1. **Example 1.** Consider an economy that extends for three dates \((t = 0, 1, 2)\). There are two consumers A and B, each endowed with one unit of asset X and one unit of asset Y. There are 4 possible states of nature, \(\omega_1, \omega_2, \omega_3, \omega_4\). The two consumers’ common information structure is as follows. At time 0, they know that the true state is among \(\omega_1, \omega_2, \omega_3, \omega_4\). At time 1, they know whether the event \(E \equiv \{\omega_1, \omega_2\}\) has or has not occurred. At time 2, they learn the true state. The assets pay dividends only at time 2. The two investors’ subjective probability measures and the dividends per unit (dpu, denominated in a single consumption good) for the two assets are summarized in the following table.

<table>
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<tr>
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<th>(\omega_2)</th>
<th>(\omega_3)</th>
<th>(\omega_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A’s prob.</td>
<td>(\frac{1}{10})</td>
<td>(\frac{2}{10})</td>
<td>(\frac{3}{10})</td>
<td>(\frac{4}{10})</td>
</tr>
<tr>
<td>B’s prob.</td>
<td>(\frac{4}{10})</td>
<td>(\frac{2}{10})</td>
<td>(\frac{3}{10})</td>
<td>(\frac{1}{10})</td>
</tr>
<tr>
<td>X’s dpu</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Y’s dpu</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

A and B only want to consume at time 2. They have a common von Neumann-Morgenstern utility function for time-2 consumption: \(u(c) = \log(c)\).

(i) Is there a Radner equilibrium for this economy? If you think that there is one, find (1) the equilibrium price processes for assets X and Y; and (2) A’s equilibrium trading strategy.

(ii) Suppose that we add one more agent C into the economy. Suppose that C has the same beliefs and preferences as A, but unlike A, C is endowed with no assets. Rather, C is endowed with \(e(t, a_t) \geq 0\) units of the consumption good at date \(t\) when event \(a_t\) occurs. You are told that in the equilibrium of this three-agent economy, the processes of prices of X and Y are the same as in part (i). Find one non-zero positive stochastic process \(e(t, a_t)\) that is consistent with this information.
Solution. First consider part (i). It is clear that the markets are complete at time 1 whether the event $E$ occurs or not. Let us conjecture that the markets are dynamically complete, and then solve the equivalent Arrow-Debreu equilibrium. We shall verify our conjecture afterwards.

Thus, imagine an equivalent economy where at time 0 there is a complete set of traded Arrow-Debreu securities. Let $a_i$ and $b_i$ be respectively A’s and B’s holdings of the Arrow-Debreu security paying one unit of consumption if and only if the time-2 state is $\omega_i$. Let $p_i$ be the time-0 price of the Arrow-Debreu security corresponding to the time-2 state $\omega_i$. Let us choose the normalization

$$\sum_{j=1}^{4} p_j = 1;$$

that is, the time-2 consumption good is taken as numeraire. A seeks to

$$\max_{a_i \in \mathbb{R}; i=1,2,3,4} \sum_{i=1}^{4} \frac{i}{10} \log(a_i)$$

subject to

$$\sum_{i=1}^{4} a_i p_i \leq \sum_{j=1}^{4} 2(1+j)p_j.$$

Similarly, B seeks to

$$\max_{b_i \in \mathbb{R}; i=1,2,3,4} \sum_{i=1}^{4} \frac{5-i}{10} \log(b_i)$$

subject to

$$\sum_{i=1}^{4} b_i p_i \leq \sum_{j=1}^{4} 2(1+j)p_j.$$

Solving the two maximization problems given $p_1, p_2, p_3, p_4$, and then solving for $p_1, p_2, p_3, p_4$ using the markets clearing condition $a_j + b_j = 4(1+j), j = 1, 2, 3, 4$, we have

$$a_1 = \frac{8}{5}, \ a_2 = \frac{24}{5}, \ a_3 = \frac{48}{5}, \ a_4 = 16.$$
\[ b_1 = \frac{32}{5}, \quad b_2 = \frac{36}{5}, \quad b_3 = \frac{32}{5}, \quad b_4 = 4, \]

and
\[ p_1 = \frac{30}{77}, \quad p_2 = \frac{20}{77}, \quad p_3 = \frac{15}{77}, \quad p_4 = \frac{12}{77}. \]

Next, imagine that markets reopen at time 1 in event \( E \). Let \( q_1 \) and \( q_2 \) be the prices of the Arrow-Debreu securities corresponding to states \( \omega_1 \) and \( \omega_2 \) respectively. A’s problem is to
\[
\max_{A_1, A_2 \in \mathbb{R}} \frac{1}{10} \log(A_1) + \frac{2}{10} \log(A_2)
\]
subject to
\[ A_1 q_1 + A_2 q_2 \leq a_1 q_1 + a_2 q_2, \]
where \( A_1 \) and \( A_2 \) are the date-2 consumption for A in respectively states \( \omega_1 \) and \( \omega_2 \). Recall that if there is a complete set of traded Arrow-Debreu securities at date 0, and if traders have rational expectations, then \( A_i = a_i \), for \( i = 1, 2 \). This implies that (using \( q_1 + q_2 = 1 \)),
\[ q_1 = \frac{3}{5}, \quad q_2 = \frac{2}{5}. \]

Similarly, imagine that markets reopen at time 1 in event \( E^c \) and let \( q_3 \) and \( q_4 \) be the prices of the Arrow-Debreu securities corresponding to states \( \omega_3 \) and \( \omega_4 \) respectively. A’s problem is to
\[
\max_{A_3, A_4 \in \mathbb{R}} \frac{3}{10} \log(A_3) + \frac{4}{10} \log(A_4)
\]
subject to
\[ A_3 q_3 + A_4 q_4 \leq a_3 q_3 + a_4 q_4, \]
where \( A_3 \) and \( A_4 \) are the date-2 consumption for A in respectively states \( \omega_3 \) and \( \omega_4 \). After imposing \( A_j = a_j \), \( j = 3, 4 \), we have (using \( q_3 + q_4 = 1 \)),
\[ q_3 = \frac{5}{9}, \quad q_4 = \frac{4}{9}. \]
Now, we consider the original Radner equilibrium. Let $P_x(t, a_t)$ and $P_y(t, a_t)$ be the equilibrium prices of assets X and Y at time $t$ in event $a_t$. Now we have

\[ P_x(2, \omega_1) = 1, \ P_x(2, \omega_2) = 2, \ P_x(2, \omega_3) = 3, \ P_x(2, \omega_4) = 4, \]

\[ P_x(1, E) = P_x(2, \omega_1) \times q_1 + P_x(2, \omega_2) \times q_2 = \frac{7}{5}, \]

\[ P_x(1, E^c) = P_x(2, \omega_3) \times q_3 + P_x(2, \omega_4) \times q_4 = \frac{31}{9}, \]

and

\[ P_x(0, \Omega) = \frac{163}{77}. \]

Similarly, we have

\[ P_y(2, \omega_1) = 3, \ P_y(2, \omega_2) = 4, \ P_y(2, \omega_3) = 5, \ P_y(2, \omega_4) = 6, \]

\[ P_y(1, E) = P_y(2, \omega_1) \times q_1 + P_y(2, \omega_2) \times q_2 = \frac{17}{5}, \]

\[ P_y(1, E^c) = P_y(2, \omega_3) \times q_3 + P_y(2, \omega_4) \times q_4 = \frac{49}{9}, \]

and

\[ P_y(0, \Omega) = \frac{317}{77}. \]

Note that the two vectors

\[
\begin{bmatrix}
P_x(1, E) \\
P_x(1, E^c)
\end{bmatrix}, \quad \begin{bmatrix}
P_y(1, E) \\
P_y(1, E^c)
\end{bmatrix}
\]

are linearly independent, and hence in the Radner equilibrium the markets are dynamically complete as we conjectured.

Next, we consider A’s trading strategy in the Radner equilibrium. This is simply the trading strategy that finances the date-2 consumption vector

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix} = \begin{bmatrix}
\frac{8}{5} \\
\frac{24}{5} \\
\frac{48}{5} \\
16
\end{bmatrix}.
\]
Let \( \theta_x(t+1, a_t) \) and \( \theta_y(t+1, a_t) \) be respectively investor A’s equilibrium holdings of assets X and Y carried into time \( t + 1 \) when event \( a_t \) occurs at time \( t \). We must have
\[
\begin{bmatrix}
1 & 3 \\
2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
\theta_x(2, E) \\
\theta_y(2, E) \\
\end{bmatrix}
=
\begin{bmatrix}
\frac{8}{5} \\
\frac{24}{5} \\
\end{bmatrix},
\]
and hence we have
\[
\begin{bmatrix}
\theta_x(2, E) \\
\theta_y(2, E) \\
\end{bmatrix}
=
\begin{bmatrix}
4 \\
-\frac{4}{5} \\
\end{bmatrix}.
\]
Similarly, we have
\[
\begin{bmatrix}
3 & 5 \\
4 & 6 \\
\end{bmatrix}
\begin{bmatrix}
\theta_x(2, E^c) \\
\theta_y(2, E^c) \\
\end{bmatrix}
=
\begin{bmatrix}
\frac{48}{5} \\
16 \\
\end{bmatrix},
\]
and hence we have
\[
\begin{bmatrix}
\theta_x(2, E^c) \\
\theta_y(2, E^c) \\
\end{bmatrix}
=
\begin{bmatrix}
\frac{56}{5} \\
-\frac{24}{5} \\
\end{bmatrix}.
\]
It follows that the value of A’s trading strategy at time 1 is
\[
P_x(1, E) \times \theta_x(2, E) + P_y(1, E) \times \theta_y(2, E) = \frac{72}{25}
\]
if event \( E \) occurs, and it is
\[
P_x(1, E^c) \times \theta_x(2, E^c) + P_y(1, E^c) \times \theta_y(2, E^c) = \frac{112}{9}
\]
if event \( E \) does not occur.
Now, we must have
\[
\begin{bmatrix}
P_x(1, E) & P_y(1, E) \\
P_x(1, E^c) & P_y(1, E^c) \\
\end{bmatrix}
\begin{bmatrix}
\theta_x(1) \\
\theta_y(1) \\
\end{bmatrix}
=
\begin{bmatrix}
\frac{72}{25} \\
\frac{112}{9} \\
\end{bmatrix},
\]
and hence we have
\[
\begin{bmatrix}
\theta_x(1) \\
\theta_y(1)
\end{bmatrix} = \begin{bmatrix}
\frac{749}{115} \\
-\frac{211}{115}
\end{bmatrix}.
\]

The value of A’s trading strategy at time 0 can easily be verified to equal A’s date-0 wealth
\[
P_x(0, \Omega) + P_y(0, \Omega).
\]

Finally, consider part (ii). Apparently, one possible solution is that
\[
e(0, \Omega) = e(1, E) = e(1, E^c) = 0,
\]
and
\[
\begin{bmatrix}
e(2, \omega_1) \\
e(2, \omega_2) \\
e(2, \omega_3) \\
e(2, \omega_4)
\end{bmatrix} = \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix} = \begin{bmatrix}
\frac{8}{5} \\
\frac{24}{5} \\
\frac{48}{5} \\
16
\end{bmatrix}.
\]

**Example 2.** Consider an economy that extends for three dates \((t = 0, 1, 2)\). There is one single *perishable* consumption good, which cannot be stored at any time. There are two consumers A and B, each endowed with one unit of stock X and 480 units of the consumption good at time 0. There are 4 possible states of nature, \(\omega_1, \omega_2, \omega_3, \omega_4\). The two consumers’ common information structure is as follows. At time 0, they know that the true state is among \(\omega_1, \omega_2, \omega_3, \omega_4\). At time 1, they know whether the event \(E \equiv \{\omega_1, \omega_2\}\) has or has not occurred. At time 2, they both learn the true state. The stock X pays a one-time liquidation dividend at time 2. The two investors’ subjective probability measures and the dividends per unit (dpu, denominated in the single consumption good) for stock X are summarized in the following table.
In addition to stock X, A and B are free to issue or purchase one particular European call option written on one unit of X which has exercise price \( K \) and will expire at time 2. For all \( t = 0, 1, 2 \), A and B have a common von Neumann-Morgenstern utility function \( u_t(\cdot) \) for time-\( t \) consumption: \( u_0(z) = u_2(z) = \log(z) \), and \( u_1(z) \equiv 0 \). Assume no discounting, so that at time 0 agent \( i \) seeks to maximize

\[
u_0(c_{i0}^t) + E_i[u_1(c_{i1}^t) + u_2(c_{i2}^t)|F_0],
\]

where \( c_{i}^t \) is agent \( i \)'s time-\( t \) consumption, and \( E^i[\cdot|F_0] \) is agent \( i \)'s expectation operator (using his subjective probabilities) based the common time-0 information \( F_0 \).

(i) Suppose that \( K \in (0, 4) \). Show that there is a Radner equilibrium and find the equilibrium price process for the above call option, or show that no Radner equilibrium can exist.

**Solution.** Consider part (i). For \( K \in (0, 4) \), the call option will always be exercised at time 2, which together with stock X produces a riskless asset for the economy. Hence we conjecture that markets are dynamically complete with the stock and the option. Thus, imagine an equivalent economy where at time 0 there is a complete set of traded Arrow-Debreu securities. Let \( a_i \) and \( b_i \) be respectively A’s and B’s holdings of the Arrow-Debreu security paying one unit of consumption if and only if the time-2 state is \( \omega_i \). Let \( p_i \) be the time-0 price of the Arrow-Debreu security corresponding to the time-2 state \( \omega_i \). Let us choose the normalization

\[
\sum_{j=1}^{4} p_j = 1;
\]

that is, the time-2 consumption good is taken as numeraire. Let \( p_0 \) be the time-0 price of spot consumption. A seeks to

\[
\max_{c_{i0}^t, a_i \in \mathbb{R}; i=1,2,3,4} \log(c_{i0}^t) + \sum_{i=1}^{4} \frac{i}{10} \log(a_i)
\]
subject to
\[ p_0 c_A^0 + \sum_{i=1}^{4} a_i p_i \leq \sum_{j=1}^{4} 2(1 + j)p_j + 480p_0. \]

Similarly, B seeks to
\[
\max_{c_B^0, b_i \in \mathbb{R}; i = 1, 2, 3, 4} \log(c_B^0) + \sum_{i=1}^{4} \frac{5 - i}{10} \log(b_i)
\]
subject to
\[ p_0 c_B^0 + \sum_{i=1}^{4} b_i p_i \leq \sum_{j=1}^{4} 2(1 + j)p_j + 480p_0. \]

Note that it would be incorrect to claim that \( c_A^0 = c_B^0 = 480 \) because the consumption good is perishable.

Solving the above two maximization problems given \( p_0, p_1, p_2, p_3, p_4, \) and then solving for \( p_0, p_1, p_2, p_3, p_4 \) using the markets clearing conditions
\[ c_A^0 + c_B^0 = 960, \quad a_j + b_j = 4(1 + j), \quad \forall j = 1, 2, 3, 4, \]
we have
\[ p_0 = 1, \]
\[ c_A^0 = c_B^0 = 480, \]
\[ a_1 = \frac{8}{5}, \quad a_2 = \frac{24}{5}, \quad a_3 = \frac{48}{5}, \quad a_4 = 16, \]
\[ b_1 = \frac{32}{5}, \quad b_2 = \frac{36}{5}, \quad b_3 = \frac{32}{5}, \quad b_4 = 4, \]
and
\[ p_1 = \frac{30}{11}, \quad p_2 = \frac{20}{11}, \quad p_3 = \frac{15}{11}, \quad p_4 = \frac{12}{11}. \]

Thus for both agents, the time-0 consumption equals the time-0 endowment. This implies that the equilibrium asset allocation and asset prices will not change if we assume instead that the agents do not have time-0 endowments. Note that the latter economy was studied in Example 1 above, where in that dynamically complete economy the two agents A and B are endowed with time-2 state-dependent endowments, which are exactly the same as in the current problem (stock X in the
current problem is equivalent to the portfolio consisting of one unit of each traded asset in the economy studied in Example 1. Hence the equilibrium price processes of the 4 Arrow-Debreu securities in the current equilibrium are the same as those obtained in Example 1. This implies that at all times the price of stock X in the Radner equilibrium of the current economy is equal to the sum of the prices of the two traded assets in the Radner equilibrium of the economy studied in Example 1.\(^1\) Hence the price process of stock X is

\[
P_x(2, \omega_1) = 4, \quad P_x(2, \omega_2) = 6, \quad P_x(2, \omega_3) = 8, \quad P_x(2, \omega_4) = 10,
\]

\[
P_x(1, E) = \frac{24}{5},
\]

\[
P_x(1, E^c) = \frac{80}{9},
\]

and

\[
P_x(0, \Omega) = \frac{480}{77}.
\]

Since the call option is simply a portfolio consisting of one unit of stock X and shorting \(K\) units of the pure discount bond maturing at time 2 with face value equal to 1 (unit of the consumption good), the price process of the option is simply

\[
P_c(2, \omega_1) = 4 - K, \quad P_c(2, \omega_2) = 6 - K, \quad P_c(2, \omega_3) = 8 - K, \quad P_c(2, \omega_4) = 10 - K,
\]

\[
P_c(1, E) = \frac{24}{5} - K,
\]

\[
P_c(1, E^c) = \frac{80}{9} - K,
\]

and

\[
P_c(0, \Omega) = \frac{480}{77} - K.
\]

\(^1\)This is no accident. The endowment 480 was chosen deliberately to make sure that those who have understood Example 1 can spare the need of going through all the computations again. Of course, if the endowed time-0 consumption is not 480 for each agent, then a Radner equilibrium can still exist in general, but we have to go through the computations all over again.
Note that with these prices, markets are indeed dynamically complete, verifying our conjecture in the first place. This concludes part (i).

Example 3. Consider an economy that extends for three dates \( t = 0, 1, 2 \). There are two consumers A and B, each endowed with one unit of asset X and one unit of asset Y. There are 4 possible states of nature, \( \omega_1, \omega_2, \omega_3, \omega_4 \). The two consumers’ common information structure is as follows. At time 0, they know that the true state is among \( \omega_1, \omega_2, \omega_3, \omega_4 \). At time 1, they know whether the event \( E \equiv \{ \omega_1, \omega_2 \} \) has or has not occurred. At time 2, they learn the true state. The assets pay dividends only at time 2. The two investors’ subjective probability measures and the dividends per unit (dpu, denominated in a single consumption good) for the two assets are summarized in the following table.

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<td>( \frac{3}{10} )</td>
<td>( \frac{4}{10} )</td>
</tr>
<tr>
<td>B’s prob.</td>
<td>( \frac{1}{10} )</td>
<td>( \frac{3}{10} )</td>
<td>( \frac{2}{10} )</td>
<td>( \frac{1}{10} )</td>
</tr>
<tr>
<td>X’s dpu</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Y’s dpu</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

A and B only want to consume at time 2. They have a common von Neumann-Morgenstern utility function for time-2 consumption: \( u(c) = \log(c) \). Is there a Radner equilibrium for this economy? If you think that there is one, find it.

Solution. There is no Radner equilibrium for this economy. To show non-existence, first assume that a Radner equilibrium exists with dynamically complete markets. Take the consumption good as the numeraire, so that

\[
P_x(1, E) + P_y(1, E) = P_x(1, E^c) + P_y(1, E^c) = P_x(0, \Omega) + P_y(0, \Omega) = 3.
\]

Now the date-1 price matrix

\[
P = \begin{bmatrix}
P_x(1, E) & P_y(1, E) \\
P_x(1, E^c) & P_y(1, E^c)
\end{bmatrix}
\]

is either non-singular or singular. First assume that \( P \) is non-singular, so that in the Radner equilibrium markets are dynamically complete.
Following the same procedure as in Problem 1, we can solve for the two agents’ equilibrium consumption plans

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
\end{bmatrix} = \begin{bmatrix}
\frac{6}{5} \\
\frac{12}{5} \\
\frac{18}{5} \\
\frac{24}{5} \\
\end{bmatrix}, \\
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix} = \begin{bmatrix}
\frac{24}{5} \\
\frac{18}{5} \\
\frac{12}{5} \\
\frac{6}{5} \\
\end{bmatrix},
\]

and the asset prices

\[P_x(1, E) = P_y(1, E) = P_x(1, E^c) = P_y(1, E^c) = P_x(0, \Omega) = P_y(0, \Omega) = \frac{3}{2},\]

apparently a contradiction to the assumption that \( P \) is non-singular. Thus we are left with the possibility that \( P \) is singular whenever a Radner equilibrium exists. For some scalar \( p \), we have

\[P = \begin{bmatrix} p & 3 - p \\ 3 - p & p \end{bmatrix},\]

which implies that both asset X and asset Y are riskless during the date-0-date-1 period. Thus regardless of whether event \( E \) occurs, A’s date-1 wealth is 3.

Note that at date 1 markets become complete, whether or not event \( E \) has occurred. Let \( q_1 \) and \( q_2 \) be the date-1 prices of the Arrow-Debreu securities corresponding to states \( \omega_1 \) and \( \omega_2 \) respectively when event \( E \) is known to have occurred. Let \( q_3 \) and \( q_4 \) be the date-1 prices of the Arrow-Debreu securities corresponding to states \( \omega_3 \) and \( \omega_4 \) respectively when event \( E^c \) is known to have occurred. Then \( a_1 \) and \( a_2 \) must solve the maximization problem

\[(M1) \quad \max_{A_1, A_2} \frac{1}{3} \log(A_1) + \frac{2}{3} \log(A_2)\]

subject to

\[q_1 A_1 + q_2 A_2 = 3;\]
\[ b_1 \text{ and } b_2 \text{ must solve the maximization problem} \]

\[ \text{(M2)} \quad \max_{B_1, B_2} \frac{4}{7} \log(B_1) + \frac{3}{7} \log(B_2) \]

subject to \[ q_1 B_1 + q_2 B_2 = 3; \]

\[ a_3 \text{ and } a_4 \text{ must solve the maximization problem} \]

\[ \text{(M3)} \quad \max_{A_3, A_4} \frac{3}{7} \log(A_3) + \frac{4}{7} \log(A_4) \]

subject to \[ q_3 A_3 + q_4 A_4 = 3; \]

and \[ b_3 \text{ and } b_4 \text{ must solve the maximization problem} \]

\[ \text{(M4)} \quad \max_{B_3, B_4} \frac{2}{3} \log(B_3) + \frac{1}{3} \log(B_4) \]

subject to \[ q_3 B_3 + q_4 B_4 = 3. \]

Comparing the first two maximization problems (M1) and (M2) to the rest two maximization problems (M3) and (M4), we conclude that

\[ q_1 = q_4, \quad q_2 = q_3, \]

which then implies, by \( P_x(1, E) = P_x(1, E^c) = p, \) that

\[ 3q_1 = P_x(1, E) = P_x(1, E^c) = q_3 + 2q_4 = q_2 + 2q_1 \Rightarrow q_1 = q_2 \Rightarrow q_1 = q_2 = q_3 = q_4. \]

However, by solving (M1) and (M2) directly, we have

\[ a_1 = \frac{1}{q_1}, \quad a_2 = \frac{2}{q_2}, \quad b_1 = \frac{12}{7q_1}, \quad b_2 = \frac{9}{7q_2}, \]

so that by the markets clearing conditions,

\[ (1 + \frac{12}{7}) \frac{1}{q_1} = 3 = (2 + \frac{9}{7}) \frac{1}{q_2} \Rightarrow q_1 \neq q_2, \]
a contradiction! Hence we conclude that $P$ cannot be singular either, and no Radner equilibrium can exist for this economy.

**Example 4.** Consider an economy that extends for three dates ($t = 0, 1, 2$). There are two consumers A and B, each endowed with one unit of asset X and one unit of asset Y. There are 4 possible states of nature, $\omega_1, \omega_2, \omega_3, \omega_4$. The two consumers’ common information structure is as follows. At time 0, they know that the true state is among $\omega_1, \omega_2, \omega_3, \omega_4$. At time 1, they know whether the event $E \equiv \{\omega_1, \omega_2\}$ has or has not occurred. At time 2, they learn the true state. The assets pay dividends only at time 2. The two investors’ subjective probability measures and the dividends per unit (dpu, denominated in a single consumption good) for the two assets are summarized in the following table.

<table>
<thead>
<tr>
<th>prob.’s; dpu’s/states</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A’s prob.</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>B’s prob.</td>
<td>$\frac{3}{5}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>X’s dpu</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Y’s dpu</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

A and B only want to consume at time 2. They have a common von Neumann-Morgenstern utility function for time-2 consumption: $u(c) = \log(c)$.

(i) Find the price processes for assets X and Y in a Radner equilibrium.

(ii) Find B’s equilibrium trading strategy.

**Solution.** First consider part (i). It is clear that the markets are complete at time 1 whether the event $E$ occurs or not. Let us conjecture that the markets are dynamically complete, and then solve the equivalent Arrow-Debreu equilibrium. We shall verify our conjecture afterwards.

Thus, imagine an equivalent economy where at time 0 there is a complete set of traded Arrow-Debreu securities. Let $a_t$ and $b_t$ be respectively A’s and B’s holdings of the Arrow-Debreu security paying one unit of consumption if and only if the time-2 state is $\omega_i$. Let $p_t$ be the time-0 price of the Arrow-Debreu security corresponding to the time-2
state $\omega_i$. Let $p_E$ and $p_{E^c}$ be the time-0 price of the Arrow-Debreu security that pays one unit of consumption at time 1 when respectively event E and event $E^c$ occur. A seeks to

$$\max_{a_i \in \mathbb{R}, i = 1, 2, 3, 4} \sum_{i=1}^{4} \frac{1}{4} \log(a_i)$$

subject to

$$\sum_{i=1}^{4} a_i p_i \leq 5 \sum_{j=1}^{4} p_j.$$ 

Solving, we have

$$a_i = \frac{5 \sum_{j=1}^{4} p_j}{4p_i}.$$ 

Similarly, B seeks to

$$\max_{b_i \in \mathbb{R}, i = 1, 2, 3, 4} \frac{1}{2} \log(b_1) + \frac{1}{6} \sum_{i=2}^{4} \log(b_i)$$

subject to

$$\sum_{i=1}^{4} b_i p_i \leq 5 \sum_{j=1}^{4} p_j.$$ 

Solving, we have

$$b_1 = \frac{5 \sum_{j=1}^{4} p_j}{2p_1}, \quad b_i = \frac{5 \sum_{j=1}^{4} p_j}{6p_i}, \quad \forall i = 2, 3, 4.$$ 

Let us choose the normalization

$$\sum_{j=1}^{4} p_j = 1;$$

that is, the time-2 consumption good is taken as numéraire. The markets clearing condition now implies that

$$p_1 = \frac{9}{24}, \quad p_2 = p_3 = p_4 = \frac{5}{24}.$$
It follows that the equilibrium consumption bundles for A and B are respectively
\[
\begin{bmatrix}
a_1 \\ a_2 \\ a_3 \\ a_4 
\end{bmatrix} = \begin{bmatrix}
\frac{10}{3} \\ 6 \\ 6 \\ 6 
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 
\end{bmatrix} = \begin{bmatrix}
\frac{20}{3} \\ 4 \\ 4 \\ 4 
\end{bmatrix}.
\]

Next, imagine that markets reopen at time 1 in event $E$. Let $q_1$ and $q_2$ be the prices of the Arrow-Debreu securities corresponding to states $\omega_1$ and $\omega_2$ respectively. A’s problem is to

\[
\max_{A_1, A_2 \in \mathbb{R}} \frac{1}{2} \sum_{i=1}^{2} \log(A_i)
\]
subject to

\[A_1 q_1 + A_2 q_2 \leq \frac{10}{3} q_1 + 6 q_2,\]

so that we have

\[A_1 = \frac{\frac{10}{3} q_1 + 6 q_2}{2q_1}, \quad A_2 = \frac{\frac{10}{3} q_1 + 6 q_2}{2q_2}.\]

For $A_1 = \frac{10}{3}$ and $A_2 = 6$, we have (normalizing $q_1 + q_2 = 1$),

\[q_1 = \frac{9}{14}, q_2 = \frac{5}{14}.\]

Similarly, imagine that markets reopen at time 1 in event $E^c$ and let $q_3$ and $q_4$ be the prices of the Arrow-Debreu securities corresponding to states $\omega_3$ and $\omega_4$ respectively. A’s problem is to

\[
\max_{A_3, A_4 \in \mathbb{R}} \frac{1}{2} \sum_{i=1}^{2} \log(A_i)
\]
subject to

\[ A_3 q_3 + A_4 q_4 \leq 6q_3 + 6q_4, \]

so that we have

\[ A_3 = \frac{6q_3 + 6q_4}{2q_3}, \quad A_4 = \frac{6q_3 + 6q_4}{2q_4}. \]

For \( A_3 = A_4 = 6 \), we have (normalizing \( q_3 + q_4 = 1 \)),

\[ q_3 = \frac{1}{2}, \quad q_4 = \frac{1}{2}. \]

Now, we consider the original Radner equilibrium. Let \( P_x(t, a_t) \) and \( P_y(t, a_t) \) be the equilibrium prices of assets \( X \) and \( Y \) at time \( t \) in event \( a_t \). Now we have

\[
\begin{align*}
P_x(2, \omega_1) &= 1, \\
P_x(2, \omega_2) &= 2, \\
P_x(2, \omega_3) &= 3, \\
P_x(2, \omega_4) &= 4, \\
P_x(1, E) &= P_x(2, \omega_1) \times q_1 + P_x(2, \omega_2) \times q_2 = \frac{19}{14}, \\
P_x(1, E^c) &= P_x(2, \omega_3) \times q_3 + P_x(2, \omega_4) \times q_4 = \frac{7}{2},
\end{align*}
\]

and

\[ P_x(0, \Omega) = \frac{54}{24}. \]

Similarly, we have

\[
\begin{align*}
P_y(2, \omega_1) &= 4, \\
P_y(2, \omega_2) &= 3, \\
P_y(2, \omega_3) &= 2, \\
P_y(2, \omega_4) &= 1, \\
P_y(1, E) &= P_y(2, \omega_1) \times q_1 + P_y(2, \omega_2) \times q_2 = \frac{51}{14}, \\
P_y(1, E^c) &= P_y(2, \omega_3) \times q_3 + P_y(2, \omega_4) \times q_4 = \frac{3}{2},
\end{align*}
\]

and

\[ P_y(0, \Omega) = \frac{66}{24}. \]

From here, we deduce \( p_E = \frac{14}{24} \) and \( p_{E^c} = \frac{10}{24} \).
Note that the two vectors
\[
\begin{bmatrix}
P_x(1, E) \\
P_x(1, E^c)
\end{bmatrix}, \quad \begin{bmatrix}
P_y(1, E) \\
P_y(1, E^c)
\end{bmatrix}
\]
are linearly independent, and hence in the Radner equilibrium the markets are dynamically complete as we conjectured.

Next, we consider B’s trading strategy in the Radner equilibrium. This is simply the trading strategy that finances the date-2 consumption vector
\[
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix} = \begin{bmatrix}
20/3 \\
4 \\
4 \\
4
\end{bmatrix}.
\]

Let \(\theta_x(t+1, a_t)\) and \(\theta_y(t+1, a_t)\) be respectively B’s equilibrium holdings of assets X and Y carried into time \(t+1\) when event \(a_t\) occurs at time \(t\). We must have
\[
\begin{bmatrix}
1 & 4 \\
2 & 3
\end{bmatrix} \begin{bmatrix}
\theta_x(2, E) \\
\theta_y(2, E)
\end{bmatrix} = \begin{bmatrix}
20/3 \\
4
\end{bmatrix},
\]
and hence we have
\[
\begin{bmatrix}
\theta_x(2, E) \\
\theta_y(2, E)
\end{bmatrix} = \begin{bmatrix}
-4/5 \\
28/15
\end{bmatrix}.
\]

Similarly, we have
\[
\begin{bmatrix}
3 & 2 \\
4 & 1
\end{bmatrix} \begin{bmatrix}
\theta_x(2, E^c) \\
\theta_y(2, E^c)
\end{bmatrix} = \begin{bmatrix}
4 \\
4
\end{bmatrix},
\]
and hence we have
\[
\begin{bmatrix}
\theta_x(2, E^c) \\
\theta_y(2, E^c)
\end{bmatrix} = \begin{bmatrix}
4/5 \\
4/5
\end{bmatrix}.
\]
It follows that the value of B’s trading strategy at time 1 is

\[ P_x(1, E) \times \theta_x(2, E) + P_y(1, E) \times \theta_y(2, E) = \frac{19}{14} \times \left(\frac{-4}{5}\right) + \frac{51}{14} \times \frac{28}{15} = \frac{40}{7} \]

if event \( E \) occurs, and it is

\[ P_x(1, E^c) \times \theta_x(2, E^c) + P_y(1, E^c) \times \theta_y(2, E^c) = \frac{7}{2} \times \frac{4}{5} + \frac{3}{2} \times \frac{4}{5} = 4 \]

if event \( E \) does not occur.

Now, we must have

\[
\begin{bmatrix}
\frac{19}{14} & \frac{51}{14} \\
\frac{7}{2} & \frac{3}{2}
\end{bmatrix}
\begin{bmatrix}
\theta_x(1) \\
\theta_y(1)
\end{bmatrix} =
\begin{bmatrix}
\frac{40}{7} \\
4
\end{bmatrix},
\]

and hence we have

\[
\begin{bmatrix}
\theta_x(1) \\
\theta_y(1)
\end{bmatrix} =
\begin{bmatrix}
\frac{28}{50} \\
\frac{68}{50}
\end{bmatrix}.
\]

The value of B’s trading strategy at time 0 is thus

\[ \frac{54}{24} \times x(1) + \frac{66}{24} \times y(1) = 5.\]

**Example 5.** Consider an economy that extends for three dates \( (t = 0, 1, 2) \) with 4 states of nature \( (\omega_1, \omega_2, \omega_3, \omega_4) \) and two long-lived assets. The common information structure for investors is as follows. At \( t = 0 \), investors know that the true state is an element of \( \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} \). At \( t = 1 \), investors know whether the true state is an element of \( E = \{\omega_1, \omega_2\} \) or an element of \( E^c = \{\omega_3, \omega_4\} \). At \( t = 2 \), investors know exactly which among \( \omega_1, \omega_2, \omega_3, \omega_4 \) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:
Mr. B is endowed with 1 million dollars at date 2. He only wants to consume in event E at date 1. He can only trade the above two assets.

(i) How much can he consume in event E at date 1?
(ii) Find the trading strategy for Mr. B so that by trading the above two assets, he can consume as he wants in part (i).

Solution. First we should recover the martingale probabilities and the short-term interest rates. Let \( \pi(a_{t+1}|a_t) \) be the martingale probability that event \( a_{t+1} \) occurs at time \( t+1 \), conditional on the fact that event \( a_t \) has occurred at time \( t \). Let \( r_t(a_t) \) be the one-period riskless interest rate at time \( t \), conditional on the fact that event \( a_t \) has occurred at time \( t \).

Then it is easy to show that

\[
\pi(E|\Omega) = \frac{1}{2}, \quad \pi(E^c|\Omega) = \frac{1}{2},
\]

\[
\pi(\omega_1|E) = \frac{1}{2}, \quad \pi(\omega_2|E) = \frac{1}{2},
\]

\[
\pi(\omega_3|E^c) = \frac{1}{3}, \quad \pi(\omega_4|E^c) = \frac{2}{3},
\]

\[r_0(\Omega) = 10\%, \quad r_1(E) = 20\%, \quad r_1(E^c) = 8\%.
\]

Let \( y \) be the amount of Mr. B’s consumption in event E at date 1. We must solve

\[
\frac{\pi(E|\Omega) \times y + \pi(E^c|\Omega) \times 0}{1 + r_0(\Omega)} = \frac{\pi(E|\Omega) \times \frac{1,000,000}{1+r_1(E)} + \pi(E^c|\Omega) \times \frac{1,000,000}{1+r_1(E^c)}}{1 + r_0(\Omega)}.
\]

Solving, we have approximately

\[y = 1,759,259.\]

This is part (i).
Now, consider part (ii). We must find a trading strategy that generates cash outflow of 1 million dollars at date 2 (to offset Mr. B’s endowed income at date 2) and cash inflow of $y$ in event E at date 1. Suppose that Mr. B needs to hold $a$ and $b$ units of assets 1 and 2 at time 1 to generate $y$ in event E at date 1. That is,

$$
\begin{bmatrix}
1100 & 880 \\
1100 & 1100
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= 
\begin{bmatrix}
y \\
0
\end{bmatrix}.
$$

Solving we have approximately

$$a = \frac{y}{220} = 7997, \quad b = -\frac{y}{220} = -7997.$$ 

Next we find the replicating strategy for the pure discount bond maturing at date 2 with one-dollar face value. To replicate this bond, suppose that one must hold $m$ and $n$ units of assets 1 and 2 in event E at date 1. Then we have

$$
\begin{bmatrix}
1320 & 1100 \\
1320 & 1012
\end{bmatrix}
\begin{bmatrix}
m \\
n
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1
\end{bmatrix}.
$$

Solving we have

$$m = \frac{1}{1320}, \quad n = 0.$$ 

Thus to replicate this bond, one must hold $\frac{1}{1320}$ and 0 units of assets 1 and 2 in event E at date 1, which costs $\frac{1100}{1320} = \frac{5}{6}$. Similarly, one can show that to replicate this bond, one must hold $\frac{1}{1188}$ and 0 units of assets 1 and 2 in event $E^c$ at date 1, which costs $\frac{1100}{1188} = \frac{25}{27}$.

Now suppose that one must hold $s$ and $t$ units of assets 1 and 2 at date 0 in order to generate $\frac{5}{6}$ in event E and $\frac{25}{27}$ in event $E^c$ at date 1. We must have

$$
\begin{bmatrix}
1100 & 880 \\
1100 & 1100
\end{bmatrix}
\begin{bmatrix}
s \\
t
\end{bmatrix}
= 
\begin{bmatrix}
\frac{5}{6} \\
\frac{25}{27}
\end{bmatrix}.
$$

Solving we have

$$s = t = \frac{1}{2376}.$$
Now we can summarize the trading strategy as follows. At time 0, buy \((a - 1,000,000s) \sim 7576\) units of asset 1 and short sell \((1,000,000t - b) \sim 8418\) units of asset 2. This makes sure that there is neither cash inflow nor outflow at time 0. At date 1, if event E occurs, then sells \(a \sim 7997\) units of asset 1 and buys 7997 units of asset 2 to get cash \(y\) and consume the cash (enjoy, Mr. B!), and then turn the rest portfolio into \(-\frac{1,000,000}{1.320} \sim -758\) units of asset 1; and if event \(E^c\) occurs, then turn the portfolio into \(-\frac{1,000,000}{1.188} = -842\) units of asset 1. Then, at date 2, the portfolio incurs a sure cash outflow, which is 1 million, and Mr. B can simply spend his endowed income to fulfill this obligation. One can verify that following this trading strategy, Mr. B gets cash \(y\) in event \(E\) at date 1, and he has no cash inflow or outflow at other dates or in other events.

Example 6. (Dynamic Programming Approach) Consider an economy that extends for three dates \((t = 0, 1, 2)\) with 4 states of nature \((\omega_1, \omega_2, \omega_3, \omega_4)\) and two long-lived assets. The common informatoin structure for investors is as follows. At \(t = 0\), investors know that the true state is an element of \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\). At \(t = 1\), investors know whether the true state is an element of \(E = \{\omega_1, \omega_2\}\) or an element of \(E^c = \{\omega_3, \omega_4\}\). At \(t = 2\), investors know exactly which among \(\omega_1, \omega_2, \omega_3, \omega_4\) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>asset/(date,event)</th>
<th>(0,Ω)</th>
<th>(1,E)</th>
<th>(1,E^c)</th>
<th>(2,ω_1)</th>
<th>(2,ω_2)</th>
<th>(2,ω_3)</th>
<th>(2,ω_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1</td>
<td>2.2</td>
<td>1</td>
<td>1.48</td>
<td>3.3</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.1</td>
<td>1.32</td>
<td>1.32</td>
<td>1.1</td>
<td>1.1</td>
<td></td>
</tr>
</tbody>
</table>

Consider Mr. A who is endowed with 80,000 shares of asset 1 and 41,000 shares of asset 2 at date 0. Mr. A derives satisfaction only from his date-2 contingent consumption \(\tilde{c}\). Let \(c_j = c(\omega_j)\) be his date-2 consumption in state \(\omega_j\). At date 0, Mr. A believes that the 4 states \(\omega_1, \omega_2, \omega_3, \omega_4\) are equally likely, and he would like to maximize \(E[u(\tilde{c})]\), where \(u(x) = \log(x)\).
(i) Consider Mr. A’s problem of solving the optimal portfolio policy at date 1 when event $E$ has occurred. Find the value function $V(1, E, \theta_1(1), \theta_2(1))$ and the optimal portfolio $(\theta^*_1(2, E, \theta_1(1), \theta_2(1)), \theta^*_2(2, E, \theta_1(1), \theta_2(1)))$.

(ii) Consider Mr. A’s problem of solving the optimal portfolio policy at date 1 when event $E^c$ has occurred. Find the value function $V(1, E^c, \theta_1(1), \theta_2(1))$ and the optimal portfolio $(\theta^*_1(2, E^c, \theta_1(1), \theta_2(1)), \theta^*_2(2, E^c, \theta_1(1), \theta_2(1)))$.

(iii) Consider Mr. A’s problem of solving the optimal portfolio policy at date 0. Find the optimal $(\theta_1^*(1), \theta_2^*(1))$, thereby obtaining the optimal $(\theta^*_1(2, E, \theta^*_1(1), \theta_2(1)), \theta^*_2(2, E, \theta^*_1(1), \theta_2(1)))$ and $(\theta^*_1(2, E^c, \theta^*_1(1), \theta_2(1)), \theta^*_2(2, E^c, \theta^*_1(1), \theta_2(1)))$.

Solution. Consider part (i). We must solve the following maximization problem:

$$V(1, E, \theta_1(1), \theta_2(1)) \equiv \max_{x_1, x_2 \in \mathbb{R}} \frac{1}{2} [\log(x_1 + 1.32x_2) + \log(1.48x_1 + 1.32x_2)]$$

subject to

$$1.1x_1 + 1.1x_2 = 1.1\theta_1(1) + 1.1\theta_2(1).$$

It is straightforward to show that the optimal solutions are

$$\theta^*_1(2, E, \theta_1(1), \theta_2(1)) \equiv x^*_1 = -\frac{33}{16}[\theta_1(1) + \theta_2(1)],$$

$$\theta^*_2(2, E, \theta_1(1), \theta_2(1)) \equiv x^*_2 = \frac{49}{16}[\theta_1(1) + \theta_2(1)],$$

and

$$V(1, E, \theta_1(1), \theta_2(1)) = \frac{1}{2} [\log(1.98) + \log(0.99)] + \log(\theta_1(1) + \theta_2(1)).$$

Part (ii) is similar. We have

$$\theta^*_1(2, E^c, \theta_1(1), \theta_2(1)) = 0,$$

$$\theta^*_2(2, E^c, \theta_1(1), \theta_2(1)) = 2\theta_1(1) + \theta_2(1),$$

and

$$V(1, E^c, \theta_1(1), \theta_2(1)) = \log(2.2\theta_1(1) + 1.1\theta_2(1)).$$
Next, consider part (iii). We must solve the following maximization problem:

\[
V(0) \equiv \max_{y_1, y_2 \in \mathbb{R}} \frac{1}{2} [V(1, E, y_1, y_2) + V(1, E^c, y_1, y_2)]
\]

subject to

\[
\frac{7}{4} y_1 + y_2 = 80,000 \times \frac{7}{4} + 41,000.
\]

The optimal solutions are

\[
\theta_1(1)^* = y_1^* = \frac{-724,000}{3}, \quad \theta_2(1)^* = y_2^* = \frac{1,810,000}{3}.
\]

Plugging \(\theta_1(1) = \theta_1(1)^*\) and \(\theta_2(1) = \theta_2(1)^*\) into \(\theta_1^*(2, E, \theta_1(1), \theta_2(1)), \theta_2^*(2, E, \theta_1(1), \theta_2(1))\), \(\theta_1^*(2, E^c, \theta_1(1), \theta_2(1))\), and \(\theta_2^*(2, E^c, \theta_1(1), \theta_2(1))\), we have

\[
\theta_1^*(2, E, \theta_1^*(1), \theta_2^*(1)) = -746,625, \quad \theta_2^*(2, E, \theta_1^*(1), \theta_2^*(1)) = 1,108,625,
\]

\[
\theta_1^*(2, E^c, \theta_1^*(1), \theta_2^*(1)) = 0, \quad \theta_2^*(2, E^c, \theta_1^*(1), \theta_2^*(1)) = \frac{362,000}{3}.
\]

**Example 7. (Martingale Approach)** Consider an economy that extends for three dates \((t = 0, 1, 2)\) with 4 states of nature \((\omega_1, \omega_2, \omega_3, \omega_4)\) and two long-lived assets. The common information structure for investors is as follows. At \(t = 0\), investors know that the true state is an element of \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\). At \(t = 1\), investors know whether the true state is an element of \(E = \{\omega_1, \omega_2\}\) or an element of \(E^c = \{\omega_3, \omega_4\}\). At \(t = 2\), investors know exactly which among \(\omega_1, \omega_2, \omega_3, \omega_4\) is the true state. The prices of the two traded long-lived assets at each time-event node on the event tree corresponding to the above information structure are summarized in the following table:

<table>
<thead>
<tr>
<th>asset/(date, event)</th>
<th>((0, \Omega))</th>
<th>((1, E))</th>
<th>((1, E^c))</th>
<th>((2, \omega_1))</th>
<th>((2, \omega_2))</th>
<th>((2, \omega_3))</th>
<th>((2, \omega_4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7/3</td>
<td>1.1</td>
<td>2.2</td>
<td>1</td>
<td>1.48</td>
<td>3.3</td>
<td>1.1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.32</td>
<td>1.32</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>
Consider Mr. A who is endowed with 80,000 shares of asset 1 and 41,000 shares of asset 2 at date 0. Mr. A derives satisfaction only from his date-2 contingent consumption \( \tilde{c} \). Let \( c_j = c(\omega_j) \) be his date-2 consumption in state \( \omega_j \). At date 0, Mr. A believes that the 4 states \( \omega_1, \omega_2, \omega_3, \) and \( \omega_4 \) are equally likely, and he would like to maximize \( E[u(\tilde{c})] \), where \( u(x) = \log(x) \).

(i) Are markets dynamically complete? Find the date-0 price of the 4 Arrow-Debreu securities that pay one dollar at date 2 in exactly one of the 4 states \( \omega_1, \omega_2, \omega_3, \omega_4 \) and nothing in other states or at other dates.

(ii) Consider Mr. A’s date-0 problem of finding the optimal consumption plan that can be financed by some admissible trading strategy 

\[(\theta_1(1), \theta_2(1), \theta_1(2, E), \theta_2(2, E), \theta_1(2, E^c), \theta_2(2, E^2)).\]

(iii) Given the consumption plan \( (c_1, c_2, c_3, c_4) \) solved in part (ii), find one admissible trading strategy to finance \( (c_1, c_2, c_3, c_4) \).

**Solution.** Consider part (i). It is straightforward to verify that markets are dynamically complete. When the money market account is taken as numeraire, it is straightforward to compute the martingale probabilities \( \pi^* \) for the 4 states, which are

\[
\begin{align*}
\pi^*(\omega_1) &= \frac{1}{4} \times \frac{1}{3}, \\
\pi^*(\omega_2) &= \frac{1}{4} \times \frac{2}{3}, \\
\pi^*(\omega_3) &= \frac{3}{4} \times \frac{1}{2}, \\
\pi^*(\omega_4) &= \frac{3}{4} \times \frac{1}{2}.
\end{align*}
\]

The corresponding short-rate process can be shown to be

\[
r_{0,1} = 10\%, \ r_{1,2}(\omega_j) = 20%1_E(\omega_j), \ j = 1, 2, 3, 4.
\]

For an asset that pays \( z(\omega_j) \) in state \( j \) at date 2, \( j = 1, 2, 3, 4 \), its date-0 price is

\[
\sum_{j=1}^{4} \frac{\pi^*(\omega_j) z(\omega_j)}{(1 + r_{0,1})(1 + r_{1,2}(\omega_j))}.
\]
It follows that the date-0 prices of the 4 Arrow-Debreu securities are
\[
\begin{align*}
\phi(\omega_1) &= \frac{25}{396}, \\
\phi(\omega_2) &= \frac{50}{396}, \\
\phi(\omega_3) &= \frac{15}{44}, \\
\phi(\omega_4) &= \frac{15}{44}.
\end{align*}
\]

Consider part (ii). We must solve the following maximization problem:
\[
\max_{c_1,c_2,c_3,c_4 \in \mathbb{R}_+} \frac{1}{4} \sum_{j=1}^{4} \log(c_j),
\]
subject to
\[
\sum_{j=1}^{4} c_j \phi(\omega_j) = 80,000 \times \frac{7}{4} + 41,000 = 181,000.
\]

It is easy to show that the optimal solutions are
\[
c_1 = 716,760, \; c_2 = 358,380, \; c_3 = c_4 = \frac{398,200}{3}.
\]

Finally, consider part (iii). A trading strategy
\[
(\theta_1(1), \theta_2(1), \theta_1(2, E), \theta_2(2, E), \theta_1(2, E^c), \theta_2(2, E^c))
\]
that finances the consumption plan must satisfy the following system of equations:
\[
\begin{align*}
1 \times \theta_1(2, E) + 1.32 \times \theta_2(2, E) &= c_1, \\
1.48 \times \theta_1(2, E) + 1.32 \times \theta_2(2, E) &= c_2, \\
3.3 \times \theta_1(2, E^c) + 1.1 \times \theta_2(2, E^c) &= c_3, \\
1.1 \times \theta_1(2, E^c) + 1.1 \times \theta_2(2, E^c) &= c_4, \\
1.1 \times \theta_1(2, E) + 1.1 \times \theta_2(2, E) &= 1.1 \times \theta_1(1) + 1.1 \times \theta_2(1),
\end{align*}
\]
\[ 2.2 \times \theta_1(2, E^c) + 1.1 \times \theta_2(2, E^c) = 2.2 \times \theta_1(1) + 1.1 \times \theta_2(1). \]

Solving, we have

\[ \theta_1(1) = -\frac{724,000}{3}, \quad \theta_2(1) = \frac{1,810,000}{3}. \]

\[ \theta_1(2, E) = -746,625, \quad \theta_2(2, E) = 1,108,625, \]

\[ \theta_1(2, E^c) = 0, \quad \theta_2(2, E^c) = \frac{362,000}{3}, \]

so that the solutions are exactly the same as those obtained in part (iii) of the preceding Example. This latter method has been proven to be computationally more efficient than the dynamic programming approach demonstrated in the preceding Example, when trade can occur continuously in time.