1. (Loyalty Program) Firm I (the incumbent) is trying to sell a product to a consumer A who has unit demand and whose reservation price for firm I’s product is one dollar. Firm I’s unit cost is \( \frac{1}{2} \). There is a potential entrant, called firm E, who can produce the same product at unit cost \( c_e \), where initially \( c_e \) is firm E’s private information; all firm I and consumer A know is that \( c_e \) is drawn from the uniform distribution on \([0, 1]\).

(i) Suppose that firm I cannot offer loyalty programs. The game proceeds as follows.

- At \( t = 0 \), firm E decides to enter or not to enter the industry. Entry is costless, and firm E gets zero profits if it does not enter. Assume that firm E chooses to stay out if it expects to get zero profits after it enters the industry.
- If firm E did not enter at \( t = 0 \), then at \( t = 1 \), which is consumer A’s shopping day, firm I offers a price \( P \) to consumer A, and consumer A can either accept or reject.
- If E has chosen to enter at \( t = 0 \), then at \( t = 1 \) the two firms’ costs become public information, and they must simultaneously offer prices to consumer A. Consumer A can either buy from the low-price firm, or not to purchase at all.

(a) Show that in the subgame where firm E enters, the equilibrium product price is \( \max\left(\frac{1}{2}, c_e\right) \).

(b) Show that firm E enters if and only if \( c_e \leq \frac{1}{2} \), and hence the probability of entry is \( \phi' = \frac{1}{2} \).

(c) Show that consumer A’s expected consumer surplus is \( \frac{1}{4} \), and firm I’s expected profit is \( \frac{1}{4} \) also.

(ii) Now suppose that firm I can offer a loyalty program at the beginning of \( t = 0 \). A loyalty program is a contract \((P, P_0)\) which says that if consumer A buys from firm I at \( t = 1 \), then the price is \( P \); but if
consumer A chooses to buy from another firm at $t = 1$, then consumer A has to pay a penalty $P_0$ to firm I. The game proceeds as follows.

- At $t = 0$, firm I offers $(P, P_0)$, which consumer A can either accept or reject.
- If $(P, P_0)$ is rejected, then the game proceeds as in the case where firm I offers no loyalty program.
- If $(P, P_0)$ is accepted, which is observed by firm E, then firm E must decide to or not to enter the industry. Then the game moves on to $t = 1$.
- Suppose that $(P, P_0)$ has been accepted by consumer A at $t = 0$. If firm E has chosen to stay out at $t = 0$, then at $t = 1$ consumer A can decide whether to pay $P$ and buy 1 unit from firm I or to purchase nothing. If firm E has chosen to enter at $t = 0$, then at $t = 1$ the two firms’ production costs become public information. Then, given $(P, P_0)$, firm E can offer consumer A a price $P'$. In this event, consumer A must decide to pay $P$ and get 1 unit from firm I, or to pay $P_0$ to firm I and to buy 1 unit from firm E, or to buy nothing.

(a) Show that in the subgame where entry has occurred, firm E will optimally choose the price $P' = P - P_0$, and hence consumer’s surplus is $1 - P$ regardless whether or not entry has occurred.

(b) Show that, given $(P, P_0)$, entry occurs with probability $\phi = \max(0, P - P_0)$.

(c) Show that the loyalty program that maximizes firm I’s expected profit is $(P^*, P_0^*) = \left(\frac{3}{4}, \frac{1}{2}\right)$, which is the solution to the following maximization program:

$$\max_{P, P_0} \phi P_0 + (1 - \phi)(P - \frac{1}{2}),$$

subject to

$$1 - P \geq \frac{1}{4}.$$  
(We assume that all players in this game are risk-neutral; firms seek to maximize expected profits and consumer A seeks to maximize expected consumer surplus.)
(d) Show that under the optimal loyalty program, the probability of entry becomes $\frac{1}{4}$ (because $P_0 > 0$ serves as an entry barrier, how?), and firm I’s expected profit becomes $\frac{1}{4} + \frac{1}{16}$.

(e) Explain why firm I did not choose $P_0 = P$ to completely block entry.

**Solution.**
Consider part (i). Consider the subgame where firm E has entered. There is a unique pure-strategy equilibrium in this subgame where both firms price at the maximum of the two firms’ unit costs with consumer A purchasing solely from the firm with a lower unit cost.\(^1\)

To see this, let $p_I$ and $p_E$ denote the prices chosen by respectively firm I and firm E in a pure-strategy equilibrium. Clearly, we must have $p_E \geq c_e$ and $p_I \geq 1/2$. Suppose first that $c_e \geq 1/2$. We claim that $p_E > c_e$ is inconsistent with an equilibrium.\(^2\) Thus assume that $p_E = c_e$. Clearly, $p_I > p_E$ is inconsistent with an equilibrium, and $p_I < p_E$ is dominated by, say, $p'_I = p_I + \frac{p_E - P_I}{2}$. It follows that $p_I = p_E$.

If consumer A does not purchase from firm I with probability one, then firm I would rather choose, say, $p''_I = p_I - \epsilon$, where $\epsilon \in (0, c_e(1-a)+\frac{a}{2})$, where $a < 1$ is the probability that consumer A purchases from firm I. Hence we conclude that when $c_e \geq \frac{1}{2}$, in the unique equilibrium, $p_E = p_I = c_e$ and consumer A must purchase from firm I with probability one. The same argument can be used to establish that when $c_e < \frac{1}{2}$, $p_E = p_I = \frac{1}{2}$ with consumer A purchasing from firm E with probability one. Consequently, the equilibrium product price is $\max(c_e, 1/2)$.

The above discussion shows that firm E’s profit is $\max(c_e, 1/2) - c_e$ after entering the market, and hence firm E should enter the market in

\(^1\)Here, we have only 1 consumer with unit demand. With a lot of consumers, it may be more reasonable to assume that the two firms get the same expected sales volume if they choose the same price. In this case there is a mixed-strategy equilibrium in which, again, the firm with the lower unit cost gets the the consumers with probability one; see Problem 5 of Homework 1.

\(^2\)To see this, suppose that $p_I > p_E > c_e$, but then firm I would choose, say, $p_E - \epsilon$, where $\epsilon \in (0, p_E - c_e)$, over $p_I$, a contradiction. What if $p_I = p_E > c_e$? In this case at least one firm would deviate regardless of consumer A’s behavior. What if $p_I < p_E$? But then firm I would still want to raise $p_I$ slightly.
the first place if and only if \( c_e < 1/2 \). Hence the probability of entry is 
\[ \phi' = \text{prob.}(c_e < 1/2) = 1/2. \]

Now, we can compute firm I’s expected profit. The preceding discussion shows that, whenever firm E enters, it must be that \( c_e < 1/2 \) so that the equilibrium product price is 1/2 and firm I earns zero profits. On the other hand, firm I becomes a monopolistic firm if firm E chooses not to enter, and in that case firm I will price at consumer A’s reservation value, which is 1. Thus before firm E makes the entry decision, firm I’s expected profit is

\[
\int_{c_e \geq 1/2} [1 - c_e] \cdot 1 dc_e = \int_{1/2}^{1} [1 - \frac{1}{2}] dc_e = \frac{1}{4}.
\]

Now we compute consumer A’s expected payoff. Note that whenever firm E enters, the equilibrium product price will be 1/2; or else, the price would be 1. Therefore consumer A’s expected consumer surplus is

\[
\text{prob.}(\{c_e < \frac{1}{2}\}) \cdot [1 - \frac{1}{2}] = \frac{1}{4}.
\]

This finishes our discussions for part (i).

Now, consider part (ii). Consider first the subgame where consumer A has joined the loyalty program \((P, P_0)\) and firm E has entered. At this time, if A buys from firm I, A must pay \( P \); and if A chooses to buy from firm E, then A needs to pay \( P' \) to firm E and \( P_0 \) to firm I. Thus consumer A will buy from firm E if and only if \( P' \leq P - P_0 \). Consequently, firm E will optimally choose \( P' = P - P_0 \).

Now we prove that once A has joined the loyalty program, A’s payoff is \( 1 - P \) whether or not firm E chooses to enter. To see this, note that A must buy from firm I if firm E does not enter, yielding the surplus \( 1 - P \); and if firm E does enter, A would feel indifferent about buying from firm E (by paying \( P - P_0 \) to firm E and \( P_0 \) to firm I) or buying from firm I (by paying \( P \) to firm I), but A is assumed to buy from firm E, which also yields for A the surplus \( 1 - P \).

Now we examine firm E’s entry decision. Given that consumer A has joined the loyalty program \((P, P_0)\), firm E’s post-entry profit is \( P - P_0 - \)
Therefore, firm E chooses to enter if and only if \( c_e < P - P_0 \), and the probability of entry is \( \phi = \text{prob}(c_e < P - P_0) = \max(0, P - P_0) \).

Now, we consider firm I’s optimal design of the loyalty program. According to part (i), consumer A’s expected consumer surplus is \( 1/4 \) if A refuses to join the loyalty program. Thus, consumer A is willing to join the loyalty program \((P, P_0)\) if and only if \( 1 - P \geq 1/4 \). Now, if A joins the loyalty program and subsequently firm \( E \) enters, firm I will earn \( P_0 \) since A will buy from firm \( E \), but if subsequently firm \( E \) chooses to stay out, then firm I will earn \( P - 1/2 \). Therefore, the optimal loyalty program \((P^*, P_0^*)\) is the solution to the following maximization program:

\[
\max_{P, P_0} \left( \phi P_0 + (1 - \phi)(P - \frac{1}{2}) \right) = \max(0, P - P_0)P_0 + (1 - \max(0, P - P_0))(P - \frac{1}{2}),
\]

subject to

\[
1 - P \geq \frac{1}{4}.
\]

We can divide the feasible programs into 2 classes.

**Class 1.** Those programs \((P, P_0)\) that totally block entry; that is, \( P_0 \geq P \).

In this case, the maximization program becomes

\[
\max_{P, P_0} P - \frac{1}{2},
\]

subject to

\[
P_0 > P, \quad 1 - P \geq \frac{1}{4}.
\]

The solution is \( P^* = \frac{3}{4} \) with any \( P_0^* > \frac{3}{4} \). Firm I’s payoff from the optimal class-1 scheme is \( \frac{1}{4} \), which is exactly what firm I makes in the absence of any loyalty program.

**Class 2.** Those programs \((P, P_0)\) that allow a positive probability of entry; that is, \( P_0 < P \).

In this case, the maximization program becomes

\[
\max_{P, P_0} \Pi_I(P, P_0) \equiv (P - P_0)P_0 + (1 - P + P_0)(P - \frac{1}{2}),
\]
subject to

\[ 1 - P \geq \frac{1}{4}. \]

Note that \( P \leq \frac{3}{4} \), and hence firm I cannot make more than \( \frac{3}{4} - \frac{1}{4} \) by retaining consumer A in the presence of firm E. This explains why it may be beneficial for firm I to allow consumer A to trade with firm E. When \( c_e \) is close to zero, the social benefit \( (1 - c_e) \) emerges from the trade between consumer A and firm E is greater than \( \frac{3}{4} - \frac{1}{4} \), and if firm I can extract most of that surplus, then letting go of consumer A is better than retaining consumer A. Indeed, given \( P \), by choosing a higher \( P_0 < P \), firm I can extract more when letting go consumer A, but that would also result in a lower probability of entry of firm E (and hence a lower probability that firm I can extract that benefit). This trade-off explains why \( \Pi_I(P, \cdot) \) is concave, and hence given \( P \), there is an optimal interior solution for \( P_0 \), which satisfies the first-order condition

\[
\frac{\partial \Pi_I}{\partial P_0} = 2P - 2P_0 - \frac{1}{2}, \Rightarrow P_0 = P - \frac{1}{4}.
\]

By replacing \( P_0 \) by \( P - 1/4 \) in \( \Pi_I(P, P_0) \), we can re-write firm I’s maximization problem as

\[
\max_P \Pi_I(P, P - \frac{1}{4}) \equiv [P - (P - \frac{1}{4})](P - \frac{1}{4}) + [1 - (P - \frac{1}{4})](P - \frac{1}{2}),
\]

subject to

\[ 1 - P \geq \frac{1}{4}. \]

Thus we have \( P^* = \frac{3}{4} \) and \( P_0^* = \frac{1}{2} \) for the optimal class-2 loyalty program. Under the optimal class-2 loyalty program \((P^*, P_0^*) = (\frac{3}{4}, \frac{1}{2})\), firm E may enter with probability \( P^* - P_0^* = \frac{1}{4} \), and firm I’s expected profit becomes

\[
\Pi_I^* = \frac{1}{4} \times \frac{1}{2} + (1 - \frac{1}{4}) \times \left( \frac{3}{4} - \frac{1}{2} \right) = \frac{5}{16} > \frac{1}{4},
\]

and hence the optimal class-1 scheme is dominated by the optimal class-2 scheme. This proves that the optimal loyalty program is indeed the above optimal class-2 scheme \((P^*, P_0^*) = (\frac{3}{4}, \frac{1}{2})\).
The optimal (class-2) loyalty program raises firm I’s expected profit because it leads to a profit transfer from firm E to firm I, and for that matter, it does not totally block the entry by firm E. The idea is that allowing consumer A to trade with firm E rather than firm I would be socially efficient when $c_e < \frac{1}{2}$, and totally blocking the entry by firm E would result in an efficiency loss. Ideally, firm I would like its loyalty program to encourage the entry of firm E whenever $c_e < \frac{1}{2}$ and to fully extract the trading gain that emerges when consumer A is allowed to switch from firm I to the low-cost firm E. This cannot be perfectly done, since (i) consumer A is strategic, and would refuse to join the loyalty program unless firm I promises to charge $P = \frac{2}{3}$ rather than 1; and (ii) firm E is strategic and an overly high $P_0$ would fail to induce much entry (and would imply an overly low probability that firm I can extract the efficiency gain generated by consumer A and firm E’s trade). The bottom line is that the loyalty program discourages entry and results in some efficiency loss (note that consumer A still buys from firm I if $c_e \in (\frac{1}{4}, \frac{1}{2})$), and in terms of welfare effects, firm I and firm E are respectively made better and worse off. Consumer A’s welfare remains unchanged. This exercise is adapted from Aghion, P, and P. Bolton, 1987, Contracts as a barrier to entry, American Economic Review, 77, 388-401.

2. (Borrowing Bank Loan or Issuing Corporate Bond?) An entrepreneur needs to invest 1 dollar to build a firm at date 0, while he has only $w < 1$ dollars. There are two types of investors in the financial market: households and commercial banks. The entrepreneur and all investors are risk neutral without time preferences; that is, they all seek to maximize expected profits and future cash flows are never discounted. The difference between the two types of investors is that banks have committed to spend a cost $c > 0$ to monitor each borrowing firm’s operations, but households do not have the expertise that is required to oversee the firm’s operations. Because banks will have to spend on monitoring, either the entrepreneur chooses to borrow $1 - w$ from households only, or he must borrow (at least partially) from banks, and in the latter case, he needs to raise $1 - w + c$, instead of $1 - w$.

After the entrepreneur gets the funding, he can incur a private cost $\phi \geq 0$ to determine the quality $p$ of the firm’s investment project at
date 1. For simplicity, suppose that a project of quality \( p \) may generate \( Y \) dollars with probability \( p \) and nothing with probability \( 1 - p \) at date 3, where the constant \( Y > 1 \). The personal cost \( \phi \) is a function of \( p \), and let us assume that, for some constant \( K > Y \),

\[
\phi(p) = \frac{K}{2} p^2, \quad \forall p \in [0, 1].
\]

If a bank lends to the firm at date 0, then it can see \( p \) after spending \( c > 0 \) before date 2. The firm can be liquidated at date 2, and its (non-negative) liquidation value is \( L < 1 \). Let \( \overline{p} \) be the first-best project quality; that is,

\[
\overline{p} = \arg \max_{p \in [0, 1]} pY - \phi(p).
\]

Assume that \( \overline{p}Y - \phi(\overline{p}) > 1 \).

(A) First suppose that banks do not exist. In this case, the entrepreneur can first decide to or not to offer a financial contract to a household, and in case he does, the household can either accept or reject it. A financial contract, or a corporate bond, is defined as \((I, Q, R)\), such that, according to this contract, (i) the household needs to give the entrepreneur \( I \) dollars at date 0; (ii) the household (or, the bondholder) can choose to or not to liquidate the firm at date 2; (iii) the household will get \( Q \in [0, L] \) in the event that the firm is liquidated at date 2; and (iv) the household will get \( R \in [0, Y] \) in the event that the firm is not liquidated at date 2, and it generates \( Y \) at date 3. Show that in equilibrium the entrepreneur chooses to borrow from a household and the household is willing to lend \( I = 1 - w \) if and only if \( w \geq w^* \) for some \( w^* \). From now on, assume the following numerical values:

\[
K = 72, \quad Y = 13, \quad L = \frac{3}{5}, \quad w = \frac{1}{2}.
\]

Let \( \pi_E \) be the entrepreneur’s expected wealth and \( p^* \) the equilibrium project quality. Show that under the optimal corporate bond \((I, Q, R)\),

\[
p^* = \frac{1}{8}, \quad R = 4, \quad \pi_E = \frac{9}{16}.
\]

Check if it is right that \( \overline{p} = \frac{Y}{K} \). Then the assumption \( \overline{p}Y - \phi(\overline{p}) > 1 \) reduces to \( Y^2 > 2K \). Given \( Y \), this last inequality holds for some \( K \) if \( Y > 2 \).

Hint: The household will lend to the entrepreneur at date 0, only if it will not liquidate the firm at date 2. Foreseeing this fact, the household will not lend to the entrepreneur.
(B) Next suppose that the entrepreneur can only borrow from banks. In this case, the entrepreneur can first decide to offer or not to offer a financial contract to a bank, and in case he does, the bank can either accept or reject it. A financial contract, or a bank loan, is defined as \((i, q, r)\), such that, according to this contract, (i) the bank needs to give the entrepreneur \(i\) dollars at date 0; (ii) the bank can choose to or not to liquidate the firm at date 2, after it sees the firm’s choice of \(p\) at date 1; (iii) the bank will get \(q \in [0, L]\) in the event that the firm is liquidated at date 2; and (iv) the bank will get \(r \in [0, Y]\) in the event that the firm is not liquidated at date 2, and it generates \(Y\) at date 3. Show that the first-best \(p\) can be attained in equilibrium if \(L \geq 1 - w + c\).

Show that, unless the amount \(1 - w\) to be lent is small enough. To solve this problem, use backward induction. First consider the bondholder’s problem at date 2 and the entrepreneur’s problem at date 1. Note that this is a simultaneous subgame at date 1! Apparently, given \((I, Q, R)\), which defines a date-1 subgame, the bondholder should optimally choose to liquidate the firm if and only if \(Q \geq p^*R\), where \(p^*\) is the entrepreneur’s equilibrium choice of \(p\) in the date-1 subgame. On the other hand, the entrepreneur will choose \(p = 0\) if he expects that the bondholder will liquidate the firm at date 2; and the entrepreneur will choose \(p^* = \arg\max_{p \in [0, 1]} p(Y - R) - \phi(p)\) if he expects that the bondholder will not liquidate the firm at date 2. In equilibrium, both parties’ conjectures about their rivals’ moves must be correct. Note that there may exist multiple Nash equilibria. Verify that it is always an equilibrium of the date-1 subgame that the bondholder always liquidates the firm and the entrepreneur always picks \(p = 0\). In case of multiple equilibria, you should assume that only the Pareto undominated equilibria may actually take place; this is an important assumption, because it helps us determine the date-0 equilibrium outcome. After finding the Pareto undominated equilibria, you should then move back to date 0, and consider whether the entrepreneur should issue the bond, and whether the household should purchase the bond if the bond is issued.

\textbf{Hint}: There is an important difference between part (A) and part (B): at date 2, the bank can see \(p\) before it decides to liquidate the firm and get \(q\), or not to liquidate the firm and get \(r\). For part (A), if a Pareto undominated equilibrium exists, it must be that the firm is never liquidated. Let \(p^*\) be the equilibrium choice made by the entrepreneur, expecting no liquidation from the household. Then it is necessary that \(Q \leq p^*R\), so that \(Q \in [0, \min(L, 1 - w)]\), because in equilibrium the household must break even; that is, \(R = \frac{1-w}{p^*}\). Now, for the entrepreneur, choosing \(p^*\) must be optimal, and hence it solves the first-order condition: \(Y - R = \phi(p^*)\), or using the fact that the household must break even, \(Y = \phi(p^*) = R \Rightarrow 1 - w = p^*[Y - \phi(p^*)]\), and the latter equation gives \(p^* = \frac{Y + \sqrt{Y^2 - 4K(1-w)}}{2K}\) or \(p^* = \frac{Y - \sqrt{Y^2 - 4K(1-w)}}{2K}\). It is easy to see that the entrepreneur will choose \(p^* = \frac{Y + \sqrt{Y^2 - 4K(1-w)}}{2K}\) as his best response. (Why?) Note that \(p^*\) is increasing
with the above specified numerical values, under the optimal bank loan contract \((i, q, r)\),

\[
p^* = \bar{p} = \frac{13}{72}, \quad \pi_E = \frac{169}{144} - \frac{1}{2} - c,
\]
so that the entrepreneur prefers borrowing bank debt to issuing a corporate bond if \(c < \frac{1}{5}\).

**Solution.** The assertions should be clear following the detailed hints. We focus on the computations then. For part (A), at the subgame where the entrepreneur is about to choose \(p\), she seeks to

\[
\max_p p(Y - R) - \phi(p),
\]

and the first-order condition to this maximization program gives rise to two solutions,

\[
\frac{Y + \sqrt{Y^2 - 4K(1-w)}}{2K} \quad \text{and} \quad \frac{Y - \sqrt{Y^2 - 4K(1-w)}}{2K},
\]

and since the creditor (whether it is a bank or a household) must break even in equilibrium, the former solution, which results in a higher project value, is preferred by the entrepreneur. Note that, as required to be a probability, indeed we have

\[
0 < p^* = \frac{Y + \sqrt{Y^2 - 4K(1-w)}}{2K} < \frac{Y + \sqrt{Y^2}}{2K} = \frac{Y}{K} \leq 1.
\]

and continuous in \(w\), and it converges to the first-best \(\bar{p}\) when \(w \uparrow 1\). Since the household will break even, the entrepreneur’s equilibrium payoff is exactly \(p^*Y - \phi(p^*) - (1-w)\), which is increasing and continuous in \(w\), because \(p^*\) is increasing and continuous in \(w\). Moreover, the entrepreneur’s equilibrium payoff will converge to \(pY - \phi(\bar{p}) - (1-w) > 1 - (1-w) = w\) as \(w \uparrow 1\). This shows that there exists \(w^*\) such that when \(w \geq w^*\), by borrowing from the household and making the household break even, the entrepreneur can get a payoff higher than or equal to \(w\), showing that the entrepreneur will proceed with this borrowing in the first place.

For part (B), show that if \(L \geq 1 - w + c\), then in equilibrium the entrepreneur will offer the following contract to the bank: \(i = q = 1 - w + c, \quad r = \frac{1 - w + c}{\bar{p}}\). Show using backward induction that, indeed, with this contract, the bank will liquidate the firm if and only if it finds out that the firm has chosen some \(p < \bar{p}\), and expecting this, the entrepreneur will choose \(p^* = \bar{p}\) in the earlier stage. Verify that no other contracts can dominate this contract.
With the specified numerical values, we have under the optimal \((I, Q, R)\),

\[ p^* = \frac{1}{8}. \]

At the optimal \((I, Q, R)\), the creditor must break even, so that

\[ 1 - w = p^* R \Rightarrow R = 4, \]

implying that

\[ \pi_E = p^*(Y - R) - \phi(p^*) = \frac{9}{16}. \]

Now, consider part (B). Under the optimal \((i, q, r)\), the first-best quality \(p = \frac{13}{72}\) will be implemented, so that the total social surplus\(^6\)

\[ pY - \phi(p) = \frac{169}{144}. \]

Thus we have

\[ \pi_E = \frac{169}{144} - (1 - w) - c = \frac{169}{144} - \frac{1}{2} - c, \]

which is greater than \(\frac{9}{16}\) if and only if \(c \leq \frac{1}{9}\).

3. (Retailer’s Opportunistic Pricing Behavior and Consumers’ Coupon Redemption.) There are two consumers with unit demand for the product produced by a firm. The firm has no production costs. The two consumers’ valuations for the product are respectively \(H\) and \(L\). The firm has already issued a cents-off coupon with face value \(v\), and to redeem the coupon the two consumers must incur costs \(T_H\) and \(T_L\). This term appears in our discussion frequently. For example, consider a seller that can produce a product at cost \(c\) and sell it to a buyer whose valuation for the product is \(v\). If \(p\) is the transaction price between the seller and the buyer, then we call \(v - p\) the buyer’s surplus and \(p - c\) the seller’s surplus (or profit) and the sum of the two, \(v - c\), the social surplus (or social benefit). Productive efficiency is said to prevail in equilibrium if production and trade take place when and only when \(v > c\).
respectively. Assume that
\[ 2L - v > H \geq L + v > L > 0, \]
and that
\[ H - v \geq H - T_H > L - T_L > v - T_L > 0. \]
The extensive game starts after the firm has already chosen \( v \), and it is described as follows.

- Seeing \( v \), the two consumers must decide independently whether to carry the coupon and redeem it on the shopping day. A consumer with valuation \( j \in \{H, L\} \) will incur a cost \( T_j \) before the shopping day if he decides to carry the coupon till the shopping day. Consumers’ decisions about whether to carry the coupon are unobservable to the firm.
- Then, on the shopping day, the firm must choose a retail price \( p \) before consumers arrive.
- Then, consumers walk in the store, see \( p \), and decide whether to make a purchase, and if they have carried a coupon till the shopping day, (it is obviously a dominant strategy at this moment) to present the coupon to the firm in order to get a price reduction equal to \( v \).

(i) Show that given that \( v \) satisfies the above conditions, this game has a unique Nash equilibrium where consumer H will never redeem the coupon while consumer L and the firm both use mixed strategies in equilibrium; that is, in equilibrium consumer L feels indifferent about redeeming and not redeeming the coupon, and the firm feels indifferent

\footnote{Therefore consumer H gets a surplus \( H - (p - v) - T_H \) if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Similarly, consumer L gets a surplus \( L - (p - v) - T_L \) if he decides to obtain the coupon and present it to the firm at the time he makes the purchase. Of course, a consumer can always forget about the coupon, and simply make the purchase. In the latter case, consumer H would get a surplus \( H - p \) and consumer L would get a surplus \( L - p \). Recall that each consumer gets zero surplus if he chooses to make no purchase.}
about two optimal prices $p_2 > p_1$.

(ii) Now, maintain all assumptions in part (i) except that the inequalities

$$2L - v > H \geq L + v$$

are replaced by

$$2L > H.$$ 

Define

$$M = 2L - kv,$$

with

$$k = \frac{L - v}{L + v} \in (0, 1).$$

Re-consider the above extensive game. Solve for the mixed-strategy NEs.

**Solution.** We shall give a detailed analysis for part (i), and then part (ii) can be analyzed analogously.

Consider part (i). Recall that $H \geq L + v$. Because of $T_H \geq v$, redeeming the coupon would reduce consumer H’s valuation. Because consumer H would not redeem the coupon, the firm would set $p = H$ if the firm wants to serve consumer H only. Because the firm can set $p = L$ to serve both consumers H and L and because $2L - v > H, p = H$ is dominated by $p = L$. (If $p = L$, the worst possible case facing the firm is the situation where L carries a coupon, so that the firm must reimburse an amount $v$ to L, implying the firm gets a revenue $2L - v$, which is still greater than $H$, which is the revenue from serving H alone.) Note that all $p < L$ are dominated by $p = L$ (both consumers will buy the product at any such $p$, with or without a coupon). Note that when consumer L sees the price $p$, the redemption cost has been sunk, and thus all $p \in (L, L + v)$ are dominated by $p = L + v$ (at any such $p$, consumer L will buy the product if and only if he is carrying a coupon; $v$ Note that the redemption cost $T_j$ is already sunk on the shopping day. If the firm expects consumer L to carry the coupon with probability one, then $p = L + v$, so that consumer L will end up with a negative consumer surplus; and if the firm expects consumer L to not carry the coupon with probability one, then $p = L$, so that consumer L actually prefers to carry the coupon before the shopping day. Show that there can be no pure strategy equilibrium. Then, argue that in a mixed strategy equilibrium, the firm randomizes over at most two prices.

---

8Note that the redemption cost $T_j$ is already sunk on the shopping day. If the firm expects consumer L to carry the coupon with probability one, then $p = L + v$, so that consumer L will end up with a negative consumer surplus; and if the firm expects consumer L to not carry the coupon with probability one, then $p = L$, so that consumer L actually prefers to carry the coupon before the shopping day. Show that there can be no pure strategy equilibrium. Then, argue that in a mixed strategy equilibrium, the firm randomizes over at most two prices.
consumer H will buy the product always if we assume that $H \geq L + v$). When $p > L + v$, only consumer H may buy the product. Because of $2L > H$, all $p > L + v$ are dominated by $p = L$. We conclude that only $p = L$ and $p = L + v$ are undominated choices for the seller.

Is there a pure-strategy equilibrium where $p = L$? Given that the firm sets price at $L$, consumer L will redeem the coupon. But given that consumer L will redeem the coupon, the firm has an incentive to raise the price to $L + v$ ($T_L$ will be sunk when consumer L sees $p$!). Thus this equilibrium cannot exist.

Is there a pure-strategy equilibrium where $p = L + v$? Given that the firm sets price at $L + v$, consumer L will not redeem the coupon, but given that consumer L does not redeem the coupon in equilibrium, the firm is better off pricing at $L$. Thus, this pure-strategy equilibrium cannot exist either. We conclude that there exist no pure-strategy equilibria for this game.

Consider the the mixed-strategy equilibrium where the firm sets $p_2 = L + v$ with probability $x$ and $p_1 = L$ with probability $1 - x$, consumer L redeems the coupon with probability $y$, and consumer H does not redeem the coupon. Note that at both $p_1$ and $p_2$, consumer H always buy the product. Since the firm uses a mixed strategy in equilibrium, it must obtain the same payoff choosing the prices $p_1$ and $p_2$. Thus,

$$(1 + y)(L + v) - yv = 2L - yv.$$

Solving the equation above, we have $y = \frac{L - v}{L + v}$. Similarly, since consumer L uses a mixed strategy in equilibrium, consumer L is indifferent about redeeming and not redeeming the coupon. Thus,

$$x(-T_L) + (1 - x)(v - T_L) = 0.$$ 

Solving the equation above, we have $x = 1 - \frac{T_L}{v}$. Therefore, there is a unique mixed-strategy equilibrium with $x = 1 - \frac{T_L}{v}$ and $y = \frac{L - v}{L + v}$.

Now, consider part (ii). Note that by assumption

$$2L > \max(H, M).$$

The equilibria can be classified as follows.
• It is easy to verify that the solution to part (i) remains valid if $L + v \leq H < M$.⁹

• If instead $H < L + v < M$, then there exists a pure-strategy equilibrium where the firm prices at $H$ with probability one.¹⁰ ¹¹

• If instead $H = M$, then we have a continuum of mixed-strategy NEs, where the firm randomizes over the three prices $L$, $L + v$, and $H$, with the probability of pricing at $L$ being $\frac{TL}{v}$, and where consumer L redeems the coupon with probability $k$.

• Finally, if $H > M$,¹² then in equilibrium the firm randomizes over $L$ and $H$, with the probability of pricing at $L$ being $\frac{TL}{v}$, and with consumer L redeeming the coupon with probability $\frac{2L^2 - H}{v}$.

4. (Signal Jamming and Cournot Competition) Consider firms 1 and 2 that engage in Cournot competition at $t = 1$ and $t = 2$, facing random demand functions at both periods. The inverse demand function at $t = 1$ is

$$\tilde{p}_1 = \tilde{a} - q_1 - q_2,$$

where $\tilde{a}$ is a positive random variable with $E[\tilde{a}] = 1$ and $q_j$ is firm $j$’s output level at $t = 1$. The inverse demand function at $t = 2$ is

$$\tilde{p}_2 = \tilde{b} - Q_1 - Q_2,$$

where $\tilde{b}$ is a positive random variable and $Q_j$ is firm $j$’s output level at $t = 2$. Each firm seeks to maximize the sum of expected profits over the two periods. That is, both firms are risk-neutral without time preferences. The game proceeds as follows.

• At the beginning of $t = 1$, both firms must simultaneously make output choices $q_1$ and $q_2$ without seeing the realization of $\tilde{a}$.

---

⁹Note that $M$ is exactly the expected profit that the firm obtains in the mixed-strategy equilibrium obtained in part (i).

¹⁰Again one can verify that the firm cannot price at either $L$ or $L + v$ in a pure-strategy equilibrium.

¹¹In this pure-strategy equilibrium, the firm’s expected profit is $2H - v$, because consumer L will redeem the coupon with probability one. The firm does not want to deviate and price at $L + v$, because $2H - v > (L + v) - v = L$.

¹²Verify that $M > L + v$ always!
At the beginning of $t = 2$, after knowing $q_j$ and the realization $p_1$ of $\tilde{p}_1$, firm $j$ must choose $Q_j$. The two firms make output choices at the same time, without seeing the realization of either $\tilde{a}$ or $\tilde{b}$. At this time, firm $j$ does not see $q_i$ that was chosen by its rival, firm $i$.

(i) First assume that $\tilde{b}$ and $\tilde{a}$ are independently and identically distributed. Solve the equilibrium output choices $(q_1^*, q_2^*, Q_1^*, Q_2^*)$ in the unique SPNE.

(ii) Ignore part (i). Now assume instead that $\tilde{b} = \lambda \tilde{a}$, where $\lambda < 2$ is a constant known to both firms. Solve the unique symmetric SPNE.

(iii) Do the two firms get higher date-1 expected profits in part (ii) or in part (i)? Why?

(iv) Suppose that $\lambda = 1$. Do the two firms get higher date-2 expected
profits in part (ii) or in part (i)? Why?\(^{13}\)

\(^{13}\)Hint: Verify that \((q_1^*, q_2^*, Q_1^*, Q_2^*) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) in part (i). For part (ii), let \((q^*, Q^*(p_1, q))\) denote the unique symmetric SPNE, where both firms choose \(q^*\) at \(t = 1\), and both choose \(Q^*(p_1, q)\) after choosing \(q\) at \(t = 1\) and subsequently learning that the realization of \(p_1\) is \(p_1\). Then in equilibrium, \(\tilde{p}_1 = \tilde{a} - 2q^*\), or \(\tilde{a} = \tilde{p}_1 + 2q^*\). At the beginning of \(t = 2\), given the realization \(p_1\) of \(\tilde{p}_1\) and its own output choice \(q_i\) at \(t = 1\), and given that firm \(j\) does not deviate from its equilibrium strategy, firm \(i\) knows that \(\tilde{a} = p_1 + q_i + q^*\). Moreover, firm \(i\) knows that that firm \(j\) would believe that \(\tilde{a} = p_1 + 2q^*\) and seek to maximize

\[
\max_Q [\lambda(p_1 + 2q^*) - Q^*(p_1, q^*) - Q]Q,
\]

where note that firm \(j\) does not know firm \(i\) has chosen \(q_i\) rather than \(q^*\). That is, firm \(i\) believes that firm \(j\) would choose the \(Q\) that satisfies

\[
Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1, q^*)}{2},
\]

which has to be \(Q^*(p_1, q^*)\) also. Hence firm \(i\) believes that firm \(j\) would choose

\[
Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.
\]

Firm \(i\), knowing that it has chosen \(q_i\) rather than \(q^*\) at \(t = 1\), seeks to maximize the following date-2 profit:

\[
\max_Q [\lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q]Q,
\]

so that given \((p_1, q_i)\), firm \(i\)'s optimal date-2 output level is

\[
Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1 + 2q^*)}{3}}{2},
\]

which yields for firm \(i\) the following date-2 profit

\[
\frac{1}{4} \left[ \frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i \right]^2.
\]

At \(t = 1\), expecting firm \(j\) to choose \(q^*\), firm \(i\) seeks to

\[
\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4} E[(\frac{2\lambda \tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],
\]

which is concave in \(q_i\) because \(\lambda < 2\). Show that the optimal \(q_i\) must satisfy the first-order condition for this maximization problem; that is,

\[
1 - q^* - 2q_i + \frac{\lambda}{6} \left( \frac{2\lambda \tilde{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i \right) = 0,
\]

or using \(E[\tilde{p}_1] = 1 - q_i - q^*\), and \(q_i = q^*\) in equilibrium, show that

\[
q^* = \frac{1}{3} + \frac{\lambda^2}{27}.
\]

Show that then \(Q^*(p_1, q^*) = \frac{\lambda}{3} q^*\).
Solution. Consider part (i). Since \( \tilde{b} \) and \( \tilde{a} \) are independent, the two firms do not care about their date-2 decisions \( Q_1 \) and \( Q_2 \) when they engage in the date-1 Cournot competition. Being risk-neutral, given \( q_j \), firm \( i \) seeks to
\[
\max_{q_i} q_i(\mathbb{E}[\tilde{a}] - q_i - q_j) = q_i(1 - q_i - q_j),
\]
so that this game has the same equilibrium as the Cournot game presented in Example 1 of Lecture 1, Part I. That is, in equilibrium,
\[
q_i^* = q_2^* = \frac{1}{3}.
\]
Similarly, at date 2, given \( Q_j \), firm \( i \) seeks to
\[
\max_{Q_i} Q_i(\mathbb{E}[\tilde{b}] - Q_i - Q_j) = Q_i(1 - Q_i - Q_j),
\]
so that this game also has the same equilibrium as the Cournot game presented in section 11 of Lecture 1, Part I. That is, in equilibrium,
\[
Q_1^* = Q_2^* = \frac{1}{3}.
\]
This finishes part (i).

Now, for part (ii), let \((q^*, Q^*(p_1,q))\) denote the unique symmetric SPNE, where both firms choose \( q^* \) at \( t = 1 \), and both choose \( Q^*(p_1,q) \) after choosing \( q \) at \( t = 1 \) and subsequently learning that the realization of \( \hat{p}_1 \) is \( p_1 \). Then in equilibrium, \( \hat{p}_1 = \hat{a} - 2q^* \), or \( \hat{a} = \hat{p}_1 + 2q^* \). At the beginning of \( t = 2 \), given the realization \( p_1 \) of \( \hat{p}_1 \) and its own output choice \( q_i \) at \( t = 1 \), and given that firm \( j \) does not deviate from its equilibrium strategy, firm \( i \) knows that \( \tilde{a} = p_1 + q_i + q^* \). Moreover, firm \( i \) knows that that firm \( j \) would believe that \( \tilde{a} = p_1 + 2q^* \) and seek to maximize
\[
\max_Q \mathbb{E}[\lambda(p_1 + 2q^*) - Q^*(p_1,q^*) - Q],
\]
where note that firm \( j \) does not know firm \( i \) has chosen \( q_i \) rather than \( q^* \). That is, firm \( i \) believes that firm \( j \) would choose the \( Q \) that satisfies
\[
Q = \frac{\lambda(p_1 + 2q^*) - Q^*(p_1,q^*)}{2},
\]
18
which has to be \( Q^*(p_1, q^*) \) also. Hence firm \( i \) believes that firm \( j \) would choose

\[
Q^*(p_1, q^*) = \frac{\lambda(p_1 + 2q^*)}{3}.
\]

Firm \( i \), knowing that it has chosen \( q_i \) rather than \( q^* \) at \( t = 1 \), seeks to maximize the following date-2 profit:

\[
\max_{Q} \left[ \lambda(p_1 + q_i + q^*) - Q^*(p_1, q^*) - Q\right],
\]

so that given \( (p_1, q_i) \), firm \( i \)'s optimal date-2 output level is

\[
Q_i = \frac{\lambda(p_1 + q_i + q^*) - \frac{\lambda(p_1 + 2q^*)}{3}}{2},
\]

which yields for firm \( i \) the following date-2 profit

\[
\frac{1}{4}\left[ \frac{2\lambda p_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i \right]^2.
\]

At \( t = 1 \), expecting firm \( j \) to choose \( q^* \), firm \( i \) seeks to

\[
\max_{q_i} [1 - q_i - q^*]q_i + \frac{1}{4}E [(\frac{2\lambda \hat{p}_1}{3} + \frac{\lambda q^*}{3} + \lambda q_i)^2],
\]

which is concave in \( q_i \) because \( \lambda < 2 \). It follows that the optimal \( q_i \) must satisfy the first-order condition for this maximization problem; that is,

\[
1 - q^* - 2q_i + \frac{\lambda}{6} \left[ \frac{2\lambda E[\hat{p}_1]}{3} + \frac{\lambda q^*}{3} + \lambda q_i \right] = 0,
\]

or using \( E[\hat{p}_1] = 1 - q_i - q^* \), and \( q_i = q^* \) in equilibrium, we have

\[
q^* = \frac{1}{3} + \frac{\lambda^2}{27}.
\]

It follows that \( Q^*(p_1, q^*) = \frac{\lambda a}{3} \).

Now, consider part (iii). Comparing part (i) to part (ii), we see that both firms make lower expected profits at date 1 in part (ii). This happens because in part (ii) firms cannot resist the temptation of expanding outputs as means of manipulating their rivals’ beliefs about the realization of \( a \). By secretly expanding its output \( q_i \), firm \( i \) wants
to make its rival $j$ believe in a lower realization of $\tilde{a}$, which implies a lower demand (whose intercept is $\lambda \tilde{a}$) at date 2, and if firm $i$ succeeds in making its rival believe in a lower date-2 demand, then it can benefit from choosing a higher date-2 output $Q_i$ given that its rival will on average choose a lower output $Q_j$. In equilibrium this incentive is correctly recognized by its rival $j$, but the incentive to engage in signal-jamming still changes the two firms’ date-1 profits. Both firms are worse off in part (ii), because of a lower product price resulting from output expansion ($q^* > \frac{1}{3}$).

Finally, consider part (iv). Note that in part (ii)

$$E[Q^*(p_1, q^*)] = \frac{\lambda E[\tilde{a}]}{3} = \frac{E[\tilde{a}]}{3} = \frac{1}{3},$$

where recall that $\frac{1}{3}$ is the two firms’ date-2 output choice in part (i). Signal-jamming does not fool any player in equilibrium (that is, both firms can infer correctly the realized $\tilde{a}$ from the realized date-1 price), but in part (ii), since $\tilde{a} = \tilde{b}$, the two firms’ common date-2 output choice depends on the realization of $\tilde{a}$. This is in sharp contrast with part (i), where $\tilde{a}$ and $\tilde{b}$ are independent, so that the firms’ date-2 output choices can never depend on the realized $\tilde{a}$. Now, since in part (ii) each firm’s date-2 expected profit is a convex function of its date-2 output $Q^*(p_1, q^*)$, and since $Q^*(p_1, q^*)$ is a mean-preserving spread of the firms’ date-2 output choice (which is $\frac{1}{3}$) in part (i), the two firms actually obtain higher expected date-2 profits in part (ii) than in part (i). Indeed, each firm gets the following expected date-2 profit in part (i),

$$\frac{1}{3}(E[\tilde{b}] - \frac{1}{3} - \frac{1}{3}) = \frac{1}{9},$$

but in part (ii) its expected date-2 profit becomes

$$E[\frac{\tilde{a}}{3}(\tilde{a} - \frac{\tilde{a}}{3} - \frac{\tilde{a}}{3})] = \frac{E[\tilde{a}^2]}{9} > \frac{(E[\tilde{a}])^2}{9} = \frac{1}{9},$$

where the inequality follows from Jensen’s inequality and the fact that the function $h(z) = z^2$ is strictly convex. Thus the two firms make higher expected date-2 profits in part (ii) than in part (i).
5. Re-consider the sequential game presented in section 3 of Lecture 1, Part II.

(i) Verify that the (reduced) strategic form of the game, where players move simultaneously and where equivalent strategies are identified, can be represented by the following bi-matrix:

<table>
<thead>
<tr>
<th>player 1/player 2</th>
<th>l</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>2,2</td>
<td>2,2</td>
</tr>
<tr>
<td>(L,A)</td>
<td>3,1</td>
<td>1,0</td>
</tr>
<tr>
<td>(L,B)</td>
<td>0,-5</td>
<td>1,0</td>
</tr>
</tbody>
</table>

(ii) Verify that given \( \epsilon > 0 \) small, if player 1 adopts the totally mixed strategy

\[
\begin{bmatrix}
R & 1 - \epsilon - \epsilon^2 \\
(L,A) & \epsilon^2 \\
(L,B) & \epsilon
\end{bmatrix},
\]

then \( r \) is player 2’s best response, but \( l \) is not. Hence if player 2’s restricted to assigning \( l \) a positive probability in an \( \epsilon \)-perfect equilibrium, then she will assign \( l \) with a probability less than \( \epsilon \) (and hence \( r \) must be assigned with a probability of at least \( 1 - \epsilon \)). Verify that given player 2’s totally mixed strategy

\[
\begin{bmatrix}
1 & \epsilon \\
r & 1 - \epsilon
\end{bmatrix},
\]

\( R \) is player 1’s best response, but \((L,A)\) and \((L,B)\) are not. Conclude that player 1’s totally mixed strategy

\[
\begin{bmatrix}
R & 1 - \epsilon - \epsilon^2 \\
(L,A) & \epsilon^2 \\
(L,B) & \epsilon
\end{bmatrix}
\]
and player 2’s totally mixed strategy

\[
\begin{bmatrix}
1 & \epsilon \\
r & 1 - \epsilon
\end{bmatrix}
\]

indeed form an \( \epsilon \)-perfect equilibrium given the specified \( \epsilon > 0 \). Since this is true for all small \( \epsilon > 0 \), by letting \( \epsilon \downarrow 0 \), verify that indeed, \((R,r)\) is a trembling-hand perfect equilibrium for the strategic game

<table>
<thead>
<tr>
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<th>r</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2,2</td>
</tr>
<tr>
<td>(L,A)</td>
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<td>1,0</td>
</tr>
<tr>
<td>(L,B)</td>
<td>0,-5</td>
<td>1,0</td>
</tr>
</tbody>
</table>

(iii) Now, let us consider a new game similar to the game above, but with one difference: “the player 1” that gets to choose between A and B after player 2 chooses l in the original game is now replaced by a new player, called player 3. In this new game, player 3 and player 1 have the same payoff function, which is the payoff function of the player 1 in the original game. To represent this three-player normal-form game, we draw two bi-matrices as follows (where we identify player 3’s payoff with player 1’s payoff):

<table>
<thead>
<tr>
<th>player 3/player 2</th>
<th>l</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3,1</td>
<td>1,0</td>
</tr>
<tr>
<td>B</td>
<td>0,-5</td>
<td>1,0</td>
</tr>
</tbody>
</table>

<table>
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<td>2,2</td>
</tr>
<tr>
<td>B</td>
<td>2,2</td>
<td>2,2</td>
</tr>
</tbody>
</table>

In this new normal-form game, the 3 players move simultaneously, with player 1 choosing one of the two bi-matrices, player 2 choosing l or r, and player 3 choosing A or B. Note that the first bi-matrix corresponds to player 1 choosing L (and the second bi-matrix corresponds to player 1 choosing R) in the original extensive game.
Show that in this modified game, the only trembling-hand perfect equilibrium is the unique SPNE in the original game, where player 1 chooses L, and then player 2 chooses l, and then player 3 chooses A.

**Solution.** Part (i) and part (ii) are self-evident. We shall then concentrate on part (iii). Note that for player 3, B is weakly dominated by A, and whenever player 1 and player 2 both adopt totally mixed strategies, player 3 strictly prefers A to B. Thus given $\epsilon > 0$ small, player 3 must adopt A with a probability exceeding $1 - \epsilon$ in the corresponding $\epsilon$-perfect equilibrium. For $\epsilon$ sufficiently small, player 2’s best response against player 3’s totally mixed strategy is l, and hence player 2 must adopt l with a probability exceeding $1 - \epsilon$. In this case, player 1’s best response is to choose the first bi-matrix with a probability exceeding $1 - \epsilon$. Clearly, the above totally mixed strategy profile constitutes an $\epsilon$-perfect equilibrium, and by letting $\epsilon \downarrow 0$, we conclude that the only possible trembling-hand perfect equilibrium for the current agent-normal-form game is $(L, l, A)$, which is exactly the equilibrium path of the unique SPNE for this game.