1. (Competitive Manufacturers May Make More Profits with Non-integrated Distribution Channels.) Recall the Cournot game in Example 1 of Lecture 1, Part I. Assume that $c = F = 0$ and the inverse demand in the relevant range is

$$P(Q) = 1 - Q, \ 0 \leq Q = q_1 + q_2 \leq 1.$$  

(i) Find the equilibrium profits for the two firms.

(ii) Now suppose that the two manufacturing firms cannot sell their products to consumers directly. Instead, firm $i$ (also referred to as manufacturer $i$) must first sell its product to retailer $R_i$. Then retailers $R_1$ and $R_2$ then compete in the Cournot game. The extensive game is now as follows.

- The two firms first announce $F_1$ and $F_2$ simultaneously, where $F_i$ is the franchise fee that firm $i$ will charge retailer $i$, which is a fixed cost of retailer $i$. $R_1$ and $R_2$ simultaneously decide to or not to turn down the offers made by the firms. Assume that firm $i$ and retailer $R_i$ both get zero payoffs if $F_i$ gets turned down by retailer $R_i$.

- Then, after knowing whether $F_1$ and $F_2$ get accepted by respectively $R_1$ and $R_2$, the two firms announce $w_1$ and $w_2$ simultaneously, where $w_i$ is the unit whole price that firm $i$ will charge retailer $i$.

- Next, in case the firms’ offers are both accepted, then given $(F_1, F_2, w_1, w_2)$, the two retailers simultaneously choose $q_1$ and $q_2$.

Show that in the unique subgame-perfect Nash equilibrium (SPNE) each manufacturing firm gets a profit of $\frac{10}{9}$. (Hint: Backward induction asks you to always start from the last-stage problem, which is the Nash equilibrium of the subgame where $R_1$ and $R_2$ play the Cournot game given some $(F_1, F_2, w_1, w_2)$. You can show that the equilibrium $(q_1^*, q_2^*)$ depend on $(w_1, w_2)$ but not on $(F_1, F_2)$, because the latter are
fixed costs. Then, you should move backwards to consider the two manufacturers’ competition in choosing \( w_1 \) and \( w_2 \), given some \((F_1, F_2)\).

Here assume that the two manufacturers know that different choices of \( w_1 \) and \( w_2 \) will subsequently affect \( R_1 \)’s and \( R_2 \)’s choices of \( q_1 \) and \( q_2 \). Finally, you can move to the first-stage of the game, where the two firms simultaneously choose \( F_1 \) and \( F_2 \). \(^1\)

**Solution.** Let us solve the SPNE using backward induction. First consider the subgame where \((F_1, F_2, w_1, w_2)\) are given, and the two retailers are about to choose \( q_1 \) and \( q_2 \). Retailer \( i \), given \( q_j \), seeks to

\[
\max_{q_i} \Pi^R_i(q_i, q_j; w_i, F_i) \equiv q_i(1 - q_i - q_j - w_i - F_i).
\]

The first-order condition gives retailer \( i \)’s reaction function

\[
r_i(q_j; w_i) = \frac{1 - q_j - w_i}{2}, \quad \forall i, j \in \{1, 2\}, \ i \neq j.
\]

Thus there is a unique NE in this subgame, which is \(^2\)

\[
(q_1^*(w_1, w_2), q_2^*(w_1, w_2)) = \left( \frac{1 - 2w_1 + w_2}{3}, \frac{1 + w_1 - 2w_2}{3} \right).
\]

Correspondingly, the two retailers’ profits are

\[
\Pi^R_1(q_1^*(w_1, w_2), q_2^*(w_1, w_2); w_1, F_1) = \frac{(1 - 2w_1 + w_2)^2}{9} - F_1
\]

and

\[
\Pi^R_2(q_2^*(w_1, w_2), q_1^*(w_1, w_2); w_2, F_2) = \frac{(1 - 2w_2 + w_1)^2}{9} - F_2.
\]

\(^1\) This exercise intends to show why employing independent retailers may be a good idea even if using a firm’s own outlets can be cheaper. Essentially, employing an independent retailer amounts to delegating the retailer the choice of output, knowing that the retailer, unlike the manufacturer, will be choosing output given a positive unit cost \( w_i \)!

A higher unit cost credibly convinces the rival retailer that less output will be produced, and with both manufacturers producing less outputs, their profits become higher.

\(^2\) Why does \( q_i^* \) increase with \( w_j \)? Again, this results from the fact that \( q_1 \) and \( q_2 \) are strategic substitutes. A higher \( w_j \) means that retailer \( j \) is faced with a higher unit cost, and hence \( q_j \) ought to be lower, which then implies that retailer \( i \) should optimally respond by choosing a higher \( q_i \).
Now, consider the stage where \((F_1, F_2)\) are given and the two manufacturers are about to choose \(w_1\) and \(w_2\). Manufacturer \(i\), given \(w_j\), seeks to

\[
\max_{w_i} F_i + w_i q_i^*(w_i, w_j), \quad \forall i, j \in \{1, 2\}, \ i \neq j.
\]

The first-order condition gives

\[
w_i = \frac{1 + w_j}{4}, \quad \forall i, j \in \{1, 2\}, \ i \neq j.
\]

Note that \(w_1\) and \(w_2\) are indeed strategic complements! Thus there is a unique NE in this subgame where the two manufacturers both set the unit wholesale price at \(\frac{1}{3}\):

\[
w_1^* = w_2^* = \frac{1}{3}.
\]

In this equilibrium, for \(i = 1, 2\), manufacturer \(i\)'s profit is

\[
F_i + \frac{6}{81}.
\]

The correspondingly profits of the two retailers are

\[
\Pi_1^R(q_1^*(\frac{1}{3}, \frac{1}{3}), q_2^*(\frac{1}{3}, \frac{1}{3}); \frac{1}{3}, F_1) = \frac{4}{81} - F_1
\]

and

\[
\Pi_2^R(q_2^*(\frac{1}{3}, \frac{1}{3}), q_1^*(\frac{1}{3}, \frac{1}{3}); \frac{1}{3}, F_2) = \frac{4}{81} - F_2.
\]

Now, consider the stage where the two manufacturers are about to choose \(F_1\) and \(F_2\). Manufacturer \(i\)'s problem is

\[
\max_{F_i} F_i + \frac{6}{81}
\]

---

3When manufacturer \(i\) expects manufacturer \(j\) to choose a higher \(w_j\), it realizes that, keeping its choice \(w_i\) unchanged, subsequently the two retailers will choose higher \(q_i^*\) and lower \(q_j^*\), which marginally encourages manufacturer \(i\) to raise \(w_i\) in the first place: the drawback of raising \(w_i\) is that it leads to a lower \(q_i^*\), and hence it is less costly to do this when \(q_i^*\) rises because of a higher \(w_j\)! This explains strategic complementarity between \(w_i\) and \(w_j\).
subject to

\[ \Pi_i^R(q_i^*(\frac{1}{3}, \frac{1}{3}), q_j^*(\frac{1}{3}, \frac{1}{3}); \frac{1}{3}; F_i) = \frac{4}{81} - F_i \geq 0. \]

There is a unique SPNE in this game where \( F_1 = F_2 = \frac{4}{81} \), and hence the two manufacturers’ equilibrium profits are both \( \frac{10}{81} \).

We must emphasize the importance of the timing of the game. That the two firms are able to first offer \( F_1 \) and \( F_2 \) to respectively R1 and R2 and then to subsequently choose \( w_1 \) and \( w_2 \) is important to the above result. If instead the two manufacturers must offer \( (F_1, w_1) \) and \( (F_2, w_2) \) to R1 and R2 at the first stage of the game, then given \( w_j \), firm \( i \) would like to choose \( w_i = 0 \), because a zero unit wholesale price can serve as a commitment that conviincs \( R_j \) that \( R_i \) would produce more given any quantity \( q_j \) (or, simply, \( R_i \)'s reaction function will be

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4
This commitment is valuable, because output choices are strategic substitutes, which implies that \( R_j \) will reduce output \( q_j \) if \( R_j \) believes that it is faced with a more aggressive reaction function. Consequently, choosing \( w_i = 0 \) can raise \( R_i \)'s profit, which in turn implies that, manufacturer \( i \) in offering \( w_i = 0 \), can choose a higher \( F_i \) to extract \( R_i \)'s profit.

In the current setting, however, given that \( F_1 \) and \( F_2 \) were offered and accepted in the preceding stage, the two firms in choosing \( w_1 \) and \( w_2 \) would never choose a zero unit wholesale price, because a zero wholesale price would result in no additional income for the manufacturer. Indeed, at this stage, as we have shown, regardless of \( F_1 \) and \( F_2 \), the

\[ \text{4In this case, given } (F_j, w_j), \text{ manufacturer } i \text{ seeks to} \]
\[
\max_{(F_i, w_i)} F_i + w_i q_i^*(w_i, w_j),
\]

subject to
\[
q_i^*(1 - q_i^* - q_j^* - w_i) - F_i \geq 0.
\]

Optimality requires that the latter constraint be binding, and hence
\[
F_i = q_i^*(1 - q_i^* - q_j^* - w_i),
\]
or equivalently, manufacturer \( i \) seeks to
\[
\max_{w_i \geq 0} q_i^*(1 - q_i^* - q_j^*) \equiv H(w_i; w_j) = \frac{1}{9}(1 - 2w_i + w_j)(1 + w_i + w_j),
\]

where the new objective function is simply the profit function facing an otherwise-identical vertically integrated channel (that is, the firm that is both manufacturer \( i \) and \( R_i \)). Since \( q_i^* \) and \( q_j^* \) are respectively decreasing and increasing in \( w_i \), it is easy to verify that this new objective function is decreasing in \( w_i \) given \( w_j \), and hence we obtain a corner solution \( w_i = 0 \). Indeed, direct differentiation yields
\[
\frac{\partial H}{\partial w_i} = \frac{1}{9}(-4w_i - w_j - 1) < 0, \Rightarrow w_i^* = 0.
\]

The same argument applies to manufacturer \( j \) as well, and hence when the two manufacturers must offer \( (F_1, w_1) \) and \( (F_2, w_2) \) to \( R_1 \) and \( R_2 \) at the first stage of the game, the latter two retailers behave just like firms 1 and 2 in Example 1 in Lecture 1, Part I (with zero production costs). As can be easily checked, in the current situation, with \( w_1 = w_2 = 0 \), the two retailers will choose \( q_1^* = q_2^* = \frac{1}{8} \).
two firms choose $w_1 = w_2 > 0$. Retailer $R_i$ can infer this fact (as we do) when it must decide whether to accept $F_i$. This explains why in equilibrium the two manufacturers are able to set $F_1 = F_2 = \frac{4}{81}$.

Note that when a single manufacturer chooses a positive unit wholesale price, it induces its downstream retailer to reduce output (because the unit wholesale price is the retailer’s unit cost, and a higher unit cost leads to a lower output choice), which, by the fact that output choices are strategic substitutes, in turn encourages the other retailer to expand output, which hurts the manufacturer’s downstream retailer. However, with both manufacturers offering positive unit wholesale prices, the net effect of positive wholesale prices is to induce both retailers to select an output level that is lower than the output level that the two manufacturers would choose in the absence of independent retailers (or, in the case of vertically integrated distribution channels). This lower output level then leads to a higher equilibrium retail price, which raises the sum of the manufacturer’s and the retailer’s profits in each distribution channel. The sum of profits of the manufacturer and the retailer coincides with the manufacturer’s equilibrium profit in the current case, because by assumption the manufacturer can offer a two-part tariff to its downstream dealer, leaving the latter with a zero profit.\(^5\)

2. Reconsider the Bertrand game discussed in class. Assume that there is only one buyer with unit demand for the homogeneous good produced by the two firms, and the buyer’s willingness to pay for the good is $v$.

\(^5\)We have assumed that the two firms have homogeneous products and the demand is linear. When the two firms’ products are differentiated or when the demand functions are not linear, raising the equilibrium product prices by using one independent retailer may reduce a manufacturer’s sales volume by too much and hence may or may not be a good idea; see Patrick Rey and Joseph Stiglitz, 1995, The Role of Exclusive Territories in Producers’ Competition, *Rand Journal of Economics*, 26, 431-451. See also T. W. McGuire and R. Staelin, 1983, An Industry Equilibrium Analysis of Downstream Vertical Integration, *Marketing Science*, 2, 161-191. Note that if a manufacturer $i$ sells through more than one retailer in a small district, then intra-brand competition between these retailers will lead to the Bertrand outcome where all retailers hired by manufacturer $i$ offer $w_i$ as the retail price—the distribution channel of manufacturer $i$ is essentially vertically integrated! This highlights the importance of hiring exactly ONE independent retailer (a practice referred to as exclusive territory), if manufacturer $i$ would like to raise its retail price by hiring independent retailers.
Let $c_j$ denote the unit production cost of firm $j$. Assume that
\[ v > c_2 > c_1 \geq 0. \]

The game proceeds as follows. First, the two firms must simultaneously
announce their prices $p_1$ and $p_2$. Then, the buyer can buy one unit of
the good from either firm 1 or firm 2, or the buyer can choose not to
buy anything at all. Assume that the buyer seeks to maximize
\[ \max(0, v - p_1, v - p_2), \]
and assume that the buyer is willing to make a purchase as long as buying
generates a non-negative consumer surplus. Find a Nash equilibrium for this game. (**Hint:** First suppose that there is a pure-strategy NE $(p_1, p_2)$. Show that a contradiction arises if $p_i > v$ for some $i$. Show that if $p_i < c_i$ for some $i$, then $i = 2$ and $j = 1$ with $c_1 \leq p_1 \leq p_2 < c_2$, and hence a contradiction arises because from firm 1’s perspective there is no optimal $p_1$ in this case. Hence conclude that for any pure strategy NE $(p_1, p_2)$, it must be that $c_i \leq p_i \leq v$, for $i = 1, 2$. Thus either $v \geq p_1 = p_2 \geq c_2$, which gives arise to another contradiction, or $p_i > p_j \geq c_j$, which implies that firm $j$ should deviate unilaterally and replace $p_j$ by $p_j' = p_j + \epsilon$ for sufficiently small $\epsilon > 0$. Conclude that this game has no pure-strategy NE. Finally, find a mixed-strategy NE in which firm 1 chooses $p_1 = c_2$ with probability one and firm 2 chooses a mixed strategy $\tilde{p}_2$, whose distribution function is $F(x)$, with $F(c_2) = 0$ and $F(v) = 1$.)

**Solution.** It is straightforward to show that a contradiction will arise
to each and every supposed pure-strategy NE. Hence this game has
no pure-strategy NEs. As suggested in the hint, let us now look for
mixed-strategy NEs. Suppose that in equilibrium firm 1 gets to serve
the buyer with probability one by setting $p_1 = c_2$. Firm 2 then prices
above $p_1$ with probability one. Assume that $\tilde{p}_2$ has a distribution function $F(\cdot)$. Since $\tilde{p}$ has realizations lying in the interval $[c_2, v]$, we have $F(c_2) = 0$ and $F(v) = 1$. Given $p_1 = c_2$, any $p_2 \in [c_2, v]$ looks equally good from firm 2’s perspective, and this explains why firm 2 wants to randomize over these prices. We must make sure that firm 1 has no incentive to unilaterally deviate from his equilibrium pure strategy.
Suppose that $F(\cdot)$ is continuous on $(c_2, v)$. Then by choosing $p_1 \in (c_2, v)$ firm 1 gets the expected profit

$$(p_1 - c_1)[1 - F(p_1)],$$

and we require that this amount is less than or equal to firm 1’s equilibrium expected profit, which is $(c_2 - c_1)$. This gives rise to the condition

$$F(x) \geq 1 - \frac{c_2 - c_1}{x - c_1}, \forall x \in (c_2, v).$$

The boundary case involves

$$F^*(x) = 1 - \frac{c_2 - c_1}{x - c_1}, \forall x \in (c_2, v).$$

Note that, as required,

$$F^*(c_2) = 0,$$

but

$$\lim_{x \uparrow v} F^*(x) < 1,$$

implying that $F^*(\cdot)$ has a jump at $v$; or the event $\{\tilde{p}_2 = v\}$ may occur with a strictly positive probability

$$1 - \lim_{x \uparrow v} F^*(x) = \frac{c_2 - c_1}{v - c_1}.$$

What happens if firm 1 deviates and chooses to price at $v$? Then with probability

$$\frac{c_2 - c_1}{v - c_1}$$

firm 2 may also price at $v$, and in this event firm 1’s expected profit is

$$\frac{1}{2} \times \frac{c_2 - c_1}{v - c_1} \times (v - c_1) < c_2 - c_1,$$

and with probability

$$1 - \frac{c_2 - c_1}{v - c_1}$$

firm 2 may price lower than $v$, and firm 1 gets nothing in this event. Hence we have verified that firm 1 has no incentive to deviate and
price at $v$ either. This proves that $(p_1 = c_2, F^*(\cdot))$ does constitute a mixed-strategy NE.

As emphasized, this is merely one such equilibrium. Any distribution function $F(\cdot)$ with

$$F(c_2) = F^*(c_2) = 0, \quad F(v) = F^*(v) = 1,$$

and

$$F(x) \geq F^*(x), \quad \forall x \in [c_2, v]$$

together with $p_1 = c_2$ forms an NE.\(^6\)

Note that in this game firm 1 gets to serve the buyer with probability one. This exercise has an exact counterpart in auction theory; see section 16 of Lecture 1, Part I. For a more detailed treatment, see my note “The Nash equilibria for Bertrand-competitive duopolists with Diverse Unit Costs.”

3. (A Strategic Role of Futures Contracts) Consider example 1 in Lecture 1, part I, where firms 1 and 2 can costlessly produce a product and engage in Cournot competition with the inverse demand being, in the relevant range,

$$P(q_1 + q_2) = 1 - q_1 - q_2.$$

This problem is a modification of the above Cournot game.

(i) Assume that there are two dates. The two firms will compete at date 1, but at date 0, both firms can correctly expect the date-1 inverse demand function, which is the $P(\cdot)$ defined above. At date 0, the futures market opens for the product produced by the two firms. There are price-competitive investors in the futures market, who, just like the two firms, are risk neutral without time preferences (that is, there will be no discounting for anyone). The extensive game is as follows.

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\(6\)The reader may wonder why firm 2 is willing to go through the trouble of randomization given that firm 2 is doomed to getting a zero profit. This has something to do with our specification about firm 2’s preference. Firm 2 cares about profits and nothing else. If, say, firm 2, besides its concern for profits, hates firm 1 because firm 1 would take everything away from firm 2, then we might expect firm 2 to choose $p_2 = c_2$: pricing higher than $c_2$ only makes firm 1 better off. Here, jealousy does not enter the picture, which is our assumption. Firm 2 is assumed to care about profits, literally. Hence firm 2 is willing to randomize over any price that leads to the same optimal profit, which is zero.
• At date 0, (only) firm 1 can sign a futures contract with the competitive investors. In the futures contract, firm 1 promises to deliver $f_1$ units of the product at date 1 to one of the investors (say, Mr. A), and Mr. A promises to pay the price $F$ (referred to as the date-0 futures price of the product). We assume that firm 1 announces $f_1$, and the competitive investors then determine the futures price $F$. Assume that investors have rational expectations; that is, upon seeing $f_1$, they can use backward induction to anticipate the date-1 price of the product (called the date-1 spot price of the product), and to rule out arbitrage opportunities, in the date-0 equilibrium, $F$ must equal the anticipated date-1 price.

• At date 1, upon seeing firm 1’s date-0 futures contract $(f_1, F)$, the two firms choose $q_1$ and $q_2$ simultaneously. Note that firm 1’s profit as a function of $q_1, q_2$ is

$$\Pi_1(q_1, q_2; f_1) = [1 - q_1 - q_2][q_1 - f_1] + F f_1.$$ 

Firm 2’s profit function is still

$$\Pi_2(q_1, q_2) = [1 - q_1 - q_2]q_2.$$ 

• Then, after firms set $q_1$ and $q_2$, the date-1 price $P(q_1, q_2)$ is realized, and firm 1 must deliver $f_1$ units of the product to Mr. A, and Mr. A must pay firm 1 $F f_1$ dollars.

Find the SPNE of this extensive game. Explain why firm 1 may benefit from futures trading.  

(ii) Now, suppose that both firms can engage in futures trading at date 0, with $f_1$ and $f_2$ units sold respectively at the futures price $F$ determined at date 0. Again, assume that all investors in the futures market

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7Hint: Use backward induction. First consider the date-1 subgame with $f_1$ given. This is just a Cournot game with the two firms’ profit functions being $\Pi_1$ and $\Pi_2$ specified above. Let the subgame equilibrium be $(q_1^*(f_1), q_2^*(f_1))$, which depends on $f_1$. Now move backwards to consider firm 1’s date-0 choice of $f_1$. Remember that the investors in the futures market can rationally expect the date-1 spot price of the product, which is $P((q_1^*(f_1), q_2^*(f_1))$, and given $f_1$, they will compete in price so that in the date-0 futures market equilibrium, $F = P((q_1^*(f_1), q_2^*(f_1))$. Given that $F = P((q_1^*(f_1), q_2^*(f_1))$, find firm 1’s optimal $f_1$. 

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have rational expectations when they compete in price to determine $F$. Re-derive the SPNE. Explain why the two firms might be hurt by the availability of futures trading.\footnote{Hint: Again, consider the date-1 subgame with $f_1, f_2$ given. Now for $i = 1, 2$, firm $i$’s profit function becomes

$$\Pi_i(q_i, q_j; f_i) = [1 - q_i - q_j][q_i - f_i] + F f_i.$$}

\textit{Solution.} Consider part (i). It is straightforward to show that the two firms’ date-1 reaction functions are

$$r^1_1(q_2; f_1) = \frac{1 + f_1 - q_2}{2}, \quad r^1_2(q_1) = \frac{1 - q_2}{2}.$$ 

Hence we have the subgame equilibrium

$$q^*_1(f_1) = \frac{1}{3} + \frac{2}{3} f_1, \quad q^*_2(f_1) = \frac{1}{3} - \frac{1}{3} f_1.$$ 

Now consider firm 1’s date-0 choice of $f_1$. Since $F = P((q^*_1(f_1), q^*_2(f_1))$ (a no-arbitrage condition!), at date 0 firm 1 seeks to

$$\max_{f_1} P((q^*_1(f_1), q^*_2(f_1))q^*_1(f_1) = \frac{1}{3}(1 - f_1)(\frac{1}{3} + \frac{2}{3} f_1),$$

for which the necessary and sufficient first-order condition gives

$$f_1 = \frac{1}{4},$$

implying that, in equilibrium,

$$F^* = P^* = \frac{1}{4}, \quad q^*_1 = \frac{1}{2}, \quad q^*_2 = \frac{1}{4}, \quad \Pi_1^* = \frac{1}{8}, \quad \Pi_2^* = \frac{1}{16}.$$ 

Compared to the Cournot equilibrium profit $\frac{1}{9}$, firm 1 is better off with futures trading. The reason is that after committing to sell $f_1$ units at
a fixed price $F$, which will not fall when firm 1 expands output at date 1, firm 1 has an incentive to choose a higher total output at date 1. This fact results in firm 2 lowering output accordingly (because output choices are strategic substitutes). In essence, firm 1’s selling futures contracts serves as a commitment that tells its rival that its reaction function is now shifted upwards. Consequently, firm 1 benefits from futures trading, which hurts firm 2 at the same time.

Next consider part (ii). Given $(f_1, f_2)$, now the subgame equilibrium becomes

$$
q_1^*(f_1, f_2) = \frac{1}{3} + \frac{2}{3} f_1 - \frac{1}{3} f_2, \quad q_2^*(f_1, f_2) = \frac{1}{3} + \frac{2}{3} f_2 - \frac{1}{3} f_1,
$$

$$
P^*(f_1, f_2) \equiv P(q_1^*(f_1, f_2), q_2^*(f_1, f_2)) = \frac{1}{3}(1 - f_1 - f_2).
$$

Now consider the date-0 futures market equilibrium. Firm $i$’s problem is to, given the conjectured $f_j$,

$$
\max_{f_i} P(q_i^*(f_i, f_j), q_j^*(f_i, f_j))q_i^*(f_i, f_j) = \frac{1}{3}(1 - f_i - f_j)(\frac{1}{3} + \frac{2}{3} f_i - \frac{1}{3} f_j).
$$

The necessary and sufficient first-order condition gives firm $i$’s date-0 reaction function

$$
r_i^0(f_j) = \frac{1 - f_j}{4}, \quad i, j = 1, 2, \quad i \neq j.
$$

Thus the date-0 equilibrium is

$$
f_1^* = f_2^* = \frac{1}{5},
$$

implying that

$$
q_1^* = q_2^* = \frac{2}{5}, \quad F^* = P^* = \frac{1}{5}, \quad \Pi_1^* = \Pi_2^* = \frac{2}{25}.
$$

Compared to the Cournot equilibrium profit, each firm is worse off. The reason is that, as in the game of prisoners’ dilemma, here each firm intends to hold a short position in the futures contract as an attempt to force its rival to produce less. With the short positions in the futures
contract, both firms are faced with a residual inverse demand with lower elasticity to their output expansion. Consequently, both firms choose to produce more in the subgame where futures contracts have been signed, leading to a lower spot and futures price for the product, and lower profit for each firm.\(^9\)

4. (A Strategic Role of Option Contracts) This exercise can be applied to joint ventures, but we shall consider a simpler interpretation. There are two players in this sequential game, a landlord (L) and a tenant (T). The landlord can first spend \(a \in [0,1]\) to build a house, and then after the tenant moves in, the tenant can spend \(b \in [0,1]\) to make improvements on the house. The resale value of the house is \(v(a,b) = af + bh\), where the constants \(f, h \in (0,1)\). (Of course the landlord charges a rent from the tenant, say \(r\), for renting the house for a given period, say a year, but this rental transaction has nothing to do with our main analysis and so we shall forget about it at this moment.) Let us call

\[
S(a, b) \equiv v(a, b) - a - b
\]

the social benefit, and the solution

\[
(a^*, b^*) = \arg \max_{a,b \in [0,1]} S(a, b)
\]

will be called the first-best investments. We shall assume that \(a, b\) can only be observed by the landlord and the tenant but not by the court of law (i.e., they are non-verifiable variables), and hence cannot be put in a legally binding contract. Moreover, \(S(a, b)\) is not verifiable either.\(^{10}\)

What L and T can do is to sign a contract to decide who owns the house. The timing of the game is as follows. The two first sign an ownership contract, and then given the contract L first chooses \(a\), and upon seeing \(a\), T must choose \(b\). Then the house is sold after the rental period, and the two people share the profits according to the ownership contract.

\(^9\)This exercise is adapted from Blaise Allaz and Jean-Luc Vila, 1993, Cournot Competition, Forward Markets and Efficiency, Journal of Economic Theory, 59, 1-16.

\(^{10}\)Otherwise, show that a simple sharing rule that gives L and T respectively \(aS\) and \((1 - a)S\) will induce L and T to invest respectively at \(a^*\) and \(b^*\).
(i) Compute $a^*, b^*$. Suppose first that $a, b$ are contractible. Show that if $L$ and $T$ are both rational, they will put $a = a^*, b = b^*$ in the contract.

From now on, return to our initial assumption that $a, b$ cannot be verified in the court of law, and hence $L$ and $T$ can only try to “implement” efficient $a, b$ by choosing a smart “ownership contract.”

(ii) Suppose that $L$ owns the house exclusively (so that $T$ cannot share a cent when the house is sold), determine the $a, b$ and $v(a, b)$ by backward induction.

(iii) Suppose that before building the house, $L$ sells the house to $T$ by making a take-it-or-leave-it offering price $q$ (so that $L$ cannot share a cent when the house is sold). Determine the $a, b$ and $v(a, b)$ by backward induction. Find $q$.

(iv) Suppose that before building the house, $T$ agrees to pay $L$ some money $z$ to jointly own the house with $L$, and $L$ and $T$ will subsequently receive respectively $\lambda v(a, b)$ and $(1 - \lambda)v(a, b)$ when selling the house (where $\lambda$ is exogenously given). Determine the $a, b$ and $v(a, b)$ by backward induction. Find $z$, assuming that $L$ has all the bargaining power in determining $z$.

(v) Finally, consider the following contingent ownership contract: $L$ owns the house initially, and he gives an option for free (why for free?) to $T$, and the option allows $T$ to buy the house at the exercise price $p = v(a^*, b^*) - b^*$ after $L$ chooses $a$ but before $T$ chooses $b$. Find the SPNE by backward induction. Determine the equilibrium $a, b$ and $v(a, b)$.

(vi) Explain why the contingent ownership contract attains the first-best efficiency, while the other ownership contracts do not.

(vii) Now suppose instead that after $L$ chooses $a$ but before $T$ decides to or not to exercise the option, $L$ can offer a new contract to $T$. (We call this re-contracting event a “renegotiation.”) This new contract will replace the existing option contract if and only if both $L$ and $T$ agree to do so. The new contract states a (probably) different exercise price $p'$ that allows $T$ to pay $p'$ to $L$ and get the house before $T$ chooses $b$. Find the equilibrium $a$ and $b$ chosen by $L$ and $T$ respectively.

Solution. Consider part (i). The first-best investment levels $(a^*, b^*)$
must solve the following maximization problem
\[
\max_{a,b} S(a, b) = v(a, b) - a - b = af + bh - a - b.
\]
The necessary and sufficient first-order conditions yield \(a^* = f^{1/(1-f)}\) and \(b^* = h^{1/(1-h)}\). Since rational people must sign a Pareto efficient contract, these will be L and T’s choices if they can sign complete contracts.

Consider part (ii). Obviously, T will choose \(b = 0\) since he cannot share the proceeds from selling the house. Thus L seeks to
\[
\max_a v(a, 0) - a = af - a.
\]
The solution is \(a = a^*\). Hence when L owns the house exclusively, \(v(a, b) = v(a^*, 0)\) and L’s payoff is \(S(a^*, 0)\).

Consider part (iii). Suppose that T has already paid \(q\) to L before L chooses \(a\). Then L will choose \(a = 0\). Thus T seeks to
\[
\max_b v(0, b) - b = bh - b,
\]
yielding \(b = b^*\). Thus, the proceeds from selling the house will be \(v(0, b^*)\). For T to be willing to pay \(q\) for the house in the first place, it must be that \(q \leq v(0, b^*) - b^*\). Thus L optimally chooses \(q = v(0, b^*) - b^*\). It follows that L’s payoff is \(S(0, b^*)\).

Consider part (iv). Consider the subgame where T has already paid \(z\) to L for the right of jointly owning the house. Given that L has chosen \(a\), T seeks to
\[
\max_b (1 - \lambda)v(a, b) - b = (1 - \lambda)(af + bh) - b.
\]
Thus T optimally chooses \(b = [(1 - \lambda)h]^{1/(1-h)} \equiv b(\lambda)\). Rationally expecting T’s behavior, in choosing \(a\), L seeks to
\[
\max_a \lambda v(a, b(\lambda)) - a = \lambda\{af + [b(\lambda)]^h\} - a.
\]
The solution is \(a = (\lambda f)^{1/(1-f)} \equiv a(\lambda)\). The proceeds from selling the house will thus be \(v(a(\lambda), b(\lambda))\). Thus T will accept \(z\) if and only
if \( z \leq (1 - \lambda)v(a(\lambda), b(\lambda)) - b(\lambda) \). Consequently, L will choose \( z = (1 - \lambda)v(a(\lambda), b(\lambda)) - b(\lambda) \), which yields for L the payoff \( S(a(\lambda), b(\lambda)) \).

Consider part (v). If T does not exercise the option, then he must choose \( b = 0 \) because he does not get to share the proceeds from selling the house. If T exercises the option, then given any \( a \) he will choose \( b \) to

\[
\max_b v(a, b) - b = a^f + b^h - b,
\]

since he exclusively owns the house. Thus T will choose \( b = b^* \) after he exercises the option.

Should T exercise the option? T knows that he will choose \( b = b^* \) if he exercises the option, and hence he chooses to exercise the option if and only if

\[
v(a, b^*) - b^* - p = v(a, b^*) - b^* - [v(a^*, b^*) - b^*] \geq 0 \iff a \geq a^*.
\]

The result is not surprising. The house value depends not only on \( b \) but also on \( a \). From T’s perspective, given the strike price, the house is worth buying only if \( a \) is large enough. Indeed, the higher the strike price chosen by L, the higher \( a \) must be in order to induce T to exercise the option. By wisely setting \( p = v(a^*, b^*) - b^* \), T will exercise the option if and only if \( L \) chooses some \( a \geq a^* \).

Now, what would be L’s choice about \( a \)? If L chooses some \( a < a^* \), then T will not exercise the option, and T will subsequently choose \( b = 0 \), leading to the payoff \( S(a, 0) \) for L. If L chooses some \( a \geq a^* \), then T will exercise the option and L’s payoff would become \( S(a, b^*) \). Thus L’s optimal choice is \( a = a^* \), which generates for L the first-best payoff \( S(a^*, b^*) \).

An interesting question here is why L offers the option for free? In fact, regardless of the strike price chosen by L, T will refuse to pay anything for the option. Why? Note that after T obtains the option, L will choose some \( a \) that makes T feel indifferent between to and not to exercise the option. In other words, L will choose some \( a \) that ensures that T makes zero profits by exercising the option. Therefore, for any strike price chosen by L, T will attach zero value to the option.

Consider part (vi). The above discussion shows that the first-best efficiency is attained in part (v) but not in parts (ii), (iii), or (iv). There
is a free-rider problem in parts (ii), (iii) and (iv), which prevents the first-best efficiency from prevailing. On the other hand, in part (v), T’s incentive to choose $b^*$ can be ensured by making T the sole owner at the time the house is sold (or equivalently, making T the sole residual claimant). For L, on the other hand, by wisely choosing the strike price for the option, L can be induced to choose $a = a^*$. This explains how the first best efficiency is attained in part (v).

Finally, note that in part (v), given the existing option contract T will not exercise the option if $a < a^*$. We have assumed there that the existing option contract cannot be renegotiated, even though such inefficiency may exist. What if L and T can renegotiate the existing option contract? Does the opportunity of renegotiating an inefficient old contract undermine our result that option contracts can help attain the first-best efficiency?

Recall from part (v) that T will exercise the option if and only if

$$v(a, b^*) - b^* - p = v(a, b^*) - b^* - [v(a^*, b^*) - b^*] \geq 0 \iff a \geq a^*.$$ 

Now, if a new contract specifies a strike price $p' > p$, T will never agree to replace the old contract $p$ by this new contract $p'$. Thus if L wants to offer a new contract to T, he must choose some $p' \leq p$. Suppose that L has already spent some $a \geq a^*$. Since T is willing to exercise the option under old contract, L will optimally choose $p' = p$ in this case, so that contract renegotiation does not arise in this case. What if L has spent some $a < a^*$? To induce T to agree to replace the old contract $p$ by this new contract $p'$, it is necessary and sufficient that the new strike price $p'$ satisfies

$$v(a, b^*) - b^* - p' \geq 0.$$ 

Hence the optimal $p' = v(a, b^*) - b^*$. Therefore, if L has chosen some $a < a^*$, he will offer a new contract that yields for L the payoff $v(a, b^*) - b^* - a = S(a, b^*)$. It follows that L should optimally choose $a = a^*$!

Our conclusion is that, allowing renegotiation does not change our main result that option contracts can help resolve the free-rider problem and attain the first-best efficiency.\[^{11}\]

\[^{11}\] This exercise is adapted from George Nöldeke and Klaus M. Schmidt, 1998, Sequential Investments and Options to Own, *Rand Journal of Economics*, 29, 633-653.