Accurate Approximation Formulas for Stock Options with Discrete Dividends

Tian-Shyr Dai
Dept. Information & Financial Management
National Chiao Tung University
Joint work with Yuh-Dauh Lyuu
Outline

• The Dividend Models for Pricing Stock Options and Their Problems
• Our Approximation Analytical Formulas
• Numerical results
The Dividend Models

• Lognormal diffusion model: \( \frac{dS}{S} = r \, dt + \sigma \, dW \)
  – Black-Scholes formula

• Continuous dividend yield model: 
  \( \frac{dS}{S} = (r - q) \, dt + \sigma \, dW \)
  – Merton (1973) extends Black-Scholes formula.
  – Almost all stocks pay discrete dividends.

• Discrete dividend setting:
  – Black (1975).
  – Pay known dividends in the future.
Models for Discrete Dividends (Frishling (2002))

1. Stock price is divided into two parts:
   - Present value of future dividends.
   - Net of dividend stock price: follows the lognormal diffusion process.

2. Cum-dividend stock price follows the lognormal process.
   - Stock price.
   - Forward values of the dividends paid from today up to maturity

3. Discontinuous process:
   - Jumps down at the exdividend date.
   - Follows lognormal price process between two exdividend dates.
Drawback of These Models

- These three models are inconsistent.
  - Ex: Model 1 generates lower price than Model 3.
  - The vol. for PV of future dividends is 0.

- Model 1 and 2 produce unreasonable prices
  - Ex: Model 1 could incorrectly render a down-and-out barrier option worthless.

- Model 3 is hard to evaluate
  - No analytical pricing formulas exist.
  - Numerical evaluations can be intractable.
Other Approximations for Discrete Dividends

The discrete dividend is approximated by

- Fixed discrete dividend yield.
  - $S=100$, $D=1 \rightarrow$ dividend yield $= 1/100 = 1\%$.

- Fixed continuous dividend yield.

These two approaches are efficient but inaccurate.
Our Analytical Approximation Formula

• Our goal: Find a dividend expression
  – that approximate the discrete dividend well.
  – that makes deriving analytical formula easy.

• The discrete dividend is approximated by a continuous dividend yield that can be represented as a function of
  – discrete dividends
  – stock returns.
Derive the Dividend Yield

\[ S(t_1) = S(0)e^{\mu t_1 + \sigma(B(t_1) - B(0))} - c_1 \equiv S(0)e^{(\mu - q_1) t_1 + \sigma(B(t_1) - B(0))} \]
\[ S(0)e^{\mu t_1 + \sigma(B(t_1) - B(0))}(1 - e^{-q_1 t_1}) = c_1 \]
\[ \Rightarrow 1 - e^{-q_1 t_1} = \frac{c_1 e^{-\mu t_1}}{S(0)}e^{-\sigma(B(t_1) - B(0))} \]

By Taylor series expansion:

\[ 1 - (1 - q_1 t_1) \approx k_1 \left(1 - \sigma(B(t_1) - B(0))\right) \]

to get \[ q_1 \approx \frac{k_1(1-\sigma(B(t_1)-B(0)))}{t_1}. \]
Approximation Analytical Formula

\[ S(T) \approx S'(0)e^{(\mu-k_1/T)T+k_1\sigma(B(t_1)-B(0))+\sigma(B(T)-B(0))} \]

follows lognormal process:

- Mean shift from \( \mu \) to \( \mu - k_1/T \).
- Vol. shift from \( \sigma \) to \( \sqrt{\frac{\text{Var}[k_1\sigma(B(t_1)-B(0))+\sigma(B(T)-B(0))]}{T}} \).

The option value can be solved by a Black-Scholes-like formula:

\[
e^{-rT}E((S(T) - X)^+)
= S(0)e^{aN(d_1)} - Xe^{-rT}N(d_2),
\]

where \( a = \frac{\sigma_1^2-\sigma^2}{2T} - k_1 \).
Improve Accuracy by Second-Order Taylor Expansion

Expand $k_1 e^{-\sigma (B(t_1) - B(0))}$ as
$$k_1 \left( 1 - \sigma [B(t_1) - B(0)] + \sigma^2 \frac{[B(t_1) - B(0)]^2}{2} \right),$$
we have:

$$q_1 \approx \frac{k_1 \left[ 1 - \sigma (B(t_1) - B(0)) + \sigma^2 \frac{(B(t_1) - B(0))^2}{2} \right]}{t_1}$$

$$\approx \frac{k_1 \left[ 1 - \sigma (B(t_1) - B(0)) \right]}{t_1} + \delta_1,$$

$S(T)$ can be expressed as

$$S'(T) \equiv S'(0) e^{(\mu - k_1/T)T} + \sigma (B(T) - B(0)) + k_1 \sigma (B(t_1) - B(0)) - \delta_1,$$

which follows lognormal distribution.
Two-Dividend Case

\[ S(t_1)e^{\mu t_2 + \sigma (B(t_1 + t_2) - B(t_1))} - c_2 \equiv S(t_1)e^{(\mu - q_2)t_2 + \sigma (B(t_1 + t_2) - B(t_1))} \]

\[ 1 - e^{-q_2 t_2} = \frac{c_2 e^{-\mu t_2}}{S(t_1)} e^{-\sigma (B(t_1 + t_2) - B(t_1))} \]

We derive that

\[ q_2 \approx \frac{k_2 [1 - (1 + k_1)\sigma (B(t_1) - B(0)) - \sigma (B(t_2) - B(t_1))] + \delta_2}{t_2} \]

\[ S(T) = \left( S(t_1)e^{\mu t_2 + \sigma (B(t_1 + t_2) - B(t_1))} - c_2 \right)e^{\mu (T - t_1 - t_2) + \sigma (B(T) - B(t_1 + t_2))} \]

\[ = S(t_1)e^{(\mu - q_2)t_2 + \sigma (B(t_1 + t_2) - B(t_1))} e^{\mu (T - t_1 - t_2) + \sigma (B(T) - B(t_1 + t_2))} \]
Multiple-Dividend Case

Derive the formula by mathematical induction.

\[
a_{i,j} = \begin{cases} 
0, & \text{if } i > j, \\
\sigma, & \text{if } i = j, \\
\sum_{h=1}^{j-1} a_{i,h} k_h + \sigma & \text{if } i < j,
\end{cases}
\]

\[
\delta_i = \frac{k_i \sum_{j=1}^{i} a_{j,i}^2 t_j}{2},
\]

\[
k_i = \frac{c_i e^{-\mu \sum_{j=1}^{i} t_i + \sum_{j=1}^{i-1} (k_j + \delta_j)}}{S'(0)},
\]

\[
\sigma_i = \sqrt{\frac{\sum_{j=1}^{i} a_{j,i+1}^2 t_j + a_{i+1,i+1}^2 (T - \sum_{h=1}^{i} t_h)}{T}}.
\]
The pricing formula of a vanilla call option is then

$$S'(0)e^{\frac{\sigma_n^2 - \sigma^2}{2}T - \sum_{i=1}^n (k_i + \delta_i)}N(d''_1) - Xe^{-rT}N(d''_2),$$

where $d''_1 = \frac{\ln \frac{S'(0)}{X} + (\mu + \sigma_n^2)T - \sum_{i=1}^n (k_i + \delta_i)}{\sigma_n \sqrt{T}}$ and

$d''_2 = d''_1 - \sigma_n \sqrt{T}$. 
## Pricing a Call Option with One Discrete Dividend

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<th>X</th>
<th>FDY</th>
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<th>Our</th>
<th>Model 3</th>
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<td>*16.336</td>
<td>17.090</td>
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