell which copula is the most satisfactory, regardless of the criterion, because all the copula functions produce option prices that are evidently different from the vanilla option prices. For example, the average absolute difference between the copula and vanilla implied volatilities is about 1\% for the one-month maturities and 2\% for the three-month maturities when historical correlations are employed.

One-month cross-rate RNDs formed on a typical date, 18th July 2000, are shown on Figure 8. The peaks of the copula densities are lower than the vanilla densities because the copula variances exceed the vanilla variances. Figure 9 shows the averages of the one-month implied volatility functions, when lognormal mixtures are used to estimate the marginal densities. All the copula functions again produce volatility smiles, although the general levels

\textsuperscript{11} Because of market liquidity or/and investors’ bank relationship, the quotes of cross-rate options may be biased from the fundamental prices.

\textsuperscript{12} When asset prices follow Geometric Brownian motion, the second-moment measures of returns, such as variances and correlations, do not depend on risk assumptions. Thus, variances and correlations are the same under measures $P$ and $Q$. However, as suggested by the theoretical work of Branger and Schlag (2004), when price processes include jump components, the change of measure will affect variances and correlations and the size of this impact varies across maturities.
of these smiles are visibly higher than the vanilla smiles. The previous remarks about the relative depth and the symmetry of the copula smiles remain valid when the implied correlation is replaced by the historical correlation.

While it is difficult to decide which copula function performs best when historical data are used to estimate correlations, the average absolute difference between the Gaussian-copula and vanilla implied volatilities are the lowest, and the average absolute difference of call prices and the K-S statistics are also comparatively low for the Gaussian copula. Also, the shapes of typical vanilla volatility smiles are more similar to those of Gaussian volatility smiles. Therefore, the Gaussian copula should perform satisfactorily if the precision of correlation estimates can be improved.

8. Concluding remarks

This paper illustrates practical methods for estimating cross-rate risk-neutral densities. We suppose there is limited information available about the prices of cross-rate options, that contrasts with an abundance of information about the prices of dollar-rate options. The theoretical cross-rate RND is derived from the bivariate RND for two dollar exchange rates. Cross-rate RNDs can be used to price any European contingent claim and they also provide insight into market expectations.

Our methods make an assumption about the number of available implied volatilities for cross-rate options. We find that cross-rate option prices are sensitive to this number, which is assumed to be either one or zero. When there are no cross-rate option prices we have to rely on historical dollar exchange rates to quantify the dependence between the two dollar rates. The historical correlation estimates are positive and on average less than those implied by option prices.

Our methods use a copula function to provide an empirical approximation to the
function defined by the bivariate RND of the dollar-rates divided by the product of the marginal RNDs. The Gaussian and Frank copulas provide satisfactory results, when we compare the cross-rate RNDs obtained from one cross-rate option price and many dollar-rate option prices with the densities that can be estimated from seven cross-rate option prices quoted by a particular bank. These comparisons show that the cross-rate RND can be estimated equally well from market-traded and over-the-counter dollar-rate option prices.
Appendix

To simplify equation (4) into equation (5) it is necessary to simplify the function

\[ h(x) = \frac{\partial^2}{\partial x^2} \left[ \int_0^\infty \int_0^\infty \max(y - xz, 0) f_s(y, z) dy dz \right]. \]

First, let

\[ g(x) = \int_0^\infty \int_0^\infty \max(y - xz, 0) f_s(y, z) dy dz = \int_0^\infty \left[ \int_0^{\max(y - xz, 0)} f_s(y, z) dy \right] dz \]

and replace \( y \) by \( tz \). Then \( dy = z dt \) and \( g(x) \) becomes

\[ g(x) = \int_0^\infty z^2 \left[ \int_x^{\infty} (t - x) f_s(tz, z) dt \right] dz. \]

Second, differentiate the inner integral twice, obtaining

\[ \frac{\partial}{\partial x} \int_x^\infty (t - x) f_s(tz, z) dt = -\int_x^\infty f_s(tz, z) dt \]

and

\[ \frac{\partial^2}{\partial x^2} \int_x^\infty (t - x) f_s(tz, z) dt = f_s(xz, z). \]

Consequently, \( h(x) \) simplifies to

\[ h(x) = \int_0^\infty z^2 f_s(xz, z) dz. \]
References


Table 1: Summary statistics for euro/pound RNDs obtained from over-the-counter option prices, including one cross-rate option price

Average values of summary statistics for euro/pound risk-neutral densities, for various distributional assumptions and dependence functions. Either lognormal mixtures or GB2 densities define the marginal dollar-rate and the vanilla cross-rate densities, all obtained from OTC option prices. The Gaussian, Frank, Plackett, and Clayton copula functions are used when defining the bivariate dollar-rate densities. The dependence parameters of the copulas are estimated by equating the theoretical at-the-money cross-rate option price with the market price.

| Panel A: Averages for 1-month RNDs, with lognormal mixtures |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| | Vanilla | Gaussian | Frank | Plackett | Clayton |
| N | 12 | 12 | 12 | 12 | 12 |
| Rho/Theta | 0.5592 | 4.2562 | 6.5222 | 1.2700 |
| Mean | 1.6399 | 1.6399 | 1.6399 | 1.6399 |
| Variance | 0.0029 | 0.0028 | 0.0030 | 0.0031 |
| Skewness | 0.1846 | 0.0903 | 0.1105 | 0.1296 |
| Kurtosis | 3.7993 | 3.1101 | 3.9153 | 4.2272 |
| K-S | 0.0155 | 0.0149 | 0.0200 | 0.0304 |
| G-call | 0.0474 | 0.0331 | 0.0398 | 0.0712 |
| G-implied | 0.0030 | 0.0025 | 0.0031 | 0.0055 |

| Panel B: Averages for 1-month RNDs, with GB2 densities |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| | Vanilla | Gaussian | Frank | Plackett | Clayton |
| N | 12 | 12 | 12 | 12 | 12 |
| Rho/Theta | 0.5609 | 4.2876 | 6.5994 | 1.2764 |
| Mean | 1.6399 | 1.6399 | 1.6399 | 1.6399 |
| Variance | 0.0030 | 0.0029 | 0.0031 | 0.0032 |
| Skewness | 0.2295 | 0.1089 | 0.1597 | 0.1824 |
| Kurtosis | 4.2845 | 3.7081 | 4.9407 | 5.2758 |
| K-S | 0.0140 | 0.0156 | 0.0213 | 0.0292 |
| G-call | 0.0436 | 0.0325 | 0.0411 | 0.0645 |
| G-implied | 0.0028 | 0.0026 | 0.0033 | 0.0053 |

| Panel C: Averages for 3-month RNDs, with lognormal mixtures |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| | Vanilla | Gaussian | Frank | Plackett | Clayton |
| N | 3 | 3 | 3 | 3 | 3 |
| Rho/Theta | 0.6569 | 5.4340 | 9.1488 | 1.7567 |
| Mean | 1.6207 | 1.6207 | 1.6207 | 1.6207 |
| Variance | 0.0083 | 0.0078 | 0.0086 | 0.0090 |
| Skewness | 0.2054 | 0.1832 | 0.2483 | 0.3003 |
| Kurtosis | 4.0336 | 3.5060 | 4.7762 | 5.3487 |
| K-S | 0.0122 | 0.0134 | 0.0211 | 0.0330 |
| G-call | 0.0349 | 0.0288 | 0.0549 | 0.0705 |
| G-implied | 0.0025 | 0.0019 | 0.0033 | 0.0057 |

| Panel D: Averages for 3-month RNDs, with GB2 densities |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| | Vanilla | Gaussian | Frank | Plackett | Clayton |
| N | 3 | 3 | 3 | 3 | 3 |
| Rho/Theta | 0.6580 | 5.4663 | 9.2415 | 1.7650 |
| Mean | 1.6207 | 1.6207 | 1.6207 | 1.6207 |
| Variance | 0.0083 | 0.0079 | 0.0086 | 0.0090 |
| Skewness | 0.2222 | 0.2042 | 0.2990 | 0.3438 |
| Kurtosis | 4.3982 | 3.8037 | 5.4548 | 5.9379 |
| K-S | 0.0120 | 0.0142 | 0.0219 | 0.0326 |
| G-call | 0.0316 | 0.0356 | 0.0615 | 0.0625 |
| G-implied | 0.0024 | 0.0022 | 0.0036 | 0.0054 |

*K-S: Kolmogrov-Smirnov statistic.
*G-call: Average absolute error of call price as a ratio of the vanilla price.
*G-implied: Average absolute error of implied volatility.
Table 2: Summary statistics for euro/pound RNDs obtained from CME option prices and one cross-rate OTC option price

Average values of summary statistics for euro/pound risk-neutral densities, for various distributional assumptions and dependence functions. Either lognormal mixtures or GB2 densities define the marginal dollar-rate obtained from CME option prices. The vanilla cross-rate densities are obtained from OTC option prices. The Gaussian, Frank, Plackett, and Clayton copula functions are used when defining the bivariate dollar-rate densities. The dependence parameters of the copulas are estimated by equating the theoretical at-the-money cross-rate option price with the market price.

Panel A: Averages for 1-month RNDs, with lognormal mixtures

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Panel B: Averages for 1-month RNDs, with GB2 densities

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Panel C: Averages for 3-month RNDs, with lognormal mixtures

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Panel D: Averages for 3-month RNDs, with GB2 densities

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*K-S: Kolmogorov-Smirnov statistic.
*G-call: Average absolute error of call price as a ratio of the vanilla price.
*G-implied: Average absolute error of implied volatility.
Table 3: Summary statistics for euro/pound RNDs obtained from over-the-counter option prices and the historical realized correlation coefficient

Average values of summary statistics for euro/pound risk-neutral densities, for various distributional assumptions and dependence functions. Either lognormal mixtures or GB2 densities define the marginal dollar-rate and the vanilla cross-rate densities, all obtained from OTC option prices. The Gaussian, Frank, Plackett, and Clayton copula functions are used when defining the bivariate dollar-rate densities. The dependence parameters of the copulas for one-month (three-month) maturities are estimated from the historical correlation for one month (three months) of thirty-minute dollar-rate returns.

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*K-S: Kolmogrov-Smirnov statistics.
*G-call: Average absolute error of call price as a ratio of the vanilla price.
*G-implied: Average absolute error of implied volatility.
Table 4: Summary statistics for euro/pound RNDs obtained from CME option prices and the historical realized correlation coefficient

Average values of summary statistics for euro/pound risk-neutral densities, for various distributional assumptions and dependence functions. Either lognormal mixtures or GB2 densities define the marginal dollar-rate obtained from CME option prices. The vanilla cross-rate densities are obtained from OTC option prices. The Gaussian, Frank, Plackett, and Clayton copula functions are used when defining the bivariate dollar-rate densities. The dependence parameters of the copulas for one-month (three-month) maturities are estimated from the historical correlation for one month (three months) of thirty-minute dollar-rate returns.

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*K-S: Kolmogorov-Smirnov statistics.
*G-call: Average absolute error of call price as a ratio of the vanilla price.
*G-implied: Average absolute error of implied volatility.
Figure 1: Bivariate copula densities

The contours and bivariate densities (PDFs) are for two random variables whose marginal densities are the standard normal density. Each bivariate density is the product of the two marginal densities and a copula function. Densities are shown for the Gaussian, Frank, Plackett and Clayton copulas when Spearman’s rho equals 0.5.
Figure 2: The empirical research structure

- **$/£ options**
  - **$/£ RND**
  - **$/€ RND**
  - **$/€ options**

- **Copulas**
  - **Bivariate RNDs $/£, $/€**

- **Dependence parameter**

- **Historical data**

- **ATM cross-rate option price**

- **All cross-rate option prices**

- **Univariate RND €/£**

**Steps**

- **Step 1**
  - **$/£ options**
  - **$/£ RND**
  - **$/€ RND**

- **Step 2**
  - **Copulas**
  - **Bivariate RNDs $/£, $/€**

- **Step 3**
  - **Univariate RND €/£**

**Variable transformation**
Typical RNDs for the euro/pound rate, respectively derived from CME and OTC dollar-rate option prices. Lognormal mixtures define the marginal dollar-rate and the vanilla cross-rate densities. The dependence parameters of the copulas are estimated by equating the theoretical at-the-money cross-rate option price with the market price.
Figure 4: Implied volatilities for the euro/pound rate from OTC option prices, when one cross-rate option price is known

Average implied volatilities, derived from OTC vanilla option prices and various copula functions. The parameters of the copulas are estimated by equating the theoretical at-the-money cross-rate option price with the market price.
Average implied volatilities, derived from CME option prices and various copula functions. The parameters of the copulas are estimated by equating the theoretical at-the-money cross-rate option price with the market price.
Figure 6: Implied volatilities for the euro/pound rate from OTC and CME option prices, when one cross-rate option price is known

Average implied volatilities, derived from OTC and CME option prices and various copula functions. The parameters of the copulas are estimated by equating the theoretical at-the-money cross-rate option price with the market price.

**Average 1-month Implied Volatility with Gaussian and Frank Copulas**

**Average 3-month Implied Volatility with Gaussian and Frank Copulas**
Figure 7: Time series of implied and realized correlation coefficients for the dollar/pound and dollar/euro rates

Each implied correlation is calculated from the one-month at-the-money implied volatilities of OTC options on the dollar/pound, dollar/euro and euro/pound rates. Each realized correlation is calculated from the previous month of thirty-minute returns, obtained from CME dollar/pound and dollar/euro futures prices.
Figure 8: Euro/pound RNDs when dependence is estimated from historical prices

Typical RNDs for the euro/pound rate, respectively derived from CME and OTC dollar-rate option prices. Lognormal mixtures define the marginal dollar-rate and the vanilla cross-rate densities. The dependence parameters of the copulas are estimated from the historical correlation for one month of thirty-minute dollar-rate returns.
Figure 9: Implied volatilities for the euro/pound rate, when dependence is estimated from historical prices

Average one-month implied volatilities, derived from OTC and CME option prices and various copula functions. The dependence parameters of the copulas are estimated from the historical correlation for one month of thirty-minute dollar-rate returns.