1. In this note we shall go over briefly the distribution-based and preference-based fund separation theorems, and two linear pricing models in static economies, namely the CAPM (Capital Asset Pricing Model) and the APT (Arbitrage Pricing Theory).

2. A developing economy typically has highly incomplete financial markets. Market incompleteness, however, will not cause much loss in welfare if, given a large number of (and hence a nearly complete set of) traded assets, when we vary investors’ initial wealth and/or preferences, investors’ optimal portfolios are always portfolios of a small number of fixed portfolios, for when that happens, most financial markets can be closed down without affecting trading efficiency, as long as those fixed portfolios are still available for trading. We shall focus on the situation where investors’ optimal portfolios are always portfolios of two fixed portfolios, and we refer to this situation by saying that two-fund separation holds. If one of the separating portfolios is riskless, then we shall refer to that riskless portfolio as money, and we say that monetary separation holds. When two-fund separation holds, a developing economy can still attain full trading efficiency if two properly chosen financial assets are available for trading.

3. We shall start with the definition and implications of distribution-based two-fund separation, which lead to the well-known Capital Asset Pricing Models (the CAPMs), and then we shall compare the CAPMs to the APT. Finally we review the conditions on the primitives of the economy that ensure that two-fund separation holds in equilibrium. In that regard, financial economists have found restrictions on return distributions alone (Ross 1978; Litzenberger and Ramaswamy 1979; Chamberlain 1983; Owen and Rabinovitch 1983), and restrictions on preferences
alone (Cass and Stiglitz 1970), that imply two-fund separation. The former and the latter restrictions lead to respectively the distribution-based and the preference-based separation theorems.

4. Consider a two-period economy with perfect financial markets, where \( I \) price-taking investors can trade \( N \) risky assets and perhaps a riskless asset also. When the riskless asset exists, we shall refer to it as asset 0. Let \( q_j \) be the net supply of asset \( j \), and \( p_j \) be the equilibrium price of asset \( j \), for all \( j = 0, 1, \ldots, N \). Assets generate consumption (or cash flows) at date 1. An investor \( i \) is endowed with some traded securities whose date-0 value is \( W_i > 0 \), and he consumes only at date 1. Thus \( W_i \) is also the date-0 value of investor \( i \)'s equilibrium asset holdings. Define

\[
W_m \equiv \sum_{i=1}^{I} W_i,
\]

which denotes the aggregate wealth at date 0.

**Definition 1** In the absence of the riskless asset, a portfolio is a vector \( w_{N \times 1} \) with

\[
w'1 = 1,
\]

where \( ' \) stands for matrix transpose, and \( 1_{N \times 1} \) is the vector of which all elements are equal to one. Also, a vector \( w_{N \times 1} \) with \( w'1 = 0 \) is referred to as an arbitrage portfolio. In the presence of a riskless asset with rate of return \( r_f \), a portfolio is any vector

\[
x_{(N+1) \times 1} = \begin{bmatrix} 1 - w'1 \\ w \end{bmatrix},
\]

\(^1\)Suppose that markets are complete, and consider how an investor’s favorite portfolio of all traded assets may vary with his initial wealth. If no matter how his initial wealth varies, his favorite portfolio can always be represented as a small number of fixed portfolios, then we say that his consumption-and-investment behavior exhibits fund separation. Here the term separation stems from the fact that, in solving for his utility-maximizing consumption and investment decisions, the investor can first find his favorite portfolio and then decide how to allocate his initial wealth to his favorite portfolio and to current consumption. That is, the investor’s optimal portfolio and his optimal consumption problems can be solved separately. See for example section 2.2 of Rubinstein, M., 1974, An Aggregation Theorem for Securities Markets, *Journal of Financial Economics*, 1, 225-244.
where the first element of $\mathbf{x}$ is the portfolio weight for the riskless asset (asset 0). Correspondingly, an arbitrage portfolio is a vector

$$\mathbf{x}_{(N+1)\times 1} = \begin{bmatrix} -\mathbf{w}' \mathbf{1} \\ \mathbf{w} \end{bmatrix}. $$

Thus a portfolio is simply a way to allocate an investor’s initial wealth on the traded assets, and an arbitrage portfolio is a way to take positions in multiple assets without costing an investor anything at date 0. Note that for all $j = 1, 2, \ldots, N$, asset $j$ can also be represented as a portfolio, which has only one non-zero element appearing at the $j$-th place.

For all $j = 1, 2, \ldots, N$, let $\tilde{r}_j$ be the rate of return on risky asset $j$, and let $\mathbf{e}_{N\times 1}$ be the column vector of which the $j$-th element is $E[\tilde{r}_j]$. Define also $\mathbf{V}_{N\times N}$ as the square matrix of which the $(k, j)$-th element (that appears on the $k$-th row and the $j$-th column) is $\text{cov}(\tilde{r}_k, \tilde{r}_j)$. We call $\mathbf{V}$ the covariance matrix for the random vector $\mathbf{\tilde{r}}$. Let $\tilde{r}_w$ be the (random) rate of return on the portfolio $\mathbf{w}$; that is, if you spend 1 dollar to hold the portfolio $\mathbf{w}$ at time 0, then you will get $1 + \tilde{r}_w$ dollars at time 1.

**Theorem 1** The following statements are true.

(i) $\mathbf{V} = E[(\mathbf{\tilde{r}} - \mathbf{e})(\mathbf{\tilde{r}} - \mathbf{e})']$.

(ii) $\mathbf{V}$ is positive semi-definite. $\mathbf{V}$ is positive definite if and only if it is impossible to form a riskless portfolio of the $N$ risky assets. In case $N = 2$, such a riskless portfolio can be constructed if and only if $\tilde{r}_1$ and $\tilde{r}_2$ are perfectly correlated.

(iii) The expected value of $\tilde{r}_w$ is $\mathbf{w}' \mathbf{e}$. The variance of $\tilde{r}_w$ is $\mathbf{w}' \mathbf{V} \mathbf{w}$. The covariance of $\tilde{r}_{w_1}$ and $\tilde{r}_{w_2}$ is $\mathbf{w}_1' \mathbf{V} \mathbf{w}_2$.

**Proof.** Consider part (i). Note that the $(k, j)$-th element of the matrix $E[(\mathbf{\tilde{r}} - \mathbf{e})(\mathbf{\tilde{r}} - \mathbf{e})']$ is

$$E[(\tilde{r}_k - E[\tilde{r}_k])(\tilde{r}_j - E[\tilde{r}_j])],$$

which is exactly the definition of $\text{cov}(\tilde{r}_k, \tilde{r}_j)$. 


Next, consider part (ii). Pick any $x_{n \times 1}$, and observe that

$$
x'Vx = x'E[(\tilde{r} - e)(\tilde{r} - e)']x
= E[x' (\tilde{r} - e)(\tilde{r} - e)']
x = E[(x'\tilde{r} - x'e)((\tilde{r}'x - e'x)]
= E[(x'\tilde{r} - E[x'\tilde{r}])^2] = \text{var}[x'\tilde{r}] \geq 0.
$$

Thus by definition, $V$ is positive semi-definite. Note also that an equivalence condition for $V$ not to be positive definite is the existence of $x_{n \times 1} \neq 0$ such that

$$\text{var}[x'\tilde{r}] = x'Vx = 0,$$

so that $x$ is either an arbitrage portfolio generating a sure return, or $\frac{x}{x'1}$ is a riskless portfolio.

Now, if $N = 2$, we have

$$V_{2 \times 2} = \begin{bmatrix}
\text{var}[\tilde{r}_1] & \text{cov}(\tilde{r}_1, \tilde{r}_2) \\
\text{cov}(\tilde{r}_1, \tilde{r}_2) & \text{var}[\tilde{r}_2]
\end{bmatrix},$$

and if $V$ is singular, then its determinant must equal zero, and so

$$\frac{\text{cov}(\tilde{r}_1, \tilde{r}_2)^2}{\text{var}[\tilde{r}_1]\text{var}[\tilde{r}_2]} = 1,$$

and so the coefficient of correlation for $(\tilde{r}_1, \tilde{r}_2)$ equals either 1 or $-1$.

Finally, consider part (iii). By definition of rate of return, we have

$$\tilde{r}_w = \frac{1 + w'\tilde{r}}{1} - 1 = w'\tilde{r} = \sum_{j=1}^{N} w_j \tilde{r}_j.$$

Hence we have

$$E[\tilde{r}_w] = E[\sum_{j=1}^{N} w_j \tilde{r}_j] = \sum_{j=1}^{N} w_j E[\tilde{r}_j] = w'e.$$

Similarly, mimicking the proof for part (ii), one can easily show that

$$\text{var}[\tilde{r}_w] = w'Vw.$$
Finally, note that
\[
\begin{align*}
    w_1' V w_2 &= E[(w_1' \tilde{r} - w_1' e)(w_2' \tilde{r} - w_2' e)'] \\
    &= E[(w_1' \tilde{r} - w_1' e)(w_2' \tilde{r} - w_2' e)] = \text{cov}(w_1' \tilde{r}, w_2' \tilde{r}).
\end{align*}
\]
This finishes the proof.

5. Now we define the distributional two-fund separation.

**Definition 2** The equilibrium rates of return $\tilde{r}$ exhibit two-fund separation if and only if there exist two portfolios $w_1$ and $w_2$ such that for any feasible portfolio $w$, there exists a constant $\lambda(w) \in \mathbb{R}$ such that
\[
\lambda(w)w_1' \tilde{r} + (1 - \lambda(w))w_2' \tilde{r} \geq_{SSD} w' \tilde{r}.
\]
That is, for each portfolio $w$, we can find a portfolio
\[
\lambda(w)w_1 + (1 - \lambda(w))w_2,
\]
which is composed of only the two portfolios $w_1$ and $w_2$, such that all risk-averse investors prefer $\lambda(w)w_1 + (1 - \lambda(w))w_2$ to $w$.

Here recall the definition and the main theorem of second-order stochastic dominance:

**Theorem 2** A random terminal wealth $\tilde{x}$ stochastically dominates another random terminal wealth $\tilde{y}$ if and only if one of the following three equivalence conditions holds:

1. $E[u(\tilde{x})] \geq E[u(\tilde{y})]$ for all concave $u : \mathbb{R} \to \mathbb{R}$;
2. $E[\tilde{x}] = E[\tilde{y}]$ and for all $z \in \mathbb{R}$, $\int_{-\infty}^{\tilde{x}} [F_x(t) - F_y(t)] dt \leq 0$, where $F_j$ is the distribution function of $j$, for $j = x, y$;
3. There exists a random variable $\tilde{e}$ such that for each and every realization $x$ of $\tilde{x}$, $E[\tilde{e}|x] = 0$, and moreover, $F_y$ is also the distribution function of $\tilde{x} + \tilde{e}$.

The following proposition follows from the preceding theorem immediately.
Proposition 1 If $\tilde{x} \geq_{SSD} \tilde{y}$, then $E[\tilde{x}] = E[\tilde{y}]$ and $\text{var}[\tilde{x}] \leq \text{var}[\tilde{y}]$.

Proof. By the preceding theorem, we know that for some $\tilde{e}$ with $E[\tilde{e}|\tilde{x}] = 0$ the two random variables $\tilde{y}$ and $\tilde{x} + \tilde{e}$ have the same distribution function, and hence they have the same variance and expected value also. Now, observe that

$$E[\tilde{y}] = E[\tilde{x}] + E[\tilde{e}] = E[\tilde{x}] + E[E[\tilde{e}|\tilde{x}]] = E[\tilde{x}],$$

where the second equality follows from the law of iterated expectations. Hence $E[\tilde{e}] = 0$. Now, observe also that

$$\text{cov}(\tilde{e}, \tilde{x}) = E[\tilde{e}\tilde{x}] - E[\tilde{x}]E[\tilde{e}]$$

$$= E[\tilde{e}\tilde{x}] = E[E[\tilde{e}\tilde{x}]|\tilde{x}] = E[\tilde{x}E[\tilde{e}|\tilde{x}]]$$

$$= E[\tilde{x} \cdot 0] = 0,$$

so that

$$\text{var}[\tilde{y}] = \text{var}[\tilde{x} + \tilde{e}] = \text{var}[\tilde{x}] + \text{var}[\tilde{e}] + 2\text{cov}(\tilde{e}, \tilde{x})$$

$$= \text{var}[\tilde{x}] + \text{var}[\tilde{e}] \geq \text{var}[\tilde{x}],$$

where the inequality follows from $\text{var}[\tilde{e}] \geq 0$. 

The definition of two-fund separation and the preceding proposition together imply that if two-fund separation holds in equilibrium, then an investor’s equilibrium portfolio must have the minimum variance of return in the class of portfolios promising the same expected rate of return. This brings us to our next task, which is to characterize the set of portfolios each having the minimum return variance among the portfolios with the same expected rate of return.\footnote{Alternatively, if each investor is endowed with a mean-variance utility function $U(E[\tilde{W}], \text{var}[\tilde{W}])$, where $U(\cdot, \cdot)$ is increasing in its first argument and decreasing in its second argument, then every investor’s optimal portfolio must be a \textit{frontier portfolio} (to be defined below). Since $U(\cdot, \cdot)$ is decreasing in its second argument, the investor is said to be \textit{variance averse}. In general, an investor’s preference over feasible investment projects may violate the independence axiom if that preference can be represented by a mean-variance utility function. The following is an example. (Notation follows from Lecture 6.)} We shall refer to
this set of portfolios the portfolio frontier, and its elements the frontier portfolios. Note that rational investors’ equilibrium choices must be frontier portfolios if two-fund separation holds in equilibrium. We shall distinguish the case where no riskless asset is present from the case where a riskless asset is present.

6. First we consider the case where there does not exist a riskless asset.

**Definition 3** A portfolio $w$ is a frontier portfolio if it has the minimum variance of return among the portfolios that promise the same expected rate of return. That is, $w$ is a frontier portfolio if and only if for all portfolios $w'$,

$$w'_e = w'e \Rightarrow w'_V w_1 \geq w'Vw.$$  

Since we have assumed that there does not exist a riskless asset, each $\tilde{r}_j$ has a positive variance. We shall further assume that there does not exist a riskless portfolio (nor a riskless arbitrage portfolio). Recall from Theorem 1 that the latter assumption implies immediately that $V$ is positive definite, and therefore non-singular.

**Theorem 3** Suppose that $e$ is not proportional to $1$. Then for all $\mu \in \mathbb{R}$, there exists a unique portfolio $w^*(\mu)$ that solves the following minimization problem:

$$\min_w \frac{1}{2} w'Vw,$$

subject to

$$w'1 = 1, \ w'e = \mu.$$  

Moreover,

$$w^*(\mu) = g + h\mu,$$

2.) Suppose that $Z = \{1, -1\}$ and the three lotteries $p, q, r$ are defined as $p(1) = \frac{4}{5}$, $q(1) = \frac{1}{4}$, $r(1) = \frac{1}{5}$, $p(-1) = \frac{1}{4}$, $q(-1) = \frac{3}{4}$, $r(-1) = \frac{2}{5}$. Recall that $P$ contains all probability distributions on $Z$. Suppose that an investor’s preference on $P$ can be defined by the mean-variance utility function $U = E[W] - \text{var}[W]$. Then it can be verified that the investor prefers $p$ to $q$, and yet she also prefers $\frac{1}{2}q + \frac{1}{2}r$ to $\frac{1}{2}p + \frac{1}{2}r$. 

7
where
\[ g = \frac{bV^{-1}e - aV^{-1}1}{b^2 - ac}, \quad h = \frac{-eV^{-1}e + bV^{-1}1}{b^2 - ac}, \]
and
\[ a \equiv e'V^{-1}e, \quad b \equiv 1'V^{-1}e = e'V^{-1}1, \quad c \equiv 1'V^{-1}1. \]

Proof. Define the Lagrangian
\[ L(w, t, s) \equiv \frac{1}{2}w'Vw - t(w'1 - 1) - s(w'e - \mu). \]
Since the Hessian of the objective function is
\[ D^2[\frac{1}{2}w'Vw] = V, \]
which is positive definite, we conclude that \( \frac{1}{2}w'Vw \) is a convex function of \( w \), so that according to the Lagrange Theorem the optimal \( w \) must satisfy the following first-order condition:
\[
D[L] = 0_{(N+2)\times 1} \iff \begin{bmatrix}
\frac{\partial L}{\partial w_1} \\
\frac{\partial L}{\partial w_2} \\
\vdots \\
\frac{\partial L}{\partial w_N} \\
\frac{\partial L}{\partial t} \\
\frac{\partial L}{\partial s}
\end{bmatrix} = 0_{(N+2)\times 1}.
\]
Thus the optimal solution must satisfy
\[ Vw = t1 + se, \quad w'1 = 1, \quad w'e = \mu. \]
From here, we obtain
\[ w = V^{-1}[t1 + se] \]
\[ \Rightarrow 1 = 1'w = t1'V^{-1}1 + s1'V^{-1}e, \quad \mu = e'w = te'V^{-1}1 + se'V^{-1}e, \]
which allows us to solve for $t$ and $s$. Replacing the explicit solutions for $t$ and $s$ into
\[ w = V^{-1}[t1 + se], \]
we obtain the optimal solution
\[ w^*(\mu) = g + h\mu, \]
where
\[ g = \frac{bV^{-1}e - aV^{-1}1}{b^2 - ac}, \quad h = \frac{-cV^{-1}e + bV^{-1}1}{b^2 - ac}, \]
and
\[ a \equiv e'V^{-1}e, \quad b \equiv 1'V^{-1}e = e'V^{-1}1, \quad c \equiv 1'V^{-1}1. \]
Observe that $h = 0$ if and only if $e = b/c$, which has been ruled out by the assumption that $e$ is not proportional to $1$. Thus each $\mu \in \mathbb{R}$ corresponds to a distinct $w^*(\mu)$.\(^3\)

7. The preceding theorem shows that, since $w^*(\mu)$ is linear in $\mu$, it can be spanned by two points $w^*(\mu_1)$ and $w^*(\mu_2)$, for any $\mu_1 \neq \mu_2$. Verify that, indeed,
\[ w^*(\mu) = \frac{\mu - \mu_2}{\mu_1 - \mu_2} w^*(\mu_1) + \frac{\mu_1 - \mu}{\mu_1 - \mu_2} w^*(\mu_2). \]
In particular, letting $\mu_1 = 1$ and $\mu_2 = 0$, we can conclude that any $w^*(\mu)$ can be spanned by $g + h = w^*(100\%)$ and $g = w^*(0\%)$. Interpreted this way, a frontier portfolio can be constructed by holding the portfolio $g = w^*(0\%)$ and then adding an arbitrage portfolio $h\mu$ to it. Recall that an arbitrage portfolio is a portfolio that costs nothing; the sum of its portfolio weights is zero.

**Proposition 2** The following minimization problem has a unique solution, referred to as the minimum variance portfolio for obvious reasons:
\[ \min_w \frac{1}{2} w'Vw, \]
\[^3\]If instead $E[\tilde{r}_j] = \frac{b}{c}$ for all $j = 1, 2, \ldots, N$, then $w^*(\mu)$ exists if and only if $\mu = \frac{b}{c}$. 
subject to
\[ \mathbf{w}' \mathbf{1} = 1. \]
The solution is denoted by \( \mathbf{w}_{mvp} \), and in fact, \( \mathbf{w}_{mvp} = \mathbf{w}^*(\frac{b}{c}) \).

Proof. Again the solution can be easily obtained by applying the Lagrange theorem. Define the Lagrangian
\[ L(\mathbf{w}, t) \equiv \frac{1}{2} \mathbf{w}' \mathbf{V} \mathbf{w} - t(\mathbf{w}' \mathbf{1} - 1). \]

The optimal solution must satisfy
\[
\begin{bmatrix}
\frac{\partial L}{\partial w_1} \\
\frac{\partial L}{\partial w_2} \\
\vdots \\
\frac{\partial L}{\partial w_N} \\
\frac{\partial L}{\partial t}
\end{bmatrix} = \mathbf{0}_{(N+1) \times 1}.
\]

Thus the optimal solution must satisfy
\[ \mathbf{V} \mathbf{w} = t \mathbf{1}, \mathbf{w}' \mathbf{1} = 1. \]

From here, we obtain
\[ \mathbf{w} = \mathbf{V}^{-1} [t \mathbf{1}] \Rightarrow 1 = 1' \mathbf{w} = t 1' \mathbf{V}^{-1} \mathbf{1} \Rightarrow t = [1' \mathbf{V}^{-1} \mathbf{1}]^{-1} \]
\[ \Rightarrow \mathbf{w}_{mvp} = \frac{\mathbf{V}^{-1} \mathbf{1}}{1' \mathbf{V}^{-1} \mathbf{1}}. \]

On the other hand, observe that
\[ \mathbf{w}^*(\frac{b}{c}) = \mathbf{g} + \mathbf{h} \frac{b}{c} = \frac{1}{b^2 - ac} \left[ b \mathbf{V}^{-1} \mathbf{e} - a \mathbf{V}^{-1} \mathbf{1} - b \mathbf{V}^{-1} \mathbf{e} + b^2 \mathbf{V}^{-1} \mathbf{1} \right] \]
\[ = \frac{1}{c(b^2 - ac)} \left[ b^2 \mathbf{V}^{-1} \mathbf{1} - ac \mathbf{V}^{-1} \mathbf{1} \right] = \frac{1}{c} \mathbf{V}^{-1} \mathbf{1} \]
\[ = \frac{\mathbf{V}^{-1} \mathbf{1}}{1' \mathbf{V}^{-1} \mathbf{1}}. \]

Hence \( \mathbf{w}_{mvp} = \mathbf{w}^*(\frac{b}{c}) \).
8. Now, define

\[ \sigma^2(\mu) \equiv w^*(\mu)'Vw^*(\mu), \]

and

\[ \sigma(\mu) \equiv \sqrt{w^*(\mu)'Vw^*(\mu)}. \]

Note that \( \sigma^2(\mu) \) is the variance of the rate of return on the frontier portfolio with expected rate of return equal to \( \mu \). The graph of the function \( \sigma^2(\cdot) \) on the \((\mu, \sigma^2)\)-space is a parabola:

\[ \sigma^2(\mu) = h'Vh \mu^2 + 2g'Vh \mu + g'Vg, \]

where, by the fact that \( V \) is positive definite, the coefficient of \( \mu^2 \) is strictly positive! Since \( w_{mvp} \neq 0 \), we know that for all \( \mu \in \mathbb{R} \),

\[ \sigma^2(\mu) \geq \sigma^2(\frac{b}{c}) > 0. \]

On the other hand, one can show that

\[ \left[ \frac{\sigma(\mu)}{\sqrt{c}} \right]^2 - \left[ \frac{\mu - \frac{b}{c}}{\sqrt{d/c^2}} \right]^2 = 1, \]

where\(^4\)

\[ d = ac - b^2, \]

\(^4\)We can show that \( d > 0 \). Verify that

\[ 0 < (be - a1)'V^{-1}(be - a1) = a(ac - b^2). \]

There is another way to see the sign of \( d \). Note that \( V^{-1} \) is symmetric:

\[ [V^{-1}]' = [V']^{-1} = [V]^{-1} = V^{-1}. \]

Note also that \( V^{-1} \) is positive definite: for each \( y \in \mathbb{R}^N \), there exists an \( x \in \mathbb{R}^N \) such that \( y = Vx \); this actually defines a one-to-one correspondence from \( \mathbb{R}^N \) to itself. Hence

\[ \forall y \neq 0 \Rightarrow x = V^{-1}y \neq 0, \quad 0 < x'Vx = x'(VV^{-1})Vx = x'V'V^{-1}Vx = (Vx)'V^{-1}(Vx) = y'V^{-1}y, \]

showing that \( V^{-1} \) is positive definite. Hence \( V^{-1} \) is a legitimate covariance matrix for some \( N \times 1 \) random vector \( \tilde{z} \). Recognizing this fact, we can now interpret \( \frac{b}{c} \) as the square of the coefficient of correlation for the random variables \( 1'\tilde{z} \) and \( e'\tilde{z} \), which is less than 1.
so that the graph of the function $\sigma(\cdot)$ on the $(\mu, \sigma)$-space is the right piece of a hyperbola.\footnote{Here by convention, the horizontal axis measures $\sigma$.}

If we assume that investors care about only the first two moments of $\tilde{W}$, with their welfare increasing and decreasing in the first and in the second moments of $\tilde{W}$ respectively, then each and every investor will end up holding a frontier portfolio with an expected rate of return greater than $\frac{b}{c}$. We shall also refer to the set of mean-variance efficient portfolios the efficient frontier. The above conclusion is that, in the absence of a riskless asset, the efficient frontier is the upper half of the right piece of a hyperbola in the $(\mu, \sigma)$-space, composed of those frontier portfolios with $\mu > \frac{b}{c}$.

9. To prepare for our next main result, we give a few propositions.

**Proposition 3** For each $\mu \neq \frac{b}{c}$, there exists a unique $\mu'(\mu) \in \mathbb{R}$ such that the covariance of rates of return on respectively $w^*(\mu)$ and $w^*(\mu'(\mu))$ is zero. Moreover, for all $\mu \neq \frac{b}{c}$,

$$
\mu'(\mu) = \frac{b}{c} - \frac{ac-b^2}{c^2 \mu - \frac{b}{c}}
$$

so that

$$(\mu - \frac{b}{c})(\mu'(\mu) - \frac{b}{c}) < 0.$$  

**Proof.** One can obtain $\mu'(\mu)$ by directly solving

$$
(w^*(\mu))'Vw^*(\mu'(\mu)) = 0,
$$

\footnote{This will be true if each investor is endowed with some mean-variance utility function. Note that if an investor is endowed with a quadratic VNM utility function $U(W) = W - \frac{1}{2}W^2$, where the constant $\rho > 0$, then the investor will optimally hold a frontier portfolio, since $E[U(\tilde{W})] = E[\tilde{W}] - \frac{\rho}{2}(E[\tilde{W}])^2 - \frac{\rho}{2} \text{var}[\tilde{W}]$ is decreasing in $\text{var}[\tilde{W}]$ given $E[\tilde{W}]$. However, note that $E[U(\tilde{W})]$ is not increasing in $E[\tilde{W}]$, and hence we cannot be sure if the investor will hold a mean-variance efficient portfolio.}
using the formula
\[ w^*(\mu) = g + h\mu, \quad w^*(\mu') = g + h\mu'. \]
Indeed, we have, using\( d = ac - b^2 > 0, \)
\[ g'Vg = \frac{1}{d^2}[a^2c - ab^2], \quad h'Vh = \frac{1}{d^2}[ac^2 - cb^2], \]
and
\[ g'Vh = \frac{1}{d^2}[b^3 - abc]. \]
Thus
\[ (w^*(\mu))'Vw^*(\mu') = 0 \iff \frac{1}{d^2}[a^2c - ab^2 + \mu\mu'(ac^2 - cb^2) + (\mu + \mu')(b^3 - abc)] = 0 \]
\[ \iff \mu' = \frac{ab^2 - a^2c + \mu(abc - b^3)}{\mu(ac^2 - cb^2) - abc + b^3} = \frac{\mu b - a}{\mu c - b} = \frac{\mu b - \frac{b^2}{c} - \frac{d}{c}}{\mu c - b} = \frac{b}{c} - \frac{d}{c} \mu - \frac{b}{c}. \]
Now the last assertion becomes transparent.

\section*{Proposition 4}
The set of mean-variance efficient portfolios is a convex set; that is, a convex combination\(^7\) of a finite number of mean-variance efficient portfolios is a mean-variance efficient portfolio.

\textit{Proof.} Note that a convex combination of a finite number of frontier portfolios is a frontier portfolio. Indeed, suppose that

\[ 0 \leq \alpha_1, \alpha_2, \ldots, \alpha_m \leq 1, \sum_{k=1}^m \alpha_k = 1, \quad \mu_1, \mu_2, \ldots, \mu_m > \frac{b}{c}. \]

\(^7\)Recall that if \( X \) is a real vector space, then for any \( x_1, x_2, \ldots, x_m \in X \) and \( \alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1] \) with \( \sum_{k=1}^m \alpha_k = 1 \), \( \sum_{k=1}^m \alpha_k x_k \) is called a convex combination of the \( m \) vectors \( x_1, x_2, \ldots, x_m \in X \). A subset \( A \subset X \) is a convex set if for all \( x_1, x_2 \in A \) and for all \( \lambda \in [0, 1] \), the convex combination \( \lambda x_1 + (1 - \lambda)x_2 \in A \).
then we have

\[ m \sum_{k=1}^m \alpha_k \mathbf{w}^*(\mu_k) = \sum_{k=1}^m \alpha_k [\mathbf{g} + \mathbf{h}\mu_k] = \mathbf{g} + \mathbf{h} \sum_{k=1}^m \alpha_k \mu_k = \mathbf{w}^*(\sum_{k=1}^m \alpha_k \mu_k), \]

so that \( \sum_{k=1}^m \alpha_k \mathbf{w}^*(\mu_k) \) is a frontier portfolio. Moreover, since

\[ \sum_{k=1}^m \alpha_k \mu_k > \sum_{k=1}^m \alpha_k \frac{b}{c} = \frac{b}{c}, \]

\( \sum_{k=1}^m \alpha_k \mathbf{w}^*(\mu_k) \) is actually a (mean-variance) efficient portfolio. ||

Let \( q_{ij} \) be the number of shares of asset \( j \) held by investor \( i \) in equilibrium, and let \( q_i \) be the \( N \times 1 \) vector of which the \( j \)-th element is \( q_{ij} \). Let \( \mathbf{q} \) be the \( N \times 1 \) vector of which the \( j \)-th element is \( q_j \), and \( \mathbf{P}_{N \times N} \) be the diagonal matrix of which the \((j,j)\)-th element is \( p_j \). Since markets clear in equilibrium, we know that for all \( j = 1, 2, \ldots, N \),

\[ \sum_{i=1}^I q_i = \mathbf{q}. \]

Note also that for all \( i = 1, 2, \ldots, I \),

\[ W_i = \mathbf{1}' \mathbf{P} \mathbf{q}_i. \]

Since \( \sum_{i=1}^I \mathbf{q}_i = \mathbf{q} \) and since \( \sum_{i=1}^I W_i = W_m \), we have

\[ W_m = \mathbf{1}' \mathbf{P} \mathbf{q}. \]

**Definition 4** The market portfolio is defined as \( \mathbf{w}_m \), of which the \( j \)-th element is

\[ w_{mj} \equiv \frac{p_j q_j}{W_m}, \quad \forall j = 1, 2, \ldots, N. \]

That is, the market portfolio is simply the market-value-weighted portfolio. Equivalently, we can write

\[ \mathbf{w}_m = \frac{\mathbf{P} \mathbf{q}}{W_m} = \frac{\mathbf{P} \mathbf{q}}{\mathbf{1}' \mathbf{P} \mathbf{q}}. \]
Proposition 5  The following equation holds:

\[ w_{m} = \sum_{i=1}^{I} \frac{W_{i}}{W_{m}} w_{i}, \]

where \( w_{i} \) is investor \( i \)'s equilibrium portfolio; that is,

\[ w_{i} = \frac{Pq_{i}}{W_{i}} = \frac{Pq_{i}}{1/Pq_{i}}. \]

Proof. Note that

\[ \sum_{i=1}^{I} \frac{W_{i}}{W_{m}} w_{i} = \sum_{i=1}^{I} \frac{W_{i}}{W_{m}} Pq_{i} = \sum_{i=1}^{I} \frac{Pq_{i}}{W_{m}} = \frac{Pq}{W_{m}} = w_{m}. \]

The preceding proposition shows that the market portfolio is a convex combination of individual investors’ equilibrium portfolios. Thus the market portfolio will be mean-variance efficient if each individual investor chooses to hold a mean-variance efficient portfolio in equilibrium.

Proposition 6  Fix any \( \mu \neq \frac{b}{c} \) and the associated \( \mu'(\mu) \). Then, for any feasible portfolio \( w \), we have

\[ w'e = \mu'(\mu) + \frac{w'Vw^*(\mu)}{\mu - \mu'(\mu)}[\mu - \mu'(\mu)]. \]

Proof. It is tedious but rather straightforward to prove this assertion. Simply use the expressions

\[ \mu'(\mu) = \frac{b}{c} - \frac{d}{\mu - \frac{b}{c}}, \]

and

\[ w^*(\mu) = g + h \mu. \]

10. Now we introduce Fischer Black’s zero-beta CAPM.
Theorem 4 Suppose that in equilibrium of the date-0 financial markets, two-fund separation holds and all investors hold mean-variance efficient portfolios. Then in the absence of the riskless asset, for each portfolio $w$,

$$w'e = \mu'(\mu_m) + \frac{w'Vw_m}{w_m'Vw_m} [\mu_m - \mu'(\mu_m)],$$

where

$$\mu_m = w_m'e = E[\bar{r}_m]$$

denotes the expected rate of return on the market portfolio.

Proof. This theorem follows from the preceding propositions directly. Since the market portfolio is a convex combination of individual investors’ equilibrium portfolios, it is mean-variance efficient, and hence the assertion follows from Proposition 6. ||

11. Now we move on to the case where the riskless asset exists. We shall assume that the riskless asset is in zero net supply, whereas all the $N$ risky assets are in strictly positive net supply.

Again, let the $(N + 1)$-vector $q_i$ be investor $i$’s equilibrium holdings of the $N$ traded assets (where asset 0 is the riskless asset). Let the $(N + 1)$-vector $q$ contain the net supplies of the $N + 1$ traded assets. Let $P_{(N+1)\times(N+1)}$ be the diagonal matrix whose $(j, j)$-th element is the price of asset $j$, $p_j$. Let investor $i$’s equilibrium portfolio be

$$x_i = \begin{bmatrix} 1 - w_i'1 \\ w_i \end{bmatrix},$$

where the $N \times 1$-vector $w_i$ contains portfolio weights that investor $i$ assigns to the $N$ risky assets. Let $x_m$ be the market portfolio. Let $W_i$ and $W_m$ be respectively investor $i$’s initial wealth and the aggregate wealth at date 0. Then we have

$$W_i = 1'Pq_i, \quad x_i = \frac{Pq_i}{W_i}, \quad W_m = 1'Pq.$$
Note that
\[ x_m = \frac{Pq}{W_m} = \sum_{i=1}^{I} \frac{Pq_i}{W_m} = \sum_{i=1}^{I} \frac{Pq_i W_i}{W_m} = \sum_{i=1}^{I} \frac{W_i}{W_m} x_i, \]
where the second equality is the markets clearing condition. Hence we have shown that the market portfolio is once again a convex combination of the individual investors’ equilibrium portfolios.

12. Again, we shall start with the formulae for the first two moments of portfolio returns.

**Proposition 7**  The expected value and variance of the rate of return on portfolio
\[ x = \begin{bmatrix} 1 - w'1 \\ w \end{bmatrix} \]
are respectively
\[ w'e + (1 - w'1)r_f \]
and
\[ w'Vw. \]

The covariance of the rates of return on respectively portfolio
\[ x_1 = \begin{bmatrix} 1 - w_1'1 \\ w_1 \end{bmatrix} \]
and portfolio
\[ x_2 = \begin{bmatrix} 1 - w_2'1 \\ w_2 \end{bmatrix} \]
is
\[ w_1'Vw_2. \]

**Proof.** The expression for the expected rate of return is obvious. The covariance of the rates of return on respectively portfolio
\[ x_1 = \begin{bmatrix} 1 - w_1'1 \\ w_1 \end{bmatrix} \]
and portfolio
\[ x_2 = \begin{bmatrix} 1 - w'_2 1 \\ w_2 \end{bmatrix} \]
is
\[ x'_1 \begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix} x_2 = \begin{bmatrix} 1 - w'_1 1 \\ w'_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 1 - w'_2 1 \\ w_2 \end{bmatrix} = w'_1 V w_2. \]

Now we can give a characterization of the portfolio frontier.

**Proposition 8** Suppose that there exists \( j \in \{1, 2, \ldots, N\} \) such that \( E[\tilde{r}_j] \neq r_f \). Then for each \( \mu \in \mathbb{R} \), there exists an \( N \times 1 \) vector \( w^*(\mu) \) such that

\[ w'e + (1-w'1)r_f = \mu = w^*(\mu)'e + (1-w^*(\mu)'1)r_f \Rightarrow w'Vw \geq w^*(\mu)'Vw^*(\mu). \]

That is,
\[ x^*(\mu) = \begin{bmatrix} 1 - w^*(\mu)' 1 \\ w^*(\mu) \end{bmatrix} \]
is the frontier portfolio with expected rate of return \( \mu \). Moreover, given \( \mu \),
\[ w^*(\mu) = \frac{(\mu - r_f)V^{-1}(e - r_f 1)}{(e - r_f 1)'V^{-1}(e - r_f 1)}, \]
and
\[ w^*(\mu)'Vw^*(\mu) = \frac{(\mu - r_f)^2}{(e - r_f 1)'V^{-1}(e - r_f 1)}. \]

**Proof.** We must solve the following minimization
\[ \min_w \frac{1}{2} w'Vw, \]
subject to
\[ w'e + (1-w'1)r_f = \mu. \]
Now the asserted formulae can be obtained by applying the Lagrange Theorem. More precisely, define the Lagrangian
\[ L(w, t) \equiv \frac{1}{2} w'Vw - t[w'e + (1-w'1)r_f - \mu], \]
where one can easily verify that the functions $f(w) = \frac{1}{2} w' V w$ and $g(w) = w' e + (1 - w' 1) r_f - \mu$ are respectively convex and affine functions of $w$. Thus we can obtain the optimal solution by setting the gradient of $L$ to the $(N + 1) \times 1$ zero vector. That is, the optimal $w^*$ must satisfy

$$V w^* = t[e - r_f 1] \Rightarrow w^* = t V^{-1} [e - r_f 1],$$

and

$$[w^*]' e + (1 - [w^*]' 1) r_f = \mu \iff [e - r_f 1]' w^* = \mu - r_f.$$

It follows that

$$\mu - r_f = [e - r_f 1]' w^* = t[e - r_f 1]' V^{-1} [e - r_f 1],$$

and hence

$$t = \frac{\mu - r_f}{[e - r_f 1]' V^{-1} [e - r_f 1]},$$

It follows that

$$w^*(\mu) = \frac{(\mu - r_f) V^{-1} (e - r_f 1)}{(e - r_f 1)' V^{-1} (e - r_f 1)},$$

and

$$w^*(\mu)' V w^*(\mu) = \frac{(\mu - r_f)^2}{(e - r_f 1)' V^{-1} (e - r_f 1)}.$$

Note that the supposition that there exists $j \in \{1, 2, \cdots, N\}$ such that $E[r_j] \neq r_f$ implies that $(e - r_f 1) \neq 0$,

so that by the fact that $V^{-1}$ is positive definite (which we showed in a preceding footnote),

$$H \equiv (e - r_f 1)' V^{-1} (e - r_f 1) > 0.\|$$

**Corollary 1** Recall that in the preceding proposition,

$$x^*(\mu) = \begin{bmatrix} 1 - w^*(\mu)' 1 \\ w^*(\mu) \end{bmatrix}, \quad w^*(\mu) = \frac{(\mu - r_f) V^{-1} (e - r_f 1)}{H},$$

and

$$H \equiv (e - r_f 1)' V^{-1} (e - r_f 1) > 0.$$

Then, the following assertions are true.
• $w^*(\mu)'1 = 0$ for all $\mu \in \mathbb{R}$ if and only if $r_f = \frac{b}{c}$.

• If $r_f \neq \frac{b}{c}$, then there exists a unique $\mu^* \in \mathbb{R}$ such that $w^*(\mu^*)'1 = 1$, where $\mu^* > r_f$ if and only if $r_f < \frac{b}{c}$.

**Proof.** It is easy to see that

$$w^*(\mu)'1 = 0, \forall \mu \in \mathbb{R}, \Leftrightarrow (\mu - r_f)1'V^{-1}(e - r_f1) = 0, \forall \mu \in \mathbb{R},$$

$$\Leftrightarrow 1'V^{-1}(e - r_f1) = 0,$$

$$\Leftrightarrow b = 1'V^{-1}e = r_f1'V^{-1}1 = r_f c,$$

$$\Leftrightarrow r_f = \frac{b}{c}.$$

Similarly, we have, if $r_f \neq \frac{b}{c}$:

$$w^*(\mu^*)'1 = 1 \Rightarrow \mu^* = \frac{H}{1'V^{-1}(e - r_f1)} + r_f,$$

$$\Rightarrow \mu^* = \frac{H}{b - r_f c} + r_f,$$

so that $\mu^* > r_f$ if and only if $r_f < \frac{b}{c}$.

13. The preceding proposition shows that, in the presence of a riskless asset, the portfolio frontier is the union of two half lines on the $\mu - \sigma$ space (where the vertical and the horizontal axes measure respectively the expected value and the standard deviation of the rate of return on a portfolio):

$$\sigma(\mu) = \begin{cases} \frac{\mu - r_f}{\sqrt{H}}, & \text{if } \mu \geq r_f; \\ \frac{-\mu - r_f}{\sqrt{H}}, & \text{if } \mu < r_f. \end{cases}$$

Apparently, the minimum variance portfolio in the current case is

$$x_{mvp} = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

that is, $x_{mvp}$ is the riskless asset. Thus the half line

$$\sigma(\mu) = \frac{\mu - r_f}{\sqrt{H}}, \forall \mu \geq r_f$$

contains all the mean-variance efficient portfolios, and is termed the capital market line (CML).
14. Since we have assumed that the riskless asset is in zero net supply and all risky assets in strictly positive supply, we have the following theorem.\footnote{Most of the preceding propositions were developed by Merton (1972).}

**Theorem 5** Suppose that the riskless asset exists and two-fund separation holds in equilibrium with every investor holding a mean-variance efficient portfolio.

- In equilibrium, \( r_f < \frac{b}{c} \).
- There exists a unique \( \mu^* \geq r_f \) such that
  \[
  w^*(\mu^*)'1 = 1;
  \]
  that is, \( x^*(\mu^*) \) contains only risky assets.
- Moreover, for all \( \mu \geq r_f \), there exists \( a(\mu) \geq 0 \) such that
  \[
  x^*(\mu) = a(\mu)x^*(\mu^*) + [1 - a(\mu)]x_{mvp}.
  \]
- In fact, \( x^*(\mu^*) = x_m \).

*Proof.* First observe that if instead \( \frac{b}{c} = r_f \), then \( w^*(\mu)'1 = 0 \) for all \( \mu \in \mathbb{R} \), and so every investor would be taking a long position in the riskless asset in equilibrium (because \( W_i > 0 \) for all \( i = 1, 2, \cdots, I \)), which implies that the market for the riskless asset cannot clear, a contradiction! Hence we know that \( \frac{b}{c} \neq r_f \). We shall prove below that \( r_f < \frac{b}{c} \).

Next, recall from the preceding Corollary that the equation

\[
1 = w^*(\mu)'1 = \frac{(\mu - r_f)'V^{-1}(e - r_f)1}{(e - r_f)'V^{-1}(e - r_f)1}
\]

has a unique solution \( \mu^* \) if and only if

\[
\frac{b}{c} - r_f = 1'V^{-1}(e - r_f)1 \neq 0,
\]
and in that case we obtain
\[ \mu^* = r_f + \frac{H}{b - cr_f} \neq r_f, \]
and
\[ w^*(\mu^*) = \frac{\mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})}{1\mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})}, \]
and moreover, \( r_f < \frac{b}{c} \) if and only if \( r_f < \mu^* \).

Now we show that the riskless asset and \( \mathbf{x}^*(\mu^*) \) span the entire portfolio frontier. Note that, for all \( \mu \in \mathbb{R} \),
\[
\mathbf{x}^*(\mu) = \begin{bmatrix} 1 - w^*(\mu)^t \mathbf{1} \\ w^*(\mu) \end{bmatrix} = a(\mu) \begin{bmatrix} 0 \\ w^*(\mu^*) \end{bmatrix} + [1 - a(\mu)] \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
where
\[
a(\mu) = w^*(\mu)^t \mathbf{1} = \frac{(\mu - r_f)\mathbf{1}^t \mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})}{(\mathbf{e} - r_f \mathbf{1})^t \mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})}.
\]
In fact, to see that the last equality holds, simply note that
\[
a(\mu)w^*(\mu^*) = w^*(\mu)^t \mathbf{1}w^*(\mu^*)
\]
\[
= (\mu - r_f)\mathbf{1}^t \mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1}) \times \frac{\mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})}{(\mathbf{e} - r_f \mathbf{1})^t \mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})}
\]
\[
= (\mu - r_f)^2 \mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})
\]
\[
= (\mathbf{e} - r_f \mathbf{1})^t \mathbf{V}^{-1}(\mathbf{e} - r_f \mathbf{1})
\]
\[
= w^*(\mu).
\]
Thus we conclude that every frontier portfolio is a portfolio of the riskless asset and \( \mathbf{x}^*(\mu^*) \).

Now we show that \( \mathbf{x}^*(\mu^*) \) is nothing but the market portfolio \( \mathbf{x}_m \). By assumption, all the \( I \) investors hold mean-variance efficient portfolios in equilibrium, and hence there must exist \( a_1, a_2, \ldots, a_I \in \mathbb{R} \), such that for all \( i = 1, 2, \ldots, I \),
\[
\mathbf{x}_i = \begin{bmatrix} 1 - a_i \\ a_i w^*(\mu^*) \end{bmatrix}.
\]
Since the riskless asset is in zero net supply, we have

\[ 0 = p_0 q_0 = \sum_{i=1}^{I} (1 - a_i) W_i \Rightarrow \sum_{i=1}^{I} a_i W_i = W_m. \]

It follows that

\[ x_m = \sum_{i=1}^{I} \frac{W_i}{W_m} x_i = \sum_{i=1}^{I} \frac{W_i}{W_m} \left[ \begin{array}{c} 1 - a_i \\ a_i w^*(\mu^*) \end{array} \right] = \left[ \begin{array}{c} 0 \\ w^*(\mu^*) \end{array} \right] = x^*(\mu^*); \]

that is, \( x^*(\mu^*) \) is exactly the market portfolio.

Finally, we show that \( \frac{b}{c} > r_f \). We have shown above that \( \frac{b}{c} \neq r_f \). Now suppose instead that \( \frac{b}{c} < r_f \). We show above that this implies that \( \mu^* < r_f \), which implies that the CML (the set of mean-variance efficient portfolios) contains portfolios that involve selling \( x_m \) short and putting all the money in the riskless asset. Since by assumption all investors hold portfolios on the CML, every investor is taking a long position in the riskless asset, which implies that the market for the riskless asset cannot clear, a contradiction. Thus it must be that \( \frac{b}{c} > r_f \) (and hence \( \mu^* > r_f \)). \( \Box \)

15. Because of the preceding theorem, from now on we re-write \( \mu^* \) by \( \mu_m \) or \( E[\tilde{r}_m] \), where \( \tilde{r}_m \) denotes the rate of return on the market portfolio. Now we are ready to introduce the (traditional) Sharpe-Lintner-Mossin CAPM.

**Theorem 6** Suppose that the riskless asset exists and two-fund separation holds with every investor holding a mean-variance efficient portfolio. Suppose that the riskless asset is in zero net supply and all risky assets in strictly positive supply. Then in equilibrium, for any feasible portfolio

\[ x_k = \left[ \begin{array}{c} 1 - w'_k 1 \\ w_k \end{array} \right], \]

if we denote its rate of return by \( \tilde{r}_k \equiv w'_k \tilde{r} + (1 - w'_k 1)r_f \), then the following CAPM equation holds:

\[ E[\tilde{r}_k] = r_f + \beta_k (E[\tilde{r}_m] - r_f), \]
where

\[ \beta_k \equiv \frac{\text{cov}(\tilde{r}_k, \tilde{r}_m)}{\text{var}(\tilde{r}_m)}. \]

**Proof.** Note that

\[
\text{cov}(\tilde{r}_k, \tilde{r}_m) = w_k' V w_m = w_k' V w^*(\mu^*) = \frac{w_k' V V^{-1}(e - r_f 1)}{1' V^{-1}(e - r_f 1)}
\]

\[
= \frac{w_k' (e - r_f 1)}{1' V^{-1}(e - r_f 1)} = \frac{w_k' e + r_f [1 - w_k' 1] - r_f}{1' V^{-1}(e - r_f 1)} = \frac{E[\tilde{r}_k] - r_f}{1' V^{-1}(e - r_f 1)},
\]

where the denominator is equal to \( c(b_c - r_f) > 0 \). Since the portfolio \( x_k \) was chosen arbitrarily, the above equation holds for the market portfolio as well. Hence we have

\[
\text{var}(\tilde{r}_m) = \frac{E[\tilde{r}_m] - r_f}{1' V^{-1}(e - r_f 1)}.
\]

Dividing \( \text{cov}(\tilde{r}_k, \tilde{r}_m) \) by \( \text{var}(\tilde{r}_m) \), we obtain

\[
\beta_k = \frac{E[\tilde{r}_k] - r_f}{E[\tilde{r}_m] - r_f},
\]

and hence the assertion follows. ||

16. The CAPM equation can be graphically demonstrated in the \( \beta - \mu \) space, which is a line referred to as the *security market line* (SML). It is seen from the SML that the expected rate of return on a portfolio depends only on the beta (with respect to the market portfolio) of that portfolio. A risky asset with a negative beta can have an expected rate of return lower than \( r_f \), for example. The idea is that a rational investor realizes that he will ultimately take a long position in the market portfolio besides lending and borrowing at the riskless rate \( r_f \). From this perspective, every portfolio \( x_k \) is just one of the ingredient assets making up the portfolio that he will be holding in equilibrium. The part of variability in the return of \( x_k \) that is un-correlated with \( \tilde{r}_m \) is not his concern; the risk contribution made by an ingredient portfolio
\( x_k \) is captured by the covariance of \( \tilde{r}_k \) and \( \tilde{r}_m \). More precisely, let the \( j \)-th element of \( w_m \) be \( w_{jm} \) (which equals \( \frac{p_j q_j}{W_m} \)), and observe that

\[
\text{var}(\tilde{r}_m) = \sum_{j=1}^{N} w_{jm} \text{cov}(\tilde{r}_j, \tilde{r}_m),
\]

and hence \( \text{cov}(\tilde{r}_j, \tilde{r}_m) \) is the risk contribution of asset \( j \) to the investor’s equilibrium portfolio. Dividing both sides of the last equation by \( \text{var}(\tilde{r}_m) \), we get

\[
100\% = \sum_{j=1}^{N} w_{jm} \beta_j,
\]

so that the share of the risk contributed by asset \( j \) to the market portfolio is equal to \( \beta_j \) times the portfolio weight \( w_{jm} \) of asset \( j \) in the market portfolio.

17. Recall that for any two random variables \( \tilde{x}, \tilde{y} \) with finite, strictly positive variances, we can find real numbers \( a, b \) and a random variable \( \tilde{e} \) such that

\[
\tilde{y} = a + b \tilde{x} + \tilde{e},
\]

with \( \text{cov}[\tilde{x}, \tilde{e}] = 0 = E[\tilde{e}] \).\(^9\) In fact, one can verify that

\[
b = \frac{\text{cov}[\tilde{x}, \tilde{y}]}{\text{var}[\tilde{x}]}.
\]

Now, for any asset or portfolio \( j \), by the above theorem we can write

\[
\tilde{r}_j = a + b \tilde{r}_m + \tilde{e},
\]

where we have \( b = \beta_j \). It follows that

\[
\text{var}[\tilde{r}_j] = \beta_j^2 \text{var}[\tilde{r}_m] + \text{var}[\tilde{e}].
\]

For obvious reasons, the left-hand side is referred to as the total risk of asset or portfolio \( j \), and the first term on the right-hand side the systematic, or non-diversifiable, or non-idiiosyncratic risk. The last term on the right-hand side is then referred to as the diversifiable risk, or

\(^9\)This is the theorem of mean-variance projection.
**Idiosyncratic risk.** The thrust of the CAPM is that only the non-diversifiable risk is priced; the diversifiable risk is irrelevant because rational investors will hold a mean-variance efficient portfolio. Moreover, since \( \text{var}[\tilde{r}_m] \) is common to any two assets or portfolios \( j \) and \( k \), in comparing the systematic risks we can focus on the comparison of \( \beta_j \) to \( \beta_k \).

18. Prior to Markowitz (1952, 1959), security analysis focused on picking undervalued securities, and a portfolio was considered nothing but an accumulation of the optimally picked securities. Markowitz was the first person to point out that merely accumulating the predicted winners is a poor portfolio selection procedure, for it ignores the effect of portfolio diversification on risk reduction. He assumed that investors’ preferences are increasing in the expected value and decreasing in the standard deviation of portfolio returns, and he defined the efficient frontier. His analysis provides a formal definition of diversification, and gives a measure (the beta) for the risk contribution of an ingredient security. He also developed rules for the construction of an efficient portfolio. Markowitz’s portfolio theory implies that a firm should evaluate investment projects in the same way that investors evaluate securities. His normative analysis was applied by Treynor (1961), Sharpe (1964), Lintner (1965), and Mossin (1966) (who and Treynor are the same person) to create a positive pricing theory of capital assets, which is the CAPM that we have developed above. The SML is the most important prediction of the CAPM, and it gives an explicit formula to compute the cross-sectional risk-return trade-off.

19. Now we give a series of examples.

**Example 1** Suppose that the riskless asset is present. Recall that for any two portfolios \( i \) and \( j \) with non-zero risk premia, we have

\[
\frac{E[\tilde{r}_i - r_f]}{E[\tilde{r}_j - r_f]} = \frac{\beta_i}{\beta_j}.
\]

Show that for any two portfolios \( i \) and \( j \) lying on the CML with non-zero risk premia, we have

\[
\frac{E[\tilde{r}_i - r_f]}{E[\tilde{r}_j - r_f]} = \frac{\beta_i}{\beta_j}.
\]
risk premia, 10
\[
\frac{E[\tilde{r}_i - r_f]}{E[\tilde{r}_j - r_f]} = \frac{\sigma_i}{\sigma_j} = \frac{\beta_i}{\beta_j}.
\]

**Example 2** Suppose that in a two-period perfect markets economy there are 3 traded assets, labeled 1, 2, and 3, where asset 1 is in zero net supply. Suppose that the following equilibrium data are valid.

\[
e = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.2 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.01 & -0.01 \\ 0 & -0.01 & 0.04 \end{bmatrix}.
\]

(i) Suppose that the Sharpe-Lintner CAPM holds. Find the portfolio weights (on the 3 traded assets) for the market portfolio.

(ii) Consider an efficient portfolio with an expected rate of return 0.12. Mr. A has $700,000. How much money should he borrow or lend (at the riskfree rate) if he intends to hold this portfolio? 11

**Hint**: Assume that for some \( \lambda_i, \lambda_j > 0, \)
\[
\tilde{r}_i = \lambda_i \tilde{r}_m + (1 - \lambda_i) r_f, \quad \tilde{r}_j = \lambda_j \tilde{r}_m + (1 - \lambda_j) r_f.
\]

Now, compute \( \sigma_i, \sigma_j, \beta_i, \) and \( \beta_j. \) In particular, verify that
\[
\sigma_i = \lambda_i \sigma_m, \quad \sigma_j = \lambda_j \sigma_m,
\]
and that
\[
\frac{\beta_i}{\beta_j} = \frac{\text{cov}[(1 - \lambda_i)r_f + \lambda_i \tilde{r}_m, \tilde{r}_m]}{\text{cov}[(1 - \lambda_j)r_f + \lambda_j \tilde{r}_m, \tilde{r}_m]}.
\]

**Hint**: Deduce that \( r_f = E[\tilde{r}_1]. \) Now, suppose that \( \tilde{r}_m = w \tilde{r}_2 + (1 - w) \tilde{r}_3. \) Apply the SML to assets 2 and 3 to get
\[
\frac{E[\tilde{r}_2] - r_f}{E[\tilde{r}_3] - r_f} = \frac{0.01w - 0.01(1 - w)}{-0.01w + 0.04(1 - w)},
\]
and hence show that \( E[\tilde{r}_m] = \frac{1}{5}. \) Now suppose that the efficient portfolio in part (ii) has an expected rate of return equal to \( \lambda r_f + (1 - \lambda)E[\tilde{r}_m] = 0.12. \) Obtain \( \lambda \) and show that Mr. A should lend $490,000.
Example 3  Assume that the Sharpe-Lintner CAPM holds and you are given the following equilibrium data about the rates of return on assets 1 and 2:

<table>
<thead>
<tr>
<th>$\tilde{r}_1/\tilde{r}_2$</th>
<th>0.15</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
<td>$\frac{1}{5}$</td>
<td>0</td>
</tr>
<tr>
<td>0.15</td>
<td>0</td>
<td>$\frac{1}{5}$</td>
</tr>
</tbody>
</table>

Suppose that $E[\tilde{r}_m] = 16\%$ and that asset 1 is an efficient portfolio.

(i) Compute $\text{cov}(\tilde{r}_2, \tilde{r}_m)$.

(ii) Is asset 2 also an efficient portfolio?\(^{12}\)

Example 4  Suppose that the Sharpe-Lintner CAPM holds. Consider an asset whose date-1 random payoff is $\tilde{x}$. What is its date-0 price $P_x$?

Note that the asset’s rate of return $\tilde{r}_x \equiv \frac{\tilde{x}}{P_x} - 1$

must satisfy

$$E(\tilde{r}_x) = r_f + \frac{\text{cov}(\tilde{r}_x, \tilde{r}_m)}{\text{var}(\tilde{r}_m)}[E(\tilde{r}_m) - r_f],$$

or

$$\frac{E(\tilde{x})}{P_x} = (1 + r_f) + \frac{\text{cov}(\tilde{x}, \tilde{r}_m)}{P_x\text{var}(\tilde{r}_m)}[E(\tilde{r}_m) - r_f] \equiv (1 + r_f) + \frac{\text{cov}(\tilde{x}, \tilde{r}_m)}{P_x}\lambda.$$

\(^{12}\)Deduce that the coefficient of correlation between $\tilde{r}_1$ and $\tilde{r}_2$ is $\rho_{1,2} = 1$. Thus show that to rule out arbitrage opportunities,

$$r_f = \frac{\sigma_2\mu_1 - \sigma_1\mu_2}{\sigma_2 - \sigma_1},$$

where $\sigma_j = \sqrt{\text{var}[\tilde{r}_j]}$ and $\mu_j = E[\tilde{r}_j]$. Now apply the CML to asset 1, and obtain $\sigma_m = \sqrt{\text{var}[\tilde{r}_m]} = 0.04$. Finally, apply the SML to asset 2 and get $\beta_2$, which together with $\sigma_m$ allows you to get $\text{cov}(\tilde{r}_2, \tilde{r}_m)$. It is straightforward to verify whether asset 2 is lying on the CML also. Indeed, verify that $r_f = 0$, and that $\frac{\sigma_2}{\sigma_2} = \frac{3}{5} = \frac{\sigma_2}{\sigma_2}$. (cf. Example 1).
Multiply both sides by $P_x$, we have

$$E(\tilde{x}) = P_x(1 + r_f) + \lambda \text{cov}(\tilde{x}, \tilde{r}_m),$$

or,

$$P_x = \frac{E(\tilde{x}) - \lambda \text{cov}(\tilde{x}, \tilde{r}_m)}{1 + r_f},$$

where the numerator of the last expression is called the certainty equivalent of this asset (or in short-hand notation, $CE_x$), and $\lambda \text{cov}(x, r_m)$ is called the risk premium in payoffs for this asset. We conclude that the following two ways of computing the price of an asset are both valid:

$$P_x = \frac{E(\tilde{x})}{1 + E(\tilde{r}_x)} = \frac{CE_x}{1 + r_f}.$$

Now, we give an application of this CE formula.

Mr. X is the CEO of a large company, considering taking one of the following two mutually exclusive investment projects, A and B. The features of these two projects can be summarized as follows.

- Both incur a date-0 cash outflow of $1,000;
- Both generate a sure date-1 cash revenue equal to $1,500;
- The two projects differ in their date-1 cash expenses (denoted by $C_A$ and $C_B$ respectively). There are three equally likely date-1 states, referred to as boom, average, and recession. The following table summarizes the date-1 cash expenses of the two projects and the realized rate of return on the market portfolio in each of the three date-1 states.

<table>
<thead>
<tr>
<th>states</th>
<th>prob.</th>
<th>$C_A$</th>
<th>$C_B$</th>
<th>$r_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>boom</td>
<td>$\frac{1}{3}$</td>
<td>500</td>
<td>600</td>
<td>20%</td>
</tr>
<tr>
<td>average</td>
<td>$\frac{1}{3}$</td>
<td>400</td>
<td>400</td>
<td>10%</td>
</tr>
<tr>
<td>recession</td>
<td>$\frac{1}{3}$</td>
<td>300</td>
<td>200</td>
<td>0%</td>
</tr>
</tbody>
</table>

Which project between A and B has a higher variance of date-1 cash flow? Which project has a higher NPV (net present value) at date 0?
**Solution.** It can be verified that both projects have the same expected date-1 cash flow, but project B’s date-1 cash flow has a higher variance. Nonetheless, the date-0 firm value is maximized when project B is chosen over project A, under the assumption that the CAPM holds at date 0. The latter assumption says that the firm’s investors all take long positions in the market portfolio and put the rest of their wealth in the riskless asset at date 0, so that they care about only the risk contribution from the firm’s equity to their optimal portfolio. That is, the investors in valuing the firm care about only the correlation of the firm’s date-1 cash earnings and the random rate of return on the market portfolio, not the variance of the firm’s date-1 cash earnings.

Now let us compute the date-0 present value $PV_j$ given that project $j$ is taken at date 0. Assuming that $-100\% < r_f < 10\% = E[r_m]$,

$$PV_j = \frac{1500 - 400 - \lambda \text{cov}(-C_j, r_m)}{1 + r_f}. $$

Since $\lambda, 1 + r_f$ are both positive by assumption, and since $\text{cov}(-C_B, r_m) < \text{cov}(-C_A, r_m)$, we have

$$PV_B > PV_A \Rightarrow NPV_B = PV_B - 1000 > PV_A - 1000 = NPV_A,$$

implying that, according to the NPV-maximization criterion, project B should be taken at date 0.

**Example 5** Consider the following probability matrix:

<table>
<thead>
<tr>
<th>$r_i / r_M$</th>
<th>0.1</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>0.12</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Suppose the CAPM holds. What is $r_f$? What is $\sigma_i$? Discuss.$^{13}$

$^{13}$**Hint:** Show that $\beta_i = 0$. 

30
Example 6 Consider the following probability matrix:

<table>
<thead>
<tr>
<th></th>
<th>0.1</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>0.4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Does the Sharpe-Lintner CAPM hold?\footnote{Hint: Show that $\beta_i = 1$. Now apply the SML to asset $i$ and compare $E[\tilde{r}_i]$ and $E[\tilde{r}_m]$.}

Solution. First we compute the first and second moments of $(r_i, r_M)$. We have

\[
E[r_i] = \frac{3}{4} \times 0 + \frac{1}{4} \times 0.4 = 0.1,
\]

\[
E[r_M] = \frac{1}{2} \times 0.1 + \frac{1}{2} \times 0.3 = 0.2,
\]

\[
\text{var}[r_i] = \frac{3}{4} \times (0 - 0.1)^2 + \frac{1}{4} \times (0.4 - 0.1)^2 = 0.03,
\]

\[
\text{var}[r_M] = \frac{1}{2} \times (0.1 - 0.2)^2 + \frac{1}{2} \times (0.3 - 0.2)^2 = 0.01,
\]

\[
\text{cov}(r_i, r_M) = \frac{1}{2} \times (0.1 - 0.2) \times (0 - 0.1) + \frac{1}{4} \times (0.3 - 0.2) \times (0 - 0.1)
+ 0 \times (0.1 - 0.2) \times (0.4 - 0.1) + \frac{1}{4} \times (0.3 - 0.2) \times (0.4 - 0.1) = 0.01.
\]

Thus we have

\[
\beta_i \equiv \frac{\text{cov}(r_i, r_M)}{\text{var}[r_M]} = \frac{0.01}{0.01} = 1.
\]

If the Sharpe-Lintner CAPM holds, then we must have

\[
0.1 = E[r_i] = r_f + \beta_i (E[r_M] - r_f) = r_f + (E[r_M] - r_f) = E[r_M] = 0.2,
\]

which is a contradiction. Hence we conclude that the CAPM does not hold.
Example 7 Suppose that the traditional CAPM holds period by period. Consider an asset that promises to pay you the following per-unit random payoff at date 2:

\[
\tilde{M}_2 \frac{\tilde{M}_2}{M_1},
\]

where \( M_t \) is the date-\( t \) value of the market portfolio of risky assets. Suppose further that the riskless rate from date 0 to date 1, \( r_{01}^f \), and that from date 1 to date 2, \( r_{12}^f \), are respectively 5% and 7%. Determine the date-0 and date-1 prices for the asset. \(^{15}\)

Example 8 Suppose that in a two-period perfect-markets economy, two risky assets (assets 1 and 2) together with a riskless asset (asset 0) are traded at date 0, which generate cash flows at date 1. Suppose that for assets 1 and 2, we have the following data:

\[
e = \begin{bmatrix} E[\tilde{r}_1] \\ E[\tilde{r}_2] \end{bmatrix} = \begin{bmatrix} 0.24 \\ 0.08 \end{bmatrix}, \quad V = \begin{bmatrix} \text{cov}(\tilde{r}_1, \tilde{r}_1) & \text{cov}(\tilde{r}_1, \tilde{r}_2) \\ \text{cov}(\tilde{r}_1, \tilde{r}_2) & \text{cov}(\tilde{r}_2, \tilde{r}_2) \end{bmatrix} = \begin{bmatrix} 0.04 & -0.01 \\ -0.01 & 0.01 \end{bmatrix}.
\]

(i) Assume that asset 0 is in zero net supply and asset 1 is in strictly positive supply. Suppose that \((w, 1 - w)\) is a portfolio of assets 1 and 2 (where \( w \in \mathbb{R} \)), and it has zero covariance with asset 1. Find \( w \).

(ii) Continue to assume that asset 0 is in zero net supply and asset 1 is in strictly positive supply. Suppose that asset 2 is also in zero net supply. Suppose that the traditional CAPM holds in equilibrium. What is \( r_f \)?

\(^{15}\)Hint: Use the certainty equivalent formula to show that at date 1, the price of that asset is

\[
P_1 = \frac{E[\tilde{M}_2]}{M_1} - \lambda \text{cov}(\tilde{M}_2, \tilde{r}_m)
\]

where

\[
\lambda = \frac{E[\tilde{r}_m] - r_{12}^f}{\text{var}[\tilde{r}_m]}, \quad \tilde{r}_m = \frac{\tilde{M}_2}{M_1} - 1.
\]

Hence conclude that the date-1 price of that asset is non-random from investors’ perspective at date 0. Conclude that at date 0, carrying that asset till date 1 and then selling it is a riskless trading strategy. Now you can compute the date-0 price of that asset accordingly:

\[
P_0 = \frac{P_1}{1 + r_{01}^f} = \frac{20}{21}.
\]
(iii) Now, ignore parts (i) and (ii). Assume instead that the riskless asset is in zero net supply and the risky assets are both in strictly positive supply. Moreover, in addition to the $e$ and $V$ given above, you are told that $r_f = 0.1$. Suppose further that the traditional CAPM holds. Find $E[r_m]$, the expected rate of return on the market portfolio.

Solution. For part (i), we solve
\[
\text{cov}(w\hat{r}_1 + (1-w)\hat{r}_2, \hat{r}_1) = 0, \Rightarrow w = \frac{1}{5}.
\]
For part (ii), we know that asset 1 is the market portfolio, and hence the risky portfolio obtained in part (i) must have a zero beta, implying that its expected rate of return equals $r_f$. Hence we have
\[
r_f = \frac{1}{5} \times 0.24 + \frac{4}{5} \times 0.08 = 0.112.
\]
Finally, for part (iii), letting $(w, 1-w)$ be the portfolio weights that the market portfolio assigns to assets 1 and 2, we have
\[
-7 = \frac{E[\hat{r}_1] - r_f}{E[\hat{r}_2] - r_f} = \frac{\text{cov}(w\hat{r}_1 + (1-w)\hat{r}_2, \hat{r}_1)}{\text{cov}(w\hat{r}_1 + (1-w)\hat{r}_2, \hat{r}_2)} = \frac{5w - 1}{1 - 2w},
\]
yielding
\[
w = \frac{2}{3}.
\]
Hence we have
\[
E[\hat{r}_m] = \frac{2}{3} \times 0.24 + \frac{1}{3} \times 0.08 = \frac{14}{15}.
\]

Example 9 Suppose that in a two-period perfect-markets economy, two risky assets (assets 1 and 2) together with a riskless asset (asset 0) are traded at date 0, which pay one-time cash flows at date 1. Suppose that for assets 1 and 2, we have the following data:
\[
e = \begin{bmatrix} 0.25 \\ 0.10 \end{bmatrix}, \quad V = \begin{bmatrix} 0.04 & z \\ z & 0.01 \end{bmatrix}.
\]
We shall assume that asset 0 is in zero net supply and assets 1 and 2 are in non-negative supply.

Suppose that the Sharpe-Lintner CAPM holds in equilibrium. Suppose that the portfolio \((\frac{1}{5}, \frac{4}{5})\), which consists of assets 1 and 2 only, has a zero covariance with the market portfolio. Suppose that the market portfolio is either asset 1 alone or asset 2 alone. Find \(z\).

**Solution.** Suppose that the market portfolio (generated by assets 1 and 2 only) consists of a fraction \(w\) of the initial wealth allocated to asset 1. Define \(y = 100z\). We have

\[
0 = \text{cov}(w\tilde{r}_1 + (1 - w)\tilde{r}_2, \frac{1}{5}\tilde{r}_1 + \frac{4}{5}\tilde{r}_2)
\]

\[
= \frac{0.01}{5}(w \times 4 + 4wy + (1 - w)y + 4(1 - w) \times 1)
\]

\[
\Rightarrow 4 + y + 3wy = 0.
\]

Recall that \(w\) equals either zero or one. If \(w = 1\), then \(y = -1\); or else, \(y = -4\). Note that if \(y = -4\), so that \(z = -0.04\), \(V\) is no longer a positive definite matrix: its determinant

\[(0.04)(0.01) - (-0.04)^2 < 0!\]

Hence we conclude that \(w = 1\), and hence

\[z = -0.01.\]

**Example 10** Re-consider the three-asset economy described in Example 8, but assume instead that the riskless asset is in zero net supply and the risky assets are both in strictly positive supply. Moreover, in addition to the \(e\) and \(V\) given above, you are told that \(r_f = \frac{136}{1300}\). Suppose that the traditional CAPM holds. Find \(E[r_m]\) (the expected rate of return on the market portfolio).
Solution  Since the riskless asset is in zero net supply, we conclude that the market portfolio is a portfolio consisting of assets 1 and 2 only. Let the market portfolio be \((0, \lambda, 1 - \lambda)\), where 0 is the portfolio weight for the riskless asset, and \(\lambda\) is the portfolio weight for asset 1. Now we solve for \(\lambda\). Since the CAPM holds, we have from the SML

\[
\frac{-11}{2} \cdot \frac{0.24 - \frac{136}{1300}}{0.08 - \frac{136}{1300}} = \frac{E[r_1] - r_f}{E[r_2] - r_f} = \frac{\beta_1(E[r_m] - r_f)}{\beta_2(E[r_m] - r_f)}
\]

\[
= \frac{\beta_1}{\beta_2} = \frac{\text{cov}(r_1, \lambda r_1 + (1 - \lambda) r_2)}{\text{cov}(r_2, \lambda r_1 + (1 - \lambda) r_2)}
\]

\[
= \frac{\lambda \text{var}(r_1) + (1 - \lambda) \text{cov}(r_1, r_2)}{\lambda \text{cov}(r_1, r_2) + (1 - \lambda) \text{var}(r_2)}
\]

\[
= \frac{\lambda(0.04) + (1 - \lambda)(-0.01)}{\lambda(-0.01) + (1 - \lambda)(0.01)}
\]

\[
\Rightarrow \lambda = \frac{3}{4}.
\]

Thus the market portfolio (of traded assets 0,1,2) is

\[
\mathbf{w}_m = \begin{bmatrix} 0 \\ \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}.
\]

It follows that

\[
E[r_m] = 0 \times r_f + \frac{3}{4} \times E[r_1] + \frac{1}{4} \times E[r_2] = 0.2.
\]

Example 11  Suppose that in the two-period economy, the markets for the \(N\) risky assets are perfect. In the market for the riskless asset, however, unlimited lending at the interest rate \(r_f\) is allowed, but borrowing is completely prohibited. Draw the efficient frontier on the \(\sigma - \mu\) space, assuming that \(r_f > E[\tilde{r}_{mvp}]\), where \(\tilde{r}_{mvp}\) is the rate of return on the minimum variance portfolio composed of risky assets only. Do we still have 2-fund separation?
Solution. Yes, we do. Recall that two-fund separation holds when all the efficient portfolios can be spanned by two fixed portfolios. In the current case, although it takes more than 2 funds to span the portfolio frontier, it only takes two funds to span the efficient frontier. See Figure 1 in the attached pdf file frontier.pdf.

Example 12 Suppose that in the two-period economy, the markets for the N risky assets are perfect. In the market for the riskless asset, however, the lending rate \( r_L \) differs from the borrowing rate \( r_B \). Assume that \( r_B > E[\tilde{r}_{mvp}] > r_L \), where \( \tilde{r}_{mvp} \) is the rate of return on the minimum variance portfolio composed of risky assets only. Draw the efficient frontier on the \( \sigma - \mu \) space.

(i) Do we still have 2-fund separation? If not, and if we have \( k \)-fund separation, what is the smallest \( k \)?

(ii) How many distinct portfolios do we need to span the entire portfolio frontier in this case?

Solution. For part (i), the answer is no. Now we need 3 funds to span the efficient frontier. On the other hand, for part (ii), it takes either 3 or 4 funds to span the portfolio frontier. See Figures 2 and 3 in the attached pdf file frontier.pdf.

Example 13 Consider two risky assets with rates of return \( \tilde{r}_1 \) and \( \tilde{r}_2 \), and denote the expected value and standard deviation of \( \tilde{r}_j \) by \( \mu_j \) and \( \sigma_j \). Suppose that \( \mu_1 > \mu_2, \sigma_1 > \sigma_2 \).

A portfolio of these two assets can be conveniently denoted by \((w, 1-w)\), where \( w \) is the portfolio weight assigned to (the percentage of the initial wealth spent on) asset 1.

(i) Find the portfolio for the two assets with the smallest variance of rate of return. From now on, we refer to this portfolio the minimum variance portfolio of assets 1 and 2, or simply the mvp.\(^{16}\) Can the mvp turn out to be asset 1 alone? If it can, when does this happen? Can it be asset 2 alone? If it can, when does this happen?

\(^{16}\)Here you must detail the first-order and second-order conditions.
(ii) Now, suppose that the two risky assets are the only two traded assets at date 0. Suppose also that every investor is endowed with a mean-variance utility function (as defined in Problem 2 of Homework 1). That is, every investor’s welfare is increasing in the expected value and decreasing in the variance of the rate of return on the portfolio that he chooses to hold at date 0. Suppose furthermore that

$$\mu_1 < \mu_2, \sigma_1 > \sigma_2.$$ 

Can there be an investor with a mean-variance utility function that is willing to take a long position in asset 1 at date 0? Does your answer depend on whether the two assets are in positive supply?

Solution. For part (i), we seek to

$$\min_{w \in \mathbb{R}} f(w) \equiv \frac{1}{2} \text{var}(w\tilde{r}_1 + (1-w)\tilde{r}_2),$$

and it can be easily verified that, with $\rho$ being the coefficient of correlation between $r_1$ and $r_2$,

$$f'(w) = w(\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2) + \rho \sigma_1 \sigma_2 - \sigma_2^2,$$

and

$$f''(w) = \sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2 \geq \sigma_1^2 - 2 \cdot 1 \cdot \sigma_1 \sigma_2 + \sigma_2^2 = (\sigma_1 - \sigma_2)^2 > 0,$$

so that $f(\cdot)$ is strictly convex. Thus the first-order condition is necessary and sufficient:

$$f'(w^*) = 0 \implies w^* = \frac{\sigma_2^2 - \rho \sigma_1 \sigma_2}{\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2}.$$ 

Apparently, asset 1 cannot be the MVP. For asset 2 to be the MVP, we need $w^* = 0$ or $\text{cov}(\tilde{r}_1, \tilde{r}_2) = \sigma_2^2$.\footnote{In general, one can show that with $N$ risky assets and one riskless asset, if $\tilde{r}_{mvp}$ stands for the random rate of return on the MVP generated by $N$ risky assets and $\tilde{r}_p$ is the rate of return on any portfolio $p$, then we have $\text{cov}(\tilde{r}_p, \tilde{r}_{mvp}) = \text{var}(\tilde{r}_{mvp})$.} This finishes part (i).

Before we examine part (ii), let us determine whether asset 1 and asset 2 are respectively mean-variance efficient.
Note that (1) both assets are inefficient if \( \mu_1 < w^* \mu_1 + (1 - w^*) \mu_2 \); (2) both are efficient if \( \mu_2 > w^* \mu_1 + (1 - w^*) \mu_2 \); and (3) asset 1 is efficient while asset 2 inefficient if \( \mu_2 < w^* \mu_1 + (1 - w^*) \mu_2 \). In case (1), since \( \mu_1 > \mu_2 \), it must be that \( w^* > 1 \). Similarly, \( w^* < 0 \) in case (2) and \( w^* \in (0,1) \) in case (3). We claim that \( w^* < 1 \). To see this, note that the sign of \( w^* - 1 \) is the sign of \( \rho - \frac{\sigma_1}{\sigma_2} \), which is negative. Thus asset 1 is efficient. For \( w^* > 0 \), it must be that \( \rho < \frac{\sigma_2}{\sigma_1} \). Thus asset 2 is efficient if and only if \( \rho > \frac{\sigma_2}{\sigma_1} \).

Now consider part (ii). Apparently, if \( \mu_1 < \mu_2 \) and \( \sigma_1 > \sigma_2 \), then asset 1 is dominated by asset 2 as a single asset. However, investors are not required to choose one single asset. In fact, they are allowed to choose any portfolio generated by the two assets. It may still happen that some efficient portfolio is a convex combination of asset 1 and asset 2, and that happens if and only if \( w^* > 0 \), or simply \( \rho < \frac{\sigma_2}{\sigma_1} \), according to our preceding discussion. There are two cases to consider: either asset 2 is efficient or it is inefficient. If asset 2 is efficient, then asset 1 must be inefficient, and this case is consistent with both assets being in positive net supply. If instead asset 2 is inefficient then asset 1 cannot be in positive net supply: every mean-variance rational investor will choose to hold an efficient portfolio in equilibrium, and so everyone is selling asset 1 short in equilibrium.

**Example 14** Suppose that in a two-period economy there are 3 traded assets, labeled 1,2 and 3. Suppose that the following data are valid and short sale is completely prohibited (that is, the portfolio weights are all required to be non-negative).

\[
e = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.4 \end{bmatrix}, \quad V = \begin{bmatrix} 0.01 & -0.02 & -0.03 \\ -0.02 & 0.04 & 0.06 \\ -0.03 & 0.06 & 0.16 \end{bmatrix}.
\]

Find the portfolio frontier (i.e. the frontier portfolio \( w^*(\mu) \) for each target expected rate of return \( \mu \in \mathbb{R} \)). (Hint: Set up a minimization program with inequality and equality constraints, and apply Kuhn-Tucker theorem.)

**Solution.** One can verify that \( V \) is positive semi-definite, but not positive definite. Comparing the first and second rows of \( V \) reveals
that, essentially, for some constant $\alpha$,

$$\tilde{r}_2 = \alpha - 2\tilde{r}_1.$$  

Comparing the first two elements of $\mathbf{e}$ reveals that

$$\alpha = 0.4.$$  

Now, define

$$\nu = 10\mu, \ \forall \mu \in [0.1, 0.4].$$

Given $\mu \in [0.1, 0.4]$, or $\nu \in [1, 4]$, we seek to

$$\min_{\mathbf{w}} 100\mathbf{w}'\mathbf{Vw}$$

subject to

$$\mathbf{w} = \begin{bmatrix} x \\ 1 - x - y \\ y \end{bmatrix},$$

$$x \geq 0, \ y \geq 0, \ x + y \leq 1,$$

and

$$0.4(1 - x - y) + (3x + 2y - 2)e_1 + ye_3 = \mu = \frac{\nu}{10},$$

where $e_1 = 0.1$ and $e_3 = 0.4$ denote respectively $E[\tilde{r}_1]$ and $E[\tilde{r}_3]$.

It follows that

$$x = 2y - \nu + 2 \geq 0 \iff y \geq \frac{\nu}{2} - 1,$$

and

$$1 - x - y \geq 0 \iff y \leq \frac{\nu - 1}{3}.$$  

Thus our minimization problem can be restated as, given $\nu \in [1, 4]$,

$$\min_y f(y) = (8y - 3\nu + 4)^2 + 16y^2 - 6y(8y - 3\nu + 4)$$

39
subject to
\[
\max(0, \frac{\nu}{2} - 1) \leq y \leq \frac{\nu - 1}{3}.
\]
The unconstrained minimum of the convex function \(f(\cdot)\) appears at
\[
y' = \frac{15\nu - 20}{32}.
\]
Note that
\[
y' \geq \frac{\nu}{2} - 1 \iff \nu \leq 12; \quad y' \geq 0 \iff \nu \geq \frac{4}{3}; \quad y' \leq \frac{\nu - 1}{3} \iff \nu \leq \frac{28}{13}.
\]
Let us now take cases.

**Case 1.** Suppose that \(1 \leq \nu \leq 2\), so that \(\max(0, \frac{\nu}{2} - 1) = 0\).
In this case, using the aforementioned properties of \(y'\) and the fact that \(f(\cdot)\) is strictly convex, we have
\[
y^*(\nu) = \begin{cases} 
0, & \text{if } 1 \leq \nu \leq \frac{4}{3}; \\
y', & \text{if } \frac{4}{3} \leq \nu \leq 2.
\end{cases}
\]

**Case 2.** Suppose that \(2 \leq \nu \leq 4\), so that \(\max(0, \frac{\nu}{2} - 1) = \frac{\nu}{2} - 1\).
In this case, using the aforementioned properties of \(y'\) and the fact that \(f(\cdot)\) is strictly convex, we have
\[
y^*(\nu) = \begin{cases} 
y', & \text{if } 2 \leq \nu \leq \frac{28}{13}; \\
\frac{\nu - 1}{3}, & \text{if } \frac{28}{13} \leq \nu \leq 4.
\end{cases}
\]
To sum up, the frontier portfolios can be characterized as follows.
Example 15 Consider a two-period perfect markets economy where there are only two traded risky assets at date 0. Let the rates of return on the two assets be \( \tilde{r}_1 \) and \( \tilde{r}_2 \), with their means, variances, and coefficient of correlation being \( \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \) and \( \rho \). You are told that the \text{mvp} \) would remain the same if \( \rho \) were replaced by any other \( \rho' \in (-1, 1) \). Suppose that \( E[\tilde{r}_\text{mvp}] = 10\% \) and \( \mu_1 = 5\% \). Suppose that Mr. A cares about only the expected value and variance of the rate of return on his portfolio as we assumed in Portfolio Theory, and he has an initial wealth $800,000. If Mr. A wants to enjoy an expected rate of return 13\%, then how much money should he spend on asset 2? (Hint: Try to get information about \( \sigma_1 \) and \( \sigma_2 \) from the statement the \text{mvp} \) would remain the same if \( \rho \) were replaced by any other \( \rho' \in (-1, 1) \). Then use the fact that \( E[\tilde{r}_\text{mvp}] = 10\% \) and \( \mu_1 = 5\% \) to deduce \( \mu_2 \).)

Solution. From Example 13, we can obtain the portfolio weight on asset 1 of the \text{mvp}, which is

\[
w^*(\mu) = \begin{cases} 
\frac{2 - 10\mu}{3}, & \text{if } 0.1 \leq \mu \leq \frac{2}{15}; \\
\frac{-10\mu + 12}{16}, & \text{if } \frac{2}{15} \leq \mu \leq \frac{14}{65}; \\
\frac{14 - 65\mu}{16}, & \text{if } \frac{14}{65} \leq \mu \leq 0.4. \\
\end{cases}
\]
and you are told that this portfolio weight remains the same when \( \rho \) is replaced by \( \rho' \), for any \( \rho, \rho' \in (-1, 1) \). Hence we obtain

\[
\sigma_1 \sigma_2 (\sigma_1^2 - \sigma_2^2) (\rho - \rho') = 0 \Rightarrow \sigma_1 = \sigma_2,
\]

which in turn implies that the \textbf{mvp} is the equally weighted portfolio of assets 1 and 2. From here, since \( E[\tilde{r}_{\text{mvp}}] = 10\% \) and \( \mu_1 = 5\% \), we conclude that \( \mu_2 = 15\% \). If the portfolio \((w, 1 - w)\) yields an expected rate of return equal to 13\%, then it must be that \( w = \frac{1}{5} \), and hence Mr. A should spend \$800,000 \times \frac{4}{5} = \$640,000 on asset 2.

**Example 16** Suppose that the Sharpe-Lintner CAPM holds in a two-period perfect markets economy where \( r_f = 0 \). Suppose that it is equally likely that \( \tilde{r}_m = 10\% \) and \( 14\% \).

(i) Recall the price of market risk defined by

\[
\lambda = \frac{E[\tilde{r}_m] - r_f}{\text{var}(\tilde{r}_m)}.
\]

Compute \( \lambda \).

(ii) Suppose that in equilibrium the following data about firm A are valid. Let \( \tilde{x} \) be the date-1 total cash earnings of firm A, and let \( D \) be the face value of firm A’s debt that will be due at date 1. Suppose that conditional on \( \tilde{r}_m = 10\% \), \( \tilde{x} \) is equally likely to take any value \( j \in \{1, 2, \cdots, 1000\} \), and conditional on \( \tilde{r}_m = 14\% \), \( \tilde{x} \) is equally likely to take any value \( k \in \{1, 2, \cdots, 100\} \). Suppose that 100 < \( D \) < 1000 and \( D \) is a positive integer. Assume that firm A is a corporation protected by limited liability. Find the date-0 debt value for firm A as a function of \( D \). (Hint: Use the certainty equivalent formula developed in Example 4. Now apply the law of iterated expectations to \( E[\min(\tilde{x}, D)] \) and \( \text{cov}(\min(\tilde{x}, D), \tilde{r}_m) \) by first conditioning these expectations on a realization of \( \tilde{r}_m \), and then taking the unconditional expectations.)

**Solution.** It is easy to show that \( \lambda = 300 \), which is part (i).

For part (ii), let \( P \) be the bond price, and from the certainty equivalent formula mentioned in the hint we can get (since \( r_f = 0 \))

\[
P = E[\tilde{y}] - \lambda \text{cov}(\tilde{y}, \tilde{r}_m),
\]

42
where $\tilde{y}$ is the date-1 payoff generated by one unit of the corporate bond. What is $\tilde{y}$? Since the firm is protected by limited liability, $\tilde{y} = \min(\hat{x}, D)$.

By the law of iterated expectations, we have

$$P = E[\tilde{y}] - \lambda \text{cov}(\tilde{y}, \tilde{r}_m) = E[\tilde{y}] - \lambda E[\tilde{y}(\tilde{r}_m - \mu_m)]$$
$$= E[\tilde{y}(1 - \lambda(\tilde{r}_m - \mu_m))]$$
$$= E[E[\tilde{y}|\tilde{r}_m](1 - \lambda(\tilde{r}_m - \mu_m))]$$

$$= \text{prob.}(\tilde{r}_m = 10\%)\{E[\min(\hat{x}, D)|\tilde{r}_m = 10\%](1 - \lambda(10\% - 12\%))\} + \text{prob.}(\tilde{r}_m = 14\%)\{E[\min(\hat{x}, D)|\tilde{r}_m = 14\%](1 - \lambda(14\% - 12\%))\}$$

$$= \frac{1}{2} \times \{\frac{D(D+1)}{2} + (1,000 - D)D}{1,000} \times 7\}$$
$$+ \frac{1}{2} \times \frac{100(100+1)}{100} \times (-5)$$
$$= \frac{1}{2}\{-\frac{7D^2}{2,000} + \frac{14,007D}{2,000}\} + \frac{1}{2}\{-\frac{505}{2}\}$$
$$= -\frac{7D^2 + 14,007D - 505,000}{4,000}.$$ 

This finishes part (ii).

**Example 17** Consider a two-period perfect markets economy where there are only two traded risky assets at date 0. Let the rates of return on the two assets be $\tilde{r}_1$ and $\tilde{r}_2$, with their means, variances, and coefficient of correlation being $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and $\rho$. Assume that $(\tilde{r}_1, \tilde{r}_2)$ are bivariate normal (meaning that any linear combination of $\tilde{r}_1$ and $\tilde{r}_2$ is again a normal random variable).\(^\ast\) Suppose that there are only two

\[^\ast\]Two random variables $\hat{x}, \hat{y}$ are bivariate normal if their joint density function

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y\sqrt{1-\rho^2}}}, \forall x, y \in \mathbb{R},$$

where $\mu_x, \mu_y, \sigma_x, \sigma_y, \rho$ are the means, the standard deviations, and the coefficient of correlation of $\hat{x}$ and $\hat{y}$.
investors at date 1, both seeking to maximize $E[-e^{-\tilde{W}}]$, where $\tilde{W}$ is an investor’s date-1 random wealth. For $i = 1, 2$, investor $i$ is endowed with one unit of asset $i$ (which is the total supply of asset $i$) and nothing else. Let the date-0 equilibrium prices of the two assets be $p_1$ and $p_2$ respectively (so that asset 1 is taken as numeraire). Find the date-0 market portfolio as a function of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and $\rho$. (Hint: Notice that investor $i$ seeks to maximize $E[\tilde{W}] - \frac{1}{2} \text{var}(\tilde{W})$ when choosing their demands for the two assets, $d^*_1(p)$ and $d^*_2(p)$. Find the demands for the two assets for each investor $i$, and then solve for the equilibrium price $p$ for asset 2 by imposing the markets clearing condition. Finally, recall the definition of the market portfolio.)

Solution. By the hint, investor $i$ given his initial wealth $W_i$ seeks to maximize

$$\max_{d^*_1, d^*_2 \in \mathbb{R}} E[d^*_1(1 + \tilde{r}_1) + d^*_2p(1 + \tilde{r}_2)] - \frac{1}{2} \text{var}[d^*_1(1 + \tilde{r}_1) + d^*_2p(1 + \tilde{r}_2)],$$

subject to

$$d^*_1 + d^*_2p = W_i,$$

where $W^1 = 1$ and $W^2 = p$. Replacing $d^*_1 = W_i - d^*_2p$ into the objective function, and calling the latter $L(d^*_2p)$, we have

$$L(x) = E[(W^i - x)(1 + \tilde{r}_1) + x(1 + \tilde{r}_2)] - \frac{1}{2} \text{var}[(W^i - x)(1 + \tilde{r}_1) + x(1 + \tilde{r}_2)],$$

where $L(\cdot)$ can be easily verified to be concave. Rewrite $L$ as

$$L(x) = E[W^i(1 + \tilde{r}_1) + x(\tilde{r}_2 - \tilde{r}_1)] - \frac{1}{2} \text{var}[W^i(1 + \tilde{r}_1) + x(\tilde{r}_2 - \tilde{r}_1)]$$

$$= W^i(1 + \mu_1) + x(\mu_2 - \mu_1) - \frac{1}{2}[(W^i)^2\sigma_1^2 + x^2(\sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2) + 2W^ix(\rho\sigma_1\sigma_2 - \sigma_1^2)].$$

Hence the optimal solution $x$ satisfies

$$L'(x) = 0 \Rightarrow \mu_2 - \mu_1 - x(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) - W^i(\rho\sigma_1\sigma_2 - \sigma_1^2) = 0,$$

or, equivalently,

$$d^*_2p = \frac{\mu_2 - \mu_1 + W^i\sigma_1^2 - W^i\rho\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2},$$

44
so that using $W^1 + W^2 = 1 + p$ and the market clearing condition

$$d_1^2 + d_2^2 = 1,$$

we obtain

$$p = \frac{2(\mu_2 - \mu_1) + (1 + p)[\sigma_1^2 - \rho \sigma_1 \sigma_2]}{\sigma_2^2 + \sigma_1^2 - 2\rho \sigma_1 \sigma_2},$$

implying that

$$p = \frac{2(\mu_2 - \mu_1) + \sigma_1^2 - \rho \sigma_1 \sigma_2}{\sigma_2^2 - \rho \sigma_1 \sigma_2}.$$

Thus $p$ is higher if $\mu_2$ is higher or if $\mu_1$ is lower. By the definition of the market portfolio, we have

$$w_m = \begin{bmatrix} \frac{1}{1+p} \\ \frac{p}{1+p} \end{bmatrix} = \begin{bmatrix} \frac{\sigma_2^2 - \rho \sigma_1 \sigma_2}{2(\mu_2 - \mu_1) + \sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2} \\ \frac{2(\mu_2 - \mu_1) + \sigma_1^2 - \rho \sigma_1 \sigma_2}{2(\mu_2 - \mu_1) + \sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2} \end{bmatrix}. \tag{1}$$

20. The equilibrium CAPM equation has a counterpart derived from a no-arbitrage argument, which is the APT (arbitrage pricing theory) pricing equation.\footnote{We have assumed thus far that there exists a competitive equilibrium, and proved that the CAPM must hold in equilibrium provided that the imposed conditions are met. See for example Nielsen (1990) for a set of conditions that ensure the existence of competitive equilibrium for an economy where each investor is endowed with a (possibly non-linear) mean-variance utility function. We have mentioned earlier that a mean-variance utility function may not be equivalent to an expected utility function. While the CAPM must obviously hold in the equilibrium of an economy where investors have mean-variance utility functions, we will show below that the CAPM need not be incompatible with the equilibrium of an economy where investors are expected utility maximizers.} The arbitrage pricing theory was first developed by Stephen A. Ross (1976), but our discussion below will follow Hubersman (1982).

21. Consider a sequence $\{\mathcal{E}_n; n \in \mathbb{Z}_+\}$ of two-period frictionless economies, where in economy $\mathcal{E}_n$, there are $n$ traded assets, of which the random rates of return $\tilde{r}^n$ are affine functions of $k$ random factors (best thought of as aggregate economic variables) and of their idiosyncratic noises; more precisely,

$$\tilde{r}^n = e^n + B^n \tilde{d} + \tilde{u}^n, \tag{1}$$
where $e_n^{n \times 1} = E[\tilde{r}^n]$, $\tilde{d}_{k \times 1}$ is a random vector containing the outcomes of $k$ aggregate economic variables, $B_{n \times k}^n$ is a non-random matrix, and $\tilde{u}^{n \times 1}$ gives the idiosyncratic risks for the $n$ traded assets in economy $E_n$. Note that as $n$ grows, the number of rows in $\tilde{r}^n, e^n, B^n$, and $\tilde{u}^n$ grows, but the random vector $\tilde{d}$ remains unchanged. (Note that we have put a superscript $n$ on $\tilde{r}, e, B,$ and $\tilde{u}$ to emphasize that these matrices are data pertaining to the $n$-th economy $E_n$.)

22. The main result below shows that as $n$ tends to infinity, in order that these return data do not admit arbitrage opportunities defined by Stephen A. Ross, for almost all traded assets, the risk premia are approximately linear functions of $B$. This result should be contrasted with the traditional CAPM, where in equilibrium the risk premium of each asset is a linear function of the asset’s beta with respect to the rate of return on the market portfolio. Thus the APT gives a multi-beta extension of the CAPM.\textsuperscript{20}

23. It will be assumed from now on that (i) $E[\tilde{u}^n] = 0_{n \times 1}$; and (ii) the covariance matrix of $\tilde{u}^n$ is

\[
V_{n \times n}^n \equiv \begin{bmatrix}
\sigma_1^2 & 0 & \cdots & 0 \\
0 & \sigma_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_n^2
\end{bmatrix},
\]

and moreover, although as $n$ grows, there will be more and more main diagonal elements in $V^n$, these main diagonal elements $\sigma_i^2$ are always bounded above by a finite positive number $T > 0$. Recall from section 17 that by performing a mean-variance projection, we can always write down equation (1) as long as the random variables contained in $\tilde{r}^n$ and $\tilde{d}$ all have finite variances. However, mean-variance projection

\textsuperscript{20}However, the no-arbitrage argument employed in the derivation of the APT is different from the equilibrium approach adopted in the development of the CAPM. For a genuine multi-beta extension of the traditional CAPM in a continuous-time setting, see Robert Merton, 1973, An Intertemporal Capital Asset Pricing Model, \textit{Econometrica}, and for the case where Merton’s multi-beta CAPM reduces to a single-beta CAPM, see Douglas Breeden, 1979, An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities, \textit{Journal of Financial Economics}.
per se does not guarantee that the random variables contained in the projection leftover \( \tilde{u}^n \) are pairwise uncorrelated! Thus that \( V^n \) is a diagonal matrix is an important and restrictive assumption made by the APT theorists, which plays a crucial role in leading to the main prediction of the theory.

24. Before we proceed further, let us consider a special case where there is a traded riskless asset, and where for all \( n \in \mathbb{Z}_+ \), \( \tilde{u}^n = 0_{n \times 1} \). In this case, for all \( n \), we have

\[
\tilde{r}^n = e^n + B^n \tilde{d}.
\]

Assume furthermore that the \( k \) random variables contained in \( \tilde{d} \) are the rates of return on \( k \) traded portfolios tracking perfectly \( k \) macroeconomic risks. Then, we claim that, in order to rule out arbitrage opportunities, we must have

\[
\tilde{r}^n = r_f 1_{n \times 1} + B^n [\tilde{d} - r_f 1],
\]

so that

\[
E[\tilde{r}^n] = r_f 1_{n \times 1} + B^n (E[d] - r_f 1),
\]

which is a multi-beta version of the CAPM.

To see what happens, fix any \( j \in \{1, 2, \ldots, n\} \) and consider the following portfolio strategy: spending \( (1 - \sum_{h=1}^k \beta_{jh}) \) on the riskless asset, and \( \beta_{jh} \) on the portfolio with rate of return \( \tilde{d}_h \), for all \( h = 1, 2, \ldots, k \). (Note that \( \beta_{jh} \) is the \((j, h)\)-element of \( B^n \).) This portfolio strategy costs 1 dollar at date 0, and it generates

\[
(1 - \sum_{h=1}^k \beta_{jh}) r_f + \sum_{h=1}^k \beta_{jh} \tilde{d}_h
\]

at date 1. If instead one spends the dollar on asset \( j \) at date 0, then the dollar will generate

\[
e_j + \sum_{h=1}^k \beta_{jh} \tilde{d}_h
\]
at date 1. Since the two strategies both cost a dollar at date 0, and since they both generate risky cash flows $\sum_{h=1}^{k} \beta_{jh} \tilde{d}_h$ at date 1, we must have

$$e_j = (1 - \sum_{h=1}^{k} \beta_{jh}) r_f$$

in order to rule out arbitrage opportunities at date 0. Hence the claim is true.

In the following, we shall show that in an economy with a countably infinite number of traded assets where no arbitrage opportunities exist, and where for all $n$, the rates of return on the first $n$ traded assets satisfy (1) with $\tilde{u}^n$ having the diagonal covariance matrix $V^n$, almost all traded assets’ expected rates of return can be approximately represented by the above multi-beta CAPM.

25. To obtain the main results of APT, we must first establish two simple lemmas, which we shall use to prove the two theorems below.

**Lemma APT-1** Given $y, x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, where $k < n$ and $X_{n \times k} \equiv [x_1, x_2, \ldots, x_k]$ has rank $k$, there is a vector $b \in \mathbb{R}^k$ and a vector $h \in \mathbb{R}^n$ such that

$$y = Xb + h,$$

and that

$$X'h = 0_{k \times 1};$$

namely, $h'x_i = 0$, for all $i = 1, 2, \ldots, k$.

**Proof.** Define

$$b \equiv (X'X)^{-1}X'y,$$

where the inverse exists because $X$ has rank $k$. Now given $b$, define

$$h \equiv y - Xb,$$

and we have

$$X'h = X'(y - Xb) = X'y - X'Xb = X'y - X'y = 0_{k \times 1}.$$

**Lemma APT-2** For all $z \in \mathbb{R}^n$, we have

$$z'Vz \leq Tz'z.$$
Proof. Simply observe that
\[ z'Vz = \sum_{i=1}^{n} \sigma_i^2 z_i^2 \leq T \sum_{i=1}^{n} z_i^2 = Tz'z. \]

26. Now, given the return data in the n-economy \( \mathcal{E}_n \),
\[ \tilde{r}^n = e^n + B^n \tilde{d} + \tilde{u}^n, \]
we apply Lemma APT-1 to write
\[ e^n = \rho^n \mathbf{1} + B^n q^n + c^n, \]
with \( e^n \) in place of \( y \) and \([\mathbf{1}, B^n]\) in place of \( X \) in Lemma APT-1. Thus \( c^n \) corresponds to the vector \( h \) in Lemma APT-1. By Lemma APT-1, \( c^n \) must be orthogonal to each and every column vector in \([\mathbf{1}, B^n]\).
That is, we have
\[ [c^n]'\mathbf{1} = 0, \quad [c^n]'B^n = 0'_{1\times k}. \]

Definition APT-1 An arbitrage portfolio in economy \( \mathcal{E}_n \) is any n-vector \( w^n \) such that \([w^n]'\mathbf{1} = 0\).

Definition APT-2 An arbitrage opportunity in the sense of Ross is a sequence \( \{w^n; n \in \mathbb{Z}_+\} \) of arbitrage portfolios in the corresponding sequence \( \{\mathcal{E}_n; n \in \mathbb{Z}_+\} \) of two-period frictionless economies, such that\(^{21}\)

\(^{21}\)To understand the definition, consider a competitive securities markets economy in which every investor is endowed with a mean-variance utility function \( U(E[\tilde{W}], \text{var}[\tilde{W}]) \), which is strictly increasing in its first argument and strictly decreasing in its second argument, and which satisfies
\[ \lim_{E \uparrow +\infty} U(E, \text{var}) = +\infty, \quad \forall \text{var} \in \mathbb{R}_+. \]

Now, if an arbitrage opportunity in the sense of Ross exists, then it is a feasible but not necessarily optimal strategy for investor \( i \) to keep the initial wealth \( W_{i0} \) on one hand and to hold the arbitrage portfolio \( w^n \) on the other hand, and this strategy will yield for investor \( i \) the utility
\[ U(E[\tilde{W}], \text{var}[\tilde{W}]) = U(W_{i0}[1 + (w^n)'\tilde{r}^n], W_{i0}^2 \text{var}[(w^n)'\tilde{r}^n]), \]
which tends to \(+\infty\) as \( n \) tends to \(+\infty\), so that there does not exist an optimal trading strategy for investor \( i \)—the return data are incompatible to a competitive equilibrium.
\[
\lim_{n \to \infty} E[(w^n)'] \hat{r}^n] = +\infty, \\
\lim_{n \to \infty} \text{var}[(w^n)'] \hat{r}^n] = 0.
\]

27. **Theorem APT-1** Suppose that the sequence \( \{E_n; n \in \mathbb{Z}_+\} \) of two-period frictionless economies does not admit any arbitrage opportunities in the sense of Ross. Then, there must exist some constant \( A > 0 \) such that for all \( n \in \mathbb{Z}_+ \),

\[
[c^n]'[c^n] = [e^n - \rho^n 1 - B^n q^n]'[e^n - \rho^n 1 - B^n q^n] \leq A.
\]

This implies that, given \( A \), the following subset of \( \mathbb{R}^{k+1} \),

\[
H^n \equiv \left\{ \begin{bmatrix} \rho \\ q \end{bmatrix} : [e^n - \rho 1 - B^n q]'[e^n - \rho 1 - B^n q] \leq A \right\},
\]

is non-empty.

**Proof.** Suppose not. Then, given \( A_1 > 0 \), there must exist \( n_1 \in \mathbb{Z}_+ \) such that

\[
[c^{n_1}]'[c^{n_1}] > A_1,
\]

and given \( A_2 > [c^{n_1}]'[c^{n_1}] > A_1 \), there must exist \( n_2 > n_1, n_2 \in \mathbb{Z}_+ \) such that

\[
[c^{n_2}]'[c^{n_2}] > A_2,
\]

and so on, and so forth.

Then, the infinite sequence \( \{[c^n]'[c^n]\} \) must contain a subsequence \( \{[c^{n_l}]'[c^{n_l}]\} \) with

\[
\lim_{n_l \to +\infty} [c^{n_l}]'[c^{n_l}] = +\infty.
\]

Define for all \( n_l \in \mathbb{Z}_+ \),

\[
a_{n_l} \equiv \{[c^{n_l}]'[c^{n_l}]\}^{-\frac{3}{4}}.
\]
Observe that \( w^n l \equiv a_n c^n l \) is an arbitrage portfolio in economy \( \mathcal{E}_{nl} \), according to Definition APT-1. We shall show that the sequence \( \{w^n l\} \) is an arbitrage opportunity in the sense of Ross, as defined in Definition APT-2, and hence we have a contradiction.

To this end, note that

\[
\lim_{n \to \infty} E[(w^n l)' \tilde{r} n l] = \lim_{n \to \infty} a_n E[(c^n l)'(e^n l - B^n \tilde{d} + \tilde{u}^n l)] = \lim_{n \to \infty} a_n [c^n l]'[c^n l] = \lim_{n \to \infty} \{[c^n l]'[c^n l]\}^{\frac{1}{2}} = +\infty.
\]

On the other hand, we have the limit of the variances of the gross returns on these arbitrage portfolios being

\[
\lim_{n \to \infty} \text{var}[(w^n l)'(1 + \tilde{r} n l)] = \lim_{n \to \infty} a_n^2 (c^n l)'V(c^n l) \leq T \lim_{n \to \infty} a_n^2 (c^n l)'(c^n l) = T \lim_{n \to \infty} \{[c^n l]'[c^n l]\}^{\frac{1}{2}} = 0.
\]

Thus we conclude that for all \( n \in \mathbb{Z}_+ \),

\[
[c^n l]'[c^n l] = [e^n - \rho^n 1 - B^n q^n]'[e^n - \rho^n 1 - B^n q^n] \leq A.
\]

The last assertion is obvious, for \( H^n \) contains at least the vector

\[
\begin{bmatrix}
\rho^n \\
q^n
\end{bmatrix}
\]

that satisfies

\[
e^n = \rho^n 1 + B^n q^n + c^n.
\]

51
28. Let $R(n)$ be the rank of $B^n$. Note that $R(n) \leq R(n + 1) \leq k$. Thus, as $n$ increases unboundedly, $R(n)$ will converge to, say $R(\pi)$; that is, $R(n) = R(\pi)$ for all $n \geq \pi$. Fix any $n \geq \pi$, we can assume that all the columns in $B^n$ are linear combinations of the first $R(\pi)$ columns of $B^n$. This fact together with the preceding theorem then implies that given $A$, for any $n \geq \pi$, the set

$$H^n \equiv \left\{ \begin{bmatrix} \rho \\ q \end{bmatrix} : [e^n - \rho \mathbf{1} - B^nq]'[e^n - \rho \mathbf{1} - B^nq] \leq A, \right\}$$

$q_{R(\pi)+1} = q_{R(\pi)+2} = \cdots = q_{k} = 0$}

is non-empty. Note that $H^{n+1} \subset H^n$ and for all $n \geq \pi$, $H^n$ is a compact set. This implies that

$$\bigcap_{n \in \mathbb{Z}_+} H^n$$

is non-empty.22 Thus there exist a constant $A$ and some $(k + 1)$ vector

$$\begin{bmatrix} \rho \\ q \end{bmatrix}$$

satisfying

$$\sum_{i=1}^{\infty} [E[\tilde{r}_i] - (\rho + \beta_{11}q_1 + \beta_{i2}q_2 + \cdots + \beta_{ik}q_k)]^2 < A.$$  

Define

$$c_i \equiv E[\tilde{r}_i] - (\rho + \beta_{11}q_1 + \beta_{i2}q_2 + \cdots + \beta_{ik}q_k).$$

Since the series $\sum_{i=1}^{\infty} c_i^2$ converges, we must have

$$\lim_{i \to \infty} c_i^2 = 0,$$

22Any decreasing sequence of non-empty closed sets in a compact space has a non-empty intersection.
or equivalently, given any $\epsilon > 0$, however small, there must exist $N(\epsilon) \in \mathbb{Z}_+$ such that

$$i > N(\epsilon) \Rightarrow |c_i| < \epsilon;$$

that is, except for the first $N(\epsilon)$ assets, all other assets $i > N(\epsilon)$ must have (recall that $e_i \equiv E[\tilde{r}_i]$)

$$|E[\tilde{r}_i] - (\rho + \beta_{i1}q_1 + \beta_{i2}q_2 + \cdots + \beta_{ik}q_k)| < \epsilon.$$

In plain words, almost all assets have their risk premia being approximately represented as affine functions of the $k$ beta’s with respect to the $k$ aggregate economic variables.\(^{23}\)

29. In the above it has been assumed that there does not exist a riskless asset. Now we introduce the riskless asset, and refer to it as asset 0, which is assumed to exist in each and every economy $\mathcal{E}_n$ (so that $\mathcal{E}_n$ has $(n + 1)$ assets). In this case, an arbitrage portfolio can be represented as an $(n + 1)$-vector

$$\begin{bmatrix} -1'c^n \\ c^n \end{bmatrix},$$

where unlike in the preceding sections, the $n$-vector $c^n$ is no longer required to have its elements sum up to one.

30. **Theorem APT-2** Suppose that the sequence $\{\mathcal{E}_n; n \in \mathbb{Z}_+\}$ of two-period frictionless economies (with the riskless asset having rate of return $r_f$) does not admit any arbitrage opportunities in the sense of Ross. Let us redefine the notation. Let $\tilde{r}^n$ now denote the excess rates of return vector. Its mean, $e^n$, becomes the risk premia vector. Note that $V^n$ remains to be the covariance matrix of $\tilde{r}^n$ under this new definition. Then, there must exist some constant $A > 0$ such that for all

\[^{23}\text{When } k = 1, \text{ the APT equation } \quad E[\tilde{r}_i] \sim \rho + \beta_{i1}q_1, \]

which holds approximately for almost all assets, should be contrasted with Fischer Black’s zero-$\beta$ CAPM; see Theorem 4. By letting $\rho = \mu'(\mu_m)$ and $q_1 = \mu_m - \mu'(\mu_m)$, the APT equation becomes the zero-$\beta$ CAPM equation.
\[ n \in \mathbb{Z}_+, \quad [e^n - B^n q^n]'[e^n - B^n q^n] \leq A. \]

**Proof** Applying Lemma APT-1, we can write

\[ e^n = B^n q^n + c^n, \]

with \( e^n \) in place of \( y \) and \( B^n \) in place of \( X \) in Lemma APT-1. Thus \( c^n \) corresponds to the vector \( h \) in Lemma APT-1. By Lemma APT-1, \( c^n \) must be orthogonal to each and every column vector in \( B^n \). That is, we have

\[ [c^n]'B^n = 0_{1 \times k}. \]

Now if the assertion fails to be true, then the infinite sequence \( \{[c^n]'[c^n]\} \) must contain a subsequence \( \{[c^{n_i}]'[c^{n_i}]\} \) with

\[ \lim_{n_i \to \infty} [c^{n_i}]'[c^{n_i}] = +\infty. \]

In this case, define for all \( n_i \in \mathbb{Z}_+ \),

\[ a_{n_i} = \{[c^{n_i}]'[c^{n_i}]\}^{-\frac{3}{2}}. \]

Observe that

\[ w^{n_i} = a_{n_i} \begin{bmatrix} -1'c^{n_i} \\ c^{n_i} \end{bmatrix} \]

is an arbitrage portfolio in economy \( E_{n_i} \), according to Definition APT-1. We shall show that the sequence \( \{w^{n_i}\} \) is an arbitrage opportunity in the sense of Ross, as defined in Definition APT-2. This will establish a contradiction.

To this end, note that the limit of the risk premium on \( w^{n_i} \)

\[ \lim_{n_i \to \infty} E[a_{n_i}(c^{n_i})'\tilde{r}^{n_i}] = \lim_{n_i \to \infty} a_{n_i}E[(c^{n_i})'(e^{n_i} + B^{n_i}\tilde{d} + \tilde{u}^{n_i})] = \lim_{n_i \to \infty} a_{n_i}E[(c^{n_i})'(B^{n_i}q^{n_i} + c^{n_i} + B^{n_i}\tilde{d} + \tilde{u}^{n_i})] \]

\[ \text{Again, if } k = 1, \text{ then the APT equation looks just like the Sharpe-Lintner CAPM equation if } q_1 = E[\tilde{r}_m] - r_f. \]
= \lim_{n_l \to \infty} a_n_l[c_n{l}]^t[c_n{l}]

= \lim_{n_l \to \infty} \{(c_n{l})^t[c_n{l}]\}^{\frac{1}{2}} = +\infty.

On the other hand, we have the limit of the variances of the (gross) returns on these arbitrage portfolios being

\[
\lim_{n_l \to \infty} \text{var}[-a_n_l 1^t c_n{l}(1 + r_f) + a_n_l(c_n{l})^t(1 + r_f 1 + \tilde{r}^n{l})]
\]

= \lim_{n_l \to \infty} a_n_l^2 (c_n{l})^t V(c_n{l})

\leq T \lim_{n_l \to \infty} a_n_l^2 (c_n{l})^t(c_n{l})

= T \lim_{n_l \to \infty} \{(c_n{l})^t[c_n{l}]\}^{-\frac{1}{2}} = 0.\|

31. We have assumed in the preceding review of APT that there exist an infinite number of traded assets whose rates of return are linearly related to a fixed number (k) of macroeconomics variables representing the systematic risks. Without specifying what the k macroeconomic variables are, we now show that the APT holds trivially in an economy with a finite number (n) of risky traded assets. In fact, in the absence of a riskless asset, if we let k = n and define

\[
\tilde{d} = \tilde{r} - e, \quad B_{n \times n} = I_{n \times n}, \quad \hat{u} = 0_{n \times 1},
\]

in economy \( E_n \), then we have

\[
e = \rho 1 + Bq.
\]

where

\[
\rho = 0, \quad q = e.
\]

Note that although the APT holds exactly (rather than approximately) for economy \( E_n \), it does not give any useful predictions regarding the cross-sectional relationships among the expected rates of return on traded assets.
32. Now, reconsider economy $E_n$ in the absence of a riskless asset. Suppose that we make the following stronger assumption

$$E[\tilde{u}_j | \tilde{d}] = 0, \forall j = 1, 2, \ldots, n.$$  

Suppose that there exists an investor with von Neumann-Morgenstern utility function $u$ such that $u'' < 0 < u'$, and that for this investor the optimal portfolio $\mathbf{w}^*$ is such that $\sum_{j=1}^n w_j^* \tilde{u}_j = 0$; that is, $\mathbf{w}^*$ contains no unsystematic risk. We claim that the APT must hold exactly. To see this, note that $\mathbf{w}^*$ must solve the following maximization problem:

$$\max_{\mathbf{w} \in \mathbb{R}^n} E[u(W_0(1 + \mathbf{w}' \tilde{r}))]$$  

subject to  

$$\mathbf{w}' \mathbf{1} = 1.$$  

Let $\gamma$ be the Lagrange multiplier for the constraint. The first-order conditions give

$$E[u'(W_0(1 + [\mathbf{w}^*]' \tilde{r})) \tilde{r}] = \gamma \mathbf{1}.$$  

Define

$$f \equiv u'(W_0(1 + [\mathbf{w}^*]' \tilde{r})) > 0,$$  

and note that $f$ depends on $\tilde{d}$ but not on $\tilde{u}$. The first-order conditions can be re-written as

$$E[f \cdot (\mathbf{e} + \mathbf{B} \tilde{d} + \tilde{u})] = \gamma \mathbf{1}.$$  

It follows that

$$\mathbf{e} E[f] = \gamma \mathbf{1} + \mathbf{B} E[f \tilde{d}] + E[f \tilde{u}]$$  

$$= \gamma \mathbf{1} + \mathbf{B} E[f \tilde{d}],$$  

since

$$E[f \tilde{u}] = E[E[f \tilde{u}^n] | \tilde{d}]$$  

$$= E[f E[\tilde{u} | \tilde{d}]] = 0.$$  

It follows that

$$\mathbf{e} = \frac{\gamma}{E[f]} \mathbf{1} + \mathbf{B} \frac{f}{E[f]} \tilde{d},$$  

which is an exact APT relationship.
33. Now we turn to the necessary and sufficient conditions for two-fund separation. The following conditions are taken from Litzenberger and Ramaswamy (1979); Ross (178) gives conditions for general k-fund separation. We now state two theorems without proofs; see respectively sections 4.4 and 4.12 of Huang and Litzenberger (1988) for the proofs.

**Theorem 7** Suppose that the riskless asset does not exist and that \( e \) is not proportional to \( 1 \). Fix \( \mu_1, \mu_2 \in \mathbb{R}, \mu_1 \neq \mu_2 \). The equilibrium rates of return \( \tilde{r} \) exhibit two-fund separation if and only if for all portfolios \( w \),

\[
\begin{align*}
\left[ \frac{w'e - \mu_2}{\mu_1 - \mu_2} w^*(\mu_1) + \frac{\mu_1 - w'e}{\mu_1 - \mu_2} w^*(\mu_2) \right]' \tilde{r} = E[w'\tilde{r} | \left[ \frac{w'e - \mu_2}{\mu_1 - \mu_2} w^*(\mu_1) + \frac{\mu_1 - w'e}{\mu_1 - \mu_2} w^*(\mu_2) \right]' \tilde{r}].
\end{align*}
\]

To understand this theorem, recall that when two-fund separation holds, given any \( \mu_1 \neq \mu_2 \), \( w^*(\mu_1) \) and \( w^*(\mu_2) \) can be the two separating funds, and given any portfolio \( w \), the portfolio

\[
w^*(w) \equiv \frac{w'e - \mu_2}{\mu_1 - \mu_2} w^*(\mu_1) + \frac{\mu_1 - w'e}{\mu_1 - \mu_2} w^*(\mu_2)
\]

stochastically dominates \( w \) in the second degree. Theorem 2 then shows that for some random variable \( \tilde{\epsilon} \) with

\[
E[\tilde{\epsilon} | w^*(w)'\tilde{r}] = 0,
\]

the two random variables \( w^*(w)'\tilde{r} + \tilde{\epsilon} \) and \( w'\tilde{r} \) have the same distribution function. Here, Theorem 7 gives the stronger result that the two random variables \( w^*(w)'\tilde{r} + \tilde{\epsilon} \) and \( w'\tilde{r} \) must be the same random variable! To obtain \( \tilde{\epsilon} \), here we can perform mean-variance projection.\(^{25}\)

\(^{25}\)Given any two random variables \( \tilde{x}, \tilde{y} \) both with finite positive variances, there exist constants \( a, b \) and random variable \( \tilde{u} \) such that \( \tilde{y} = a + b\tilde{x} + \tilde{u} \) with

\[
E[\tilde{u}] = \text{cov}(\tilde{u}, \tilde{x}) = 0.
\]

We refer to \( b\tilde{x} \) the projected value of \( \tilde{y} \) on \( \tilde{x} \), and \( a + \tilde{u} \) the residual from that projection.
of \( w'\tilde{r} \) on \( w^*(w)'\tilde{r} \), and the residual from the projection is exactly the \( \tilde{\epsilon} \) that we were looking for; that is, it satisfies

\[
E[\tilde{\epsilon}| w^*(w)'\tilde{r}] = 0.
\]

**Theorem 8** Suppose that the riskless asset exists and is in zero net supply with equilibrium rate of return \( r_f \), that the \( N \) risky assets are in strictly positive supply, and that there exists \( j \in \{1, 2, \ldots, N\} \) such that \( E[\tilde{r}_j] \neq r_f \). The equilibrium rates of return \( r_f \) and \( \tilde{r} \) exhibit two-fund separation if and only if for all portfolios \( w \),

\[
E[w'\tilde{r}+(1-w'1)r_f - \frac{w'e - w'1r_f}{E[\tilde{r}_m] - r_f} \tilde{r}_m + \frac{E[\tilde{r}_m] - w'e - (1-w'1)r_f}{E[\tilde{r}_m] - r_f} r_f | \tilde{r}_m] = 0.
\]

The idea of Theorem 8 is similar to that of Theorem 7. We simply replace \( w^*(\mu_1) \) and \( w^*(\mu_2) \) in Theorem 7 by the market portfolio and the riskless asset, for the latter can be the two separating funds in the presence of the riskless asset.

34. A special class of distributions is of particular interest, because those distributions not only imply two-fund separation (and hence they satisfy respectively Theorems 7 and 8), but they also identify every expected utility maximizer with a mean-variance utility maximizer.

**Definition 5** A random vector \( \tilde{x}_{n \times 1} \) is **elliptically distributed** if its density takes the form

\[
f(x) = |\Omega|^{-\frac{1}{2}}g((x - e)'\Omega^{-1}(x - e); n],
\]

where \( g(\cdot) \) is some univariate function with parameter \( n \), the \( n \times n \) positive definite matrix \( \Omega \) is called the dispersion matrix, and \( e \) is, again, the \( n \times 1 \) mean vector.

It can be shown that if the variance-covariance matrix \( V \) of \( \tilde{x} \) is well-defined,\(^{26}\) then it is proportional to \( \Omega \) in the sense that there is some constant \( k > 0 \) such that

\[
V = k\Omega.
\]

\(^{26}\) The multivariate Cauchy distribution is elliptical, but its mean and variance are not finite.
It can be shown that the characteristic function of $\tilde{x}$ has the form

$$\phi_n(t) \equiv E[e^{it'x}] = e^{it'\Phi(t'\Omega)}$$

for some function $\Phi(\cdot)$ which does not depend on $n$. Multivariate normal random vectors are merely an example; they are many members in this class with fatter tails than the multivariate normal random vectors.

Now we show that elliptical distributions are stable under linear combinations. Given any conformable matrix $T$ define $\tilde{z} = T\tilde{x}$. Then

$$m \equiv E(\tilde{z}) = Te, \quad V_{\tilde{z}} = E[(\tilde{z} - m)(\tilde{z} - m)'] = TVT' = kT\Omega T'.$$

Note that

$$\phi_{\tilde{z}}(t) = E[e^{it'Tx}] = \phi_x(T't) = e^{it'm\Phi(\frac{1}{k}t'V_{\tilde{z}}t)},$$

proving that $\tilde{z}$, like $\tilde{x}$, also has an elliptical distribution.

Now, to show that mean-variance analysis is fully consistent with expected utility maximization under elliptical distributions, it suffices to show that dispersion is disliked. Given an investor’s VNM utility function $u$, with $u' > 0 > u''$, define

$$U(\mu, \omega) \equiv E[u(\tilde{z}_p)],$$

where $\tilde{z}_p$ is the date-1 terminal wealth generated by portfolio $p$, and $\mu$ and $\omega$ are respectively its mean and dispersion. We can rewrite

$$\tilde{z}_p = \mu + \omega\tilde{\xi},$$

where $\tilde{\xi}$ is elliptically distributed with unit dispersion (which would be standard normal if $\tilde{z}_p$ is normal). We want to determine the sign of $\frac{\partial U}{\partial \omega}$.

Note that

$$\frac{\partial U}{\partial \omega} = \int_{-\infty}^{\infty} u'(\mu + \omega\xi)g(\xi^2; 1)d\xi$$

$$= \int_0^{\infty} \xi g(\xi^2; 1)[u'(\mu + \omega\xi) - u'(\mu - \omega\xi)]d\xi < 0,$$

where the above second equality uses symmetry and the inequality follows from concavity of $u(\cdot)$. 59
Since any nonnegative function \( g(\cdot) \), when properly normalized, can be used as a basis of an elliptical distribution, the problem of unlimited liability pertaining to multivariate normality does not extend to all elliptical distributions.\footnote{The multivariate student \( t \) distribution with \( \nu \) degree of freedom is also elliptical, and the corresponding \( g(s; n) = (\nu + s)^{-\frac{n+\nu}{2}} \).} Note that for an elliptical random variable, all odd central moments are zero, and even (say \( 2q \)) central moments are proportional to the dispersion raised to the \( q \)-th power. In particular, recall that if \( \tilde{x} \sim N(\mu, \sigma) \), then for any positive integer \( q \),

\[
E[(\tilde{x} - \mu)^{2q}] = 1 \cdot 3 \cdot 5 \cdots (2q - 1)\sigma^{2q},
\]

and

\[
E[(\tilde{x} - \mu)^{2q-1}] = 0.
\]

Now we state here without proofs two theorems, which are due to Chamberlain (1983).

**Theorem 9** Suppose that the riskless asset does not exist. Suppose that \( e \) is not proportional to \( 1 \). Then every expected utility function with VNM utility function \( u \) satisfying \( u' > 0 > u'' \) is in fact a mean-variance utility function if and only if there exists a non-singular matrix \( T_{N \times N} \) such that in

\[
T[1 + \tilde{r}] = \begin{bmatrix} \tilde{m} \\ \tilde{z}_{(N-1) \times 1} \end{bmatrix},
\]

conditional on \( \tilde{m}, \tilde{z} \) is elliptically distributed around \( 0_{(N-1) \times 1} \).

**Theorem 10** Suppose that a riskless asset exists. Then every expected utility function with VNM utility function \( u \) satisfying \( u' > 0 > u'' \) is in fact a mean-variance utility function if and only if there exists a non-singular matrix \( T_{N \times N} \) such that

\[
T(\tilde{r} - e)
\]

is elliptically distributed around \( 0_{N \times 1} \).
To learn more about elliptical distributions, the reader can refer to Owen and Rabinovitch (1983) and look up Chapter 4, Appendix B, of Ingersoll (1987).

35. Now, we turn to the conditions on preferences that imply two-fund separation regardless of the properties of return distributions. The results depend on whether markets are complete or not, and so we again summarize two preference-based fund-separation theorems without proofs. The reader can find the proofs in Cass and Stiglitz (1970).

**Theorem 11** We have the following results when the date-0 markets are complete.

- Given a complete market at date 0, an investor’s optimal portfolio is a portfolio of two fixed portfolios no matter how his initial wealth changes, if and only if his VNM utility function $u$ satisfies either of the following two differential equations:
  
  $$ Au'(W) + Bu'(W)^\beta = W, $$

  or

  $$ u'(W)^\alpha [A + B \log(u'(W))] = W, $$

  where $A, B, \alpha, \beta$ are appropriate constants that ensure $u' > 0 > u''$.

- Given a complete market at date 0, an investor’s optimal portfolio is a portfolio of the riskless asset and a fixed risky portfolio no matter how his initial wealth changes, if and only if his VNM utility function $u$ satisfies either of the following two differential equations:

  $$ u'(W) = (a + bW)^c, $$

  or

  $$ u'(W) = ae^{bW}; $$

  that is, $u$ is either an extended power function (this class includes quadratic, the logarithmic, and general CRRA utility functions) or the CARA utility function.
Theorem 12 We have the following results when the date-0 markets are incomplete.

- Given an incomplete market at date 0, an investor’s optimal portfolio is a portfolio of two fixed portfolios no matter how his initial wealth changes, if and only if his VNM utility function $u$ is either a quadratic function or a CRRA function.

- Given an incomplete market at date 0, an investor’s optimal portfolio is a portfolio of the riskless asset and a fixed risky portfolio no matter how his initial wealth changes, if and only if his VNM utility function $u$ satisfies either of the following two differential equations:
  
  $$u'(W) = (a + bW)^c,$$

  or

  $$u'(W) = ae^{bW};$$

  that is, $u$ is either an extended power function (this class includes quadratic, the logarithmic, and general CRRA utility functions) or the CARA utility function.

Thus the preference-based two-fund separation holds only if investors’ VNM utility functions are members in the hyperbolic absolute risk aversion (HARA) class. There is apparently no reason to believe that the real-world investors are endowed with these special VNM utility functions, and therefore we should not expect to observe fund separation in real-world asset allocation practices. Some authors have tried to show that the real-world practices are compatible with the assumption that investors have HARA VNM utility functions; see for example Bajeux-Besnainou, Jordan and Portait (2001).

Before we end this note, we must record an associated result about asset demands which is rather important.

---

28 Given a VNM utility function $u$ with $u' > 0 > u''$, the function $-\frac{u'}{u''}$ is referred to as the risk tolerance associated with $u$. We say that $u$ is a HARA function if and only if its risk tolerance function is an affine function. One can verify easily that the CARA, CRRA (including the logarithmic) and quadratic functions are all members in the HARA class.
Theorem 13 Suppose that markets are incomplete at date 0 and yet preference-based two-fund separation holds. Let $Z_j$ be the dollar-amount that an investor $i$ spends on asset $j$. Then, there exist constants $\{A_j, B_j; j = 1, 2, \cdots, N\}$ such that $Z_j$ as a function of $W_i$ is linear:

$$Z_j(W_i) = A_jW_i + B_j, \quad \forall j = 1, 2, \cdots, N.$$ 

It can be verified that for an investor with a CRRA VNM utility function, he actually holds one fixed portfolio regardless how his initial wealth changes; and that for an investor with a CARA VNM utility function, he actually spends a fixed amount of money in each risky asset, regardless how his initial wealth changes.

References


