Ito's Lemma:

Continuous functions of Ito variables also follow Ito process if at least once differentiable in t and twice differentiable and twice differentiable in x.

$$dx = a(x,t)dt + b(x,t)dz$$
$$F(x,t) \sim \text{Ito Process}$$

$$dF = [a(x,t)F_x + F_t + \tfrac{1}{2}b^2(x,t)F_{xx}]dt + F_x b(x,t)dz$$

Examples:

(i)

$$ds = \mu s dt + \sigma(t) s dW$$
$$F(s,t) = \ln s$$
$$F_s = 1/s$$
$$F_{ss} = -1/s^2$$
$$F_t = 0$$

$$dF = [\mu s \tfrac{1}{s} + 0 + \tfrac{1}{2}(\sigma(t)s)^2(-\tfrac{1}{s^2})]dt + \sigma(t)s\tfrac{1}{s}dW = (\mu - \tfrac{1}{2}\sigma^2(t))dt + \sigma(t)dW$$

(ii)

Suppose

$$ds = \mu s dt + \sigma(t) s dW$$
$$F(s,t) = \ln s$$

(iii) two random terms

Black-Scholes Formula:

(I)Assumptions:

(i)The stock price follows geometric Brownian motion.

(ii)The short selling of securities with full use of proceeds is permitted

(iii)There are no transactions costs or taxes. All securities are perfectly divisible.

(iv)There are no riskless arbitrage opportunities.

(v)Security trading is continuous.

(vi)There are no dividends during the life of the derivative.

(vii)The risk-free rate of interest, r, is constant and the same for all maturities.

(II)BS PDE:

(i)Non-dividend Case:

We can form a risk-free portfolio by:

-1: derivative

$+\frac{\partial f}{\partial s}$**: shares**

The value of the portfolio is

$$\Pi = -f + \frac{\partial f}{\partial s} s$$

And the change in value is

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial s}\Delta s = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right)\Delta t$$

Because this equation does not involve random term, the portfolio must be riskless during time Δt**. Hence, the following equation must be satisfied.**

$$\Delta\Pi = r\Pi\Delta t$$

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial s}s\right)\Delta t$$

So that we have the Black-Scholes PDE

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf$$

For European call option, the boundary condition is

$f = \max(s - X, 0)$, when t=T

For European put option, the boundary condition is

$f = \max(X - s, 0)$, when t=T

(ii)Dividend Case:

Similar to the non-dividend case, we can form a portfolio by:

-1: derivative

$+\frac{\partial f}{\partial s}$**: shares**

And we have the following:

$$\Pi = -f + \frac{\partial f}{\partial s} s$$

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial s}\Delta s = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right)\Delta t$$

In time Δt **the holder of the portfolio earns capital gains equal to** $\Delta\Pi$ **and dividends on the stock position equal to**

$$qs\frac{\partial f}{\partial s}\Delta t$$

Define ΔW **as the change in the wealth of the portfolio holder in time** Δt**. It follows that**

$$\Delta W = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2 + qs\frac{\partial f}{\partial s}\right)\Delta t$$

Because this expression is independent of the Wiener process , the portfolio is instantaneously riskless. Hence

$$\Delta W = r\Pi\Delta t$$

So that

$$\frac{\partial f}{\partial t} + (r-q)s\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 f}{\partial s^2} = rf$$

This is the Black-Scholes PDE for dividend case.

Proof of Black-Scholes Formula:

Key Result:

If V is a lognormally distributed and the standard deviation of lnV is s then

$$E[\max(V-X,0)] = E(V)N(d_1) - XN(d_2)$$

where

$$d_1 = \frac{\ln[E(V)/X] + s^2/2}{s}$$

$$d_2 = \frac{\ln[E(V)/X] - s^2/2}{s}$$

and E denotes expected value.

Proof of Key Result:

Define g(V) as the probability density function of V. It follows that

$$E[\max(V-X),0] = \int_X^\infty (V-X)g(V)dV$$

The variable lnV is normally distributed with standard deviation s. From the properties of the lognormal distribution the mean of lnV is m where

$$m = \ln[E(V)] - s^2/2$$

Define a new variable

$$Q = \frac{\ln V - m}{s}$$

This variable is normally distributed with a mean of zero and a standard deviation of 1. Denote the density function for Q by h(Q) so that

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$$

By change of variable, we get:

$$\hat{E}[\max(V-X),0] = \int_{(\ln X - m)/s}^\infty (e^{Qs+m} - X)h(Q)dQ$$

Now

$$e^{Qs+m}h(Q) = \frac{1}{\sqrt{2\pi}} e^{(-Q^2+2Qs+2m)/2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{(-(Q-s)^2+s^2+2m)/2}$$

$$= \frac{e^{m+s^2/2}}{\sqrt{2\pi}} e^{[-(Q-s)^2]/2}$$

$$= e^{m+s^2/2}h(Q-s)$$

This means

$$\hat{E}[\max(V - X), 0] = e^{m+s^2/2}\int_{(\ln X - m)/s}^{\infty} h(Q-s)dQ - X\int_{(\ln X - m)/s}^{\infty} h(Q)dQ \ldots(*)$$

If we define N(x) as the probability that a variable with a mean of zero and a standard deviation of 1 is less than x, the first integral in the above equation (*) is

$$1 - N[(\ln X - m)/s - s]$$

or

$$N[(-\ln X + m)/s + s]$$

Substituting for m, this becomes

$$N[\frac{\ln[E(V)/X] + s^2/2}{s}] = N(d_1)$$

Similarly the second integral in equation (*) is $N(d_2)$**.**

Therefore

$$\hat{E}[\max(V - X, 0)] = e^{m+s^2/2}N(d_1) - XN(d_2)$$

Substituting for m, the key result follows.

The Black-Scholes Result:

We now consider a call option on a non-dividend-paying stock maturing at time T. The strike price is X, the risk-free rate is r, the current stock price is S_0**, and the volatility is** σ**. We know**

$$C = e^{-rT}\hat{E}[\max(S_T - X, 0)]$$

where S_T **is the stock price at time T and** $\hat{E}$ **denotes expectations in a risk-neutral world. Under the stochastic process assumed by BS,** S_T **is lognormal. Also we know** $\hat{E}(S_T) = S_0 e^{rT}$ **and the standard deviation of** $\ln S_T$ **is** $\sigma\sqrt{T}$ **.**

From the key result just proved, we have:

$$C = S_0 N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$