Jump and Volatility Risk Premiums Implied by VIX

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The literature

- Incorporating jumps into the stochastic volatility model has long been advocated in the empirical option pricing literature; for example, Bakshi, Cao and Chen (1997), Bates (2000), Chernov and Ghysel (2000), Duffie, Pan, and Singleton (2000), Pan (2002), Eraker (2004), and Broadie, Chernov and Johannes (2006).
- Anderson, Benzoni and Lund (2002) and Eraker, Johannes and Polson (2003) concluded that allowing jumps in prices can improve the fitting for the time-series of equity returns. However, Bakshi, Cao and Chen (1997), Bates (2000), Pan (2002) and Eraker (2004) offered different and inconsistent results in terms of improvement on option pricing. There is no joint significance in the volatility and jump risk premium estimates in most cases.

Implementation challenges

- Broadie, Chernov, and Johannes (2006) attributed the contradictory findings to the short sample period and/or limited option contracts used in those papers. But using options over a wide range of strike prices over a long time span in estimation will quickly create an unmanageable computational burden.
- Stochastic volatility being a latent variable contributes to the methodological challenge in testing and applications.

Key features of the proposed approach

- Derive a new theoretical link (allowing for price jumps) between the latent volatility and the VIX index (a CBOE volatility index for the S&P500 index targeting the 30-day maturity using a model-free volatility construction).
- Use this link to devise a maximum likelihood estimation method for the stochastic volatility model with/without jumps in order to obtain the volatility and jump risk premiums among other parameters.
- This approach only uses two time series: price and VIX, and thus bypasses the numerically demanding step of valuing options. The VIX index has in effect summarized all critical information in options over the entire spectrum of strike prices.

A summary of the empirical findings

1. Incorporating a jump risk factor is critically important.
2. Both the jump and volatility risks are priced.
3. The popular square-root stochastic volatility process is a poor model specification irrespective of allowing for price jumps or not.

A class of jump-diffusions with stochastic volatility

Under the physical probability measure P,

$$d\ln S_t = \left[r - q + \delta_S V_t - \frac{V_t}{2}\right] dt + \sqrt{V_t} dW_t + J_t dN_t - \lambda\mu_J dt$$
$$dV_t = \kappa(\theta - V_t)dt + \upsilon V_t^{\gamma} dB_t$$

- W_t and B_t are two correlated Wiener processes with the correlation coefficient ρ.
- N_t is a Poisson process with intensity λ and independent of W_t and B_t.
- J_t is an independent normal random variable with mean μ_J and standard deviation σ_J.
- dW_t and $J_t dN_t$ have respective variances equal to dt and $\lambda(\mu_J^2 + \sigma_J^2)dt$. Thus, $V_t + \lambda(\mu_J^2 + \sigma_J^2)$ is the variance rate of the asset price process.

The model contains commonly used stochastic volatility models with or without jumps.

- Scott (1987) and Heston (1993): square-root volatility without price jumps, i.e., setting $\gamma = \frac{1}{2}$ and $\lambda = 0$.
- Hull and White (1987): linear volatility without price jumps, i.e., setting $\gamma = 1$, $\lambda = 0$ and $\theta = 0$. (Note: The volatility does not mean-revert because $\theta = 0$.)
- Bates (2000) and Pan (2002): square-root volatility with price jumps, i.e., setting $\gamma = 1/2$.

Risk-neutral jump-diffusions with stochastic volatility

Adopting a pricing kernel similar to that in Pan (2002), the system under the risk-neutral probability measure Q becomes,

$$
\begin{aligned}
d\ln S_t &= \left[r - q - \frac{V_t}{2} + \lambda^* \left(\mu_J^* + 1 - e^{\mu_J^* + \frac{\sigma_J^2}{2}}\right)\right] dt + \sqrt{V_t} dW_t^* \\
&\quad + J_t^* dN_t^* - \lambda^* \mu_J^* dt \\
dV_t &= (\kappa\theta - \kappa^* V_t)\, dt + v V_t^{\gamma} dB_t^*
\end{aligned}
$$

where $\kappa^* = \kappa + \delta_V$ and $B_t^* = B_t + \frac{\delta_V}{v} \int_0^t V_s^{1-\gamma} ds$ with δ_V being interpreted as the volatility risk premium.

Note: It can be easily verified by applying Ito's lemma that $E_t^Q \left(\frac{dS_t}{S_t}\right) = (r-q)dt$. Thus, the expected return under measure Q indeed equals the risk-free rate minus the dividend yield.

Fact 1: The VIX portfolio of options

Consider an option portfolio:

$$\begin{aligned}
&\Pi_{t+\tau}(K_0, t+\tau) \\
\equiv\ & \int_0^{K_0} \frac{P_{t+\tau}(K; t+\tau)}{K^2} dK + \int_{K_0}^{\infty} \frac{C_{t+\tau}(K; t+\tau)}{K^2} dK \\
=\ & \frac{S_{t+\tau} - K_0}{K_0} - \ln \frac{S_t}{K_0} - \ln \frac{S_{t+\tau}}{S_t}
\end{aligned}$$

Thus, taking the risk-neutral expectation gives rise to

$$e^{r\tau}\Pi_t(K_0, t+\tau) = \frac{F_t(t+\tau) - K_0}{K_0} - \ln \frac{S_t}{K_0} - E_t^Q\left(\ln \frac{S_{t+\tau}}{S_t}\right)$$

where $F_t(t+\tau)$ denotes the forward price at time t with a maturity at time $t+\tau$.

Fact 2: The risk-neutral expected cumulative return

$$
\begin{aligned}
& E_t^Q\left(\ln \frac{S_{t+\tau}}{S_t}\right) \\
= \; & (r-q)\tau - \frac{1}{2}\int_t^{t+\tau} E_t^Q\left(V_s\right) ds \\
& + \int_t^{t+\tau} \lambda^* E_t^Q\left(\mu_J^* + 1 - e^{\mu_J^* + \frac{\sigma_J^2}{2}}\right) ds \\
= \; & \left[r - q - \lambda^*\left(e^{\mu_J^* + \frac{\sigma_J^2}{2}} - (\mu_J^* + 1)\right)\right]\tau - \frac{1}{2}\int_t^{t+\tau} E_t^Q\left(V_s\right) ds
\end{aligned}
$$

where

$$
\int_t^{t+\tau} E_t^Q\left(V_s\right) ds = \frac{\kappa\theta}{\kappa^*}\left(\tau - \frac{1 - e^{-\kappa^*\tau}}{\kappa^*}\right) + \frac{1 - e^{-\kappa^*\tau}}{\kappa^*} V_t.
$$

A new theoretical link

CBOE launched the new VIX in 2003 using the following definition:

$$\mathrm{VIX}_t^2(\tau) \equiv \frac{2}{\tau} e^{r\tau} \Pi_t(F_t(t+\tau), t+\tau) + \text{adjustment terms.}$$

Using Facts 1 and 2 yields

$$\begin{aligned}\mathrm{VIX}_t^2(\tau) &= 2\phi^* + \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(V_s)\,ds \\ &= 2\phi^* + \frac{\kappa\theta}{\kappa^*}\left(\tau - \frac{1-e^{-\kappa^*\tau}}{\kappa^*}\right) + \frac{1-e^{-\kappa^*\tau}}{\kappa^*}V_t\end{aligned}$$

where $\phi^* = \lambda^*\left(e^{\mu_J^* + \sigma_J^2/2} - 1 - \mu_J^*\right)$.

Note: The extra term, ϕ^*, is entirely due to jumps. If the jump magnitude is small, this term is negligible.

Parameter identification

- Similar to an observation made in Pan (2002), λ^* and μ_J^* cannot be separately identified. Pan (2002) simply assumed $\lambda^* = \lambda$. Equally acceptable is to assume $\mu_J^* = \mu_J$.
- Instead of forcing an equality on a specific pair of parameters, we use the composite parameter ϕ^* to define the jump risk premium. Specifically, the jump risk premium is regarded as $\delta_J = \phi^* - \phi$, where $\phi = \lambda \left(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J \right)$.
- The parameters to be estimated are $\Theta = (\kappa, \theta, \lambda, \mu_J, \sigma_J, \upsilon, \rho, \gamma, \delta_S, \kappa^*, \phi^*)$.

Log-likelihood function

Denote the observed data series by $X_{t_i} = (\ln S_{t_i}, \mathrm{VIX}_{t_i})$. Let $\widehat{Y}_{t_i}(\Theta) = (\ln S_{t_i}, \widehat{V}_{t_i}(\Theta))$ where $\widehat{V}_{t_i}(\Theta)$ is the inverted value evaluated at parameter value Θ.

$$\begin{aligned} & \mathcal{L}\left(\Theta; X_{t_1}, \cdots, X_{t_N}\right) \\ = \; & \sum_{i=1}^{N} \ln f\left(\widehat{Y}_{t_i}(\Theta) \middle| \widehat{Y}_{t_{i-1}}(\Theta); \Theta\right) - N \ln\left(\frac{1 - e^{-\kappa^* \tau}}{\kappa^* \tau}\right) \end{aligned}$$

where

$$f\left(\widehat{Y}_{t_i}(\Theta) \middle| \widehat{Y}_{t_{i-1}}(\Theta); \Theta\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda h_i} (\lambda h_i)^j}{j!} g\left(\mathbf{w}_{t_i}(j, \Theta); \mathbf{0}, \mathbf{\Omega}_{t_i}(j, \Theta)\right),$$

The data

The S&P 500 index values, the CBOE's VIX index values and the one-month LIBOR rates on daily frequency over the period from January 2, 1990 to August 31, 2007.

	S&P500 return	VIX
Mean	0.00032	18.9148
Standard deviation	0.0099	6.4125
Skewness	-0.1230	0.9981
Excess Kurtosis	3.8780	0.8217
Maximum	0.0557	45.7400
Minimum	-0.0711	9.3100

The S&P 500 index, the VIX index and the corresponding realized volatility

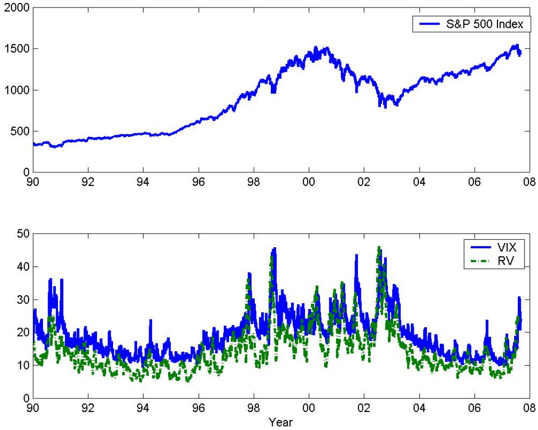

Table 2. MLE Results for SV models without jumps (the whole sample)

	q	κ	θ	v	ρ	γ	δ_S	κ^*	δ_V	LR
Sample period: 1990/1/2 – 2007/8/31										
SV0	-0.0788	0.8309	0.0472	1.3873	-0.6916	0.8936	-2.0863	-10.7595	-11.5905	
	(0.0378)	(0.6342)	(0.0334)	(0.0523)	(0.0059)	(0.0116)	(2.1030)	(0.4877)	(0.6279)	
SV1	-0.1632	0.0202	0.6077	1.9993	-0.6894	1	-4.3949	-11.9671	-11.9873	43.4414
	(0.0403)	(0.5860)	(17.6066)	(0.0194)	(0.0059)		(2.0686)	(0.4143)	(0.6069)	($p < 0.01$)
SV2	0.0812	5.2337	0.0265	0.3883	-0.6699	1/2	4.8969	-5.4592	-10.6928	923.7597
	(0.0258)	(0.5102)	(0.0026)	(0.0071)	(0.0064)		(2.1071)	(0.5577)	(0.6801)	($p < 0.01$)

Note:

- SV0 denotes the stochastic volatility model with unconstrained γ.
- SV1 denotes the stochastic volatility model with $\gamma = 1$.
- SV2 denotes the stochastic volatility model with fixed $\gamma = 1/2$.
- The volatility risk premium δ_V is computed as $\kappa^* - \kappa$.

Table 3. MLE Results for SV models with jumps (the whole sample)

	SV0	SVJ0	SVJ1	SVJ2
q	-0.0788	-0.0433	-0.0422	0.0039
	(0.0378)	(0.0540)	(0.0588)	(0.0367)
κ	0.8309	2.7245	2.7417	1.9449
	(0.6342)	(0.9331)	(0.9349)	(0.6987)
θ	0.0472	0.0228	0.0226	0.0472
	(0.0334)	(0.0050)	(0.0046)	(0.0140)
λ		54.3639	35.2252	43.9476
		(9.7152)	(6.8539)	(6.4716)
$\mu_J(\%)$		0.3696	0.4715	0.2825
		(0.0619)	(0.0836)	(0.0525)
$\sigma_J(\%)$		0.6634	0.7857	0.6284
		(0.0410)	(0.0513)	(0.0435)
υ	1.3873	1.4524	1.8942	0.4285
	(0.0523)	(0.0638)	(0.0193)	(0.0081)
ρ	-0.6916	-0.7895	-0.7813	-0.7517
	(0.0059)	(0.0082)	(0.0078)	(0.0076)
γ	0.8936	0.9098	1	1/2
	(0.0116)	(0.0131)		
δ_S	-2.0863	-0.1960	-0.0398	-0.1299
	(2.1030)	(2.7461)	(2.8559)	(2.1412)
κ^*	-10.7595	-13.4369	-14.8067	-4.2866
	(0.4877)	(0.5411)	(0.4935)	(0.6333)
$\phi^*(\%)$		-0.0892	-0.2322	0.2187
		(0.0393)	(0.0371)	(0.0424)

	SV0	SVJ0	SVJ1	SVJ2
δ_V	-11.5905	-16.1614	-17.5484	-6.2315
	(0.6279)	(1.0231)	(0.9801)	(0.9555)
$\delta_J(\%)$		-0.2464	-0.3806	0.1142
		(0.0637)	(0.0613)	(0.0522)
Log-Lik	37313.0192	38899.5289	38893.5489	38463.1233

Note:

- The reported estimates for μ_J, σ_J, ϕ^* and δ_J have been multiplied by 100.
- SVJ0 denotes the stochastic volatility model with jumps and an unconstrained γ.
- SVJ1 denotes the stochastic volatility model with jumps and $\gamma = 1$.
- SVJ2 denotes the stochastic volatility model with jumps and $\gamma = 1/2$.
- δ_V and δ_J are computed by $\kappa^* - \kappa$ and $\phi^* - \lambda(e^{\mu_J + \sigma_J^2/2} - 1 - \mu_J)$.

Conclusions

1. Incorporating a jump risk factor is critically important.
2. Both the jump and volatility risks are priced.
3. The popular square-root stochastic volatility process is a poor model specification irrespective of allowing for price jumps or not.