Bounds and Prices of Currency Cross-Rate Options

San-Lin Chung* and Yaw-Huei Wang**

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This paper derives pricing bounds of a currency cross-rate option using the option prices of two related dollar rates via a copula theory and presents the analytical properties of the bounds under the Gaussian framework. Our option pricing bounds are very general and do not rely on the distribution assumptions of the state variables or on the selection of the copula function. The empirical results suggest that there are persistent and stable relationships between the market prices and the estimated bounds of the cross-rate options and that our option pricing bounds (obtained from the market prices of options on two dollar rates) and the historical correlation of two dollar rates are highly informative for determining the prices of the cross-rate options. In addition, all analytical properties of the bounds are verified empirically. Given the estimated pricing bounds in terms of implied volatility, we show how to generate the delta bounds.

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* Department of Finance, National Taiwan University, 85, Section 4, Roosevelt Road, Taipei 106, Taiwan, R.O.C.. Tel: 886-2-3366-1084. Email: chungs@management.ntu.edu.tw.

** Corresponding author. Department of Finance, National Central University, 300, Chungda Road, Chungli 320, Taiwan, R.O.C.. Tel: 886-3-4227151 ext. 66255. Email: yhwang@mgt.ncu.edu.tw.

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1. Introduction

In the option pricing literature, researchers are not only interested in pricing but also interested in bounding the option values. There are many useful techniques that can be employed to derive option pricing bounds. For example, Merton (1973), Garman (1976), Levy (1985), and Grundy (1991) use the arbitrage-free approach to derive option pricing bounds. The fundamental idea behind this approach is that it is not possible to formulate a dominant portfolio using the underlying stock, the risk-free bond, and the options if the market is absent of arbitrage opportunities. Ritchken (1985), Ritchken and Kuo (1989), Basso and Pianco (1997), Mathur and Ritchken (2000), and Ryan (2003) use the linear programming methods to derive option pricing bounds. These studies model option pricing bounds as a linear programming problem with a discrete state space, which involves complicated calculations. In addition to the above two types of techniques, some other approaches, such as the optimization methods and the restrictions on the volatility of the pricing kernel, have also been used in the literature.

Most if not all of the previous studies derive option pricing bounds by directly using the price information (such as the price distribution or price process) of the underlying asset. In contrast to the previous literature, this study uses the option prices of the related dollar rates to derive the pricing bounds for the cross-rate option. In other words, we bound cross-rate option values using the market prices of the dollar-rate options.¹ From this sense, the idea of this paper is close to that in the static hedge literature where the exotic options are priced (and hedged) in terms of the prices of standard options.²

Since there is a triangular relationship between the foreign exchange rates among three currencies, Taylor and Wang (2005) show that it is plausible to estimate risk-neutral densities

¹ The motivation for doing this is as follows. It is generally observed that options on dollar-denominated exchange rates are traded under satisfactory liquidity, while cross-rate option markets are much less liquid. Thus, the pricing bounds obtained from the liquid market prices of dollar-rate options are useful for pricing, hedging, and arbitraging.
² See Carr, Ellis, and Gupta (1998) for an example of static hedge.
and option prices of a cross-rate under the correct numeraire\(^3\) using the market prices of two related dollar-rate options.\(^4\) Instead of directly exploring the option pricing formula, this paper provides pricing bounds for the cross-rate options. Compared with previous studies, our objective is not to derive very tight bounds, but to generate informative bounds from useful market information (dollar-rate option prices). Although our pricing bounds are not very tight, they are very general, do not rely on the distribution assumptions of the state variables, and are never violated. Moreover, our pricing bounds have economic meanings because they are portfolios composed of the dollar-rate options (and sometimes also composed of spot dollar rates) and are highly informative to the prices of the cross-rate options.

Since a cross-rate option under the dollar measure is equivalent to an option that allows the buyer to exchange one currency for the other currency, this study (using the risk-neutral pricing approach) derives the price bounds for cross-rate options by utilizing the exchange option price bounds implied in the copula theory.\(^5\) Nonetheless, the bounds do not rely on the selection of the copula function. Some analytical properties of the bounds under the Gaussian framework are also presented. Given the estimated pricing bounds in terms of implied volatility, we follow Bergman, Grundy and Wiener (1996) to derive the bounds on the delta of the cross-rate option.

Using the prices of options on foreign exchange rates among the US dollar, euro, and pound sterling, we empirically test the relationship between the market prices and the estimated bounds of the cross-rate options and verify their analytical properties. Our results suggest that there are strong and stable relationships between the market prices of cross-rate options and option prices of a cross-rate under the correct numeraire using the market prices of two related dollar-rate options. Instead of directly exploring the option pricing formula, this paper provides pricing bounds for the cross-rate options. Compared with previous studies, our objective is not to derive very tight bounds, but to generate informative bounds from useful market information (dollar-rate option prices). Although our pricing bounds are not very tight, they are very general, do not rely on the distribution assumptions of the state variables, and are never violated. Moreover, our pricing bounds have economic meanings because they are portfolios composed of the dollar-rate options (and sometimes also composed of spot dollar rates) and are highly informative to the prices of the cross-rate options.

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Using the prices of options on foreign exchange rates among the US dollar, euro, and pound sterling, we empirically test the relationship between the market prices and the estimated bounds of the cross-rate options and verify their analytical properties. Our results suggest that there are strong and stable relationships between the market prices of cross-rate options.

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\(^3\) In order to utilize the concept of the exchange option, both Taylor and Wang (2005) and this paper commence the analyses with the foreign measure of a country. As implied by Bakshi, Carr and Wu (2005), it is better to specify the generic pricing kernels in each country to derive the prices of derivatives. Nonetheless, Taylor and Wang (2005) analytically show that the prices of derivatives under different measures are equivalent when the law of one price holds.

\(^4\) The implementation of the formula for cross-rate RNDs and prices in Taylor and Wang (2005) relies on the selection of a copula function.

\(^5\) The details of the copula theory can be found in Joe (1997) and Nelsen (1999). Cherubini, Luciano, and Vechiato (2004) first applied the copula theory to derive the pricing bounds for the exchange options.
tions and the pricing bounds obtained from the market prices of options on the two dollar rates. These strong and stable relationships do not depend on deltas.

Because the correlation between two risky assets of an exchange option affects the pricing of this option, we expect that the correlation between two dollar rates, whose information is not included in our bounds, has the explanatory power for the market prices of the corresponding cross-rate options. Our empirical results show that the explanatory power increases substantially when the historical correlation is added into the regression model. In general, the pricing bounds estimated from the option prices and the correlation of two dollar rates can provide about 85% of information for determining the prices of the cross-rate options.

We also infer the cross-rate option prices from our pricing bounds and the historical correlation using a regression model. The inferred prices are very close to the market prices of the cross-rate options across deltas and the volatility of inference errors (about 0.3%) is also very small (about 0.3%). Therefore, in comparison to the traditional option pricing methods which use the underlying asset price information only, this study provides an effective approach to infer the prices of cross-rate options.

Finally, given the estimated pricing bounds in terms of implied volatility, we empirically show how to bound the cross-rate option’s delta.

The remainder of this paper is organized as follows. Section 2 derives option pricing bounds for the cross-rate option, presents the analytical properties of the bounds under the Gaussian framework, and shows how to bound the delta of the cross-rate option provided the estimated pricing bounds. Data and the empirical methodologies for generating the risk-neutral densities and option pricing bounds are presented in Section 3. Section 4 discusses the empirical results while Section 5 concludes the paper.

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6 Our results are in line with the analysis of Driessen, Maenhout, and Vilkov (2005) who find that the risk of changes in equity correlations is priced, using data on S&P 100 options and options on all the stocks in the index.
2. Pricing Bounds of the Cross-Rate Options

By applying the Fréchet bounds in the copula theory, Cherubini et al. (2004) show that the super-replication bounds of the option to exchange one asset for the other asset are composed of the prices of the univariate options on the two individual exchanged assets. We first show that the payoff of a cross-rate option under the dollar risk-neutral measure is equivalent to that of an exchange option where the two risky assets are the corresponding dollar rates. Following the same logic, we then derive the pricing bounds for the cross-rate option, which are composed of the dollar-rate options (and sometimes also composed of spot dollar-rates).

Consider options whose payoffs depend on the exchange rates among the following three currencies: US dollars (\$, USD), British pounds (£, GBP), and euros (€, EUR). We denote the dollar price of one pound at time \( t \) by \( S_t^{S/£} \) and likewise the dollar price of one euro at the same time is denoted by \( S_t^{S/€} \). The cross-rate price of one pound in euros is then given by
\[
\frac{S_t^{S/£}}{S_t^{S/€}} = \frac{S_t^{S/£}}{S_t^{S/€}}
\]
under the no-arbitrage argument.

Now consider a European call option where the holder has the right to buy £1 for €\( K \) at time \( T \). Under the dollar measure (or from the viewpoint of U.S. residents), the above option is identical to an option to exchange \( KS_T^{S/£} \) dollars for \( S_T^{S/£} \) dollars at time \( T \). Hence, a cross-rate call option under the dollar numeraire is equivalent to an option to exchange one asset for the other asset and its dollar payoff equals \( \max(S_T^{S/£} - KS_T^{S/£}, 0) \). This payoff can be re-arranged as the following:
\[
S_T^{S/£} - \max[\min(S_T^{S/£}, KS_T^{S/£}), 0].
\]

Hence, the current price of an exchange option is determined by the risk-neutral pricing approach as follows:
\[
Call_t^{£/€} = S_t^{£/€} e^{-\gamma_t(T-t)} - \text{Call}_{\min}(S_t^{S/£}, KS_T^{S/£}, 0, t, T),
\]
(2)
where \( \text{Call}_{\min}(S_1, S_2, 0, t, T) \) represents the price at time \( t \) of a call option on the minimum of \( S_1 \) and \( S_2 \) with strike price 0 and maturity time \( T \). Applying the Fréchet bounds in the copula theory, we are able to derive the upper (lower) bound of the minimum call option price and thus the lower (upper) bound of the cross-rate option price as follows.

**Proposition 1.** The upper bound of the cross-rate option price is as follows:

\[
\text{Call}^{\text{E/E}+}_{S} = \text{Call}(S^{\text{S/E}}, K^{**}, t, T) + K \text{Put}(S^{\text{S/E}}, K^{*}, t, T),
\]

where \( K^{**} \) is a constant satisfying that \( \text{FS}_{S/E}(K^{**}) + K \text{FS}_{S/E}(K^{**}) = 1 \), \( \text{F}^+_i(x) = 1 - F^+_i(x), F^-_i(x) \) is the cumulative distribution function, and \( K^{**} = K^{**}/K \). Let \( K^* \) be a constant which solves \( \text{FS}_{S/E}(K^*) = \text{FS}_{K^{**}E}(K^*) \). Then the lower bound of the cross-rate option price is as follows:

\[
\text{Call}^{\text{E/E}-}_{S} = \begin{cases} 
\text{Call}(S^{\text{S/E}}, K^*, t, T) - K \text{Call}(S^{\text{S/E}}, K^*, t, T) & \text{if } \text{FS}_{S/E}(u) < \text{FS}_{K^{**}E}(u) \\
\text{S}_t^S e^{-r_{1-t}} - K \text{S}_t^S e^{-r_{1-t}} & \text{for } u < K^* \\
K \text{Call}(S^{\text{S/E}}, K^*, t, T) - \text{Call}(S^{\text{S/E}}, K^*, t, T) & \text{otherwise.}
\end{cases}
\]

where \( K^* = K^*/K \).

**Proof:** Please see Appendix A.

From equations (3) and (4), we observe that our pricing bounds for cross-rate options are portfolios of the corresponding dollar-rate options (and may be also of the spot assets). Therefore, different from most option pricing bounds in the literature, the derived pricing bounds have economic meanings. Moreover, the derivation of our cross-rate option pricing bounds does not rely on the distribution assumption of two dollar rates and the selection of an appropriate copula function. Therefore, one can apply the technique utilized here to derive the price bounds for any European-style derivatives whose payoffs can be rearranged as the same type as that of an exchange option.

**2.1. Analytical Properties of the Pricing Bounds under the Gaussian Framework**

When the two dollar rates follow a bivariate lognormal distribution, under the triangular arbitrage relation the cross-rate also follows a lognormal distribution. Thus there exist closed-
from solutions for the option prices of two dollar rates and our pricing bounds. To have some insights on our pricing bounds, we investigate the analytical properties of these bounds when the two dollar rates follow a bivariate lognormal distribution.\textsuperscript{7}

Denote the closed-form solutions of two dollar-rate option prices at time $t$, denominated in US dollars, as $C_{BS}(S_t^{£/£}, K, r_s, r_s, \sigma_{S/£}, \tau)$ and $C_{BS}(S_t^{£/£}, K, r_s, r_s, \sigma_{S/£}, \tau)$, respectively. Proposition 2 shows that the upper and lower bounds have closed-form solutions under the bivariate lognormal distribution assumption.

\textbf{Proposition 2.} Assume that two dollar rates $S^{£/£}$ and $S^{£/€}$ follow bivariate lognormal distribution with a correlation coefficient of $\rho$ and volatilities per year of $\sigma_{S/£}$ and $\sigma_{S/€}$, respectively. Then the upper and lower bound, denominated in euros, of the cross-rate call option with a strike price of $K$ have closed-form solutions of $C_{BS}(S_t^{£/€}, K, r_e, r_e, \sigma_{S/£} + \sigma_{S/€}, \tau)$ and $C_{BS}(S_t^{£/€}, K, r_e, r_e, |\sigma_{S/£} - \sigma_{S/€}|, \tau)$, respectively.

\textbf{Proof:} Please see Appendix B.

Under the bivariate lognormal distribution assumption, the triangular arbitrage relation implies that the cross-rate option price is $C_{BS}(S_t^{£/€}, K, r_e, r_e, \sigma_{S/£}, \tau)$, where $\sigma_{S/£}^2 = \sigma_{S/£}^2 + \sigma_{S/€}^2 - 2\rho\sigma_{S/£}\sigma_{S/€}$. Therefore Proposition 2 is intuitively true because $|\sigma_{S/£} - \sigma_{S/€}| \leq \sigma_{S/£} \leq \sigma_{S/£} + \sigma_{S/€}$. When the correlation between two dollar rates is higher (lower), the cross-rate option price is closer to the lower (upper) bound. Moreover, Proposition 2 implies that when the implied volatility curves of two dollar-rate option prices are flat, the implied volatility curves of our upper and lower bounds for cross-rate options are also flat.

\subsection*{2.2. Bounds on the Delta of Cross-rate Options}

Given the estimated pricing bounds in terms of implied volatilities, it is plausible to derive bounds on the cross-rate option’s delta using the Proposition 5 of Bergman et al. (1996).\textsuperscript{8} As-

\textsuperscript{7}We thank the referee for the suggestion of showing the relevance of the bounds analytically when the two dollar rates follow a bivariate lognormal distribution.
sume that the volatility function, $\sigma(s, t)$, be a function of the underlying asset price $s$ and time $t$ only. Let $\underline{\sigma}$ and $\overline{\sigma}$ respectively denote the lower and upper bounds on volatility, $c(s, t)$ and $c_1(s, t)$ respectively represent the market (or accurate) call price and its delta, and $c_{bs}(s, t)$ and $c_{1bs}(s, t)$ respectively stand for the Black-Scholes call price and its delta. Bergman et al. (1996) derived bounds on the option’s delta as follows.

**Proposition 5 of Bergman et al. (1996).** If for all $s$ and $t$, $\underline{\sigma}(t) \leq \sigma(s, t) \leq \overline{\sigma}(t)$, then

$$
c_{1bs}(s^*, t) \leq c_1(s, t) \leq c_{1bs}(s', t) \quad \text{where } s^* \text{ solves } c_{bs}(s, t) = c_{bs}(s, t) - c_{1bs}(s', t)(s' - s).$$

The delta bounds of Bergman et al. (1996) are true for general Markovian diffusion processes. When the cross-rate option’s value today is known, Bergman et al. (1996) show that the bounds on its delta can be strengthened as follows.

**Proposition 6 of Bergman et al. (1996).** If for all $s$ and $t$, $\sigma(s, t) \leq \overline{\sigma}(t)$, then for any $s$ and $t$ such that one knows $c(s, t)$, $c_{1bs}(s^*, t) \leq c_1(s, t) \leq c_{1bs}(s', t) \quad \text{where } s^* \text{ solves}$

$$
c(s, t) = c_{bs}(s, t) + c_{1bs}(s^*, t)(s - s^*) \quad \text{and } s' \text{ solves } c(s, t) = c_{bs}(s', t) - c_{1bs}(s', t)(s' - s).$$

Our pricing bounds are directly applicable to the Proposition 5 of Bergman et al. (1996) and thus the bounds on the deltas can be obtained straightforward. For instance, the implied volatility of our lower (upper) bound provides an estimate of $\sigma(t)$ ($\overline{\sigma}(t)$) for applying Proposition 5 of Bergman et al. (1996). When the market price of the cross-rate option is known, our upper bound can be used with to the Proposition 6 of Bergman et al. (1996) to provide tighter bounds for deltas. Later we will show the tightness of bounds on the deltas when our pricing bounds are applied to the Propositions 5 and 6 of Bergman et al. (1996).

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*We thank the referee for providing the reference of Bergman et al. (1996) and the suggestion of developing bounds on the deltas.*
3. Data and Empirical Methodologies

3.1. Data

The primary data used in this article are daily option prices quoted as Black-Scholes implied volatilities for three currency options ($/£, $/€, and €/£). We make use of a confidential file of OTC option price mid-quotes, supplied by the trading desk of an investment bank in London. Our currency option data cover the period from 15 March 1999 to 11 January 2001. The OTC quotes are for all three foreign exchange options, recorded at the end of the day in London. The data include option prices for seven exercise prices, based upon “deltas” equal to 0.1, 0.25, 0.37, 0.5, 0.63, 0.75, and 0.9. The maturity of the options is one month, with which options in the OTC market are most frequently traded. We also use the spot exchange rates of $/£, $/€, and €/£ and the euro-currency interest rates (proxies of risk-free rates) of $, £, and € recorded by DataStream as the inputs of all relevant calculations.

The summary statistics of the quoted implied volatilities are shown in Table 1. All implied volatility functions exhibit a smile shape with the level for the $/€ options being the highest while the level for the $/£ options being the lowest. The low standard deviations imply that the levels of implied volatilities for these three exchange rate options do not change much as time goes. The skewness is positive and the kurtosis is close to 3, which does not depend on deltas.

3.2. Empirical Methodologies for Generating the Bounds

Because $K^*$ and $K^{**}$ are determined by the risk-neutral densities of two dollar rates, we use the observed market prices of European call options on $/£ and $/€ and a parametric distribution specification to estimate their risk-neutral densities. Once the risk-neutral densities are obtained, $K^*$ and $K^{**}$ can be easily calculated with a numerical method (such as the Newton-

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9 Some settlement prices are available for cross-rate options traded in the Chicago Mercantile Exchange, but they correspond to almost no trading volume. Consequently, we rely on over-the-counter (OTC) option prices, with which we have the same time-to-maturity option data every day. Such prices are not in the public domain to the best of our knowledge.
Raphson method) to solve \( \bar{F}_{S^S/\varepsilon} (K^*) = \bar{F}_{KS^S/\varepsilon} (K^*) \) and \( \bar{F}_{S^S/\varepsilon} (K^{**}) + \bar{F}_{KS^S/\varepsilon} (K^{**}) = 1 \), respectively. Then, we are able to price dollar-rate options with all strikes and to get pricing bounds of the cross-rate options using equations (3) and (4).

In this paper we use the generalized beta density of the second kind (GB2) to estimate the RNDs of two dollar rates.\(^\text{10}\) The GB2 density has few parameters, but preserves many desirable properties: general levels of skewness and kurtosis are allowed, the shapes of the tails are fat relative to the lognormal density, and there are analytic formulae for the density, its moments, and the prices of options. Furthermore, the parameter estimation of the GB2 density is easy and the estimated densities are never negative. The details of the estimation of the GB2 density can be found in Bookstaber and MacDonald (1987).

4. Empirical Results

The empirical studies in this article contain five parts. We first analyze the properties of our pricing bounds and their relationships with the market prices of the cross-rate options. Next, we investigate the explanatory powers of the pricing bounds and the correlation between two dollar rates for the market prices of the cross-rate options. Then, this study examines the accuracy of our empirical models for inferring the prices of the cross-rate options. Moreover, some robust analyses for the accuracy of our results are provided. Finally, given the estimated price bounds of the cross-rate options, we demonstrate how to bound their deltas using the approach proposed by Bergman et al. (1996).

\(^{10}\) Many types of univariate RNDs have been proposed, including lognormal mixtures (Ritchey (1990) and Melick and Thomas (1997)), generalized beta densities (Bookstaber and MacDonald (1987)), multi-parameter discrete distributions (Jackwerth and Rubinstein (1996)), and densities derived from fitting spline functions to implied volatilities (Bliss and Panigirtzoglou (2002)). Providing that options are traded for a range of exercise prices that encompass most area of the risk-neutral distribution, it is documented that several flexible density families provide similar empirical estimates. The bounds estimated with the lognormal mixtures RNDs will be compared with for the robustness check.
4.1. Empirical Pricing Bounds of the Cross-Rate Options

In order to have a standardized comparison, all the market prices and pricing bounds are converted to the Black-Scholes implied volatilities. As suggested by analysis in Appendix B, all lower bounds are determined by the second alternative of equation (4) because the implied volatilities of $/€ are always larger than those of $/£ during our sample period. The descriptive statistics of the estimated pricing bounds and the market implied volatilities across deltas are shown in Table 2. The processes of the estimated pricing bounds and the market implied volatilities for deltas 0.9, 0.5, and 0.1 are shown in Figure 1.

As shown in Figure 1, the market implied volatility always lies within the estimated bounds and the evolution of the market implied volatility of the cross-rate (€/£) option exhibits a similar pattern to those of the estimated bounds across deltas. As the foreign exchange market became more volatile from 1999 to 2000, the bound range, defined as the difference between the upper bound and the lower bound, turned wider as time went by during the period.

Table 2 as well as Figure 1 suggests that the level, the mean, and the volatility of the upper bounds are almost the same across deltas with an extremely shallow smile. In contrast, the lower bound and the market implied volatilities exhibit clearer smile shapes across deltas with the lower bound smile being deeper than the market implied smile.

To deeply explore the relationships between the option market prices and the estimated bounds, we further look at the behavior of the difference between the upper bound and the market implied (upper range) and the difference between the lower bound and the market implied (lower range). Their descriptive statistics are illustrated in Table 3. Both of the level and variation of the upper ranges are larger than those of the lower ranges across deltas. We explore the relationships between the ranges and the correlation of two dollar-rates as Proposition 2 suggests that the higher the correlation is, the closer the market implied volatility is to
the lower bound. We regress the upper range and the lower range respectively on the correlation and report the slope coefficient estimates in Table 3. The results clearly indicate that the upper (lower) range is significantly positively (negatively) associated with the correlation of two dollar rates across deltas; i.e. the higher the correlation is, the closer the market implied is to the lower bound. This finding is consistent with our theoretical analysis.

In summary, the lower bounds exhibit a slight smile shape while the upper bounds and the market implied volatilities are relatively flat across deltas. Both the upper and lower bounds exhibit tractable and persistent relationships with the market prices of cross-rate options.

4.2. Pricing Bounds, Correlation, and the Cross-Rate Option Prices

Since there are persistent relationships between the market prices of the cross-rate options and the pricing bounds estimated from the option prices of two corresponding dollar rates, we further use a regression model to measure the extent where the cross-rate option prices can be explained by our pricing bounds. We regress the market implied volatilities on the upper and lower bounds. The regression model is specified as follow:

\[ MIV_t = c + \beta_1 UB_t + \beta_2 LB_t + \epsilon_t, \]  

where \( MIV_t \), \( UB_t \), and \( LB_t \) respectively denote the market implied volatility of the cross-rate option on €/£, the upper bound, and the lower bound at day \( t \), and \( \epsilon_t \) is the residual term. The estimates for this model are shown in Panel 1 of Table 4.

From Panel 1 of Table 4, we find highly significant regression coefficients (\( \beta_1 \) and \( \beta_2 \)). The adjusted \( R^2 \)'s are very high and range from 0.72 to 0.77 across deltas. It is noticeable that

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The correlation coefficients are estimated using the dynamic conditional correlation (DCC) multivariate GARCH model proposed by Engle (2002) using the historical time series data of two dollar spot rates. The fact that correlations between financial assets are usually time-varying has crucial important implication in many ways such as portfolio hedging and multivariate asset pricing. This model overcomes the complexity of conventional multivariate GARCH models in computation by directly modeling the time-varying correlation as a conditional process. The procedure of using the DCC GARCH model to generate the time-varying correlation series is detailed in Engle (2002).
$\beta_1$ is adhered to a small range (between 0.33 and 0.39) while $\beta_2$ ranges from 0.26 to 0.59. In other words, the upper bound contains almost same level of information content for the cross-rate options across delta, while the lower bound contains different levels of information content across deltas. In conclusion, we confirm that there are strong and stable relationships between the market prices of cross-rate options and the pricing bounds estimated from the market prices of the options on two dollar rates.

By analyzing the relationship between the prices of stock index options and the prices of individual stock options included in the index, Driessen et al. (2005) show the relevance of correlation risk and the associated premium for stock index options pricing.\(^{12}\) Moreover, according to compositions of the bounds in equations (3) and (4), we find that no correlation information is used in the calculation of the pricing bounds of the cross-rate options, for which we only utilize the price information of the options on two dollar rates individually. As a result, this paper includes an extra explanatory variable, the historical correlation of two dollar rates, into Model 1 to see whether the correlation is able to provide any additional explanatory power. Thus, the regression model is modified as the following:

$$Model 2: \text{MIV}_t = c + \beta_1 \text{UB}_t + \beta_2 \text{LB}_t + \beta_3 \text{Corr}_t + \varepsilon_t, \quad (6)$$

where $\text{Corr}_t$ is the correlation coefficients of two dollar rates at day $t$.\(^{13}\) When the correlation of two dollar rates increases, the variance of the cross rate decreases and thus the cross-rate option price also decreases. Therefore, the regression coefficient of the historical correlation ($\beta_3$) is expected to be negative.

The regression results for Model 2 are shown in Panel 2 of Table 4. It is clearly seen that the correlations of two dollar rates provide incremental information in deciding cross-rate option prices as all adjusted $R^2$'s increase by about 10%. The regression coefficients for the

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\(^{12}\) In the same vein, a currency cross-rate can be regarded as an equal weight index of two dollar rates since its log return can be decomposed into the sum of the log returns of the two dollar rates.

\(^{13}\) As before, the correlation coefficients are estimated using the DCC model of Engle (2002).
correlation across deltas are significantly negative and consistent with our expectation. Furthermore, our results are in line with the analyses and findings of Driessen et al.(2005).

In summary, the pricing bounds estimated from option prices of two dollar rates and the correlation of two dollar rates can provide very highly proportional information for determining the cross-rate option prices across deltas.

4.3. Inferring the Prices of Cross-Rate Options

Owing to the significant explanatory power of the estimated bounds and correlation to the market prices of cross-rate options, we are interested in the accuracy of our empirical models for inferring the prices of cross-rate options. Given the estimated parameters of the previous model (Model 2), we infer the current implied volatility for the cross-rate (€/£) options from the current market prices of the dollar-rate options and the historical correlation. The descriptive statistics of the estimation errors, defined as the absolute values of the actual values minus the inferred values, across deltas are shown in Table 5. The actual and inferred implied volatilities of the cross-rate options for deltas 0.9, 0.5, and 0.1 are shown in Figure 2.

From Table 5 we observe that the inference errors from Model 2 are very small. The average errors across deltas are about 0.3%, which is much lower than the bid-ask spread in the OTC market. In addition, the volatilities of the errors are very small as well (about 0.3%), implying that the inference performs consistently well across time.

The results are valuable since this inference provides an effective way to determine prices of cross-rate options when we only have the real-time information of dollar-rate spot and option prices. In other words, the inferred prices (or portfolios) are applicable to practical usage not only for price determination, but also for hedging, particularly when the real-time cross-rate option prices are unobservable.

4.4. Robustness Analysis
To investigate whether our results are robust, we first check whether the estimated bounds rely on the assumption of the RND distribution for the dollar rates, and then analyze whether the accuracy of the inference of cross-rate option prices from Model 2 is sensitive to sample selection, and the implied volatility, the implied skewness, and the implied kurtosis levels.

To check whether the estimated bounds depend on the assumption of the dollar-rate RND distribution, we re-assume the lognormal mixtures distribution for the RNDs of two dollar rates and compare the bounds estimated under this assumption with those under the GB2 assumption. As shown in Table 6, the differences between the bounds estimated with these two different RND assumptions are statistically insignificant across deltas at the 10% significance level although the differences of the lower bounds are larger than those of the upper bounds.

To check whether sample selection affects our findings, we redo the option price inference for two evenly divided sub-samples. The average inference errors across deltas are shown in Panel 1 of Table 7. Although the inference errors are slightly higher in the second sub-periods, the patterns across deltas are basically the same. In other words, our finding does not depend on the sample selection.

As the volatility of exchange rates increases over our sample period, it is natural to check whether the increasing volatility changes the accuracy of information provided by our pricing bounds. To further check whether the price inference error depends on volatility, we run the following regression model.

\[
E_t = c + \alpha E_{t-1} + \beta Vol_t, \tag{7}
\]

where \(E_t\) and \(Vol_t\) respectively denote the percentage pricing error and the volatility level estimated using the approach of Bakshi, Kapadia, and Madan (2003) at time \(t\). The AR(1) specification is motivated by the high first-order autocorrelation of inference errors. The estimates are reported in Panel 2 of Table 7. All \(\beta\) coefficients are insignificant under the 10% signifi-
cance level. In summary, the accuracy of information provided by our pricing bounds is immune to the volatility change. Other proxies of the market volatility level, such as the implied volatilities for different moneyness levels are also used and the results (not reported here) are almost unchanged.

As discussed before, the average implied volatilities of all exchange rates exhibit a smile shape. However the slopes of the implied volatility curves vary from negatively sloped to positively sloped during our sample period. This implies that risk neutral skewness and kurtosis change substantially everyday. In order to investigate the impact of changes in implied volatility curves on our results, we first calculate the implied skewness and kurtosis using the Theorem 1 of Bakshi et al. (2003). We then run the regression model (7) with the implied volatility being replaced by the implied skewness and kurtosis, respectively.

Figure 3 indicates that the risk neutral distributions of the cross-rates are fat-tailed (average kurtosis equals 3.31) and slightly negatively skewed (average skewness equals -0.13).\textsuperscript{14} Figure 3 also shows that the implied skewness changes noticeably over time. Nevertheless, Panel 3 of Table 7 suggests that there is no clear evidence supporting that the price inference errors across deltas are affected even though implied skewness changes much. Similarly, Panel 4 of Table 7 shows that the implied kurtosis has little impact on the inference errors across deltas.

In conclusion, our results are robust across distribution assumptions of dollar-rate RNDs, sample selections, and different levels and slopes of implied volatility curves.

4.5. Bounds on Delta of Cross-rate Options

Given the estimated pricing bounds in terms of implied volatilities, it is plausible to bound the cross-rate option’s delta with Propositions 5 or 6 of Bergman et al. (1996) respectively when the call price of the cross-rate is known or not.

\textsuperscript{14} The risk neutral distributions of two dollar rates also exhibit the same pattern, i.e. fat-tailed and slightly negatively skewed.
We take the cross-rate call option contracts quoted on June 29, 1999 as examples and show the delta bounds for the ATM call option in Figure 4.\textsuperscript{15} The solid lines in the descending order are the Black-Scholes cross-rate call prices respectively with the upper bound, the market value, and the lower bound of volatility across asset prices. When the cross-rate call price is unknown, its delta is bounded between 0.0261 and 0.9704 (dashed lines). When the call price is known, the bound range becomes smaller and ranges from 0.1248 to 0.8775 (dotted lines).\textsuperscript{16}

5. Concluding Remarks

Instead of pricing cross-rate options directly, this study relates the option pricing bounds to the prices of the corresponding dollar-rate options. Our pricing bounds are derived from a general result of the copula theory and thus do not rely on the distribution assumptions of state variables. Different from most option pricing bounds in the literature, our cross-rate option bounds are functions of the option prices (and sometimes also the spot prices) of two dollar rates.

Using the prices of options on foreign exchange rates among US dollar, euro, and pound sterling for the empirical tests, we show the persistent relationships between the market prices of the cross-rate (€/£) options and the estimated bounds. The dollar-rate option prices and the correlation between two dollar rates provide almost perfect information in deciding the prices of the cross-rate options. Owing to the almost perfect explanatory power of the option bounds and the correlation to the market prices of the cross-rate options, we successfully infer the prices of the cross-rate options under the circumstance where the current option prices of two dollar rates are available. Therefore, our results are useful for risk management and derivative pricing, particularly for those having cross-rate risk exposures.

\textsuperscript{15} The moneyness level is defined as the strike prices divided by the forward price.
\textsuperscript{16} The delta of the call option with the moneyness level 1.0325 (OTM) ranges from 0.0003 and 0.5986 and from 0.0127 and 0.5719 respectively when the option price is unknown and known.
Besides, the analytical properties of the bounds are empirically verified. Given the estimated pricing bounds in terms of implied volatility, we also demonstrate how to bound the cross-rate option’s delta both analytically and empirically.

The technique utilized to derive our cross-rate option bounds can be applied to any European derivative security whose payoff can be rearranged as the same type as that of an exchange option. As an example, we also derive the price bounds for quanto options using the same copula approach. Due to the lack of data, the empirical tests of the information efficiency of our quanto option price bounds are left to interested readers for future research.

References

of correlation risk.” Working paper, University of Amsterdam & INSEAD.
Appendix

A. Derivation of the Price Bounds for the Cross-Rate Option

Let \( \Pr \) denote the probability, \( F_i(x) \) the cumulative distribution function, and \( r \) the dollar risk-free interest rate. With probability distribution techniques, the price of an option on the minimum of two risky assets can be expressed as:

\[
Call_{\min}(S^{S/E}, K^{S/E}, 0, t, T) = e^{-r(T-t)} \int_0^\infty \Pr(\min(S^{S/E}, K^{S/E}) > x)dx
\]

\[
= e^{-r(T-t)} \int_0^\infty \Pr(S^{S/E} > x, K^{S/E} > x)dx
\]

\[
= e^{-r(T-t)} \int_0^\infty \overline{C}(F_{S^{S/E}}(x), F_{K^{S/E}}(x))dx,
\]

where \( \overline{C} \) is a survival copula\(^{17} \) and \( F_i(x) = 1 - F_i(x) \). According to the Fréchet bounds in the copula theory, it is true that \( \max(u + v - 1, 0) \leq \overline{C}(u, v) \leq \min(u, v) \) since \( \overline{C}(u, v) \) is a copula. Consequently, the upper and lower bounds of the minimum option are given as the following, respectively:

\[
Call^+_{\min}(S^{S/E}, K^{S/E}, 0, t, T) = e^{-r(T-t)} \int_0^\infty \min(F_{S^{S/E}}(x), F_{K^{S/E}}(x))dx,
\]

\[
Call^-_{\min}(S^{S/E}, K^{S/E}, 0, t, T) = e^{-r(T-t)} \int_0^\infty \max(F_{S^{S/E}}(x) + F_{K^{S/E}}(x) - 1, 0)dx.
\]

Since \( F_i(u) \) is a decreasing function of \( u \) and \( K^{**} \) is a constant which solves \( F_{S^{S/E}}(K^{**}) + F_{K^{S/E}}(K^{**}) = 1 \), it is true that \( F_{S^{S/E}}(u) + F_{K^{S/E}}(u) \geq 1 \) for \( u \leq K^{**} \). Therefore, the lower bound of the minimum option is

\[
Call^-_{\min}(S^{S/E}, K^{S/E}, 0, t, T) = e^{-r(T-t)} \int_0^\infty \max(F_{S^{S/E}}(u) + F_{K^{S/E}}(u) - 1, 0)du
\]

\[
= e^{-r(T-t)} \int_0^{K^{**}} F_{S^{S/E}}(u)du + e^{-r(T-t)} \int_0^{K^{**}} F_{K^{S/E}}(u)du - e^{-r(T-t)} \int_0^{K^{**}} du
\]

\[
= e^{-r(T-t)} \int_0^{K^{**}} F_{S^{S/E}}(u)du + e^{-r(T-t)} \int_0^{K^{**}} F_{K^{S/E}}(u)du
\]

\[
- e^{-r(T-t)} \int_0^{K^{**}} F_{K^{S/E}}(u)du - e^{-r(T-t)} \int_0^{K^{**}} du
\]

\[
= S^{S/E}_t e^{-r(T-t)} - Call(S^{S/E}, K^{**}, t, T) + KS^{S/E}_t e^{-r(T-t)} - Call(K^{S/E}, K^{**}, t, T) - e^{-r(T-t)} K^{**}.
\]

\(^{17} \) If two uniform variables \( U \) and \( V \) are jointed with a copula function \( C \), then the joint probability that \( U \) and \( V \) are greater than \( u \) and \( v \), respectively, is given by a survival function:

\( \Pr(U > u, V > v) = 1 - u - v + C(u, v) = \overline{C}(1 - u, 1 - v) \).
Substituting equation (A.3) into equation (2) and applying the put-call parity, we obtain the upper bound of the cross-rate option price as follows:

\[
\begin{align*}
\text{Call}_S^{E/L^+} &= S_t^{S/E} e^{-r_t(T-t)} - \text{Call}_{\min}(S_t^{S/E}, K^{S/E}, 0, t, T) \\
&= \text{Call}(S_t^{S/E}, K^{**, t, T}) + \text{Put}(K_S^{S/E}, K^{**, t, T}) \\
&= \text{Call}(S_t^{S/E}, K^{**, t, T}) + K \text{ Put}(S_t^{S/E}, K^{**, t, T})
\end{align*}
\]

where \( K'' = K^{**} / K \).

Assume that there exists a constant \( K^* \) such that \( \overline{F}_{S^{E/L}}(K^*) = \overline{F}_{K^{S/E}}(K^*) \). If \( \overline{F}_{S^{E/L}}(u) < \overline{F}_{K^{S/E}}(u) \) for \( u < K^* \), then it is straightforward to show that the upper bound of the minimum option is:

\[
\text{Call}_{\min}^*(S_t^{S/E}, K_S^{S/E}, 0, t, T) = e^{-r_t(T-t)} \int_0^e \min(\overline{F}_{S^{E/L}}(u), \overline{F}_{K^{S/E}}(u)) du \]

\[
= e^{-r_t(T-t)} \int_0^K \overline{F}_{S^{E/L}}(u) du + e^{-r_t(T-t)} \int_K^{e} \overline{F}_{K^{S/E}}(u) du
\]

\[
= S_t^{S/E} e^{-r_t(T-t)} - \text{Call}(S_t^{S/E}, K^*, t, T) + K \text{ Call}(S_t^{S/E}, K^{'}, t, T),
\]

where \( K' = K^* / K \). Substituting equation (A.5) into equation (2) yields the lower price bound of the cross-rate option as follows:

\[
\begin{align*}
\text{Call}_S^{E/L^-} &= S_t^{S/E} e^{-r_t(T-t)} - \text{Call}_{\min}^*(S_t^{S/E}, K_S^{S/E}, 0, t, T) \\
&= \text{Call}(S_t^{S/E}, K^{'}, t, T) - K \text{ Call}(S_t^{S/E}, K^{'}, t, T).
\end{align*}
\]

Similarly, if \( \overline{F}_{S^{E/L}}(u) > \overline{F}_{K^{S/E}}(u) \) for \( u < K^* \), then one can derive that:

\[
\begin{align*}
\text{Call}_{\min}^+(S_t^{S/E}, K_S^{S/E}, 0, t, T) &= K S_t^{S/E} e^{-r_t(T-t)} - K \text{ Call}(S_t^{S/E}, K^{'}, t, T) \\
&+ \text{Call}(S_t^{S/E}, K^*, t, T),
\end{align*}
\]

\[
\begin{align*}
\text{Call}_S^{E/L^-} &= S_t^{S/E} e^{-r_t(T-t)} - K S_t^{S/E} e^{-r_t(T-t)} + K \text{ Call}(S_t^{S/E}, K^{'}, t, T) \\
&- \text{Call}(S_t^{S/E}, K^*, t, T).
\end{align*}
\]

B. Proof of Proposition 2

When the two dollar rates follow a bivariate lognormal distribution,

\[
C_{BS}(S_t^{S/E}, K, r_S, r_E, \sigma_{S/E}, \tau) = S_t^{S/E} e^{-r_t \tau} N(d_1(S_t^{S/E}, K, r_S, r_E, \sigma_{S/E}, \tau)) \\
- Ke^{-r \tau} N(d_2(S_t^{S/E}, K, r_S, r_E, \sigma_{S/E}, \tau)),
\]
where \( \tau = T - t \), \( d_1(.) = \frac{\ln \left( \frac{S_t^{S/E}}{K} \right) + (r_5 - r_e + \frac{1}{2} \sigma_{S/E}^2)\tau}{\sigma_{S/E} \sqrt{\tau}} \), and \( d_2(.) = d_1(.) - \sigma_{S/E} \sqrt{\tau} \).

From Proposition 1, we know that \( K^{**} \) is a constant satisfying that
\[
N(d_2(S_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)) + N(d_2(KS_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)) = 1.
\]
Since \( N(x) + N(-x) = 1 \) is true, one can show that
\[
d_2(S_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau) = -d_2(KS_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau). \tag{A.8}
\]
Thus the solution of for \( K^{**} \) is:
\[
K^{**} = (KS_t^{S/E})^{\sigma_{S/E} / \sigma_{S/E}} (S_t^{S/E})^{-\sigma_{S/E} / \sigma_{S/E}} \exp \left[ r_5 - \left( \frac{\sigma_{S/E}}{\sigma_{S/E} + \sigma_{S/E}} \right) r_e - \left( \frac{\sigma_{S/E}}{\sigma_{S/E} + \sigma_{S/E}} \right) \sigma_{S/E} \sigma_{S/E} \tau \right]. \tag{A.9}
\]
From equations (3) and (A.8), the upper bound is given by:
\[
S_t^{S/E} e^{-r_e \tau} N(d_1(S_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)) - K^{**} e^{-r_e \tau} N(d_2(S_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau))
+ K^{**} e^{-r_e \tau} N(-d_2(KS_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)) - KS_t^{S/E} e^{-r_e \tau} N(-d_1(KS_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau))
= S_t^{S/E} e^{-r_e \tau} N(d_1(S_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)) - KS_t^{S/E} e^{-r_e \tau} N(-d_1(KS_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau))
\]
that the above upper bound is denominated in US dollars and its value in euros is:
\[
S_t^{E/E} e^{-r_e \tau} N(d_1(S_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)) - Ke^{-r_e \tau} N(-d_1(KS_t^{S/E}, K^{**}, r_5, r_e, \sigma_{S/E}, \tau)). \tag{A.10}
\]
Substituting equation (A.9) into equation (A.10) yields:
\[
S_t^{E/E} e^{-r_e \tau} N(d_1(S_t^{S/E}, K, r_e, r_e, \sigma_{S/E}, \tau + \sigma_{S/E}, \tau)) - Ke^{-r_e \tau} N(d_2(S_t^{S/E}, K, r_e, r_e, \sigma_{S/E}, \tau))
= C_{BS} (S_t^{S/E}, K, r_e, r_e, \sigma_{S/E}, \tau).
\]
For brevity, we derive the lower bound only for the case where \( \sigma_{S/E} > \sigma_{S/E} \). Since \( K^* \) satisfies that
\[
N(d_2(S_t^{S/E}, K^*, r_5, r_e, \sigma_{S/E}, \tau)) = N(d_2(KS_t^{S/E}, K^*, r_5, r_e, \sigma_{S/E}, \tau)),
\]
one can show that its solution is:
\[
K^* = (KS_t^{S/E})^{\sigma_{S/E} / \sigma_{S/E}} (S_t^{S/E})^{-\sigma_{S/E} / \sigma_{S/E}} \exp \left[ r_5 + \left( \frac{\sigma_{S/E}}{\sigma_{S/E} - \sigma_{S/E}} \right) r_e - \left( \frac{\sigma_{S/E}}{\sigma_{S/E} - \sigma_{S/E}} \right) \sigma_{S/E} \sigma_{S/E} \tau \right]. \tag{A.11}
\]
Since \( \sigma_{S/E} > \sigma_{S/E} \), it is straightforward to show that when \( u < K^* \),

\[
N(d_2(S^{S/E}_t, u, r_s, r_e, \sigma_{S/E}, \tau)) = N(d_2(S^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau) + \frac{\ln(K^*/u)}{\sigma_{S/E} \sqrt{\tau}})
\]

\[
< N(d_2(KS^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau) + \frac{\ln(K^*/u)}{\sigma_{S/E} \sqrt{\tau}}) = N(d_2(KS^{S/E}_t, u, r_s, r_e, \sigma_{S/E}, \tau)).
\]

Therefore the lower bound is determined by the first alternative of equation (4),\(^{18}\) i.e.

\[
S^{S/E}_t e^{-r_s \tau} N(d_1(S^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau)) - K e^{-r_s \tau} N(d_2(S^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau))
- K S^{S/E}_t e^{-r_s \tau} N(d_1(KS^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau)) + K^* e^{-r_e \tau} N(d_2(KS^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau))
= S^{S/E}_t e^{-r_s \tau} N(d_1(S^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau)) - KS^{S/E}_t e^{-r_s \tau} N(d_1(KS^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau)).
\]

the above lower bound is denominated in US dollars and its value in euros is:

\[
S^{S/E}_t e^{-r_s \tau} N(d_1(S^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau)) - Ke^{-r_e \tau} N(d_1(KS^{S/E}_t, K^*, r_s, r_e, \sigma_{S/E}, \tau)). \tag{A.12}
\]

Substituting equation (A.11) into equation (A.12) yields:

\[
S^{S/E}_t e^{-r_s \tau} N(d_1(S^{S/E}_t, K, r_e, r_s, \sigma_{S/E} - \sigma_{S/E}, \tau)) - Ke^{-r_e \tau} N(d_2(S^{S/E}_t, K, r_e, r_s, \sigma_{S/E} - \sigma_{S/E}, \tau))
= C_{BS}(S^{S/E}_t, K, r_e, r_s, \sigma_{S/E} - \sigma_{S/E}, \tau).
\]

---

\(^{18}\) On the other hand, if the volatility of \$/£ is smaller than that of \$/€, the lower bound is determined by the second alternative of equation (4).
Figure 1: The Implied Volatilities from Market Prices and the Estimated Bounds for Cross-Rate Options

This figure shows the evolutions of the Black-Scholes implied volatilities from the market prices and the estimated bounds of the cross-rate (€/£) options across deltas. The option bounds are estimated by calibrating equations (3) and (4) using the option prices of two dollar rates, $/£ and $/€.
Figure 2: Actual and Inferred Implied Volatilities for Cross-Rate Options

This figure consists of the evolutions of the actual and inferred Black-Scholes implied volatilities of the cross-rate (€/£) options across deltas. The actual implied volatilities are backed out from the market prices of options. The inferred implied volatilities are obtained from Model 2 in Section 4 using the current option prices and historical DCC correlation of two dollar rates, $/€ and $/£.
Figure 3: Implied Skewness and Kurtosis for Cross-Rate Options

This figure consists of the evolutions of the implied skewness and kurtosis of the cross-rate (€/£) options. The implied skewness and kurtosis are calculated using Theorem 1 of Bakshi et al. (2003). The results indicate that the risk neutral distributions of the cross-rates are fat-tailed (average kurtosis equals 3.31) and slightly negatively skewed (average skewness equals -0.13).
Figure 4: Delta Bounds of a Cross-Rate Option

This figure consists of the upper and lower bounds on the delta of the ATM cross-rate (€/£) call option quoted on June 29, 1999 when the option price is either unknown or known, respectively represented by the dashed and dotted lines. The delta is bounded using the Propositions 5 and 6 of Bergman et al. (1996).
Table 1: Summary Statistics of Market Implied Volatilities for the Foreign Exchange Options

<table>
<thead>
<tr>
<th>Panel 1: USD/GBP</th>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
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<tbody>
<tr>
<td>Mean</td>
<td>0.0926</td>
<td>0.0878</td>
<td>0.0870</td>
<td>0.0866</td>
<td>0.0881</td>
<td>0.0904</td>
<td>0.0964</td>
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<tr>
<td>Std. Dev.</td>
<td>0.0130</td>
<td>0.0130</td>
<td>0.0134</td>
<td>0.0136</td>
<td>0.0139</td>
<td>0.0142</td>
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<tr>
<td>Skewness</td>
<td>0.5772</td>
<td>0.6170</td>
<td>0.6473</td>
<td>0.6703</td>
<td>0.6869</td>
<td>0.7037</td>
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</tr>
<tr>
<td>Kurtosis</td>
<td>2.6181</td>
<td>2.6621</td>
<td>2.6942</td>
<td>2.7323</td>
<td>2.7607</td>
<td>2.8066</td>
<td>2.8542</td>
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<table>
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<tr>
<th>Panel 2: USD/EUR</th>
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<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
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<tbody>
<tr>
<td>Mean</td>
<td>0.1276</td>
<td>0.1193</td>
<td>0.1172</td>
<td>0.1158</td>
<td>0.1166</td>
<td>0.1179</td>
<td>0.1248</td>
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<tr>
<td>Std. Dev.</td>
<td>0.0208</td>
<td>0.0213</td>
<td>0.0212</td>
<td>0.0212</td>
<td>0.0213</td>
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<tr>
<td>Skewness</td>
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<table>
<thead>
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<th>Panel 3: EUR/GBP</th>
<th>Delta</th>
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<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.1054</td>
<td>0.0996</td>
<td>0.0980</td>
<td>0.0972</td>
<td>0.0979</td>
<td>0.0995</td>
<td>0.1050</td>
<td></td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0169</td>
<td>0.0173</td>
<td>0.0176</td>
<td>0.0176</td>
<td>0.0180</td>
<td>0.0180</td>
<td>0.0183</td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
<td>0.2457</td>
<td>0.2441</td>
<td>0.2288</td>
<td>0.2257</td>
<td>0.2222</td>
<td>0.2321</td>
<td>0.2484</td>
<td></td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.9339</td>
<td>2.7199</td>
<td>2.5933</td>
<td>2.5686</td>
<td>2.4881</td>
<td>2.4722</td>
<td>2.4511</td>
<td></td>
</tr>
</tbody>
</table>

This table consists of the summary statistics of the market implied volatilities from the options of three exchange rates ($/£, $/€, and €/£). All implied volatilities are quoted in the OTC market.

Table 2: Summary Statistics of the Implied Volatilities from the Market Prices and the Estimated Bounds

<table>
<thead>
<tr>
<th>Panel 1: Upper Bounds</th>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.2087</td>
<td>0.2061</td>
<td>0.2056</td>
<td>0.2058</td>
<td>0.2064</td>
<td>0.2077</td>
<td>0.2118</td>
<td></td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0325</td>
<td>0.0329</td>
<td>0.0330</td>
<td>0.0331</td>
<td>0.0332</td>
<td>0.0333</td>
<td>0.0334</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel 2: Market Implieds</th>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.1054</td>
<td>0.0996</td>
<td>0.0980</td>
<td>0.0972</td>
<td>0.0979</td>
<td>0.0995</td>
<td>0.1050</td>
<td></td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0169</td>
<td>0.0173</td>
<td>0.0176</td>
<td>0.0176</td>
<td>0.0180</td>
<td>0.0180</td>
<td>0.0183</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel 3: Lower Bounds</th>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0405</td>
<td>0.0319</td>
<td>0.0302</td>
<td>0.0315</td>
<td>0.0352</td>
<td>0.0396</td>
<td>0.0478</td>
<td></td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0095</td>
<td>0.0117</td>
<td>0.0124</td>
<td>0.0122</td>
<td>0.0113</td>
<td>0.0108</td>
<td>0.0110</td>
<td></td>
</tr>
</tbody>
</table>

This table consists of the summary statistics of the implied volatilities from the market prices and estimated upper and lower bounds of the cross-rate €/£ options across deltas. The option bounds are estimated by calibrating equations (3) and (4) with the option prices of two dollar-rates, $/£ and $/€.
Table 3: Summary Statistics of the Estimated Upper Ranges and Lower Ranges

**Panel 1: Upper Ranges**

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.1033</td>
<td>0.1065</td>
<td>0.1076</td>
<td>0.1086</td>
<td>0.1085</td>
<td>0.1082</td>
<td>0.1069</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0208</td>
<td>0.0208</td>
<td>0.0206</td>
<td>0.0207</td>
<td>0.0205</td>
<td>0.0206</td>
<td>0.0206</td>
</tr>
<tr>
<td>β</td>
<td>0.1431</td>
<td>0.1478</td>
<td>0.1468</td>
<td>0.1454</td>
<td>0.1435</td>
<td>0.1405</td>
<td>0.1320</td>
</tr>
<tr>
<td></td>
<td>(8.02)</td>
<td>(8.35)</td>
<td>(8.37)</td>
<td>(8.24)</td>
<td>(8.18)</td>
<td>(7.96)</td>
<td>(7.44)</td>
</tr>
</tbody>
</table>

**Panel 2: Lower Ranges**

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0648</td>
<td>0.0678</td>
<td>0.0678</td>
<td>0.0656</td>
<td>0.0627</td>
<td>0.0599</td>
<td>0.0571</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0125</td>
<td>0.0149</td>
<td>0.0144</td>
<td>0.0131</td>
<td>0.0124</td>
<td>0.0120</td>
<td>0.0116</td>
</tr>
<tr>
<td>β</td>
<td>-0.0855</td>
<td>-0.1121</td>
<td>-0.1049</td>
<td>-0.0958</td>
<td>-0.0905</td>
<td>-0.0869</td>
<td>-0.0818</td>
</tr>
<tr>
<td></td>
<td>(-7.98)</td>
<td>(-8.93)</td>
<td>(-8.59)</td>
<td>(-8.65)</td>
<td>(-8.62)</td>
<td>(-8.50)</td>
<td>(-8.24)</td>
</tr>
</tbody>
</table>

This table consists of the summary statistics of the estimated upper ranges and lower ranges of the cross-rate (€/£) options across deltas. The upper ranges and lower ranges are the distances between the upper bounds and market implieds and between the lower bounds and market implieds, respectively. In addition, the parameter estimates of the following regression model are provided.

\[
R_t = \alpha + \beta \text{Corr}_t + \varepsilon_t
\]

where \( R_t \) is the upper or lower range, \( \text{Corr}_t \) denotes the correlation between the two dollar-rates, and \( \varepsilon_t \) is the residual term at time \( t \). The correlations are generated by the DCC model of Engle (2002). The numbers in the parentheses are t-statistics.
### Table 4: Explanatory Power of Estimated Bounds and Correlation to Market Implied Volatility

#### Panel 1: Model 1

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\beta}_1$</td>
<td>0.3388</td>
<td>0.3948</td>
<td>0.3871</td>
<td>0.3594</td>
<td>0.3409</td>
<td>0.3245</td>
<td>0.3088</td>
</tr>
<tr>
<td></td>
<td>(22.79)</td>
<td>(27.01)</td>
<td>(26.93)</td>
<td>(24.60)</td>
<td>(22.17)</td>
<td>(19.22)</td>
<td>(15.64)</td>
</tr>
<tr>
<td>$\hat{\beta}_2$</td>
<td>0.5279</td>
<td>0.2611</td>
<td>0.3229</td>
<td>0.4180</td>
<td>0.5248</td>
<td>0.5730</td>
<td>0.5886</td>
</tr>
<tr>
<td></td>
<td>(10.40)</td>
<td>(6.34)</td>
<td>(8.41)</td>
<td>(10.50)</td>
<td>(11.61)</td>
<td>(11.06)</td>
<td>(9.82)</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.7421</td>
<td>0.7162</td>
<td>0.7400</td>
<td>0.7576</td>
<td>0.7706</td>
<td>0.7636</td>
<td>0.7543</td>
</tr>
</tbody>
</table>

#### Panel 2: Model 2

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\beta}_1$</td>
<td>0.3720</td>
<td>0.4079</td>
<td>0.4033</td>
<td>0.3815</td>
<td>0.3700</td>
<td>0.3618</td>
<td>0.3598</td>
</tr>
<tr>
<td></td>
<td>(32.43)</td>
<td>(38.10)</td>
<td>(38.30)</td>
<td>(35.19)</td>
<td>(31.53)</td>
<td>(26.93)</td>
<td>(21.72)</td>
</tr>
<tr>
<td>$\hat{\beta}_2$</td>
<td>0.3872</td>
<td>0.2506</td>
<td>0.2967</td>
<td>0.3629</td>
<td>0.4354</td>
<td>0.64470</td>
<td>0.4133</td>
</tr>
<tr>
<td></td>
<td>(9.82)</td>
<td>(8.32)</td>
<td>(10.57)</td>
<td>(12.30)</td>
<td>(12.62)</td>
<td>(10.81)</td>
<td>(8.17)</td>
</tr>
<tr>
<td>$\hat{\beta}_3$</td>
<td>-0.1127</td>
<td>-0.1256</td>
<td>-0.1225</td>
<td>-0.1174</td>
<td>-0.1131</td>
<td>-0.1100</td>
<td>-0.1055</td>
</tr>
<tr>
<td></td>
<td>(-18.56)</td>
<td>(-20.38)</td>
<td>(-20.43)</td>
<td>(-19.97)</td>
<td>(-18.88)</td>
<td>(-17.23)</td>
<td>(-15.17)</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.8502</td>
<td>0.8483</td>
<td>0.8613</td>
<td>0.8680</td>
<td>0.8687</td>
<td>0.8542</td>
<td>0.8341</td>
</tr>
</tbody>
</table>

This table consists of the regression results of the following three models:

Model 1: $MIV_t = c + \hat{\beta}_1 UB_t + \hat{\beta}_2 LB_t + \hat{\beta}_3 Corr_t + \varepsilon_t$

Model 2: $MIV_t = c + \beta_1 UB_t + \beta_2 LB_t + \beta_3 Corr_t + \varepsilon_t$

Here, $MIV_t$, $UB_t$, $LB_t$, and $Corr_t$ denote respectively the market implied volatility of an option on €/£, the upper bound, the lower bound, and the historical DCC correlation between S/€ and $/£ at day t$, and $\varepsilon_t$ is the residual term. The numbers in the parentheses are t-statistics.

### Table 5: Summary Statistics of Price Inference Errors for Model 2

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0032</td>
<td>0.0035</td>
<td>0.0033</td>
<td>0.0033</td>
<td>0.0033</td>
<td>0.0034</td>
<td>0.0036</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0032</td>
<td>0.0030</td>
<td>0.0029</td>
<td>0.0029</td>
<td>0.0028</td>
<td>0.0029</td>
<td>0.0033</td>
</tr>
</tbody>
</table>

This table consists of the summary statistics of the price inference errors for the cross-rate (€/£) options. The inference errors are defined as the absolute values of the actual values minus the inferred values of implied volatilities. The actual implied volatilities are backed out from the market prices of options. The inferred implied volatilities are inferred from Model 2 in Section 4 using the current market prices of options on two dollar rates ($/£ and $/€) and their historical DCC correlations.
### Table 6: Robustness Analysis for Option Bounds

#### Panel 1: Upper Bounds

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>GB2</td>
<td>0.2087</td>
<td>0.2061</td>
<td>0.2056</td>
<td>0.2058</td>
<td>0.2064</td>
<td>0.2077</td>
<td>0.2118</td>
</tr>
<tr>
<td>Mixtures</td>
<td>0.2086</td>
<td>0.2060</td>
<td>0.2056</td>
<td>0.2058</td>
<td>0.2065</td>
<td>0.2078</td>
<td>0.2119</td>
</tr>
<tr>
<td>p-value</td>
<td>0.9741</td>
<td>0.9811</td>
<td>0.9925</td>
<td>0.9996</td>
<td>0.9936</td>
<td>0.9883</td>
<td>0.9763</td>
</tr>
</tbody>
</table>

#### Panel 2: Lower Bounds

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>GB2</td>
<td>0.0405</td>
<td>0.0319</td>
<td>0.0302</td>
<td>0.0315</td>
<td>0.0352</td>
<td>0.0396</td>
<td>0.0478</td>
</tr>
<tr>
<td>Mixtures</td>
<td>0.0412</td>
<td>0.0325</td>
<td>0.0309</td>
<td>0.0316</td>
<td>0.0353</td>
<td>0.0399</td>
<td>0.0476</td>
</tr>
<tr>
<td>p-value</td>
<td>0.3566</td>
<td>0.4237</td>
<td>0.3274</td>
<td>0.9205</td>
<td>0.8876</td>
<td>0.5895</td>
<td>0.7957</td>
</tr>
</tbody>
</table>

This table consists of the means of the upper and lower bounds of the cross-rate (€/£) options across deltas, which are estimated using two different RND distribution assumptions, GB2 and log-normal mixtures, for the dollar-rates. In addition, the p-values of the mean equality tests for the two assumed distributions are provided.

### Table 7: Robustness Analysis for Price Inference Errors

#### Panel 1: Mean Errors for Sub-samples

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>First half sample</td>
<td>0.0027</td>
<td>0.0034</td>
<td>0.0032</td>
<td>0.0032</td>
<td>0.0030</td>
<td>0.0031</td>
<td>0.0031</td>
</tr>
<tr>
<td>Second half sample</td>
<td>0.0036</td>
<td>0.0036</td>
<td>0.0034</td>
<td>0.0033</td>
<td>0.0034</td>
<td>0.0036</td>
<td>0.0040</td>
</tr>
</tbody>
</table>

#### Panel 2: Regression of Errors on Implied Volatility

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.0133</td>
<td>-0.0651</td>
<td>-0.1138</td>
<td>-0.1174</td>
<td>-0.0495</td>
<td>-0.0128</td>
<td>0.0990</td>
</tr>
<tr>
<td>t-stat</td>
<td>(0.10)</td>
<td>(-0.55)</td>
<td>(-1.00)</td>
<td>(-1.02)</td>
<td>(-0.46)</td>
<td>(-0.12)</td>
<td>(0.91)</td>
</tr>
</tbody>
</table>

#### Panel 3: Regression of Errors on Implied Skewness

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.0099</td>
<td>0.0027</td>
<td>0.0104</td>
<td>0.0119</td>
<td>0.0242</td>
<td>0.0399</td>
<td>0.0612</td>
</tr>
<tr>
<td>t-stat</td>
<td>(0.40)</td>
<td>(0.12)</td>
<td>(0.48)</td>
<td>(0.54)</td>
<td>(1.18)</td>
<td>(1.96)</td>
<td>(3.07)</td>
</tr>
</tbody>
</table>

#### Panel 4: Regression of Errors on Implied Kurtosis

<table>
<thead>
<tr>
<th>Delta</th>
<th>0.9</th>
<th>0.75</th>
<th>0.63</th>
<th>0.5</th>
<th>0.37</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>-0.0027</td>
<td>0.0121</td>
<td>0.0153</td>
<td>0.0149</td>
<td>-0.0018</td>
<td>-0.0136</td>
<td>-0.0264</td>
</tr>
<tr>
<td>t-stat</td>
<td>(0.12)</td>
<td>(0.58)</td>
<td>(0.75)</td>
<td>(0.73)</td>
<td>(-0.09)</td>
<td>(-0.71)</td>
<td>(-1.39)</td>
</tr>
</tbody>
</table>

This table consists of the summary statistics used to check the sensitivity of the accuracy of option price inference for Model 2. The means of price inference errors for two evenly divided sub-samples are shown in Panel 1. In addition, we run the following regression model to analyze whether price inference errors depend on volatility, skewness or kurtosis:

$$E_t = c + \alpha E_{t-1} + \beta X_t,$$

where $E_t$ denotes the price inference error in percentage and $X_t$ is the implied volatility, implied skewness or implied kurtosis at time $t$ estimated using the Theorem 1 of Bakshi et al. (2003). The numbers in the parentheses are t-statistics.