Lecture 8

Exotic Options

Types of Exotics

- **Package**
- **Nonstandard American options**
- **Forward start options**
- **Compound options**
- **Chooser options**
- **Barrier options**
- **Binary options**
- **Lookback options**
- **Shout options**
- **Asian options**
- **Options to exchange one asset for another**
- **Options involving several assets**
- **Volatility and Variance swaps**

Packages

- **Portfolios of standard options**
- **Examples from Chapter 10: bull spreads, bear spreads, straddles, etc**
- **Often structured to have zero cost**
- **One popular package is a range forward contract**

Non-Standard American Options

- **Exercisable only on specific dates (Bermudans)**
- **Early exercise allowed during only part of life (initial "lock out" period)**
- **Strike price changes over the life (warrants, convertibles)**

Forward Start Options

- **Option starts at a future time, T_1**
- **Implicit in employee stock option plans**
- **Often structured so that strike price equals asset price at time T_1**

Compound Option

- **Option to buy or sell an option**
  - **Call on call**
  - **Put on call**
  - **Call on put**
  - **Put on put**
- **Can be valued analytically**
- **Price is quite low compared with a regular option**

Chooser Option "As You Like It"

- **Option starts at time 0, matures at T_2**
- **At T1 ($0 < T_1 < T_2$) buyer chooses whether it is a put or call**
- **This is a package!**

Chooser Option as a Package

At time T_1 the value is max(c, p).

From the put-call parity, $p = c+e^{\wedge}(-r(T_2-T_1)K-S_1e^{\wedge}(-q(T_2-T_1))$

The value at time T_1 is therefore

$c+e^{\wedge}(-q(T_2-T_1))\max(0, Ke^{\wedge}(-(r-q)(T_2-T_1))-S_1)$

This is a call maturing at time T_2 plus a put maturing at time T_1.

Barrier Options

- **Option comes into existence only if stock price hits barrier before option maturity**
  - **'In' options**
- **Option dies if stock price hits barrier before option maturity**
  - **'Out' options**
- **Stock price must hit barrier from below**
  - **'Up' options**
- **Stock price must hit barrier from above**
  - **'Down' options**
- **Option may be a put or a call**
- **Eight possible combinations**

Parity Relations

$C = C_{ui} + C_{uo}$

$C = C_{di} + C_{do}$

$P = P_{ui} + P_{uo}$

$P = P_{di} + P_{do}$

Binary Options

- **Cash-or-nothing: pays Q if ST > K, otherwise pays nothing.**
  - **Value = $e^{(-rT)} Q N(d_2)$**
- **Asset-or-nothing: pays ST if ST > K, otherwise pays nothing.**
  - **Value = $S_0 e^{(-qT)} N(d_1)$**

Decomposition of a Call Option

Long Asset-or-Nothing option

Short Cash-or-Nothing option where payoff is K

Value = $S_0 e^{(-qT)} N(d_1) - e^{(-rT)} KN(d_2)$

Lookback Options

- **Floating lookback call pays $S_T - S_{min}$ at time T (Allows buyer to buy stock at lowest observed price in some interval of time)**
- **Floating lookback put pays $S_{max} - S_T$ at time T**

(Allows buyer to sell stock at highest observed price in some interval of time)

- **Fixed lookback call pays $\max(S_{max} - K, 0)$**
- **Fixed lookback put pays $\max(K - S_{min}, 0)$**
- **Analytic valuation for all types**

Shout Options

- **Buyer can 'shout' once during option life**
- **Final payoff is either**
  - **Usual option payoff, $\max(S_T - K, 0)$, or**
  - **Intrinsic value at time of shout, $S_t - K$**
- **Payoff: $\max(S_T - S_t, 0) + S_t - K$**
- **Similar to lookback option but cheaper**

- **How can a binomial tree be used to value a shout option?**

Asian Options

- **Payoff related to average stock price**
- **Average Price options pay:**
  - **Call: max(S_{ave} – K, 0)**
  - **Put: max(K – S_{ave} , 0)**
- **Average Strike options pay:**
  - **Call: max(S_T – S_{ave} , 0)**
  - **Put: max(S_{ave} – S_T , 0)**
- **No exact analytic valuation**
- **Can be approximately valued by assuming that the average stock price is lognormally distributed**
- **Closed form solution for arithmetic average price options:**

$$M_1 = \frac{e^{(r-q)T} - 1}{(r-q)T} S_0$$

$$M_2 = \frac{2e^{(2(r-q)+\sigma^2)T} S_0{}^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0{}^2}{(r-q)T^2}\left(\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right)$$

Let

$$F_0 = M_1$$

$$\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1}\right)$$

(Note these 2 σ s are not equivalent!)

We have

$$c = e^{-rT}(F_0 N(d_1) - KN(d_2))$$

$$p = e^{-rT}(KN(-d_2) - F_0 N(-d_1))$$

where

$$d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Convertible Bonds

- **Convertible bonds are regular bonds that can be exchanged for equity at certain times in the future according to a predetermined exchange ratio**
- **Very often a convertible is callable**
- **The call provision is a way in which the issuer can force conversion at a time earlier than the holder might otherwise choose**

Exchange Options

- **Option to exchange one asset for another**
- **For example, an option to exchange one unit of U for one unit of V**
- **Payoff is max(V_T – U_T, 0)**

Basket Options

- **A basket option is an option to buy or sell a portfolio of assets**
- **This can be valued by calculating the first two moments of the value of the basket and then assuming it is lognormal**

Volatility and Variance Swaps

- **Agreement to exchange the realized volatility between time 0 and time T for a prespecified fixed volatility with both being multiplied by a prespecified principal**
- **Variance swap is agreement to exchange the realized variance rate between time 0 and time T for a prespecified fixed variance rate with both being multiplied by a prespecified principal**
- **Daily return is assumed to be zero in calculating the volatility or variance rate**

Variance Swaps

- **The (risk-neutral) expected variance rate between times 0 and T can be calculated from the prices of European call and put options with different strikes and maturity T**
- **Variance swaps can therefore be valued analytically if enough options trade**
- **For a volatility swap it is necessary to use the approximate relation**

$$\hat{E}(\bar{\sigma}) = \sqrt{\hat{E}\left(\bar{V}\right)}\left\{1 - \frac{1}{8}\left[\frac{\operatorname{var}(\bar{V})}{\hat{E}(\bar{V})^2}\right]\right\}$$

VIX Index

- **The expected value of the variance of the S&P 500 over 30 days is calculated from the CBOE market prices of European put and call options on the S&P 500**
- **This is then multiplied by 365/30 and the VIX index is set equal to the square root of the result**

How Difficult is it to Hedge Exotic Options?

- **In some cases exotic options are easier to hedge than the corresponding vanilla options (e.g., Asian options)**
- **In other cases they are more difficult to hedge (e.g., barrier options)**

Static Options Replication

- **This involves approximately replicating an exotic option with a portfolio of vanilla options**
- **Underlying principle: if we match the value of an exotic option on some boundary , we have matched it at all interior points of the boundary**
- **Static options replication can be contrasted with dynamic options replication where we have to trade continuously to match the option**

Example

- **A 9-month up-and-out call option an a non-dividend paying stock where S0 = 50, K = 50, the barrier is 60, r = 10%, and s = 30%**
- **Any boundary can be chosen but the natural one is**
  c (S, 0.75) = MAX(S – 50, 0) when S < 60
  c (60, t) = 0 when 0 <= t <= 0.75
  We might try to match the following points on the boundary
  c(S , 0.75) = MAX(S – 50, 0) for S < 60
  c(60, 0.50) = 0
  c(60, 0.25) = 0
  c(60, 0.00) = 0
  We can do this as follows:
  +1.00 call with maturity 0.75 & strike 50
  –2.66 call with maturity 0.75 & strike 60
  +0.97 call with maturity 0.50 & strike 60
  +0.28 call with maturity 0.25 & strike 60
- **This portfolio is worth 0.73 at time zero compared with 0.31 for the**

up-and out option

- **As we use more options the value of the replicating portfolio converges to the value of the exotic option**
- **For example, with 18 points matched on the horizontal boundary the value of the replicating portfolio reduces to 0.38; with 100 points being matched it reduces to 0.32**

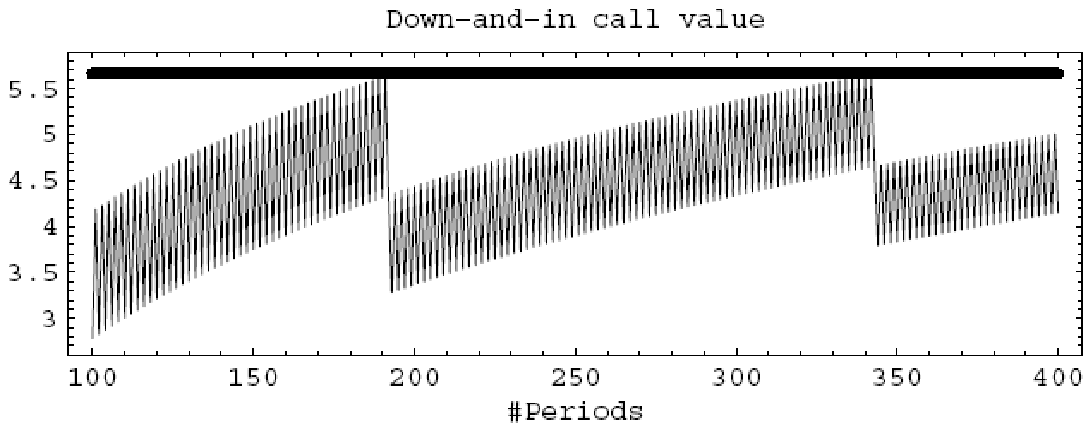
Using Static Options Replication

- **To hedge an exotic option we short the portfolio that replicates the boundary conditions**
- **The portfolio must be unwound when any part of the boundary is reached**

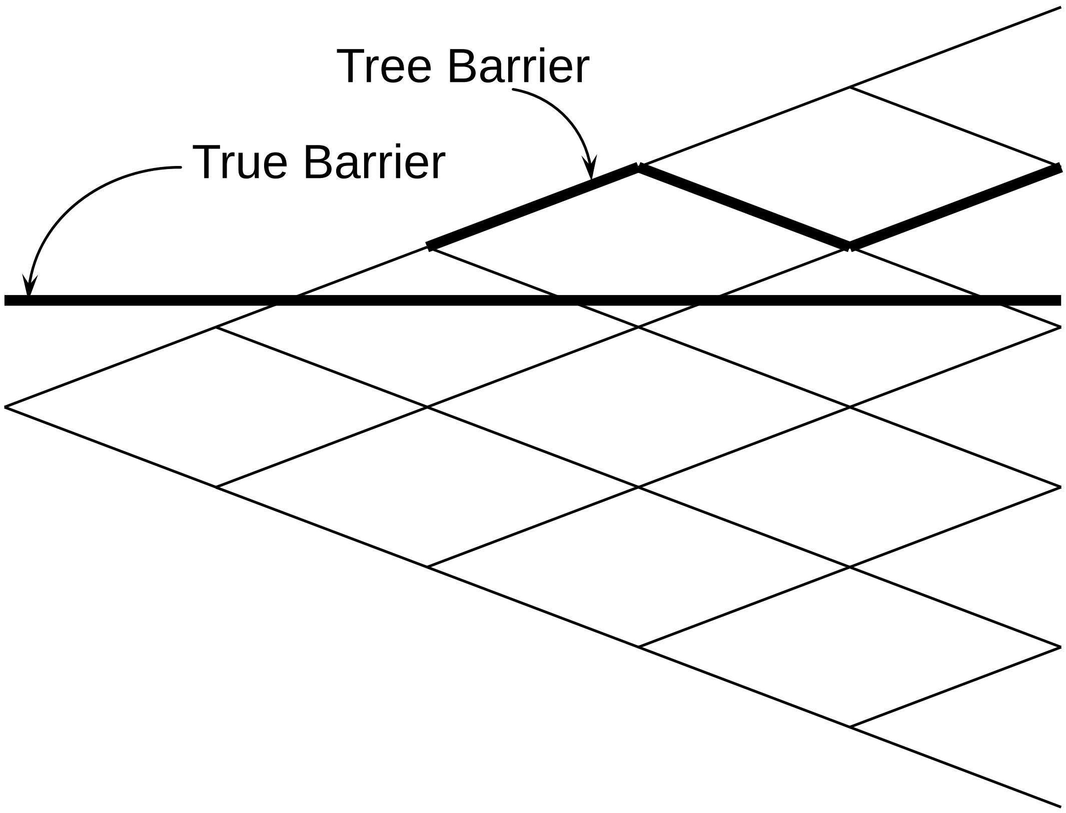
Numerical Procedures

Using Trees with Barriers

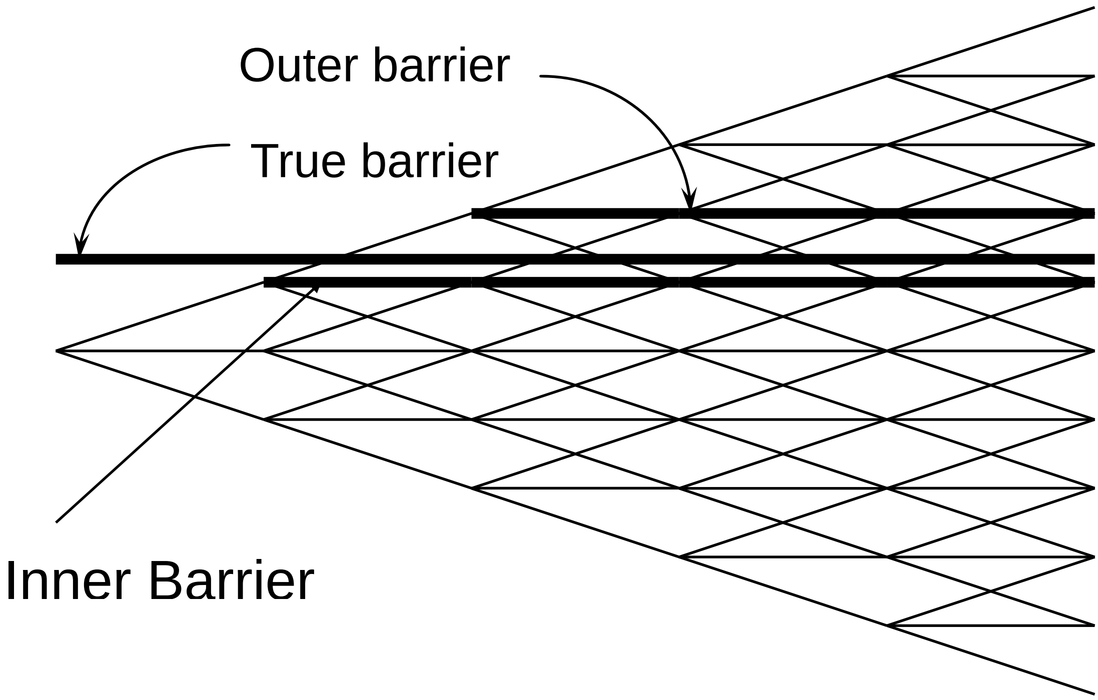
- **When trees are used to value options with barriers, convergence tends to be slow**
- **The slow convergence arises from the fact that the barrier is inaccurately specified by the tree**



True Barrier vs. Tree Barrier for a Knockout Option: The Binomial Tree Case



Inner and Outer Barriers for Trinomial Trees



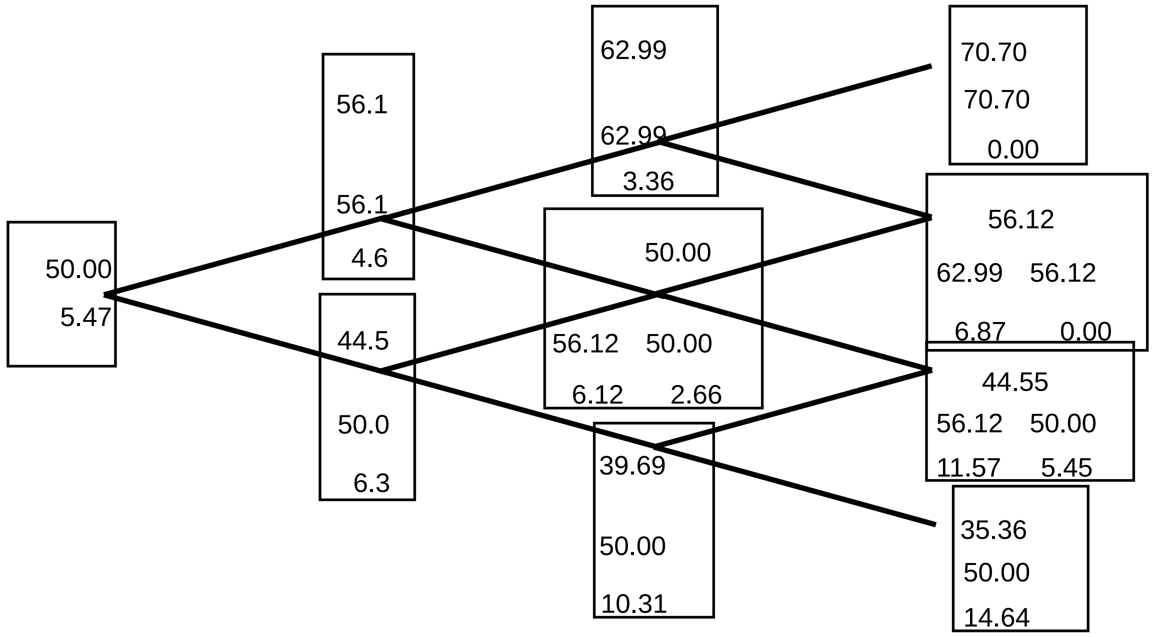
Alternative Solutions to Valuing Barrier Options

- **Interpolate between value when inner barrier is assumed and value when outer barrier is assumed**
- **Ensure that nodes always lie on the barriers**
- **Use adaptive mesh methodology**

In all cases a trinomial tree is preferable to a binomial tree

Lookback Options

- **Consider an American lookback put on a stock where S = 50, s = 40%, r = 10%, Δt = 1 month & the life of the option is 3 months**
- **Payoff is $S_{max}-S_T$**
- **We can value the deal by considering all possible values of the maximum stock price at each node**



Why the Approach Works

This approach works for lookback options because

- **The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a "path function")**
- **The value of the path function at a node can be calculated from the stock price at the node & from the value of the function at the immediately preceding node**
- **The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree**