1. Recall the economy described in Lecture 3, where there is only one perishable consumption good, and where there are \( m \) assets traded in date-0 perfect financial markets, which generate returns (in amounts of the single consumption good) at date 1. (We shall write \( t = 0 \) and \( t = 1 \) for date 0 and date 1.) More precisely, suppose that there are \( n \) possible states at date 1, with each occurring with a positive probability, and for all \( j = 1, 2, \ldots, m \), one share of asset \( j \) will generate \( x_{kj} \) units of consumption in state \( k \), for all \( k = 1, 2, \ldots, n \). Although the single consumption good is perishable, we shall abuse terminology a little and refer to it as cash, and refer to the asset returns as cash 
flows.

Let \( X_{n \times m} \) be the matrix of which the \((k, j)\)-th element is \( x_{kj} \). We shall denote asset \( j \) by the \((n \times 1)\)-vector \( x_j \), which is the \( j \)-th column of matrix \( X \). Correspondingly, let \( p_j \) be the date-0 market-clearing price of asset \( j \), and let the \((m \times 1)\)-vector \( p \) be such that its \( j \)-th element is \( p_j \). The pair \((X, p)\) will be referred to as a price system, where note that \( X \) and \( p \) are respectively exogenous and endogenous variables, and they contain respectively date-1 and date-0 cash flows.

There are many price-taking investors, who would like to consume as much as they can at dates 0 and 1.

2. In the above economy, any conceivable asset can be represented by an \((n \times 1)\)-vector \( z \), of which the \( k \)-th element \( z_k \) is the per-share cash flow generated by asset \( z \) at date 1. Thus an asset is described by the units of consumption that it generates for investors in each and every date-1 state. For this reason, we shall adopt the following terminology.

**Definition 1** An asset \( z \) is also referred to as a consumption plan, or a contingent claim.
Recall the following definitions made in Lecture 3:

**Definition 2** The date-0 asset markets are **complete** if for each conceivable asset $z \in \mathbb{R}^n$, there correspondingly exists some portfolio $y_{m \times 1}$ of the $m$ traded assets such that $z = Xy$. We say that the date-0 asset markets are **incomplete** if they are not complete.

**Definition 3** A type-0 arbitrage opportunity is a portfolio $q_0$ of the $m$ traded assets such that $p'q_0 < 0$ and $Xq_0 \geq 0_{n \times 1}$. A type-1 arbitrage opportunity is a portfolio $q_1$ such that $p'q_1 \leq 0$ and $Xq_1 > 0_{n \times 1}$. If the price system $(X, p)$ admits no type-0 or type-1 arbitrage opportunities, then the price system is **arbitrage free**.

In plain words, a type-0 arbitrage opportunity is a portfolio strategy that allows an investor to make money at date 0, and to never lose money at date 1; and a type-1 arbitrage opportunity is a portfolio strategy that allows an investor to not lose money at date 0, and to make money in at least one state at date 1.

3. We shall focus on the complete-markets case throughout this note. We shall first consider the above two-period case, and then generalize the analysis to the three-period case. Thus we shall assume from now on that the price system $(X, p)$ is arbitrage-free, and that $X$ contains $n$ linearly independent column vectors. Without loss of generality, we shall re-define $X$ as the square matrix containing exactly those $n$ linearly independent column vectors, and re-define $p$ accordingly. We shall refer to these $n$ assets as the **underlying assets**, and any other asset a **derivative asset**.

4. To begin, recall the following theorem from Linear Algebra:

---

1We write $z_{n \times 1} \geq 0_{n \times 1}$ if for all $j = 1, 2, \cdots, n$, the $j$-th element $z_j$ of $z$ is greater than or equal to zero. We write $z > 0$, if $z \geq 0$ and $z \neq 0$. We write $z >> 0$ if for all $j = 1, 2, \cdots, n$, the $j$-th element $z_j$ of $z$ is greater than zero.
Theorem 1 A mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is linear \(^2\) if and only if there exists a matrix \( A_{k \times n} \) such that the function \( f \) can be written
\[
f(x) = Ax, \quad \forall x \in \mathbb{R}^n.
\]

Proof. Consider necessity. Let \( e_i \) be the \((n \times 1)\) vector such that its \( i\)-th element is one and all other elements are zero. Then, given \( f \) being linear, we define \( A_{k \times n} \) as such that its \( j\)-th column is \( f(e_j) \). Now, for all \( x \in \mathbb{R}^n \),
\[
Ax = [f(e_1) \ f(e_2) \ \cdots \ f(e_n)]x
= [f(e_1) \ f(e_2) \ \cdots \ f(e_n)] \sum_{j=1}^{n} x_j e_j
= \sum_{j=1}^{n} x_j f(e_j)
= f(\sum_{j=1}^{n} x_j e_j) = f(x),
\]
where the third equality follows from the fact that \( Ae_j \) equals the \( j\)-th column of \( A \).

Next consider sufficiency. If \( f \) is defined as \( f(x) = Ax \) for some matrix \( A_{k \times n} \), then for all \( \alpha, \beta \in \mathbb{R} \) and all \( x, y \in \mathbb{R}^n \),
\[
f(\alpha x + \beta y) = A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \alpha f(x) + \beta f(y),
\]
and hence \( f \) is linear. \( \Box \)

5. Now, given any consumption plan \( z_{n \times 1} \), denote its date-0 no-arbitrage price by \( f(z) \). We claim that in the absence of arbitrage opportunities, \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) must be a linear function. To prove this claim, in light of the preceding theorem, we only need to show the existence of a vector

\(^2\)This means that for all \( \alpha, \beta \in \mathbb{R} \) and all \( x, y \in \mathbb{R}^n \),
\[
f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).
\]
Note that necessarily, \( f(0_{n \times 1}) = 0_{k \times 1} \). From the theory of general competitive equilibrium, the equilibrium price functional comes from the topological dual space of the commodity space. That is, the price functional is linear and continuous. In the current case where the commodity space is \( \mathbb{R}^n \), linearity ensures continuity. This result in general does not obtain when the commodity space is infinite-dimensional.
\( a_{n \times 1} \) such that \( f(z) = a'z \), for all \( z \in \mathbb{R}^n \). We claim that given \( X \) and \( p \),

\[
a' = p'X^{-1}.
\]

To see this, note that given \( z \) it is possible to form a portfolio of the \( n \) underlying assets which has identical dividends per share at \( t = 1 \) as \( z \) does. Suppose to this end we need to hold \( q_j \) units of the \( j \)-th underlying asset (if \( q_j < 0 \) we are selling the \( j \)-th underlying asset short). Define \( q_{n \times 1} \) as such that its \( j \)-th element is \( q_j \). Then,

\[
Xq = z, \quad q = X^{-1}z.
\]

This portfolio \( q \) is called a replicating portfolio for asset \( z \). The date-0 market value of this portfolio is

\[
p'q = p'X^{-1}z.
\]

Now \( p'X^{-1}z \) must also be the price for \( z \) to rule out arbitrage opportunities. To see this, suppose instead that the price of \( z \), denoted \( f(z) \), is strictly higher than \( p'X^{-1}z \). Then one can sell short one unit of \( z \) and purchase \( q_j \) units of the \( j \)-th underlying asset for all \( j = 1, 2, \ldots, n \). With this trading strategy, the investor has a date-0 cash inflow

\[
f(z) - p'X^{-1}z > 0
\]

at \( t = 0 \), and he has no cash in- or out-flows at \( t = 1 \). This is a type-0 arbitrage opportunity, which contradicts the assumption that the price system is arbitrage-free. The case where \( f(z) < p'X^{-1}z \) is similar. Hence, to rule out arbitrage opportunities, we must have \( f(z) = p'X^{-1}z \) for all \( z \in \mathbb{R}^n \).

6. Two more definitions follow:

**Definition 4** A security that pays 1 dollar at \( t = 1 \) if state \( k \) occurs and nothing otherwise is called the \( k \)-th Arrow-Debreu security.\(^3\) With \( n \) possible states at \( t = 1 \), there are \( n \) Arrow-Debreu securities in total.

\(^3\)Here the term one dollar is interchangeable for one unit of consumption in the single commodity.
Note that all consumption plans are portfolios of Arrow-Debreu securities. Since the ("no-arbitrage") price functional $f$ is linear, the date-0 cost of a consumption plan is the sum of the costs of those Arrow-Debreu securities making up that consumption plan.

**Lemma 1** In the absence of arbitrage opportunities, the date-0 price of the $k$-th Arrow-Debreu security $f(e_k) = p'X^{-1}e_k > 0$.

To see that the above lemma must be true, note that if instead $f(e_k) \leq 0$, then holding 1 unit of the $j$-th Arrow-Debreu security is a type-1 arbitrage opportunity, which contradicts the assumption that there are no arbitrage opportunities at date 0.

**Definition 5** A riskless asset is one whose per-share cash flows are invariant across all $n$ date-1 states. A (riskless) pure discount bond (or zero-coupon bond) with face value equal to a dollar is a riskless asset which pays 1 dollar at $t = 1$ no matter which state occurs. The dollar is called the face value of the pure discount bond.

**Lemma 2** In the absence of arbitrage opportunities, the date-0 price of the above pure discount bond is $f(1) = p'X^{-1}1 \equiv B > 0$, where $1 = \sum_{i=1}^{n} e_i$.

Again, if $f(1) \leq 0$, then holding the above pure discount bond is itself a type-1 arbitrage opportunity, which is a contradiction.

Given $X$ and $p$, what is the riskless rate of interest, denoted $r_f$? The answer is $\frac{1}{p'X^{-1}1} - 1$.

7. Define a set of artificial probabilities $b_i$ as follows:

$$\forall i = 1, 2, \ldots, n, \quad b_i = \frac{f(e_i)}{f(1)}.$$

These will be referred to as the martingale probabilities and sometimes the risk neutral probabilities.\(^4\)

\(^4\)Under these probabilities, the price process of any asset becomes a martingale when the pure discount bond is taken as the numeraire.
Theorem 2 (Martingale Pricing Theorem) If the price system \((X, p)\) admits no arbitrage opportunities, there must exist a set of probabilities \(b = [b_1 \, b_2 \, \cdots \, b_n]'\) such that for all \(i, b_i > 0\), and that for all assets \(z\), the price \(f(z)\) is given by

\[
f(z) = \frac{E^*[\tilde{z}]}{1 + r_f} = \frac{b'z}{1 + r_f},
\]

where the expectation \(E^*[\cdot]\) is taken using the probabilities \(\{b_i; i = 1, 2, \cdots, n\}\). Conversely, if there exists a set of martingale probabilities \(b > 0\) such that every consumption plan \(z\) has a date-0 price equal to \(\frac{b'z}{1 + r_f}\), then the price system \((X, p)\) admits no arbitrage opportunities.

Proof Suppose that \((X, p)\) admits no arbitrage opportunities. Then the date-0 price of any consumption plan \(z\) is given by \(f(z) = p'X^{-1}z\). Note that, with \(b\) being defined by

\[
\forall i = 1, 2, \cdots, n, \quad b_i = \frac{f(e_i)}{f(1)},
\]

we have

\[
f(z) = p'X^{-1}z = \sum_{i=1}^{n} z_i p'X^{-1}e_i
\]

\[
= \sum_{i=1}^{n} z_i f(e_i) = \sum_{i=1}^{n} z_i \frac{f(e_i)}{f(1)} f(1)
\]

\[
= [b'z] f(1) = \frac{b'z}{1 + r_f}.
\]

Conversely, suppose that there exists a set \(b\) of strictly positive probabilities such that for all \(z \in \mathbb{R}^n\),

\[
f(z) = \frac{b'z}{1 + r_f}.
\]

We claim that \((X, p)\) admits no arbitrage opportunities. To see this, note that if every element of \(z\) is non-negative and at least one of its elements is strictly positive, then we must have \(f(z) > 0\), so that the type-1 arbitrage opportunities do not exist. Similarly, if \(f(z) < 0\), then since every element of \(b\) is strictly positive, at least one element of \(z\) must be strictly negative, proving that the type-0 arbitrage opportunities do not exist either. This finishes the proof. ||
8. Now we give a series of examples. Suppose that there are two date-1 states, $\omega_1$ and $\omega_2$, and two traded assets at date 0, called assets 1 and 2. Asset 1’s date-1 payoff (per unit) is

$$\begin{bmatrix} 30 \\ 10 \end{bmatrix},$$

and asset 2’s date-1 payoff (per unit) is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The prices of assets 1 and 2 are respectively 20 and 1. Thus in our standard notation,

$$\begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 30 & 1 \\ 10 & 1 \end{bmatrix}.$$

• Now, are markets complete at date 0? The answer is yes, because the two vectors

$$\begin{bmatrix} 30 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are linearly independent, and hence the rank of $X$ is two.

• Consider a new asset

$$\begin{bmatrix} 22 \\ 14 \end{bmatrix}.$$

What is the no-arbitrage price for $z$?

To answer this question, we only need to find the martingale probabilities $b_1, b_2 = 1 - b_1$, and the riskless rate $r_f$. (This follows from Theorem 2.) Using the information about assets 1 and 2, we have

$$p_1 = 20 = \frac{b_1 \times 30 + (1 - b_1) \times 10}{1 + r_f}, \quad p_2 = 1 = \frac{b_1 \times 1 + (1 - b_1) \times 1}{1 + r_f}.$$

Thus we have two equations with two unknowns $b_1$ and $r_f$. Solving, we have

$$b_1 = \frac{1}{2}, \quad r_f = 0.$$
Thus the price of \( z \) must equal
\[
\frac{b_1 \times 22 + (1 - b_1) \times 14}{1 + r_f} = 18.
\]

- What can you do to obtain a riskless profit, if the date-0 price of the above asset \( z \) is equal to 20?

Let us first determine the portfolio of assets 1 and 2 that replicates asset \( z \). We must solve the following system of equations
\[
\begin{bmatrix}
30 & 1 \\
10 & 1 
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 
\end{bmatrix}
=
\begin{bmatrix}
22 \\
14 
\end{bmatrix}.
\]

Solving, we have
\[
q_1 = \frac{2}{5}, \quad q_2 = 10.
\]

Thus the answer to the above question is that you should borrow 1 unit of asset \( z \), sell it, and get 20, and then put 2 dollars in your pocket (enjoy it!), and spend 18 to buy \( \frac{2}{5} \) units of asset 1 and 10 units of asset 2. As you can verify easily, the cost of the latter portfolio is indeed 18 dollars. Note that this trading strategy is a type-0 arbitrage opportunity, which generates a sure profit of 2 dollars for you at date 0.

- Now, consider a new asset, which gives the following right (which is not an obligation) to the investor that holds it till date 1: the investor can buy 1 unit of asset 1 at date 1 at the fixed price \( k = 12 \).

We shall call this new asset a European call option written on 1 unit of asset 1 with exercise (strike) price equal to \( k \) and expiration date set to date 1. What is the \( z \) vector for this new asset?

Note that in state \( \omega_1 \), the date-1 price of asset 1 is 30, and so if you have 1 unit of the call option, then you should exercise the right to get a payoff \( 30 - k = 18 \); that is, you spend 12 and get an asset worth 30. In state \( \omega_2 \), the price of asset 1 is only 10, and you should do nothing, so that the call option generates a zero payoff in this case. Thus, the \( z \) vector for the above European call option is
\[
\begin{bmatrix}
18 \\
0 
\end{bmatrix}.
\]
Now we know the date-0 price of this call option (called its *premium*), which is
\[
\frac{b_1 \times 18 + (1 - b_1) \times 0}{1 + r_f} = 9.
\]

• Now, consider a *European put option* written on 1 unit of asset 1 with exercise (strike) price equal to \( k \) and expiration date set to date 1. Suppose that you buy this option from a bank, then you are given the right to sell 1 unit of asset 1 to that bank at date 1 at the fixed price \( k \). What is the \( z \) vector for this new asset if \( k = 12 \)?

Note that the put option gives you a right, which is *not* an obligation, and so at date 1 you should exercise that right only in state \( \omega_2 \). Why? Because in state \( \omega_1 \), you can sell 1 unit of asset 1 at the price 30 in the market! In state \( \omega_2 \), however, you can sell something worth only 10 at the price \( k = 12 \), which generates a cash inflow of 2. Thus the \( z \) vector associated with the put option is
\[
\begin{bmatrix}
0 \\
2
\end{bmatrix},
\]
so that the date-0 price of the put option is equal to
\[
\frac{b_1 \times 0 + (1 - b_1) \times 2}{1 + r_f} = 1.
\]

• Now consider the date-0 *forward price* for asset 1. Suppose that you want to hold 1 unit of asset 1 at date 1. There are two ways to do it. First, you can wait till date 1 and buy 1 unit of asset 1. Note that the date-1 price of asset 1 is random, from the perspective of date-0 information. That is, if you wait till date 1, then in order to obtain 1 unit of asset 1 you have to pay 30 in state \( \omega_1 \) and 10 in state \( \omega_2 \).

Second, you can sign a contract with a bank, called a *forward contract*, at date 0, and state in the contract that you agree to pay a price \( G \) in exchange for 1 unit of asset 1 delivered by the bank at date 1. This is called a *forward transaction*, unlike the *spot transaction* described in the first way.

9
This price $G$, which appears in the above forward contract, is called \textit{the date-0 forward price} for asset 1. Note that unlike an option, which is a right but not an obligation, a forward contract is a right and an obligation; after you sign the forward contract, you have to pay $G$ and buy 1 unit of asset 1 at date 1.\footnote{Note that the date-1 \textbf{z}-vector for a call option or for a put option must be such that $z \geq 0$, since an option is a right but not an obligation. The same is not true for the forward contract.}

To determine the forward price $G$, observe that no money changes hands at date 0 when the contract is signed. That is, at date 0 when you sign the forward contract with a bank, you do not pay the bank anything, nor does the bank pay you anything. This implies that the forward contract must have a zero value at date 0, for both you and the bank! (Otherwise, if for example the contract is worth $v > 0$ to you, then you have to pay $v$ to the bank at date 0!)

Notice that after signing the forward contract with the bank, the contract generates the following date-1 payoff for you:

$$\begin{bmatrix} 30 - G \\ 10 - G \end{bmatrix},$$

and this date-1 payoff must have a date-0 value equal to zero! It follows that

$$0 = p'X^{-1} \begin{bmatrix} 30 - G \\ 10 - G \end{bmatrix} = p'X^{-1} \begin{bmatrix} 30 \\ 10 \end{bmatrix} - Gp'X^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= p_1 - Gp'X^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow G = \frac{p_1}{p'X^{-1}1} = \frac{p_1}{B} = p_1(1 + r_f),$$

where recall that $p'X^{-1}1$ is the date-0 price of a pure discount bond (or zero-coupon bond) maturing at date 1 with face value equal to 1 dollar, and recall also that it must equal $\frac{1}{1 + r_f}$; in fact, this is how we find the riskless rate of interest $r_f$ from the data $p$ and $X$. We conclude that the date-0 forward price is the date-0 spot price multiplied by $(1 + r_f)$.\footnote{Intuitively, longing one forward contract described here at date 0 is the same as buying one unit of asset 1 and borrowing $\frac{G}{1 + r_f}$ at date 0, with the latter positions being cleared at date 1. Since the value of the latter positions must equal zero, we have $G = p_1(1 + r_f)$.}
• When markets are complete, we can also price *corporate securities*, like common stocks, preferred stocks and corporate bonds. There is no difference between the pricing of corporate securities and the pricing of options and forward contracts. Let us demonstrate an example.

Consider a firm that has borrowed some debt. The debt will mature at date 1, with face value $F > 0$. Recall that $F$ is the amount that the shareholders promise to repay the debtholders at date 1, which is not necessarily the amount that the debtholders really get at date 1. Why? Because we assume that the firm is protected by limited liability, which means that the firm can choose to go bankrupt instead of paying $F$ to the debtholders. More precisely, let $\tilde{x}$ be the firm’s total earnings at date 1. The debtholders get $F$ if the firm does not go bankrupt, but they get $\tilde{x}$ in the event of bankruptcy. Certainly, the smart shareholders will declare bankruptcy if and only if $\tilde{x} \leq F$. This implies that, although the face value of debt is what the firm promises to repay debtholders at date 1, debtholders’ date-1 payoff is $\min(\tilde{x}, F)$ actually. On the other hand, shareholders’ date-1 payoff is

$$\tilde{x} - \min(\tilde{x}, F) = \max(\tilde{x} - F, 0).$$

Note that debt is *senior* to equity, as these mathematical expressions indicate.

Now, suppose that the above asset 1 is firm A’s date-1 earnings, and firm A has borrowed some debt which will mature at date 1 with a face value $F = 12$. What is the date-0 equity value of firm A? (The equity value is the total value of firm A’s common stock.) To answer this question, we again look for the $z$ vector associated with firm A’s equity. Since asset 1 represents firm A’s earnings, shareholders’ date-1 payoff is

$$\begin{bmatrix}
\max(30 - 12, 0) \\
\max(10 - 12, 0)
\end{bmatrix}
= \begin{bmatrix}
18 \\
0
\end{bmatrix},$$

so that the date-0 equity value of firm A is

$$\frac{b_1 \times 18 + (1 - b_1) \times 0}{1 + r_f} = 9,$$
which turns out to be the date-0 price of the call option with exercise price $k = 12$ that we solved above! This is no coincidence: firm A’s equity can be regarded as a call option written on its date-1 earnings, because the shareholders, being protected by limited liability, can decide whether or not to pay the face value of debt (exercise price) and buy back the date-1 earnings.

• Continue with the above corporate securities example. What is the date-0 price for firm A’s debt? To answer this question, we only need to find the associated $z$ vector for firm A’s debt, which is

\[
\begin{bmatrix}
\min(30, 12) \\
\min(10, 12)
\end{bmatrix}
= \begin{bmatrix}
12 \\
10
\end{bmatrix},
\]

so that the date-0 debt value of firm A is

\[
\frac{b_1 \times 12 + (1 - b_1) \times 10}{1 + r_f} = 11.
\]

As you can see, the sum of equity value and debt value at date 0, which is $9 + 11 = 20$, must be the date-0 value of firm A, and by our assumption, equal to $p_1$.

• Continue with the above corporate securities example, but now, consider a callable bond issued by firm A (or more precisely, firm A’s equityholders). The callable bond is a debt which is identical to the straight debt considered above except that at date 1, one minute before the debt matures, the firm has (more precisely, the shareholders have) the right to buy it back at the price 11. Now, what is the date-0 price of this callable bond?

Again, we find the $z$ vector associated with the callable bond. Note that the firm should call the bond (or buy it back at the exercise price 11) at date 1, if and only if the date-1 state is $\omega_1$: if the firm does not call the bond, then the shareholders will have to pay 12 to the debtholders, which is greater than 11. Calling the bond in state $\omega_2$ is, apparently, not optimal. Hence the $z$ vector associated with the callable bond is

\[
\begin{bmatrix}
\min(11, \min(30, 12)) \\
\min(11, \min(10, 12))
\end{bmatrix}
= \begin{bmatrix}
11 \\
10
\end{bmatrix},
\]

12
so that the date-0 value of firm A’s callable bond is
\[
\frac{b_1 \times 11 + (1 - b_1) \times 10}{1 + r_f} = 10.5.
\]

• Continue with the above corporate securities example, but now, consider a *convertible bond* issued by firm A. The convertible bond is a debt which is identical to the *straight* debt considered above except that at date 1, one minute before the debt matures, the bondholders (debtholders) have the right to exchange the bond for 50% of the firm’s equity. Now, what is the date-0 price of this convertible bond?

Again, we find the \( z \) vector associated with the convertible bond. Let us assume that there is one single debtholder (bondholder). Note that the debtholder should convert the bond into equity at date 1 if and only if the date-1 state is \( \omega_1 \): the date-1 payoff to the debtholder is 12 if there is no conversion, which is less than 15 = 50% × 30. Converting the bond into equity in state \( \omega_2 \), on the other hand, is not wise because without conversion the debtholder can get all the 10 dollars in the firm (recall that debt is senior than equity!). Hence the \( z \) vector associated with the convertible bond is
\[
\begin{bmatrix}
\max(50\% \times 30, \min(30, 12)) \\
\max(50\% \times 10, \min(10, 12))
\end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix},
\]
so that the date-0 value of firm A’s convertible bond is
\[
\frac{b_1 \times 15 + (1 - b_1) \times 10}{1 + r_f} = 12.5.
\]

• Note that it is incorrect to argue that from the shareholders’ perspective, issuing a callable bond is better than issuing a convertible bond. It is true that at date 0 the equity value is 9.5 when a callable bond is issued and only 7.5 when a convertible bond is issued, but don’t forget that at date 0 the revenue of selling a convertible bond is more than the revenue of selling a callable bond. In fact, with complete markets, firm A’s market value is independent of its *capital structure*. That is, the equity value plus debt value is always 20, regardless what kind of debt is issued, and
in fact, regardless of how much debt is issued. If the firm chooses not to issue any debt, for example, then at date 0 its equity value equals the total firm value, which is, again, 20. This is the famous Modigliani-Miller Proposition 1, which you will encounter again in the course Financial Management. It says that when markets are perfect and complete, given a firm’s investment decisions (so that its date-1 random earnings are also given), the firm’s date-0 market value is independent of its capital structure.

9. Suppose that there are 3 possible states at date 1 and there are three traded assets at date 0. You are given the following data:

\[
p = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad X = \begin{bmatrix}
1 & 2 & 3 \\
1 & 0 & 2 \\
1 & 1 & 1
\end{bmatrix}.
\]

Show that if the price system \((X, p)\) admits no arbitrage opportunities, then

\[
\frac{4}{3} < \pi < \frac{7}{3}.
\]

(Hint: Consider the set \(A\) of portfolios consisting of assets 1 and 2 only which promise more cash flows than asset 3 in each and every state at date 1. Any such portfolio must have a date-0 price higher than \(\pi\). Find the cheapest portfolio in \(A\). Then, consider the set \(B\) of portfolios consisting of assets 1 and 2 only which promise less cash flows than asset 3 in each and every state at date 1. Any such portfolio must have a date-0 price lower than \(\pi\). Find the most expensive portfolio in \(B\).)\(^7\)

\(^7\)More precisely, first consider the following maximization problem:

\[
\max_{q_1, q_2} q_1 + \frac{2}{3} q_2
\]

subject to

\[
q_1 + 2q_2 \leq 3, \quad q_1 \leq 2, \quad q_1 + q_2 \leq 1.
\]

Show that the solution is \((q_1^*, q_2^*) = (2, -1)\), and hence the maximum value of the objective function is \(\frac{4}{3}\). Next, consider the following minimization problem:

\[
\min_{q_1, q_2} q_1 + \frac{2}{3} q_2
\]

14
10. Some economists have assumed that the date-0 forward price of an asset $z$ is the expected value of the asset’s date-1 spot price; that is, \emph{the forward price of an asset is the unbiased estimator of that asset’s future spot price}. Let us verify if this is true. Suppose that there are $n$ possible states at date 1, and all investors in the financial markets agree that $a_i > 0$ is the probability that state $i$ may occur at date 1. Let

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

so that $a'1 = 1$. Recall the martingale probabilities

$$b_i = \frac{f(e_i)}{f(1)} > 0, \ i = 1, 2, \cdots, n,$$

where $f(\cdot)$ is the asset pricing functional. Let

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Show that the date-0 forward price of asset $z$ is equal to the expected value of the asset’s date-1 spot price if and only if $z$ is orthogonal to $a - b$. In particular, if $a = b$, then the assumption that \emph{the forward price of an asset is the unbiased estimator of that asset’s future spot price} holds true for every asset $z$.

11. In a two-period economy with complete financial markets, Chen owns a firm at date 0, which has $22$ cash as its only asset. There are two equally likely date-1 states, $u$ and $d$. Chen has two mutually exclusive subject to

$$q_1 + 2q_2 \geq 3, \ q_1 \geq 2, \ q_1 + q_2 \geq 1.$$ 

Show that the solution is $(q_1^*, q_2^*) = (2, \frac{1}{2})$, and hence the minimum value of the objective function is $\frac{7}{2}$.
investment projects at hand, called G and B. (The term mutually exclusive means that the firm can only choose one project to invest but not both.) Each of the two projects needs a date-0 expenditure of $80. Chen decides to borrow the $58 (= 80 − 22) from public investors in the financial market. It is known that the riskfree rate is \( r_f = 25\% \).

The differences between projects G and B are as follows. If G is taken at date 0, then $80 must be spent at date 0, and at date 1, the project generates $150 and $50 in respectively state u and state d. If B is taken at date 0, then again $80 must be spent at date 0, and at date 1, the project generates $160 and $0 in respectively state u and state d.

(i) First suppose that only project G is available to the firm at date 0. Suppose that when borrowing the $58 at date 0, Chen promises that he will repay up to $\( F \) at date 1 (recall that \( F \) is referred to as the face value of the debt, and that the firm has limited liability). You are told that after the borrowing takes place at date 0, the date-0 firm value and equity value are respectively $100 and $42. Find \( F \)!

(ii) Now suppose that both G and B are available to the firm at date 0. Suppose that the asset prices determined by the no-arbitrage condition are independent of Chen’s choice between G and B; that is, there is no moral hazard problem. Which project will Chen choose to invest, if his choice cannot be observed by the public investors?\(^8\) Suppose that public investors are very smart, so that when lending to Chen, they can rationally expect Chen’s choice between G and B after Chen gets the $58. What is the equilibrium face value of debt that Chen has to promise to his debtholder at date 0 in order to get the $58?

**Solution.** Consider part (i). Apparently, \( F \geq 58 \), and hence the firm will default if the date-1 cash flow is $50. Can \( F \geq 150 \)? If so, then the value of the firm is equal to the value of debt, which is $100, which is much greater than $58! Thus if \( F \) is such that the date-0 value of the debt is exactly $58, we must have

\[
50 < F < 150.
\]

With this observation, we conclude that the date-1 cash flow accrued to the shareholders is \( 150 − F \) when the date-1 cash flow to the firm is

\(^8\)Hence markets are imperfect; we assume however that the non-observability of Chen’s project choice is the only imperfection in the markets.
150, and 0 when the date-1 cash flow to the firm is 50.

Applying the martingale pricing theorem, and letting $b$ be the martingale probability for the event that the date-1 cash flow to the firm is 150, we have

$$100 = \frac{150 \times b + 50 \times (1 - b)}{1 + 25\%},$$

and

$$42 = \frac{(150 - F) \times b + 0 \times (1 - b)}{1 + 25\%}.$$

We thus have

$$b = \frac{3}{4}, \quad F = 80.$$

Next, consider part (ii). Consider Chen’s investment decision at date 0, after raising $58 by issuing a debt with face value $F$. If G is chosen, then Chen’s equity is worth

$$\frac{(150 - F) \times b + 0 \times (1 - b)}{1 + 25\%};$$

whereas his equity is worth

$$\frac{(160 - F) \times b + 0 \times (1 - b)}{1 + 25\%},$$

if B is chosen! Thus Chen will choose B. (Note that the firm value under G is 100, as stated in part (i), but the firm value under B is only

$$\frac{160 \times b + 0 \times (1 - b)}{1 + 25\%} = 96,$$

and this is why we have referred to the latter project as project “B,” where B stands for Bad.) With rational expectations, the public investors know that Chen will choose project B when he gets the $58, and hence the fair face value $F$ of debt, which ensures that the date-0 value of debt is exactly $58, must be such that

$$58 = \frac{F \times b + 0 \times (1 - b)}{1 + 25\%} \Rightarrow F = \frac{290}{3}.$$
It remains to check if Chen is better off giving up borrowing in the first place. By borrowing at face value \( F = \frac{290}{3} \) and then investing in project B, Chen’s equity is worth

\[
\frac{(160 - \frac{290}{3}) \times b + 0 \times (1 - b)}{1 + 25\%} = 38 > 22,
\]

and hence Chen will go ahead borrowing and then investing in project B. We conclude that the equilibrium face value of debt is indeed \( F = \frac{290}{3} \).

12. Consider two firms A and B operating at date 0 in perfect financial markets. Suppose that A and B have exactly the same assets (this means that the assets will generate exactly the same random cash flows at date 1). Suppose that these assets may generate $154 and $110 with equal probability at date 1. Firm A has (straight) debt that will mature at date 1 with face value \( F_A \). The date-0 prices of equity and debt of firm A are respectively $14 and $114. Firm B has (straight) debt that will mature at date 1 with face value $121, and at date 0 the expected rate of return on firm B’s debt is \( \frac{17}{214} \).

(i) Suppose that the value of firm B at date 0 is $130. Design a trading strategy at date 0 to capture an arbitrage profit.

(ii) Now, suppose that firm A would like to change its debt into a convertible bond, and issue the bond to Mr. C. With the convertible bond, Mr. C is given a right (but not an obligation) to convert the bond into 86% of the ownership of firm A right before the convertible bond matures at date 1. By how much will the value of firm A’s equity be changed because of the issuance of the convertible bond?

**Solution.** Let \( b \) be the martingale probability for the event that the firms’ date-1 cash flow is 154. We have

\[
14 = \frac{\max(154 - F_A, 0) \times b + \max(110 - F_A, 0) \times (1 - b)}{1 + r_f},
\]

\[
114 = \frac{\min(154, F_A) \times b + \min(110, F_A) \times (1 - b)}{1 + r_f},
\]

\[
1 + \frac{17}{214} = \frac{\frac{1}{2} \times 121 + \frac{1}{2} \times 110}{\min(154, 121) \times b + \min(110, 121) \times (1 - b)}.
\]
The above is a system of 3 equations with 3 unknowns. Solving, we have
\[ r_f = 0.1, \quad F_A = 132, \quad b = 0.7. \]

Consider part (i). There are many feasible arbitrage strategies. One of them is to short sell a fraction \( \pi \) (say 1%) of firm B’s debt and equity, which will generate cash \( 130\pi \) at date 0. Put \( 2\pi \) in your own pocket, and spend the rest \( 128\pi \) on the same fraction \( \pi \) of firm A’s debt and equity. This way, you have earned a riskless profit of \( 2\pi \), and your date-1 cash flow is
\[ 154\pi - 154\pi = 0 \]
in the event that the firms’ date-1 cash flow is 154, and your date-1 cash flow is
\[ 110\pi - 110\pi = 0 \]
in the event that the firms’ date-1 cash flow is 110. Hence it is an arbitrage strategy.

Consider part (ii). Apparently, Mr. C should not convert the debt in the event that the date-1 cash flow to the firm is 110. Should he convert the debt when the date-1 cash flow to the firm is 154? Yes, because
\[ 0.86 > \frac{132}{154}. \]

The reduction in the firm’s equity value resulting from the replacement of the straight debt by the convertible debt is exactly the increase in the debt value. Thus we compute the latter:
\[
\frac{(0.86 - \frac{132}{154}) \times 154 \times b + 0 \times (1 - b)}{1 + r_f} = 0.28.
\]
Thus the equity value will be reduced by 0.28.

13. At date 0, perfect financial markets are in equilibrium. Firm A’s assets are worth $125 at date 0, which may either go up to $210 or go down to $105 at date 1. It is known that the put option written on firm A’s total assets expiring at date 1 with exercise price \( K = $147 \) is sold for $30 at date 0. Firm A has borrowed some debt at date 0, and its debt
is due at date 1. Firm A’s equity is worth only $10 at date 0. Find the face value of firm A’s debt.

**Solution.** The following table summarizes the cum-dividend price processes of firm A’s total assets, the put option written on those assets, and the debt in question, where $\omega_1$ and $\omega_2$ are the two date-1 states where firm A’s assets are worth respectively 210 and 105.

<table>
<thead>
<tr>
<th>assets/(time,event)</th>
<th>(0, $\Omega$)</th>
<th>(1, $\omega_1$)</th>
<th>(1, $\omega_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total assets</td>
<td>125</td>
<td>210</td>
<td>105</td>
</tr>
<tr>
<td>Put option</td>
<td>30</td>
<td>max(0, 147 - 210)</td>
<td>max(0, 147 - 105)</td>
</tr>
<tr>
<td>Debt</td>
<td>(125-10)</td>
<td>min(210, $F$)</td>
<td>min(105, $F$)</td>
</tr>
</tbody>
</table>

Let $b$ be the martingale probability for state $\omega_1$, and $r_f$ the riskless rate of interest from date 0 to date 1. From the first two assets’ data, we have

$$125 = \frac{b \times 210 + (1 - b) \times 105}{1 + r_f} \Rightarrow 250(1 + r_f) = 210 + 210b,$$

$$30 = \frac{b \times 0 + (1 - b) \times 42}{1 + r_f} \Rightarrow 150(1 + r_f) = 210 - 210b,$$

so that

$$r_f = 5\%, \ b = \frac{1}{4}.$$  

Now there are three possibilities about the unknown $F$.

- Suppose that $F > 210$. In this case, the last row in the above table becomes

<table>
<thead>
<tr>
<th>assets/(time,event)</th>
<th>(0, $\Omega$)</th>
<th>(1, $\omega_1$)</th>
<th>(1, $\omega_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>(125-10)</td>
<td>210</td>
<td>105</td>
</tr>
</tbody>
</table>

and to rule out arbitrage opportunities, we need

$$115 = \frac{b \times 210 + (1 - b) \times 105}{1 + r_f} = \frac{1}{4} \times 210 + \frac{3}{4} \times 105 = \frac{5 \times 525}{21} = 125,$$

a contradiction.
• Suppose that $F < 105$. In this case, the last row in the above table becomes

<table>
<thead>
<tr>
<th>assets/(time,event)</th>
<th>(0, $\Omega$)</th>
<th>(1, $\omega_1$)</th>
<th>(1, $\omega_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>(125-10)</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

and to rule out arbitrage opportunities, we need

$$115 = \frac{b \times F + (1 - b) \times F}{1 + r_f} = \frac{F}{20} \Rightarrow F = 120.75 > 105,$$
	her another contradiction.

• Thus we are left with the last possibility that $210 \geq F \geq 105$. This implies that the last row in the above table becomes

<table>
<thead>
<tr>
<th>assets/(time,event)</th>
<th>(0, $\Omega$)</th>
<th>(1, $\omega_1$)</th>
<th>(1, $\omega_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>(125-10)</td>
<td>$105$</td>
<td>$105$</td>
</tr>
</tbody>
</table>

and to rule out arbitrage opportunities, we need

$$115 = \frac{b \times F + (1 - b) \times 105}{1 + r_f} \Rightarrow 483 = F + 3 \times 105 \Rightarrow F = 168.$$

Since $210 \geq 168 \geq 105$, the face value of the debt is indeed $F = 168$.

14. Currently (at date 0), 25 NT dollars can be exchanged for 1 US dollar. In six months (at date 1) it may take either 27 NT dollars or 24 dollars to exchange for 1 US dollar (both events are equally likely). The spot rates of interest for a six-month loan are respectively 3% and 2.5% in Taiwan and in the states. Determine the premium for a European call option expiring in six months which allows its holder to exchange 25 NT dollars for 1 US dollar for a total amount of 200,000 NT dollars.

**Solution.** The following table summarizes the cum-dividend price processes, (denominated in NT dollar) of the 6-month USD deposit account, the 6-month NTD deposit account, and the call option, where $\omega_1$ and $\omega_2$ stand for the two date-1 states where the exchange rates become 27 and 24 respectively.
Let $b$ be the martingale probability for state $\omega_1$, and $r_f$ the riskless rate of interest from date 0 to date 1. From the first two assets’ data, we have

$$25 = \frac{b \times 27 \times (1.025) + (1 - b) \times 24 \times (1.025)}{1 + r_f},$$

$$1 = \frac{b \times 1.03 + (1 - b) \times 1.03}{1 + r_f} \Rightarrow r_f = 0.03,$$

so that

$$b = \frac{1150}{3075} = \frac{46}{123}.$$  

Thus to rule out arbitrage opportunities, we must have

$$c = \frac{46 \times \left[ \frac{200,000}{25} \times 27 - 200,000 \right]}{123 \times (1 + 3\%)} = 5809.46.$$  

15. Let us give an example with incomplete markets. Two assets are traded at date 0, with

$$p = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$  

Thus there are three date-1 states. Now, suppose that the above price system remains valid after we introduce a derivative asset

$$z = \begin{bmatrix} 5 \\ \frac{1}{4} \end{bmatrix}.$$  

<table>
<thead>
<tr>
<th></th>
<th>$(0, \Omega)$</th>
<th>$(1, \omega_1)$</th>
<th>$(1, \omega_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD deposit</td>
<td>25</td>
<td>$\frac{25}{25}(1 + 2.5%) \times 27$</td>
<td>$\frac{25}{25}(1 + 2.5%) \times 24$</td>
</tr>
<tr>
<td>NTD deposit</td>
<td>1</td>
<td>1.03</td>
<td>1.03</td>
</tr>
<tr>
<td>Call option</td>
<td>$c$</td>
<td>$\max(0, \frac{200,000}{25} \times 27 - 200,000)$</td>
<td>$\max(0, \frac{200,000}{25} \times 24 - 200,000)$</td>
</tr>
</tbody>
</table>
(i) Verify that to rule out arbitrage opportunities, we must have (a necessary condition)
\[ \frac{12}{8} < f(\mathbf{z}) < \frac{27}{8} \].

(ii) Find \( \mathcal{F} \) and \( f \) such that no arbitrage opportunities exist if and only if (a necessary and sufficient condition)
\[ f < f(\mathbf{z}) < \mathcal{F}. \]

**Solution.** For part (i), note that \( \mathbf{z} \) dominates 2 units of asset 2, and hence
\[ f(\mathbf{z}) > \frac{6}{8} \times 2 = \frac{12}{8}. \]
Moreover, \( \mathbf{z} \) is dominated by the portfolio consisting of 3 units of asset 1 and 2 units of asset 2, and hence
\[ f(\mathbf{z}) < 3 \times \frac{5}{8} + 2 \times \frac{6}{8} = \frac{27}{8}. \]

Now, for part (ii) to look for the least upper bound \( \mathcal{F} \) for \( f(\mathbf{z}) \), we shall solve the following minimization problem.

\[
\min_{\mathbf{q} \in \mathbb{R}^2} \left[ \begin{array}{c}
\frac{5}{8} \\
\frac{3}{4}
\end{array} \right] \mathbf{q}
\]
subject to
\[
\begin{bmatrix}
1 & 1 \\
\frac{1}{2} & 0 \\
0 & 1
\end{bmatrix} \mathbf{q} \geq \mathbf{z} = \begin{bmatrix}
\frac{5}{4} \\
\frac{1}{2}
\end{bmatrix}.
\]
The solution is \( \mathbf{q}' = (q_1, q_2) = (3, 2) \), with
\[ \mathcal{F} = \left[ \begin{array}{c}
\frac{5}{8} \\
\frac{3}{4}
\end{array} \right] \mathbf{q} = \frac{27}{8}. \]
Similarly, to look for the greatest lower bound \( f \) for \( f(\mathbf{z}) \), we shall solve the following maximization problem.

\[
\max_{\mathbf{q} \in \mathbb{R}^2} \left[ \begin{array}{c}
\frac{5}{8} \\
\frac{3}{4}
\end{array} \right] \mathbf{q}
\]
subject to
\[
\begin{bmatrix}
1 & 1 \\
\frac{1}{2} & 0 \\
0 & 1
\end{bmatrix} q \leq z = \begin{bmatrix}
5 \\
\frac{1}{4} \\
2
\end{bmatrix}.
\]

The solution is \( q' = (q_1, q_2) = (\frac{1}{2}, 2) \), with
\[
f = \begin{bmatrix}
\frac{5}{8} \\
\frac{3}{4}
\end{bmatrix} q = \frac{29}{16}.
\]

We arbitrage opportunities do not exist if and only if
\[
\frac{29}{16} < f(z) < \frac{54}{16}.
\]

16. Suppose that firms A, B, C, and D hold the same assets in a two-period perfect markets economy. Firm A is all-equity financed, and its common stock is worth 1,750 dollars at date 0, which may rise to 2,400 or drop to 1,200 at date 1 with equal probabilities. Firm B has debt due at date 1, with face value 1,320. Firm B’s equity is worth 675 at date 0. Firm C also has debt due at date 1 with face value 1,320, but the debt is a *callable bond*, in the sense that right before the minute its debt matures at date 1, firm C has the right to pay the bondholder an amount of 1,296 to redeem the debt. Firm D also has debt due at date 1 with face value 1,320, but its debt is a *convertible bond*, in the sense that right before the moment that the debt matures at date 1, the bondholder has the right to convert the bond into 60% of the total equity. Compute the debt value for respectively firm C and firm D.

**Solution.** By Modigliani and Miller’s Proposition 1, the date-0 market values of firms A, B, C and D should all equal. Of course, that the four firms have the same assets imply that their date-1 total earnings are the same. Let \( b \) be the martingale probability for the date-1 event that firm A’s equity value rises to 2,400. Let \( r_f \) be the riskless rate of interest from date 0 to date 1. We have, from the data about the equity of firm A and the equity of firm B,
\[
1,750 = \frac{2,400 \times b + 1,200 \times (1 - b)}{1 + r_f}, \quad 675 = \frac{(2,400 - 1,320) \times b + 0 \times (1 - b)}{1 + r_f}
\]
\[
\Rightarrow b = \frac{3}{4}, \quad r_f = 20\%.
\]
Now the $z$-vector associated with firm C’s debt is, by the fact that $1,320 > 1,296 > 1,200$,

$$\begin{bmatrix} \min(1296, 1320) \\ \min(1296, 1200) \end{bmatrix} = \begin{bmatrix} 1,296 \\ 1,200 \end{bmatrix},$$

and hence the date-0 value of firm C’s debt is

$$\frac{1,296 \times b + 1,200 \times (1 - b)}{1 + r_f} = \frac{972 + 300}{1.2} = \frac{5 \times (972 + 300)}{6} = 1,060.$$

Similarly, the $z$-vector associated with firm D’s debt is, by the fact that $1,320 < 2,400 \times 60\%$ and $1,200 > 1,200 \times 60\%$,

$$\begin{bmatrix} \max(1320, 2400 \times 60\%) \\ \max(1200, 1200 \times 60\%) \end{bmatrix} = \begin{bmatrix} 2,400 \times 60\% \\ 1,200 \end{bmatrix} = \begin{bmatrix} 1,440 \\ 1,200 \end{bmatrix},$$

and hence the date-0 value of firm C’s debt is

$$\frac{1,440 \times b + 1,200 \times (1 - b)}{1 + r_f} = \frac{1,080 + 300}{1.2} = \frac{5 \times (1,080 + 300)}{6} = 1,150.$$

It would be incorrect to claim that the firm should issue a convertible bond given the above analysis. Note that by issuing a convertible bond (rather than a callable bond, say), the firm does receive more proceeds at date 0, but it also promises higher payoffs to the bondholder at date 1. All securities are fairly priced, and in the above complete, perfect markets setting, the firm has no reason to prefer a convertible debt to a callable debt.\footnote{Things are different if market imperfections exist. For example, with corporate insiders’ (CEO and large shareholders’) incentive problems, the firm may benefit from issuing a convertible debt: if the insiders may make an investment decision that benefits themselves at the expense of the outside bondholders, the latter, being rational, may refuse to buy the bond in the first place; but if the bond is made convertible, then the bondholders can protect themselves by turning themselves into shareholders whenever the investment decision intends to benefit the shareholders and hurt the bondholders. Thus issuing a convertible debt helps convince the outside investors that the firm will not make investment decisions against bondholders’ interest, which then facilitates the firm’s fund raising at date 0. Similarly, in the presence of information asymmetry between the insiders and outsiders, the insiders may use a callable debt to signal their private information. However, the reader should bear in mind that in the perfect and complete markets setting, the firm has no reason to prefer a callable or a convertible debt.}

25
17. In reality, derivative assets do not appear to be redundant. First the economy can spare the costs of setting up a new market for a redundant derivative asset, and even if the asset is traded, we shall not expect to see a regularly high trading volume. In fact, derivative assets are introduced exactly because they are expected to have lower transaction costs than their replicating portfolios of underlying assets. Moreover, in the presence of transaction costs, it makes sense for a large institution to write a derivative security and then to utilize some dynamic hedging techniques to reduce or even remove the risk exposure. This would not happen if markets are perfect, as the institution will not make profits by going through all this trouble. Recognizing that the derivative assets may not be redundant in reality indicates that the current derivative assets pricing theory based on the idea of forming replicating portfolios may have a limited applicability! If the so-called derivative assets are no different than the so-called underlying assets, and the only way to price them correctly is to resort to the equilibrium pricing approach (such as the CAPM).

Recall from Lecture 3 that when markets are complete, Walrasian (competitive) equilibrium allocations are Pareto efficient. With incomplete markets, competitive equilibrium (called Radner equilibrium) allocations are inefficient for two reasons. First, there is the missing market problem, meaning that some asset is not available for trading. Second, the price-taking investors do not recognize the effects of their trading at one date on the wealth distribution at a future date, which is important in determining the equilibrium asset prices at that future date. There are in general two ways to improve efficiency in this case: government intervention and asset innovation. The problem with the first way is that the government may not possess enough information to make the right intervention, even if the government is benevolent (a large body of literature studies the government’s incentive problems). The problem with the latter is that, unless we can create enough assets to complete the markets once and for all, examples exist where creating a few more assets makes everyone in the economy worse off; see for example Stiglitz and Newbery (1984, *Review of Economic Studies*). Thus, selecting the right markets to open is a very important issue in incomplete-markets economies.
There are some special cases where in spite of market incompleteness, competitive equilibrium allocations are still Pareto efficient. For example, if all investors have HARA von Neumann-Morgenstern utility functions with identical risk cautiousness,\(^{10}\) then two-fund separation holds, and it takes only one risky fund and one riskless asset to span the Pareto optimal allocations; see Huang and Litzenberger (1988).

Another possibility is that markets for some long-lived assets can stay open for nearly every instant of time. The idea is that, when trading can take place very frequently, then it can take place more frequently than the speed of uncertainty resolution. In that case the markets are essentially complete during each small time interval, and by rebalancing one’s portfolio frequently, one can attain every consumption plan as he wishes. This idea (referred to as dynamic completeness) was foreshadowed by Arrow completeness, and formally proved by Harrison and Kreps (1979, *Journal of Economic Theory*). In the famous Black-Scholes economy, two long-lived assets are traded continuously in time, and the economy turns out to be dynamically complete.\(^ {11}\)

18. Now we extend the above analysis to a 3-period economy. We shall illustrate the ideas in a series of examples.

**Example 1.** Consider an economy that extends for three dates \((t = 0, 1, 2)\) with 4 states of nature \((\omega_1, \omega_2, \omega_3, \omega_4)\) and two long-lived assets. The common information structure for investors is as follows. At \(t = 0\), investors know that the true state is an element of \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\). At \(t = 1\), investors know whether the true state is an element of \(E = \{\omega_1, \omega_2, \omega_3, \omega_4\}\).

---

\(^{10}\)Recall the Arrow-Pratt measure for absolute risk aversion \(R_A(z) = -\frac{u''(z)}{u'(z)}\) at wealth \(z\). The investor’s risk tolerance \(T(z)\) at wealth \(z\) is defined as the reciprocal of \(R_A(z)\); i.e. \(T(z) = -\frac{u'(z)}{u''(z)}\). The first derivative of \(T(z)\), or \(T'(z)\), is referred to as the investor’s risk cautiousness at wealth \(z\).

\(^{11}\)How likely is it that the markets may appear to be dynamically complete? As we mentioned above, it depends on how fast information arrives to the market. The slower the information arrival, the more likely that a small set of long-lived assets can span a commodity space of a large dimension. The recent work of Berk and Uhlig (1993, *Journal of Economic Theory*), however, showed that if there is enough heterogeneity among the investors in the economy, then there is always a portion of investors who want to purchase and release information to the public to speed up the information arrival, so that it looks pessimistic that markets can actually be dynamically complete.
\{\omega_1, \omega_2\} or an element of \(E^c = \{\omega_3, \omega_4\}\). At \(t = 2\), investors know exactly which among \(\omega_1, \omega_2, \omega_3, \omega_4\) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>asset/(date,event)</th>
<th>(0, \Omega)</th>
<th>(1,E)</th>
<th>(1,E^c)</th>
<th>(2,\omega_1)</th>
<th>(2,\omega_2)</th>
<th>(2,\omega_3)</th>
<th>(2,\omega_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>\frac{1}{3}</td>
<td>1.1</td>
<td>2.2</td>
<td>1</td>
<td>1.48</td>
<td>3.3</td>
<td>1.1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.32</td>
<td>1.32</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Let me give some preliminary definitions. At first, observe that there are only two traded assets, but there are 4 different states. Hence markets are not complete by our earlier definition. However, markets are dynamically complete in the following sense: At each date \(t\), there are only two possible date \(t + 1\) events, and so the two linearly independent traded assets are enough to make the markets complete during the date-\(t\)-date-(\(t + 1\)) period. Formally, we say that:

*Markets are dynamically complete if at each date \(t\), the markets are complete during date-\(t\)-date-(\(t + 1\)) period.*

By this definition, the above 3-period economy is dynamically complete. When markets are dynamically complete, again, every asset can be priced by no arbitrage.

First note that if we assume that assets generate cash flows only at date 2, then every asset can be represented as a \(z\) vector in \(\mathbb{R}^4\).\(^{12}\) Note that at date 1, when event \(E\) occurs, there are only two possible states \(\omega_1\) and \(\omega_2\); at date 1, when event \(E^c\) occurs, there are again two possible states \(\omega_3\) and \(\omega_4\). Since assets 1 and 2 are available for trading at date 1, regardless whether event \(E\) occurs or not, and since these two assets give linearly independent payoffs at date 2, we know that every asset \(z\) can be priced at date 1. More precisely, we can focus only on the two-period economy where date 1 is treated as the first period, and \(\omega_1\) and \(\omega_2\) are treated as the two possible states of the second period, and

---

\(^{12}\)In general, an asset may generate cash flows at date 1 also. We shall consider this case later on.
apply the martingale pricing theorem to determine the price $P_z(1, E)$ of the payoff 
\[
\begin{bmatrix}
z(\omega_1) \\
z(\omega_2)
\end{bmatrix}
\]
at date 1 in event $E$. Similarly, we can focus only on the two-period economy where date 1 is treated as the first period, and $\omega_3$ and $\omega_4$ are treated as the two possible states of the second period, and apply the martingale pricing theorem to determine the price $P_z(1, E^c)$ of the payoff 
\[
\begin{bmatrix}
z(\omega_3) \\
z(\omega_4)
\end{bmatrix}
\]
at date 1 in event $E^c$. Then, consider the trading strategy of buying asset $z$ at date 0 and selling it at date 1. We can focus only on the two-period economy where date 0 is treated as the first period, and events $E$ and $E^c$ are treated as the two possible states of the second period, and apply the martingale pricing theorem to determine the price $P_z(0)$ of the payoff 
\[
\begin{bmatrix}
P_z(1, E) \\
P_z(1, E^c)
\end{bmatrix}
\]
at date 0.

An example will make everything clear. Consider a (riskless) pure discount bond that matures at date 2 with face value equal to 1. That is, the associated $z$ vector is
\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\]

For this particular asset, I will write $P_z$ as $B$. That is, for the pure discount bond, let its date-1 price be $B(1, E)$ if event $E$ occurs and $B(1, E^c)$ if event $E^c$ occurs, and let its date-0 price be $B(0)$.

We shall solve $B(1, E)$ first, and then $B(1, E^c)$, and finally $B(0)$. Consider the two-period economy where date 1 is treated as the first period, and we assume that event $E$ has occurred, and we treat $\omega_1$ and $\omega_2$ as
the two possible states of the second period. There are two underlying assets traded in the first period, and there are two possible states in the second period, so that markets are complete in this two-period economy. Let the riskless rate for this two-period economy be denoted by \( r_f(1, E) \). Let the martingale probability for state \( \omega_1 \) be denoted by \( \pi^*(\omega_1|E) \). Then the martingale pricing theorem tells us that

\[
1.1 = p_1(1, E) = \frac{1 \times \pi^*(\omega_1|E) + 1.48 \times [1 - \pi^*(\omega_1|E)]}{1 + r_f(1, E)},
\]

\[
1.1 = p_2(1, E) = \frac{1.32 \times \pi^*(\omega_1|E) + 1.32 \times [1 - \pi^*(\omega_1|E)]}{1 + r_f(1, E)},
\]

Solving, we have

\[
\pi^*(\omega_1|E) = \frac{1}{3}, \quad r_f(1, E) = 0.2.
\]

Hence we have

\[
P_z(1, E) = \frac{z(\omega_1) \times \pi^*(\omega_1|E) + z(\omega_2) \times [1 - \pi^*(\omega_1|E)]}{1 + r_f(1, E)} = \frac{z(\omega_1) \times \frac{1}{3} + z(\omega_2) \times \frac{2}{3}}{1.2},
\]

for all assets \( z \in \mathbb{R}^4 \). In particular, we have

\[
B(1, E) = \frac{1}{1.2} = \frac{5}{6}.
\]

Next, consider the two-period economy where date 1 is treated as the first period, and we assume that event \( E^c \) has occurred, and we treat \( \omega_3 \) and \( \omega_4 \) as the two possible states of the second period. There are two underlying assets traded in the first period, and there are two possible states in the second period, so that markets are complete in this two-period economy. Let the riskless rate for this two-period economy be denoted by \( r_f(1, E^c) \). Let the martingale probability for state \( \omega_3 \) be denoted by \( \pi^*(\omega_3|E^c) \). Then the martingale pricing theorem tells us that

\[
2.2 = p_1(1, E^c) = \frac{3.3 \times \pi^*(\omega_3|E^c) + 1.1 \times [1 - \pi^*(\omega_3|E^c)]}{1 + r_f(1, E^c)},
\]
1.1 = p_2(1, E^c) = \frac{1.1 \times \pi^*(\omega_3|E^c) + 1.1 \times [1 - \pi^*(\omega_3|E^c)]}{1 + r_f(1, E^c)}.

Solving, we have

\pi^*(\omega_3|E^c) = \frac{1}{2}, \ r_f(1, E^c) = 0.

Hence we have

P_z(1, E^c) = z(\omega_3) \times \pi^*(\omega_3|E^c) + z(\omega_4) \times [1 - \pi^*(\omega_3|E^c)] = z(\omega_1) \times \frac{1}{2} + z(\omega_2) \times \frac{1}{2},

for all assets \( z \in \mathbb{R}^4 \). In particular, we have

B(1, E^c) = \frac{1}{1} = 1.

Finally, consider the two-period economy where date 0 is treated as the first period, and we treat \( E \) and \( E^c \) as the two possible states of the second period. There are two underlying assets traded in the first period, and there are two possible states in the second period, so that markets will be complete in this two-period economy, if the two traded assets’ date-1 prices over the two events \( E \) and \( E^c \) are linearly independent. Let the riskless rate for this two-period economy be denoted by \( r_f(0) \). Let the martingale probability for event \( E \) be denoted by \( \pi^*(E) \). Then the martingale pricing theorem tells us that

\[ \frac{7}{4} = p_1(0) = \frac{1.1 \times \pi^*(E) + 2.2 \times [1 - \pi^*(E)]}{1 + r_f(0)}, \]

\[ 1 = p_2(0) = \frac{1.1 \times \pi^*(E) + 1.1 \times [1 - \pi^*(E)]}{1 + r_f(0)}, \]

Solving, we have

\pi^*(E) = \frac{1}{4}, \ r_f(0) = 0.1.

Hence we have

P_z(0) = \frac{P_z(1, E) \times \pi^*(E) + P_z(1, E^c) \times [1 - \pi^*(E)]}{1 + r_f(0)} = \frac{P_z(1, E) \times \frac{1}{4} + P_z(1, E^c) \times \frac{3}{4}}{1.1},
for all assets $z \in \mathbb{R}^4$. In particular, we have

$$B(0) = \frac{115}{132}.$$  

Now, consider an American put option written on one unit of asset 1 that expires at date 2 with exercise price equal to 2.2. An American put option allows its holder to exercise the right at either date 0, or date 1, or date 2. To obtain the price process of the American put option, we again use backward induction. First consider date 2. At date 2, the put option either has or has not been exercised. If it has not been exercised, then it will be except in state $\omega_3$. Hence the date-2 (cum-dividend) price of the put option is

$$P(2, \omega_1) = 1.2, \quad P(2, \omega_2) = 0.72, \quad P(2, \omega_3) = 0, \quad P(2, \omega_4) = 1.1.$$  

Now, return to date 1. First consider the case where event $E$ has occurred. In this case, if the option has not been exercised, it can be either exercised now (i.e., date 1) or carried into date 2 (selling the option at date 1 has a value equal to the maximum of the values generated by these two alternatives, and hence we do not have to consider it). The value of the option if left un-exercised until date 2 is the same as the price of a European put option expiring at date 2, which is $\frac{11}{15}$. Since exercising immediately yields $1.1 > \frac{11}{15}$, if the option has not been exercised at date 1, it should be exercised immediately when event $E$ has occurred.

Similarly, one can verify that the option should not be exercised at date 1 if event $E^c$ has occurred. This implies that the American put has a price equal to the price of its European counterpart in this case, which is $\frac{11}{20}$.

Finally, consider date 0. The value of carrying the option into date 1 is

$$\frac{1}{3} \times 1.1 + \frac{3}{4} \times \frac{11}{20} = 5 \frac{5}{8},$$

which is higher than $\frac{9}{20} = 2.2 - \frac{7}{4}$, the payoff generated by immediate exercising. Hence the put option has a date-0 price equal to $\frac{5}{8}$.
Now, we consider a date-$t$ forward contract written on 1 unit of asset 1 to be delivered at date 2, where $t = 0, 1$. Let $G(t)$ be the price stated in the forward contract signed at date $t$. (I shall also use $G(t)$ to denote the forward contract signed at date $t$.) Note that by longing one contract $G(t)$ at date $t$, an investor promises to pay $G(t)$ to her counterparty at date 2 in exchange for 1 unit of asset 1. Hence the corresponding $z$-vector is

$$
\begin{bmatrix}
1 - G(t) \\
1.48 - G(t) \\
3.3 - G(t) \\
1.1 - G(t)
\end{bmatrix}.
$$

Since by longing one contract $G(t)$ at date $t$, the investor does not pay anything to nor receive anything from the other contracting party, the above $z$-vector must have a zero date-$t$ value. Note that the date-$t$ value of the above $z$-vector is simply the difference between the date-$t$ price of asset 1, $p_1(t)$, and $G(t)B(t)$, we conclude that

$$p_1(t) - G(t)B(t) = 0 \Rightarrow G(t) = \frac{p_1(t)}{B(t)}.
$$

Thus we have

$$G(0) = \frac{3}{B(0)}, \quad G(1, E) = \frac{1.1}{B(1, E)}, \quad G(1, E^c) = \frac{2.2}{B(1, E^c)}.
$$

Now, we consider a date-$t$ futures contract $H(t)$ written on 1 unit of asset 1 to be delivered at date 2, where $t = 0, 1$. (Again, $H(t)$ will also denote the price stated in the contract $H(t)$.) Such a futures contract can be defined recursively. At date 1, the futures contract is the same as a forward contract. At date 0, by longing 1 contract $H(0)$, at date 1 an investor will receive a date-1 cash flow $\tilde{H}(1) - H(0)$ and then the old contract $H(0)$ will be replaced by the new contract $\tilde{H}(1)$. More precisely, if at date 1 event $E$ occurs, then the above investor will receive a date-1 cash flow $G(1, E) - H(0)$ and must long 1 contract $G(1, E)$; and if event $E^c$ occurs, then she will receive a date-1 cash flow $G(1, E^c) - H(0)$ and must long 1 contract $G(1, E^c)$. On the other
hand, an investor shorting the contract $H(0)$ at date 0 will receive cash flow $H(0) - G(1, E)$ and must then short 1 contract $G(1, E)$ at date 1 in case event $E$ occurs; and she will receive $H(0) - G(1, E^c)$ and then short 1 contract $G(1, E^c)$ at date 1 in case event $E^c$ occurs. Now, note that an investor entering into the contract $H(0)$, whether she is longing or shorting, does not receive money from or paying money to the other contracting party at date 0. This implies that the date-0 value of the date-1 cash flow $\hat{H}(1) - H(0)$ must be zero! We thus conclude that

$$H(0) = \pi^*(E)H(1, E) + \pi^*(E^c)H(1, E^c);$$

that is, the futures price is a martingale under the martingale probabilities. With our numerical values, we hence have

$$H(0) = \frac{1}{4} \times \frac{1.1}{B(1, E)} + \frac{3}{4} \times \frac{2.2}{B(1, E^c)}.$$

19. **Example 2.** Consider an economy that extends for three dates ($t = 0, 1, 2$) with 4 states of nature ($\omega_1, \omega_2, \omega_3, \omega_4$) and two long-lived assets. The common information structure for investors is as follows. At $t = 0$, investors know that the true state is an element of $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. At $t = 1$, investors know whether the true state is an element of $E = \{\omega_1, \omega_2\}$ or an element of $E^c = \{\omega_3, \omega_4\}$. At $t = 2$, investors know exactly which among $\omega_1, \omega_2, \omega_3, \omega_4$ is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>Assets/(Date,Event)</th>
<th>(0, $\Omega$)</th>
<th>(1, $E$)</th>
<th>(1, $E^c$)</th>
<th>(2, $\omega_1$)</th>
<th>(2, $\omega_2$)</th>
<th>(2, $\omega_3$)</th>
<th>(2, $\omega_4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14,010</td>
<td>5,040</td>
<td>20,736</td>
<td>11,520</td>
<td>0</td>
<td>34,560</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>525</td>
<td>630</td>
<td>630</td>
<td>720</td>
<td>720</td>
<td>700</td>
<td>700</td>
</tr>
</tbody>
</table>

Find the futures price process $H(t)$ written on one unit of asset 1 with date 2 as the delivery date. Find the price process $c(t)$ of a European call option written on one unit of asset 1, expiring at date 2 with an exercise price 23,040.

**Solution.** It is straightforward to obtain the following martingale probabilities and short term interest rates:
It follows from  

\[ c(2, \omega_1) = 0, \quad c(2, \omega_2) = 0, \quad c(2, \omega_3) = 11,520, \quad c(2, \omega_4) = 0 \]

that  

\[ c(1, E) = \frac{0 \times \frac{1}{5} + 0 \times \frac{1}{5}}{1 + \frac{1}{5}} = 0, \quad c(1, E^c) = \frac{11,520 \times \frac{2}{5} + 0 \times \frac{1}{5}}{1 + \frac{3}{5}} = 6,912. \]

From here, we obtain  

\[ c(0, \Omega) = \frac{0 \times \frac{1}{5} + 6,912 \times \frac{3}{4}}{1 + \frac{1}{5}} = 4,320. \]

On the other hand, by the fact that \( H(\cdot) \) converges to the spot price of the underlying asset at date 2, we have  

\[ H(2, \omega_1) = 11,520, \quad H(2, \omega_2) = 0, \quad H(2, \omega_3) = 34,560, \quad H(2, \omega_4) = 0. \]

By the fact that \( H(1) \) is the same as the forward price \( G(1) \), and by the fact that  

\[ G(t) = \frac{p(t)}{B(t)}, \]

where \( p(t) \) is the date-\( t \) price of the underlying asset (here, asset 1), and \( B(t) \) is the date-\( t \) price of a pure discount bond with face value equal to 1 dollar maturing at date 2,\(^{13}\) we have  

\[ H(1, E) = 5,040 \times (1 + \frac{1}{7}) = 5,760, \quad H(1, E^c) = 20,736 \times (1 + \frac{1}{9}) = 23,040. \]

\(^{13}\)This happens because \( p(2) - G(t) \) must have a zero value at date \( t \), where \( p(2) \) is the date-2 (typically, random) value of the underlying asset, and \( G(t) \) is the non-random forward price determined at date \( t < 2 \). Note that the date-\( t \) value of \( p(2) \) is exactly \( p(t) \), and the date-\( t \) value of \( G(t) \) is \( G(t)B(t) \).
Finally, we solve for $H(0, \Omega)$. By the fact that $H(t)$ is a martingale under the martingale probabilities,$^{14}$ we have

$$H(0, \Omega) = H(1, E) \times \frac{1}{4} + H(1, E^c) \times \frac{3}{4} = 18,720.$$ 

20. Example 3. Consider the economy described in Example 2. Let the date-2 (spot) price of an underlying asset (that pays no dividends before date 2) be denoted by $\tilde{z}$. Let $\hat{f}_1$ and $f_0$ be the date-1 and date-0 spot prices of the underlying asset. Let the one-period riskless rate be denoted by $r_{0,1}$ and $\hat{r}_{1,2}$ for respectively the date-0-date-1 and the date-1-date-2 periods. Let $B(0)$ and $\hat{B}(1)$ be respectively the date-0 and date-1 prices of the pure discount bond maturing at date 2 with face value equal to one dollar. Let $G(0)$ and $\hat{G}(1)$ be the forward price of the underlying asset determined at date 0 and date 1 respectively. Let $H(0)$ and $\hat{H}(1)$ be the futures price of the underlying asset determined at date 0 and date 1 respectively. Note that from the date-0 perspective, anything that may happen at or after date 1 is uncertain (it may depend on whether $E$ or $E^c$ happens at date 1, or it may depend on which among $\omega_1, \omega_2, \omega_3, \omega_4$ is the true state at date 2), and hence we have put a tilde $\tilde{\cdot}$ on it. Let $E^*[\cdot]$ be the expectation operator under the martingale probabilities taken at date 0. Similarly, let $\text{cov}^*(\cdot, \cdot)$ denote the covariance operator under the martingale probabilities. Show that$^{15}$

$$G(0) - H(0) = \frac{\text{cov}^*(\hat{G}(1), \hat{B}(1))}{B(0)(1 + r_{0,1})}.$$ 

---

$^{14}$This happens because (i) the date-$t$ price innovation $H(t-1) - H(t)$ must have a zero value at date $t-1$; (ii) the date-$(t-1)$ value of the date-$t$ non-random cash flow $H(t-1)$ is $H(t-1)B(t-1, t)$, where for all $s < \tau$, $B(s, \tau)$ is the date-$s$ price of a pure discount bond with face value equal to 1 dollar maturing at date $\tau$; and (iii) the date-$(t-1)$ value of the date-$t$ random cash flow $H(t)$ is $E^*[H(t)|\mathcal{F}_{t-1}]B(t-1, t)$.

$^{15}$Hint: Recall from our earlier discussions that

$$G(0) = \frac{f_0}{B(0)}, \quad \hat{H}(1) = \hat{G}(1) = \frac{\hat{f}_1}{\hat{B}(1)}.$$ 

From here, show that

$$G(0) = \frac{E^*[\hat{f}_1]}{E^*[\hat{f}_1|_{\{\mathcal{F}_{t-1,2}\}}]}.$$ 

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Solution. Following the hint, we have

\[ G(0) = \frac{f_0}{B(0)}, \quad \tilde{H}(1) = \tilde{G}(1) = \frac{\tilde{f}_1}{B(1)}, \]

and hence we have

\[ G(0) = \frac{f_0}{B(0)} = \frac{E^*[\tilde{f}_1]}{E^*[\frac{1}{1+r_{0.1}}]} = \frac{E^*[\frac{\tilde{f}_1}{1+r_{1.2}}]}{E^*[\frac{1}{1+r_{1.2}}]}. \]

On the other hand, we have

\[ H(0) = E^*[\tilde{H}(1)] = E^*[\tilde{G}(1)] = E^*[\frac{\tilde{f}_1}{B(1)}] = E^*[\frac{\tilde{f}_1}{1+\tilde{r}_{1.2}}]. \]

Thus letting \( \text{cov}^*(\cdot, \cdot) \) denote the covariance operator under the martingale probabilities, we have

\[ G(0) - H(0) = \frac{\text{cov}^*(\frac{\tilde{f}_1}{1+\tilde{r}_{1.2}})}{B(0)(1 + r_{0.1})}, \]

so that the date-0 forward price is greater than the date-0 futures price if and only if the date-1 forward price and the date-1 bond price are positively correlated (under the martingale probabilities). An important consequence of this result is that the forward price must coincide with the futures price if the interest rates are non-random. This result can be generalized for a discrete-time \( N \)-date model.\(^{16}\)

\[ H(0) = E^*[\frac{\tilde{f}_1}{1+(\tilde{r}_{1.2})}]. \]

Finally, use the definition of covariance.

\(^{16}\)To see the intuition, assume first that investors are risk neutral so that the martingale probability measure coincides with the original probability measure. Note that the futures differs from the forward contract because at date 1, the gain \( \tilde{H}(1) - H(0) \) can be re-invested at the interest rate \( \tilde{r}_{1.2} \). Recall that \( \tilde{H}(1) = \tilde{G}(1) \). If when \( \tilde{G}(1) \) gets higher, \( \tilde{r}_{1.2} \) gets lower, then the futures does not have an advantage over the forward contract. In this
Example 4. Consider an economy that extends for three dates \((t = 0, 1, 2)\) with 4 states of nature \(\{\omega_1, \omega_2, \omega_3, \omega_4\}\) and two long-lived assets.

The common information structure for investors is as follows. At \(t = 0\), investors know that the true state is an element of \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\). At \(t = 1\), investors know whether the true state is an element of \(E = \{\omega_1, \omega_2\}\) or an element of \(E^c = \{\omega_3, \omega_4\}\). At \(t = 2\), investors know exactly which among \(\omega_1, \omega_2, \omega_3, \omega_4\) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>Assets/(Date,Event)</th>
<th>(0, (\Omega))</th>
<th>(1, (E))</th>
<th>(1, (E^c))</th>
<th>(2, (\omega_1))</th>
<th>(2, (\omega_2))</th>
<th>(2, (\omega_3))</th>
<th>(2, (\omega_4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70,050</td>
<td>25,200</td>
<td>103,680</td>
<td>57,600</td>
<td>0</td>
<td>172,800</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1,050</td>
<td>1,260</td>
<td>1,260</td>
<td>1,440</td>
<td>1,440</td>
<td>1,400</td>
<td>1,400</td>
</tr>
</tbody>
</table>

The total supply of asset 1 is one unit. Asset 1 is the single asset owned by firm X. Thus the value of asset 1 is also the value of firm X. Suppose that firm X has issued a convertible debt at date 0, which matures at date 2 with face value 54,000. The convertible debt is owned by Mr. Y, and it can be converted into 36% of the firm’s equity either at date 0 or at date 1 (but not at date 2). The choice is up to Mr. Y. If the debt is not converted into equity before date 2, then Mr. Y will be paid as a debtholder at date 2; or else, there would be no debt outstanding at date 2, and Mr. Y will receive 36% of the date-2 value of asset 1.

(i) Compute the no-arbitrage prices of the convertible debt at \((0, \Omega)\), \((1, E)\), and \((1, E^c)\).

(ii) Now suppose that the date-0 price of the convertible debt is actually 25,000. Design a dynamic trading strategy to capture an arbitrage profit.

Solution. Consider part (i). From the data given in the above table,
we can compute the martingale probabilities for the four states and the (random) short-term interest rates. The results are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>(0, Ω)</th>
<th>(1, E)</th>
<th>(1, E\textsuperscript{c})</th>
<th>(2, ω\textsubscript{1})</th>
<th>(2, ω\textsubscript{2})</th>
<th>(2, ω\textsubscript{3})</th>
<th>(2, ω\textsubscript{4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>martingale prob.</td>
<td>-</td>
<td>\frac{1}{3}</td>
<td>\frac{2}{4} \times \frac{1}{2}</td>
<td>\frac{1}{4} \times \frac{1}{2}</td>
<td>\frac{2}{4} \times \frac{2}{3}</td>
<td>\frac{3}{4} \times \frac{1}{3}</td>
<td></td>
</tr>
<tr>
<td>one-period ( r_f )</td>
<td>\frac{1}{5}</td>
<td>\frac{1}{5}</td>
<td>\frac{2}{5}</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1

First consider the date-2 \( z \)-vector of a straight bond with face value 54,000. Apparently, the vector is, assuming that the firm has limited liability,

\[
\begin{bmatrix}
54,000 \\
0 \\
54,000 \\
0
\end{bmatrix}.
\]

Let us first compute the price process of this straight bond. Using Table 1, we have

<table>
<thead>
<tr>
<th></th>
<th>(0, Ω)</th>
<th>(1, E)</th>
<th>(1, E\textsuperscript{c})</th>
<th>(2, ω\textsubscript{1})</th>
<th>(2, ω\textsubscript{2})</th>
<th>(2, ω\textsubscript{3})</th>
<th>(2, ω\textsubscript{4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>25,171.875</td>
<td>23,625</td>
<td>32,400</td>
<td>54,000</td>
<td>0</td>
<td>54,000</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

We conclude that at date 1, if the convertible bond has not been converted yet, then it should be converted if and only if event \( E \) does not occur. To see this, note that if at date 1 event \( E \) occurs, then the straight bond, which represents the value of getting 54,000 in state \( ω_1 \), is worth \( 23,625 \), which is \( 93.75\% \) of the firm value \( 25,200 \) at date 1 in event \( E \). On the other hand, if at date 1 event \( E^c \) occurs, then the firm value is \( 103,680 \), and the straight bond, which represents the value of getting 54,000 in state \( ω_3 \), is worth only \( 32,400 \), accounting for simply \( 31.25\% \) of the firm value. Thus the optimal date-1 conversion strategy is to convert if and only if event \( E^c \) occurs.

At date 0, when the firm is worth \( 70,050 \), there are two alternatives for Mr. Y: to exercise immediately, or to wait until date 1, and exercise if and only if event \( E^c \) occurs at date 1. We conclude that the price process for the convertible bond is either 36\% of the price process of asset 1 or
Note that exercising immediately implies a date-0 value of
\[ 36\% \times 70,050 = 25,218, \]
while waiting until date 1 implies a value of
\[
\frac{23,625 \times \frac{1}{2} + (103,680 \times 36\%) \times \frac{3}{4}}{1 + \frac{5}{6}} = \frac{23,625 \times \frac{1}{2} + 37,324.8 \times \frac{3}{4}}{1 + \frac{5}{6}} = 28,499.875.
\]
Thus the optimal strategy at date 0 is to do nothing and wait until date 1. We conclude that the no-arbitrage price process of the convertible bond is
\[
\begin{array}{ccccccc}
(0, \Omega) & (1, E) & (1, E^c) & (2, \omega_1) & (2, \omega_2) & (2, \omega_3) & (2, \omega_4) \\
23,625 & 103,680 \times 36\% & 54,000 & 0 & 172,800 \times 36\% & 0 \times 36\%
\end{array}
\]

This finishes part (i).

Next, consider part (ii). We must first find the replicating strategy for the convertible bond. At date 1, in case event \( E \) occurs, let \( \theta_i(2, E) \) be the number of shares of asset \( i \) that should be carried into date 2, \( i = 1, 2 \), in order to produce 54,000 in state \( \omega_1 \) and nothing in state \( \omega_2 \). That is, we solve for
\[
\begin{align*}
\theta_1(2, E) \times 57,600 + \theta_2(2, E) \times 1,440 &= 54,000; \\
\theta_1(2, E) \times 0 + \theta_2(2, E) \times 1,440 &= 0,
\end{align*}
\]

implying that
\[
\theta_2(2, E) = 0, \quad \theta_1(2, E) = \frac{54,000}{57,600} = 0.9375.
\]
Similarly, at date 1, in case event \( E^c \) occurs, we can solve for \( \theta_i(2, E^c) \) from
\[
\begin{align*}
\theta_1(2, E^c) \times 172,800 + \theta_2(2, E^c) \times 1,400 &= 172,800 \times 36\% = 62,208; \\
\theta_1(2, E^c) \times 0 + \theta_2(2, E^c) \times 1,400 &= 0,
\end{align*}
\]
and the solutions are obviously
\[ \theta_2(2, E^c) = 0, \quad \theta_1(2, E^c) = 0.36. \]

Back to date 0, we can compute \( \theta_i(1) \) from
\[
\begin{cases}
\theta_1(1) \times 25,200 + \theta_2(1) \times 1,260 = 23,625; \\
\theta_1(1) \times 103,680 + \theta_2(1) \times 1,260 = 103,680 \times 0.36 = 37,324.8,
\end{cases}
\]

and the solutions are
\[
\theta_2(1) \sim 15.258716, \\
\theta_1(1) = \frac{103,680 \times 0.36 - 23,625}{103,680 - 25,200} = \frac{13,699.8}{78,480} \sim 0.1745642.
\]

Now here is our solution for part (ii).

- At date 0, you can short sell 15.258716 units of asset 2, 0.1745642 units of firm X’s common stock, and 0.1745642 units of firm X’s convertible debt, and this will generate 28,249.875 cash.\(^{17} \) Then, put (28,249.875-25,000) in your pocket, and use 25,000 to buy one unit of firm X’s convertible debt. Thus at the end of date 0, you are holding 3,249.875 in cash and 0.8254358 units of firm X’s convertible debt, and shorting 0.1745642 units of firm X’s common stock and 15.258716 units of asset 2.

- At date 1, and if event \( E \) occurs, then you should sell the 0.8254358 units of firm X’s convertible debt to generate cash
\[
0.8254358 \times 23,625 \sim 19,500.93,
\]
and to buy back 0.1745642 units of firm X’s common stock at the cost
\[
0.1745642 \times (25,200 - 23,625) = 274.93861,
\]
and 15.258716 units of asset 2 at the cost
\[
15.258716 \times 1,260 = 19,225.982.
\]

\(^{17}\)Equivalently, you will be short selling 15.258716 units of asset 2 and 0.1745642 units of asset 1, but recall from firm X’s balance sheet that one unit of asset 1 is the same as one unit of firm X’s debt plus one unit of firm X’s equity!
Note that

\[ 19,500.93 = 274.93861 + 19,225.982. \]

Now you can deliver the 0.1745642 units of firm X’s common stock and the 15.258716 units of asset 2 to exactly clear the short positions that you created at date 0.

- At date 1, in case event \(E^c\) occurs instead, first sell the 0.8254358 units of firm X’s convertible debt to obtain cash

\[
0.8254358 \times (103,680 \times 36\%) = 0.8254358 \times 37,324.8 \approx 30,809.224.
\]

Now use the cash to buy back 0.1745642 units of firm X’s common stock at the cost\(^{18}\)

\[
0.1745642 \times (103,680 \times 64\%) = 11,583.242,
\]

and 15.258716 units of asset 2 at the cost

\[
15.258716 \times 1,260 = 19,225.982.
\]

Note that

\[ 30,809.224 = 11,583.242 + 19,225.982. \]

Now you can deliver the 0.1745642 units of firm X’s common stock and the 15.258716 units of asset 2 to exactly clear the short positions that you created at date 0.

It is easy to verify that with this dynamic trading strategy, you will not have cash inflows or outflows at date 1 and date 2, but you have earned a riskless amount of \(28,249.875 - 25,000 = 3,249.875\) at date 0, which by definition is an arbitrage profit.

\(^{18}\)Note that these transactions are assumed to be completed before the convertible debt is converted. This is inessential, for if the short position in firm X’s common stock is to be cleared after the convertible debt is converted, then the number of shares you need to deliver to clear that short position may increase, but the value of that delivery is still 11,583.242. Think about it!
Example 5. Consider an economy that extends for three dates \( t = 0, 1, 2 \) with 4 states of nature \( (\omega_1, \omega_2, \omega_3, \omega_4) \) and two long-lived assets. The common information structure for investors is as follows. At \( t = 0 \), investors know that the true state is an element of \( \Omega = \{ \omega_1, \omega_2, \omega_3, \text{ and } \omega_4 \} \). At \( t = 1 \), investors know whether the true state is an element of \( E = \{ \omega_1, \omega_2 \} \) or an element of \( E^c = \{ \omega_3, \omega_4 \} \). At \( t = 2 \), investors know exactly which among \( \omega_1, \omega_2, \omega_3, \omega_4 \) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>Asset price/(Date,Event)</th>
<th>(0, ( \Omega ))</th>
<th>(1, ( E ))</th>
<th>(1, ( E^c ))</th>
<th>(2, ( \omega_1 ))</th>
<th>(2, ( \omega_2 ))</th>
<th>(2, ( \omega_3 ))</th>
<th>(2, ( \omega_4 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td>1000</td>
<td>1100</td>
<td>1100</td>
<td>1320</td>
<td>1320</td>
<td>1188</td>
<td>1188</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>900</td>
<td>880</td>
<td>1100</td>
<td>1100</td>
<td>1012</td>
<td>1208</td>
<td>1178</td>
</tr>
</tbody>
</table>

Mr. B is endowed with 1 million dollars at date 2. He only wants to consume in event \( E \) at date 1. He can only trade the above two assets.

(i) How much can he consume in event \( E \) at date 1?
(ii) Find the trading strategy for Mr. B so that by trading the above two assets, he can consume as he wants in part (i).

Solution. First we should recover the martingale probabilities and the short-term interest rates. Let \( \pi(a_{t+1}|a_t) \) be the martingale probability that event \( a_{t+1} \) occurs at time \( t+1 \), conditional on the fact that event \( a_t \) has occurred at time \( t \). Let \( r_t(a_t) \) be the one-period riskless interest rate at time \( t \), conditional on the fact that event \( a_t \) has occurred at time \( t \).

Then it is easy to show that

\[
\pi(E|\Omega) = \frac{1}{2}, \quad \pi(E^c|\Omega) = \frac{1}{2},
\]

\[
\pi(\omega_1|E) = \frac{1}{2}, \quad \pi(\omega_2|E) = \frac{1}{2},
\]

\[
\pi(\omega_3|E^c) = \frac{1}{3}, \quad \pi(\omega_4|E^c) = \frac{2}{3},
\]

\[
r_0(\Omega) = 10\%, \quad r_1(E) = 20\%, \quad r_1(E^c) = 8\%.
\]
Let $y$ be the amount of Mr. B’s consumption in event $E$ at date 1. We must solve

$$\frac{\pi(E|\Omega) \times y + \pi(E^c|\Omega) \times 0}{1 + r_0(\Omega)} = \frac{\pi(E|\Omega) \times \frac{1,000,000}{1+r_1(E)} + \pi(E^c|\Omega) \times \frac{1,000,000}{1+r_1(E^c)}}{1 + r_0(\Omega)}.$$ 

Solving, we have approximately

$$y = 1,759,259.$$ 

This is part (i).

Now, consider part (ii). We must find a trading strategy that generates cash outflow of 1 million dollars at date 2 (to offset Mr. B’s endowed income at date 2) and cash inflow of $y$ in event $E$ at date 1. Suppose that Mr. B needs to hold $a$ and $b$ units of assets 1 and 2 at time 1 to generate $y$ in event $E$ at date 1. That is,

$$\begin{bmatrix} 1100 & 880 \\ 1100 & 1100 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}.$$ 

Solving we have approximately

$$a = \frac{y}{220} = 7997, \quad b = -\frac{y}{220} = -7997.$$ 

Next we find the replicating strategy for the pure discount bond maturing at date 2 with one-dollar face value. To replicate this bond, suppose that one must hold $m$ and $n$ units of assets 1 and 2 in event $E$ at date 1. Then we have

$$\begin{bmatrix} 1320 & 1100 \\ 1320 & 1012 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

Solving we have

$$m = \frac{1}{1320}, \quad n = 0.$$ 

Thus to replicate this bond, one must hold $\frac{1}{1320}$ and 0 units of assets 1 and 2 in event $E$ at date 1, which costs $\frac{1}{1320} + \frac{5}{6} = \frac{5}{6}$. Similarly, one
can show that to replicate this bond, one must hold $\frac{1}{1188}$ and 0 units of assets 1 and 2 in event $E^c$ at date 1, which costs $\frac{1100}{1188} = \frac{25}{27}$. Now suppose that one must hold $s$ and $t$ units of assets 1 and 2 at date 0 in order to generate $\frac{5}{6}$ in event E and $\frac{25}{27}$ in event $E^c$ at date 1. We must have

$$\begin{bmatrix} 1100 & 880 \\ 1100 & 1100 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{25}{27} \end{bmatrix}.$$  

Solving we have

$$s = t = \frac{1}{2376}.$$  

Now we can summarize the trading strategy as follows. At time 0, buy $(a - 1,000,000s) \sim 7576$ units of asset 1 and short sell $(1,000,000t - b) \sim 8418$ units of asset 2. This makes sure that there is neither cash inflow nor outflow at time 0. At date 1, if event E occurs, then sell $a \sim 7997$ units of asset 1 and buy 7997 units of asset 2 to get cash $y$ and consume the cash (enjoy, Mr. B!), and then turn the rest portfolio into $-\frac{1,000,000}{1,320} \sim -758$ units of asset 1; and if event $E^c$ occurs, then turn the portfolio into $-\frac{1,000,000}{1,188} = -842$ units of asset 1. Then, at date 2, the portfolio incurs a sure cash outflow, which is 1 million, and Mr. B can simply spend his endowed income to fulfill this obligation. One can verify that following this trading strategy, Mr. B gets cash $y$ in event $E$ at date 1, and he has no cash inflow or outflow at other dates or in other events.

Example 6. Reconsider the economy described in Example 5. Let $p_j(\omega_i)$ be the date-2 price of asset $j$ in state $\omega_i$, for all $j = 1, 2$ and $i = 1, 2, 3, 4$. For example, $p_1(\omega_3) = 1188$ and $p_2(\omega_2) = 1012$. Now, consider a derivative asset that pays $\max(p_1(\omega_i), p_2(\omega_i))$ at date 2, for all $i = 1, 2, 3, 4$. 

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(i) Compute the vector 
\[
\mathbf{z}_{4 \times 1} = \begin{bmatrix}
\max(p_1(\omega_1), p_2(\omega_1)) \\
\max(p_1(\omega_2), p_2(\omega_2)) \\
\max(p_1(\omega_3), p_2(\omega_3)) \\
\max(p_1(\omega_4), p_2(\omega_4))
\end{bmatrix}
\]

(ii) Find the no-arbitrage price process for asset \( \mathbf{z} \); that is, find its price at date 0, its price in event \( E \) at date 1, and its price in event \( E^c \) at date 1.

\textit{Solution.} For part (i), it is easy to get
\[
\mathbf{z}_{4 \times 1} = \begin{bmatrix}
1320 \\
1320 \\
1208 \\
1188
\end{bmatrix}.
\]

For part (ii), let \( q(t, a_t) \) be the asset’s price at time \( t \) given that event \( a_t \) occurs at time \( t \). It follows that
\[
q(1, E) = \frac{1320 \times \frac{1}{2} + 1320 \times \frac{1}{2}}{1 + r_1(E)} = 1100,
\]
\[
q(1, E^c) = \frac{1208 \times \frac{1}{3} + 1188 \times \frac{2}{3}}{1 + r_1(E^c)} = 1106.1727,
\]
and
\[
q(0, \Omega) = \frac{q(1, E) \times \frac{1}{2} + q(1, E^c) \times \frac{1}{2}}{1 + r_0(\Omega)} = 1002.8,
\]
and these are the prices we have been looking for.

\textbf{Example 7.} Consider an economy that extends for three dates \( t = 0, 1, 2 \) with 4 states of nature \( (\omega_1, \omega_2, \omega_3, \omega_4) \) and two long-lived assets.
The common information structure for investors is as follows. At \( t = 0 \), investors know that the true state is an element of \( \Omega = \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \). At \( t = 1 \), investors know whether the true state is an element of \( E = \{ \omega_1, \omega_2 \} \) or an element of \( E^c = \{ \omega_3, \omega_4 \} \). At \( t = 2 \), investors know exactly which among \( \omega_1, \omega_2, \omega_3, \omega_4 \) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>asset/(date,event)</th>
<th>(0, ( \Omega ))</th>
<th>(1,E)</th>
<th>(1,E^c)</th>
<th>(2,( \omega_1 ))</th>
<th>(2,( \omega_2 ))</th>
<th>(2,( \omega_3 ))</th>
<th>(2,( \omega_4 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{4}{3} )</td>
<td>1.1</td>
<td>2.2</td>
<td>1</td>
<td>1.48</td>
<td>3.3</td>
<td>1.1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.32</td>
<td>1.32</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Consider Mr. A who is endowed with 80,000 shares of asset 1 and 41,000 shares of asset 2 at date 0. Mr. A derives satisfaction only from his date-2 contingent consumption \( \tilde{c} \). Let \( c_j = c(\omega_j) \) be his date-2 consumption in state \( \omega_j \). At date 0, Mr. A believes that the 4 states \( \omega_1, \omega_2, \omega_3, \omega_4 \) are equally likely, and he would like to maximize \( E[u(\tilde{c})] \), where \( u(x) = \log(x) \).

(i) Are markets dynamically complete? Find the date-0 price of the 4 Arrow-Debreu securities that pay one dollar at date 2 in exactly one of the 4 states \( \omega_1, \omega_2, \omega_3, \omega_4 \) and nothing in other states or at other dates.

(ii) Consider Mr. A’s date-0 problem of finding the optimal consumption plan that can be financed by some admissible trading strategy

\[
(\theta_1(1), \theta_2(1), \theta_1(2,E), \theta_2(2,E), \theta_1(2,E^c), \theta_2(2,E^2))
\]

(iii) Given the consumption plan \( (c_1, c_2, c_3, c_4) \) solved in part (ii), find one admissible trading strategy to finance \( (c_1, c_2, c_3, c_4) \).

Solution. Consider part (i). It is straightforward to verify that markets are dynamically complete. When the money market account is taken as numeraire, it is straightforward to compute the martingale probabilities \( \pi^* \) for the 4 states, which are

\[
\pi^*(\omega_1) = \frac{1}{4} \times \frac{1}{3},
\]

\[
\pi^*(\omega_2) = \frac{1}{4} \times \frac{2}{3},
\]

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\[
\pi^*(\omega_3) = \frac{3}{4} \times \frac{1}{2}, \\
\pi^*(\omega_4) = \frac{3}{4} \times \frac{1}{2}.
\]

The corresponding short-rate process can be shown to be
\[
r_{0,1} = 10\%, \quad r_{1,2}(\omega_j) = 20\%1_E(\omega_j), \quad j = 1, 2, 3, 4.
\]

For an asset that pays \(z(\omega_j)\) in state \(j\) at date 2, \(j = 1, 2, 3, 4\), its date-0 price is
\[
\sum_{j=1}^{4} \frac{\pi^*(\omega_j) z(\omega_j)}{(1 + r_{0,1})(1 + r_{1,2}(\omega_j))}.
\]

It follows that the date-0 prices of the 4 Arrow-Debreu securities are
\[
\phi(\omega_1) = \frac{25}{396}, \quad \phi(\omega_2) = \frac{50}{396}, \\
\phi(\omega_3) = \frac{15}{44}, \quad \phi(\omega_4) = \frac{15}{44}.
\]

Consider part (ii). We must solve the following maximization problem:
\[
\max_{c_1, c_2, c_3, c_4 \in \mathbb{R}^+} \frac{1}{4} \sum_{j=1}^{4} \log(c_j),
\]
subject to
\[
\sum_{j=1}^{4} c_j \phi(\omega_j) = 80,000 \times \frac{7}{4} + 41,000 = 181,000.
\]

It is easy to show that the optimal solutions are
\[
c_1 = 716,760, \quad c_2 = 358,380, \quad c_3 = c_4 = \frac{398,200}{3}.
\]
Finally, consider part (iii). A trading strategy

\((\theta_1(1), \theta_2(1), \theta_1(2, E), \theta_2(2, E), \theta_1(2, E^c), \theta_2(2, E^c))\)

that finances the consumption plan must satisfy the following system of equations:

\[
\begin{align*}
1 \times \theta_1(2, E) + 1.32 \times \theta_2(2, E) &= c_1, \\
1.48 \times \theta_1(2, E) + 1.32 \times \theta_2(2, E) &= c_2, \\
3.3 \times \theta_1(2, E^c) + 1.1 \times \theta_2(2, E^c) &= c_3, \\
1.1 \times \theta_1(2, E^c) + 1.1 \times \theta_2(2, E^c) &= c_4,
\end{align*}
\]

\(1.1 \times \theta_1(2, E) + 1.1 \times \theta_2(2, E) = 1.1 \times \theta_1(1) + 1.1 \times \theta_2(1),
\]

\(2.2 \times \theta_1(2, E^c) + 1.1 \times \theta_2(2, E^c) = 2.2 \times \theta_1(1) + 1.1 \times \theta_2(1).
\]

Solving, we have

\[
\begin{align*}
\theta_1(1) &= -\frac{724,000}{3}, \quad \theta_2(1) = \frac{1,810,000}{3}, \\
\theta_1(2, E) &= -746,625, \quad \theta_2(2, E) = 1,108,625, \\
\theta_1(2, E^c) &= 0, \quad \theta_2(2, E^c) = \frac{362,000}{3},
\end{align*}
\]

so that the solutions are exactly the same as those obtained in part (iii) of the preceding Example. This latter method has been proven to be computationally more efficient than the dynamic programming approach demonstrated in the preceding Example, when trade can occur continuously in time.

22. **Example 8.** Recall the 3-period economy discussed in Example 1, where there are 3 trading dates \((t = 0, 1, 2)\), 4 states of nature \((\omega_1, \omega_2, \omega_3, \omega_4)\), and 2 long-lived assets (assets 1 and 2). The common information structure for investors is as follows. At \(t = 0\), investors know that the true state is an element of \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\). At \(t = 1\), investors know whether the true state is an element of \(E = \{\omega_1, \omega_2\}\) or an element of \(E^c = \{\omega_3, \omega_4\}\). At \(t = 2\), investors know exactly which among \(\omega_1, \omega_2, \omega_3, \omega_4\) is the true state. The prices of the two traded long-lived assets at each time-event node are summarized in the following table:
Recall that the martingale probabilities $\pi^*$ for the 4 states are respectively
\begin{align*}
\pi^*(\omega_1) &= \frac{1}{4} \times \frac{1}{3}, \\
\pi^*(\omega_2) &= \frac{1}{4} \times \frac{2}{3}, \\
\pi^*(\omega_3) &= \frac{3}{4} \times \frac{1}{2}, \\
\pi^*(\omega_4) &= \frac{3}{4} \times \frac{1}{2}.
\end{align*}

The short-rate process is
\[ r_{0,1} = 10\%, \quad r_{1,2}(\omega_j) = 20\%1_{E}(\omega_j), \quad j = 1, 2, 3, 4. \]

We shall first discuss how to create an arbitrage strategy when the forward price deviates from its no-arbitrage level, and then consider the case where the futures price deviates from its no-arbitrage level.

Recall that $G(0)$, $G(1, E)$, $G(1, E^c)$ are respectively the date-0 and date-1 forward prices for one unit of asset 1 to be exchanged at date 2. To rule out arbitrage opportunities, it must be that
\[ G(0) = \frac{p_1(0)}{B(0)}, \quad G(1, E) = \frac{p_1(1, E)}{B(1, E)}, \quad G(1, E^c) = \frac{p_1(1, E^c)}{B(1, E^c)}, \]
with $B$ standing for the price of the zero-coupon bond maturing at date 2 with face value equal to 1 dollar. Carrying out the computations, we get
\[ G(0) = \frac{231}{115}, \quad G(1, E) = 1.32, \quad G(1, E^c) = 2.2. \]

What if the forward price actually prevailing at date 0 is greater than $\frac{231}{115}$? How can we trade to realize an arbitrage profit when this happens?
Observe that when the date-0 forward price $G(0)$ is greater than $\frac{231}{115}$, then signing a forward contract to sell one unit of asset 1 at date 2 (or, simply, selling one forward contract at date 0) is a good idea. This should not be surprising: asset 1 is overpriced in the date-0 forward transaction. Note that selling one forward contract at date 0 generates no date-0 cash flow, but it will generate a date-2 risky cash flow

$$
\begin{bmatrix}
G(0) - 1 \\
G(0) - 1.48 \\
G(0) - 3.3 \\
G(0) - 1.1
\end{bmatrix}.
$$

Observe that this date-2 cash flow may be negative, especially in state $\omega_3$. To remove the risk, and to generate an arbitrage profit, at date 0 you can first sell $G(0)$ units of bond $B(0)$ short, which generates $G(0)B(0) > p_1(0) = \frac{7}{4}$ at date 0. You can put $G(0)B(0) - \frac{7}{4}$ in your pocket (enjoy it!), and use the rest $\frac{7}{4}$ to buy 1 unit of asset 1 and hold the latter till date 2. At date 2, you will have to pay $G(0) \times 1 = G(0)$ to those who lend the bond $B(0)$ to you at date 0, and at the same time your long position in asset 1 generates the following payoff

$$
\begin{bmatrix}
1 \\
1.48 \\
3.3 \\
1.1
\end{bmatrix}.
$$

Hence together with the forward contract that you signed at date 0, your date-2 cash flow is exactly zero! However, you have already obtained an arbitrage profit $G(0)B(0) - \frac{7}{4}$ at date 0.

To sum up, when the forward price $G(0)$ is actually greater than $\frac{231}{115}$, you can do three things at date 0 to create an arbitrage opportunity: (1) selling 1 forward contract; (2) selling $G(0)$ units of the zero-coupon bond maturing at date 2; and (3) buying and holding 1 unit of asset 1 till date 2.

Now, continue with the above 3-period economy, and recall that $H(0)$, $H(1, E)$, $H(1, E^c)$ are respectively the date-0 and date-1 futures prices for one unit of asset 1 to be exchanged at date 2. Recall that to rule out
arbitrage opportunities, $H(0)$ must equal the expected value of $\tilde{H}(1)$ under the martingale probabilities. That is,

$$H(0) = \frac{1}{4} \times H(1, E) + \frac{3}{4} \times H(1, E^c) = \frac{1}{4} \times G(1, E) + \frac{3}{4} \times G(1, E^c) = 1.98.$$ 

How can we construct an arbitrage strategy when the actual $H(0)$ differs from 1.98, say when $H(0) = 2 > 1.98$?

Note that the entire no-arbitrage pricing theory is based on the belief that except for the current deviation of the price from its no-arbitrage level, the price will stay at its no-arbitrage level from the next trading date on. Following this belief, we assume that $H(1, E) = 1.32$ and $H(1, E^c) = 2.2$, even though $H(0) = 2 > 1.98$.

Define $(\theta_1, \theta_2)$ as the solution to the following system of equations:

$$\begin{bmatrix} 1.1 & 1.1 \\ 2.2 & 1.1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} H(1, E) - H(0) \\ H(1, E^c) - H(0) \end{bmatrix}.$$ 

Note that $(\theta_1, \theta_2)$ is the portfolio of assets 1 and 2 that replicates the date-1 payoff generated by the strategy of buying one futures contract at date 0. Solving, we have

$$\theta_1 = \frac{4}{5}, \quad \theta_2 = -\frac{78}{55}.$$ 

The date-0 cost of this portfolio is

$$\frac{7}{4} \times \frac{4}{5} - 1 \times \frac{78}{55} = -\frac{1}{55}.$$ 

Consider the following trading strategy: (1) Selling one futures contract at date 0, and buying 1 futures contract at date 1; (2) selling $\frac{78}{55}$ units

\textsuperscript{19}Undoubtedly, one should wonder why the current price has deviated from its no-arbitrage level in the first place. If this has happened right now, why should we have faith that the same will not occur in the future? This is one major logical flaw of the no-arbitrage pricing theory; see Barberis and Thaler, 2002. A Survey of Behavioral Finance, in Handbook of the Economics of Finance, edited by G. Constantinides, Milton Harris, and Rene Stulz, Amsterdam: Elsevier.
of asset 2 short and buying \( \frac{4}{5} \) units of asset 1 at date 0; and (3) selling the \( \frac{4}{5} \) units of asset 1 and buying and then returning the \( \frac{28}{35} \) units of asset 2 that somebody loaned you at date 0. Following this strategy, you will have a net date-0 cash inflow, which is equal to \( \frac{1}{55} \) (enjoy it!), and your date-1 and date-2 cash flows are zero for sure. Thus we have an arbitrage opportunity.

More precisely, the date-1 payoff generated by the above strategy results from (1) and (3), where the date-1 payoff generated by (1) is

\[
\begin{bmatrix}
H(0) - H(1, E) \\
H(0) - H(1, E^c)
\end{bmatrix},
\]

and the date-1 payoff generated by (3) is

\[
\begin{bmatrix}
H(1, E) - H(0) \\
H(1, E^c) - H(0)
\end{bmatrix};
\]

whereas the date-0 payoff generated by the above strategy results from (2) only, which is \( \frac{1}{55} \). What about the date-2 payoff generated by the above strategy? Note that you will have to sell 1 futures contract at date 1, according to (1), and hence there will be no further payoffs generated by futures trading after date 1. Similarly, the strategy asks you to clear your positions in assets 1 and 2 at date 1, according to (3), so that after date 1 there will be no further payoffs generated by taking positions in assets 1 and 2 either.

References


\[20\] Recall that when you buy one futures contract at date 0, you will get \( \hat{H}(1) - H(0) \) at date 1 and \( \hat{H}(2) - \hat{H}(1) \) at date 2. To see that the date-1 payoff for the buyer is \( \hat{H}(1) - H(0) \) instead of \( H(0) - \hat{H}(1) \), note that when \( \hat{H}(1) - H(0) > 0 \), then being able to commit to a low price \( H(0) \) at date 0 is a gain to the buyer and a loss to the seller.


