Definition of Random Variable

In simple terms, a random variable (also referred to as a stochastic variable) is a real-valued set function whose value is a real number determined by the outcome of an experiment. The range of a random variable is the set of all the values it can assume. More formally, in measure theoretic terms, a random variable X is a real-valued function that maps $\mathcal{S}$ into $\mathcal{R}$ and satisfies the condition that for every Borel set B $\in \mathcal{B}$ the inverse image $X^{-1}(B) \in \mathcal{F}$, where $X^{-1}(B) = \{s : s \in \mathcal{S}$ and X(s) $\in$ B}

Definition of Cumulative Density Function

The real-valued function F(x) such that F(x)= $P_x\{(-\infty, x]\} = P(X <= x)$ for each x $\in \mathcal{R}$ is called the distribution function, also known as the cumulative distribution/density function, or CDF.

Definition of Probability Density Function

For a random variable X if there exists a nonnegative function f(x), defined on the real line, such that for an interval B, $P(X \in B) = \int_B f(x)dx$, then X is said to have a continuous distribution and the function f(x) is called the probability density function or simply the density function (or PDF).

Moreover, for a continuous random variable, we can find the following properties

(i) $F(x) = \int_{-\infty}^{x} f(u)du$

(ii) $f(x) = F'(x)$

(iii) $\int_{-\infty}^{\infty} f(u)du = 1$

(iv) $F(b) - F(a) = \int_{a}^{b} f(u)du$

Definition of Normal Distribution

Normal distribution has the following density:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{(x-\mu)^2}{2\sigma^2}], \quad x \in \mathcal{R}$$

The distribution is written as $X \sim N(\mu, \sigma^2)$. The special case of the normal distribution for μ=0 and σ=1 is known as the standard normal and it has the following properties:

(i) $E(X) = 0$

(ii) $V(X) = 1$

(iii) $E(X^3) = 0$

(iv) $E(X^4) = 3$

Definition of Lognormal Distribution

The Lognormal distribution can be extended by letting $Y = \ln X \sim N(\mu, \sigma^2)$. X has the following PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{(\ln x - \mu)^2}{2\sigma^2}]\frac{1}{x}, \quad x >= 0$$

Furthermore, we have:

(i) $E(X) = e^{\mu + \sigma^2/2}$

(ii) $V(X) = e^{2\mu}[e^{2\sigma^2} - e^{\sigma^2}]$

Definition of Convergence in Distribution

Given a sequence of random variables X_n whose CDF is $F_n(x)$, and a CDF $F_X(x)$ corresponding to the random variable X, we say that X_n converges in distribution to X, and write $X_n \xrightarrow{d} X$, if $\lim_{n\to\infty} F_n(x) = F_X(x)$ at all points x at which $F_n(x)$ is continuous.

Definition of Convergence in Probability

The sequence of random variables X_n is said to convergence in probability to the real number x if $\lim_{n\to\infty} P[|X_n - x| > \varepsilon] = 0$ for each $\varepsilon > 0$. Thus it becomes less and less likely that the random variable $X_n - x$ lies outside the interval $(-\varepsilon, +\varepsilon)$. The sequence if random variables X_n is said to converge in probability to the random variable X if the sequence of their difference $X_n - X$ converges in probability to 0.

We write $X_n \xrightarrow{p} X$.

Definition of Convergence in mean-squared

The sequence of random variables X_n is said to convergence in mean-squared to X and designated $X_n \xrightarrow{(2)} X$ or $X_n \xrightarrow{M.S.} X$, if $E[|X_n - X|^2]$ exists and $\lim_{n\to\infty} E[|X_n - X|^2] = 0$, that is, if the 2nd moment of the difference tends to zero.

Definition of Convergence Almost Surely

The sequence of random variables X_n is said to convergence almost surely to the real numbers x, and is written as $X_n \xrightarrow{a.s.} x$, if $P[\lim X_n = x] = 1$. In other words, the sequence X_n may not converge everywhere to x, but the points where it does not converge form a set of measure zero in the probability sense. X_n is said to converge almost surely to the random variable X if $X_n - X \xrightarrow{a.s.} 0$.

Relationships among Modes of Convergence

We have the following relationships among the above convergence modes (the black arrows from A to B mean that "A implies B"):

$$
\begin{tikzcd}
\boxed{\text{Convergence almost surely}} \arrow[dr] & & \boxed{\text{Convergence in mean-squared}} \arrow[dl] \\
& \boxed{\text{Convergence in probability}} \arrow[d] & \\
& \boxed{\text{Convergence in distribution}} &
\end{tikzcd}
$$

Counter examples:

(i) Suppose the sample space is $S = [0,1]$ and X_n is a sequence of random variables defined as follows:

$$X_n(s) = \begin{cases} 1, & 0 \le s \le \frac{1}{2} \\ 0, & \frac{1}{2} < s \le 1 \end{cases}$$

With a probability of one-half for each case. The corresponding sequence of distribution functions is

$$F_{X_n}(s) = \begin{cases} 0, & s < 0 \\ \frac{1}{2}, & 0 \le s \le \frac{1}{2} \\ 1, & s > \frac{1}{2} \end{cases}$$

Next suppose the random variable X is defined as

$$X(s) = \begin{cases} 0, & 0 \le s \le \frac{1}{2} \\ 1, & \frac{1}{2} < s \le 1 \end{cases}$$

with a probability of one-half for each case. It's easily verified that the corresponding CDF is the same as that for X_n, which means that $X_n \xrightarrow{d} X$. However, $X_n - X$ is always equal to 1 and hence X_n does not converge in probability to X.

(ii) $X \sim U[0,1]$ and binary intervals $I_1 = [0,1], I_2 = [0,\frac{1}{2}], \ldots$ so that

$$I_{2^m+i} = [\tfrac{i}{2^m}, \tfrac{i+1}{2^m}]$$

for m=0,1,2,… and i=0,1,…, 2^m-1. Therefore, the 2^m intervals of length 2^{-m} cover the interval [0,1].

Let $Y_n = 1$ if $X \in I_n$ and Y_n =0 if $X \notin I_n$. The sequence $Y_1, Y_2, \ldots$ converges in probability to Y=0 since, for all $0 < \varepsilon <= 1$, $P(|Y_n - 0| >= \varepsilon) = \Pr(Y_n = 1) = P(X \in I_n)$ =length of interval $I_n \to 0, when$ $n \to \infty$. Y_n does not converge almost surely to zero since any X is in only one of the 2^n intervals of length 2^{-n}, for all n. In other words, for all $\omega \in \Omega$, $Y(\omega)$ assumes the value 1 for an infinite number of n's and assumes the value 0 for an infinite number of n's. Hence, for each $\omega \in \Omega$, $Y(\omega)$ does not converge.

(iii) Let $X_n = \begin{cases} 0, p = 1 - \frac{1}{n} \\ n, q = \frac{1}{n} \end{cases}$ hence we have $X_n \xrightarrow{p} X$. However,

$E[(X_n - 0)^2] = n^2 \frac{1}{n} \to \infty$, as $n \to \infty$.

(iv) Let $\omega \sim U(0,1)$, and $X_n(\omega) = n1_{\{(0,\frac{1}{n})\}}(\omega)$. $\forall \omega \in [0,1]$ and $\varepsilon > 0$, there $\exists N_0$ s.t. $n >= N_0$ with $\frac{1}{n} < \omega$ and $|X_n(\omega) - 0| < \varepsilon$. It implies that $X_n \xrightarrow{a.s.} 0$. However, $E[|X_n - 0|^2] = 1 \not\to 0$. Thus, X_n does not converge to 0 in mean-squared sense.

The Ito Integral

We are sufficient to prove $\sum(\mathrm{dw})^2 \xrightarrow{\mathrm{M.S.}} \mathrm{t}$.

Let $\mathrm{Q} = \sum(\mathrm{dw})^2 = \sum \Delta \mathrm{z}^2$, where $\mathrm{dw} = \sqrt{\Delta}\mathrm{z}$ $\quad \mathrm{z} \sim \mathrm{N}(0,1)$ And $\Delta = \frac{\mathrm{t}}{\mathrm{n}}$

$E[Q] = t$

$\mathrm{Var}(\mathrm{Q}) = \mathrm{nVar}(\Delta \mathrm{z}^2) = \mathrm{n}\Delta^2\mathrm{Var}(\mathrm{z}^2)$

$\mathrm{Var}(\mathrm{z}^2) = \mathrm{E}(\mathrm{z}^4) - \mathrm{E}^2(\mathrm{z}^2)$

$\qquad = 3 - 1 = 2$

$\therefore \mathrm{Var}(\mathrm{Q}) = 2\mathrm{n}\Delta^2 = 2\mathrm{n}\left(\frac{\mathrm{t}}{\mathrm{n}}\right)^2 \xrightarrow{\mathrm{n}\to\infty} 0$

Because $Var[Q] = E((Q-t)^2) \underset{n\to\infty}{\to} 0$, $\quad \sum(\mathrm{dw})^2 \xrightarrow{\mathrm{M.S.}} \mathrm{t}$