What is CAPM without Short Selling?

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Abstract

This paper follows Sharpe's idea in his classic proof of the CAPM and proves a generalized CAPM. In this sense, we propose a question: What is the CAPM without short selling. We obtain the generalized CAPM in the case without short selling and we remark that if the market does not permit the short selling, then in Sharpe's sense, the expected rate of return on any asset in the market would be less than that in the "traditional" CAPM.

Since the classical papers: Sharpe [16], Lintner [7] and Mossin [13], Capital Asset Pricing Model (CAPM) became a foundation stone of the financial economics. There were several proofs of the CAPM in different standpoints. One class of proofs is from the consumption-based model (see, for example, Cochrane [1], pp. 152–166); another class is in a general equilibrium setting (see, Duffie [3], Jarrow [4], Magill and Quinzii [8], etc.). However, the most popularized proof is that of Sharpe [16].

Although he used a geometric argument, Sharpe’s proof of the CAPM is not very mathematical and formal (see Appendix 1). Indeed, his proof is based on the mean-variance efficient frontier of the assets and Tobin’s two

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fund separation theorem. The idea could describe as follows. (1) The mean-variance efficient frontier of all the assets in the market is the capital market line, which passes two points representing the riskless asset $f$ and a mean-variance efficient portfolio (“combination”) $m$ in the risk-return plane and which is also the mean-variance efficient frontier of the asset set $\{f, m\}$; (2) for any asset $i$, if the mean-variance efficient frontier of the set $\{i, f, m\}$ is the same as the capital market line, then the CAPM holds; that is

$$E[r_i] - \mu_0 = \beta_{im}(E[r_m] - \mu_0), \quad \beta_{im} = \frac{\text{Cov}[r_i, r_m]}{\text{Var}[r_m]},$$

where $r_i$ and $r_m$ are the random rates of return on the asset $i$ and on the efficient portfolio $m$ respectively, and $\mu_0$ is the constant rate of return on the riskless asset. Sharpe’s geometric argument was used only for (2). Here, $m$ is only an efficient portfolio and not necessarily the market portfolio. The “traditional” CAPM is in the case that $m$ is the market portfolio, which is mean-variance efficient (see [17]).

This paper attempts to discuss the CAPM from (2). It follows such a general problem: Let $S_n = \{1, 2, \ldots, n\}$ be a set of assets and $S_k = \{1, 2, \ldots, k\} \subset S_n$ be its subset. What is a necessary and sufficient condition for that the mean-variance efficient frontier on $S_k$ is the same as that on $S_n$? We will call $S_k$ an efficient subset of $S_n$ if $S_k$ possesses this property. This problem is not new. In fact, Huberman and Kandel [5] studied the same problem. Also see Ross [15], Szegö [20] and Merton [11], [12]. However, in all these work there were some assumptions such as the covariance matrix of the rates of return on $S_n$ is nonsingular and then it would be not applicable directly to the CAPM case.

The paper is organized as follows. In Section 1, we show a necessary and sufficient condition for that $S_k$ is an efficient subset of $S_n$. This result in the CAPM case is just a new version of Sharpe’s proof of the CAPM. It also includes the case of “Zero-β CAPM”. For emphasizing the CAPM, the title of Section 1 is “A Generalized CAPM”. Section 2 studies the same problem, but without short selling. We also show a necessary and sufficient condition for that $S_k$ has the same mean-variance frontier as that of $S_n$ without short selling. Although in Markowitz [9], [10], the portfolio selection problems are always in the case without short selling, in Sharpe [16], the CAPM is considered only for the case with short selling; otherwise, the capital market line would be not a straight line. So, we could entitle Section 2 “What is CAPM without Short Selling?” as the title of the paper. Mathematically, the proof is quite complex and it will be in Appendix 2. Section 3 is “Conclusion”. The main mathematical results of this paper
were discussed in Shi and Yang [18], [19]. This paper desires to show their economic means associated the CAPM.

1 A Generalized CAPM

Let \( S_n = \{1, 2, \ldots, n\} \) be a set of \( n \) assets and \( r_i \) be the random rate of return of the asset \( i, i = 1, 2, \ldots, n \). Let

\[
e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n,
\]

\[
\mu_i = E[r_i], \quad i = 1, 2, \ldots, n, \quad \mu = (\mu_1, \mu_2, \ldots, \mu_n)^T,
\]

\[
V = (V_{ij})_{i,j=1,2,\ldots,n} = (\text{Cov}[r_i, r_j])_{i,j=1,2,\ldots,n},
\]

and \( w = (w_1, w_2, \ldots, w_n)^T \in \mathbb{R}^n \) is called to be a portfolio, \( \mu_w = w^T \mu \) to be the return of portfolio and \( \sigma_w = (w^T V w)^{1/2} \) to be the risk of portfolio. Then Markowitz’s mean-variance portfolio selection problem is

\[
\begin{aligned}
\min & \quad \sigma_w^2 = w^T V w = \sum_{i,j=1}^n V_{ij} w_i w_j \\
\text{s.t.} & \quad w^T e = w_1 + w_2 + \cdots + w_n = 1 \\
& \quad \mu_w = w^T \mu = w_1 \mu_1 + w_2 \mu_2 + \cdots + w_n \mu_n = \bar{\mu}
\end{aligned}
\]

(1)

A solution \( \bar{w} \in \mathbb{R}^n \) of (1) with some \( \bar{\mu} \) is called to be a frontier portfolio of \( S_n \). The set of all frontier portfolios of \( S_n \) is called to be the portfolio frontier of \( S_n \).

The problem (1) is a quadratic convex programming with two constraints of linear equalities. Its Lagrange multipliers always exist ([14], p. 207). Therefore, it is solved by the Lagrange multiplier method, i.e., \( \bar{w} \) and \( \lambda_1, \lambda_2 \) are its solution and its Lagrange multipliers if and only if they satisfy

\[
\begin{aligned}
2V \bar{w} - \lambda_1 \mu - \lambda_2 e & = 0, \\
w_1 + w_2 + \cdots + w_n & = 1, \\
w_1 \mu_1 + w_2 \mu_2 + \cdots + w_n \mu_n & = \bar{\mu}.
\end{aligned}
\]

(2)

Our main theorem of this section is

**Theorem 1.1** Let \( S_n = \{1, 2, \ldots, n\} \) be the set of \( n \) assets and \( r_1, r_2, \ldots, r_n \) be their random rates of return respectively. \( S_k = \{1, 2, \ldots, k\} \subset S_n \). Then \( S_k \) is an efficient subset of \( S_n \) if and only if

\[
\text{rank}
\begin{pmatrix}
V_{11} & V_{12} & \cdots & V_{1k} & \mu_1 & 1 \\
V_{21} & V_{22} & \cdots & V_{2k} & \mu_2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
V_{k1} & V_{k2} & \cdots & V_{kk} & \mu_k & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
V_{11} & V_{12} & \cdots & V_{1k} & \mu_1 & 1 \\
V_{21} & V_{22} & \cdots & V_{2k} & \mu_2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
V_{k1} & V_{k2} & \cdots & V_{kk} & \mu_k & 1 \\
V_{k+1,1} & V_{k+1,2} & \cdots & V_{k+1,k} & \mu_{k+1} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
V_{n1} & V_{n2} & \cdots & V_{nk} & \mu_n & 1 \\
\end{pmatrix}
\]

where \( \mu_i = E[r_i], V_{ij} = \text{Cov}[r_i, r_j], i, j = 1, 2, \ldots, n. \)

**Proof:** **Necessity.** Let \( \bar{w}^k = (\bar{w}_1, \ldots, \bar{w}_k) \in \mathbb{R}^k \) be a frontier portfolio of \( S_k. \) Then it must be a solution of problem (1) with \( k \) instead of \( n. \) Hence, from (2) with \( k \) instead of \( n, \bar{w}_1, \ldots, \bar{w}_k \) and Lagrange multiplier \( \lambda_{k1} \) and \( \lambda_{k2} \) must satisfy the following system of linear equations:

\[
\bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_k = 1 \\
\bar{w}_1 \mu_1 + \bar{w}_2 \mu_2 + \cdots + w_k \mu_k = \bar{\mu} \\
V_{11} \bar{w}_1 + V_{12} \bar{w}_2 + \cdots + V_{1k} \bar{w}_k - \frac{1}{2} \lambda_{k1} \mu_1 - \frac{1}{2} \lambda_{k2} = 0 \\
V_{21} \bar{w}_1 + V_{22} \bar{w}_2 + \cdots + V_{2k} \bar{w}_k - \frac{1}{2} \lambda_{k1} \mu_2 - \frac{1}{2} \lambda_{k2} = 0 \\
\vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\
V_{k1} \bar{w}_1 + V_{k2} \bar{w}_2 + \cdots + V_{kk} \bar{w}_k - \frac{1}{2} \lambda_{k1} \mu_k - \frac{1}{2} \lambda_{k2} = 0 \\
\]

If \( S_k \) is an efficient subset of \( S_n, \) then the portfolio frontier of \( S_k \) is the same as that of \( S_n. \) Then \( \bar{w} = (\bar{w}_1, \ldots, \bar{w}_k, 0, \ldots, 0)^T \in \mathbb{R}^n \) is a solution of (1). Thus, from (2), \( \bar{w}_1, \ldots, \bar{w}_k \) and Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) must satisfy the following system of linear equations:

\[
\bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_k = 1 \\
\bar{w}_1 \mu_1 + \bar{w}_2 \mu_2 + \cdots + w_k \mu_k = \bar{\mu} \\
V_{11} \bar{w}_1 + V_{12} \bar{w}_2 + \cdots + V_{1k} \bar{w}_k - \frac{1}{2} \lambda_1 \mu_1 - \frac{1}{2} \lambda_2 = 0 \\
V_{21} \bar{w}_1 + V_{22} \bar{w}_2 + \cdots + V_{2k} \bar{w}_k - \frac{1}{2} \lambda_1 \mu_2 - \frac{1}{2} \lambda_2 = 0 \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
V_{k1} \bar{w}_1 + V_{k2} \bar{w}_2 + \cdots + V_{kk} \bar{w}_k - \frac{1}{2} \lambda_1 \mu_k - \frac{1}{2} \lambda_2 = 0 \\
\]

4
Without loss of generality, suppose that the portfolio frontier of $S_k$ is not a singleton (otherwise, all $\mu_i$ are the same and this is a simple case). Then we can assume that for instance, $\mu_1 \neq \mu_2$. By comparing the third and fourth equations of two equation systems, it is immediately that $\lambda_{k1} = \lambda_1$, $\lambda_{k2} = \lambda_2$. That is, a solution of the first equation system is always a solution of the second system.

On the other hand, since $\bar{\mu}$ can take any value, in these two systems, we can eliminate the second equation and the above property holds again. Therefore, it is possible only if

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & 1 \\
V_{11} & V_{12} & \cdots & V_{1k} & \mu_1 & 1 & 0 \\
V_{21} & V_{22} & \cdots & V_{2k} & \mu_2 & 1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots \\
V_{k1} & V_{k2} & \cdots & V_{kk} & \mu_k & 1 & 0 \\
\end{pmatrix}
$$

$$
= \text{rank} \begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & 1 \\
V_{11} & V_{12} & \cdots & V_{1k} & \mu_1 & 1 & 0 \\
V_{21} & V_{22} & \cdots & V_{2k} & \mu_2 & 1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots \\
V_{k1} & V_{k2} & \cdots & V_{kk} & \mu_k & 1 & 0 \\
V_{k+1,1} & V_{k+1,2} & \cdots & V_{k+1,k} & \mu_{k+1} & 1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots \\
V_{n1} & V_{n2} & \cdots & V_{nk} & \mu_n & 1 & 0 \\
\end{pmatrix}
$$

(4)

which is obviously equivalent to (3).

**Sufficiency.** If (3) holds, then (4) holds also. Noting that $\bar{\mu}$ can take any value, it means that two systems of linear equations in the proof of Necessity have the same solution space. In particular, any frontier portfolio of $S_k$ must be that of $S_n$. \qed

**Remark** In the proof of this theorem there is no assumption on the covariance matrix of the random rates of return on $S_n$ or on $S_k$. Therefore, it is suitable for both the case without a riskless asset and that with a riskless asset. When $V$ is nonsingular, from the theorem, it is easy to obtain some more precise expressions, which are the cases in [5] and [20].

Joining up with Theorem 1.1 and two fund separation theorem, we obtain the following generalized CAPM:

**Theorem 1.2** Let $S_n = \{1, 2, \ldots, n\}$ be the set of $n$ assets, the portfolios $w_p$ and $w_q$ are two solutions of the mean-variance portfolio selection problem (1) for the expected rate of return $\bar{\mu}$ being $\mu_p$ and $\mu_q$ respectively, and $\mu_p \neq \mu_q$. 

5
The corresponding rates of return to \( w_p \) and \( w_q \) are \( r_{wp} \) and \( r_{wq} \) respectively. Then any asset \( u \) with the rate of return \( r' \) does not change the efficient frontier on \( S_n \) if and only if there exists \( w \in \mathbb{R} \) such that

\[
E[r'] = (1-w)E[r_{wp}] + wE[r_{wq}], \tag{5}
\]

\[
\begin{align*}
\text{Cov}[r', r_{wp}] &= (1-w)\text{Var}[r_{wp}] + w\text{Cov}[r_{wp}, r_{wq}] \\
\text{Cov}[r', r_{wq}] &= (1-w)\text{Cov}[r_{wp}, r_{wq}] + w\text{Var}[r_{wq}] \end{align*} \tag{6}
\]

In particular, if \( \text{Cov}[r_{wp}, r_{wq}] = 0 \), then any asset \( u \) with the rate of return \( r' \) does not change the efficient frontier on \( S_n \) if and only if the following “zero-\( \beta \) CAPM” holds:

\[
E[r'] - E[r_{wq}] = \beta_{r, q}^{*}(E[r_{wp}] - E[r_{wq}])),
\]

\[
\beta_{r, q}^{*} = \text{Cov}[r', r_{wp}]/\text{Var}[r_{wp}] = 1 - \text{Cov}[r', r_{wq}]/\text{Var}[r_{wq}].
\]

If a riskless asset with a constant rate of return \( \mu_0 \) and a risky asset with a random rate of return \( r_m \) are both frontier portfolios of \( S_n \), then the condition above reduces to that the following CAPM holds:

\[
E[r'] - \mu_0 = \beta_{r, m}^{*}(E[r_m] - \mu_0),
\]

\[
\beta_{r, m}^{*} = \text{Cov}[r', r_{m}]/\text{Var}[r_{m}].
\]

**Proof:** In fact, we suppose that there are two frontier assets with the rates of return \( r_{wp} \) and \( r_{wq} \), and we label them \( n + 1 \) and \( n + 2 \). Then, by using the two fund separation theorem, the union of \( S_n \) and these two assets, \( S_{n+2} \), has an efficient subset \( S_2 = \{n + 1, n + 2 \} \), and the same portfolio frontier as that of \( S_n \). Further more, appending the asset \( u \) with the rate of return \( r' \) to \( S_{n+2} \), we form a new asset set. Thus, if \( u \) does not change the portfolio frontier of \( S_n \), then \( S_2 \) is yet a efficient subset of this new asset set. The covariance matrix of the rates of return on \( S_2 \) is obviously nonsingular. By Theorem 1.1,

\[
\text{rank} \begin{pmatrix} \text{Var}[r_{wp}] & \text{Cov}[r_{wp}, r_{wq}] & E[r_{wp}] & 1 \\ \text{Cov}[r_{wp}, r_{wq}] & \text{Var}[r_{wq}] & E[r_{wq}] & 1 \\ \text{Cov}[r', r_{wp}] & \text{Cov}[r', r_{wq}] & E[r'] & 1 \end{pmatrix} = 2.
\]

which is equivalent to that there exists \( w \in \mathbb{R} \) such that the conditions (5) and (6) hold. Conversely, if there is no such a \( w \), then also from Theorem 1.1, \( S_2 \) is not a efficient subset of the new asset set, and it is possible only if \( u \) changes the portfolio frontier of \( S_n \). The last two parts of the theorem are immediate consequences of its preceding part. \( \square \)

This theorem is a generalization of the idea in Sharpe’s proof of the CAPM. In particular, if the market portfolio is mean-variance efficient, then form this theorem, the “traditional” CAPM holds.
2 What is CAPM without Short Selling?

If the market does not permit the short selling, then Markowitz’s mean-variance portfolio selection problem becomes

\[
\begin{align*}
\min \quad \sigma_w^2 &= w^T V w = \sum_{i,j=1}^{n} V_{ij} w_i w_j \\
\text{s.t.} \quad w^T e &= w_1 + w_2 + \cdots + w_n = 1 \\
\mu_w &= w^T \mu = w_1 \mu_1 + w_2 \mu_2 + \cdots + w_n \mu_n = \bar{\mu} \\
\mu_i &
\begin{cases}
2V \bar{w} - \lambda_1 \mu - \lambda_2 \sigma^2 - \nu = 0, \\
\bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_n = 1, \\
\bar{w}_1 \mu_1 + \bar{w}_2 \mu_2 + \cdots + \bar{w}_n \mu_n = \bar{\mu}, \\
\nu_i \geq 0, \bar{w}_i \geq 0, \nu_i \bar{w}_i = 0, \\
i = 1, 2, \ldots, n.
\end{cases}
\end{align*}
\]

The problem (7) is also a quadratic convex programming, but with two constraints of linear equalities and \(n\) constraints of linear inequalities. Its Lagrange multipliers also always exist ([14], p. 279). Therefore, it is also solved by the Lagrange multiplier method, i.e., \(\bar{w}\) and \(\lambda_1, \lambda_2, \nu = (\nu_1, \ldots, \nu_n)^T\), are its solution and its Lagrange (or Kuhn-Tucker) multipliers if and only if they satisfy the following Kuhn-Tucker condition

\[
\begin{align*}
2V \bar{w} - \lambda_1 \mu - \lambda_2 \sigma^2 - \nu &= 0, \\
\bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_n &= 1, \\
\bar{w}_1 \mu_1 + \bar{w}_2 \mu_2 + \cdots + \bar{w}_n \mu_n &= \bar{\mu}, \\
\nu_i &\geq 0, \bar{w}_i \geq 0, \nu_i \bar{w}_i = 0, \\
i = 1, 2, \ldots, n.
\end{align*}
\]

In this case, we cannot conclude that the original CAPM in Sharpe’s sense holds, because the two fund separation theorem no longer holds and the capital market line is no longer a straight line.

So, we can propose a question: What is the CAPM in a market which does not permit the short selling? By the same idea, we can pose the same problem: Let \(S_n = \{1, 2, \ldots, n\}\) be a set of assets and \(S_k = \{1, 2, \ldots, k\} \subset S_n\) be its subset. What is a necessary and sufficient condition for that the mean-variance portfolio frontier without short selling of \(S_k\) is the same as that of \(S_n\)? We will call \(S_k\) a frontier subset of \(S_n\) if \(S_k\) possesses this property. This is a new and quite difficult problem. Our result is as follows.

For facilitating the expression of the theorem, we call \(S_J = \{j_1, \ldots, j_t\} \subset S_k\) to be a composing set of a frontier portfolio of \(S_k\), if the following system of linear inequations has a solution \((w_{j_1}, \ldots, w_{j_t}, \lambda_1, \lambda_2)\):

\[
\begin{align*}
w_{j_1} + w_{j_2} + \cdots + w_{j_t} &= 1 \\
V_{j_1j_1} w_{j_1} + V_{j_1j_2} w_{j_2} + \cdots + V_{j_1j_t} w_{j_t} - \frac{1}{2} \lambda_1 \mu_{j_1} - \frac{1}{2} \lambda_2 &= 0 \\
V_{j_2j_1} w_{j_1} + V_{j_2j_2} w_{j_2} + \cdots + V_{j_2j_t} w_{j_t} - \frac{1}{2} \lambda_1 \mu_{j_2} - \frac{1}{2} \lambda_2 &= 0 \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]

(9)
\[ V_{ji}w_{j1} + V_{ji}w_{j2} + \cdots + V_{ji}w_{ji} - \frac{1}{2}\lambda_1\mu_i - \frac{1}{2}\lambda_2 \geq 0, \quad \forall i \in S_k \setminus S_j, \]
\[ w_{j1}, w_{j2}, \ldots, w_{ji} > 0 \]

**Theorem 2.1** Let \( S_n = \{1, 2, \ldots, n\} \) be a set of \( n \) assets. \( r_1, r_2, \ldots, r_n \) be their random rates of return respectively. \( \mu_i = E[r_i], \) \( V_{ij} = \text{Cov}[r_i, r_j], \) \( i, j = 1, 2, \ldots, n. \) \( S_k = \{1, 2, \ldots, k\} \subset S_n. \) Then in the case without short selling, \( S_k \) is a frontier subset of \( S_n \) if and only if

i) \[
\max_{1 \leq i \leq k} \mu_i = \max_{1 \leq i \leq n} \mu_i, \quad \min_{1 \leq i \leq k} \mu_i = \min_{1 \leq i \leq n} \mu_i.
\]

ii) for any composing set \( S_j(\subset S_k) \) of a frontier portfolio of \( S_k, \) and any \( i > k, \) there exist \( t_{j1}^i, t_{j2}^i, \ldots, t_{jk}^i \in \mathbb{R} \) such that

1. For any \( l \notin S_j, t_{jl}^i \geq 0, \)
2. \( t_{j1}^i + t_{j2}^i + \cdots + t_{jk}^i = 1, \) \( t_{j1}^i\mu_1 + t_{j2}^i\mu_2 + \cdots + t_{jk}^i\mu_k = \mu_i, \)
3. for any \( j \in S_j, V_{ij} \geq t_{j1}^iV_{ij} + t_{j2}^iV_{2j} + \cdots + t_{jk}^iV_{kj}. \)

The proof is in Appendix 2.

**Corollary 2.1** If \( S_k \) satisfies i) and for any \( i > k, \) there exists \( t_1^i, t_2^i, \ldots, t_k^i \geq 0 \) such that

1. \( t_1^i + t_2^i + \cdots + t_k^i = 1, \) \( t_1^i\mu_1 + t_2^i\mu_2 + \cdots + t_k^i\mu_k = \mu_i, \)
2. for any \( j = 1, \ldots, k, V_{ij} \geq t_1^iV_{1j} + t_2^iV_{2j} + \cdots + t_k^iV_{kj}, \)

then in the case without short selling, \( S_k \) is a frontier subset of \( S_n. \)

From Theorem 2.1, we have no longer a similar Theorem 1.2; i.e., in the case without short selling, there is no the CAPM “in Sharpe’s sense”. However, we have the following theorem:

**Theorem 2.2** Let \( S_2 = \{1, 2\} \) be a set of 2 assets, \( r_1 \) and \( r_2 \) be their random rates of return respectively, and \( E[r_1] \neq E[r_2]. \) Then any asset \( u \) with the rate of return \( v \) does not change the portfolio frontier of \( S_2 \) in the case without short selling if and only if there exists \( w \in [0, 1] \) such that

\[
E[r] = (1 - w)E[r_1] + wE[r_2],
\]
\[
\begin{align*}
\text{Cov}[r, r_1] & \geq (1 - w)\text{Var}[r_1] + w\text{Cov}[r_1, r_2] \\
\text{Cov}[r, r_2] & \geq (1 - w)\text{Cov}[r_1, r_2] + w\text{Var}[r_2]
\end{align*}
\] (11)

In particular, if \(\text{Cov}[r_1, r_2] = 0\), then any asset \(u\) with the rate of return \(r\) does not change the portfolio frontier of \(S_2\) in the case without short selling if and only if the following “quasi-zero-\(\beta\) CAPM” holds:

\[
E[r] - E[r_2] = \beta_{r2}(E[r_1] - E[r_2]),
\]

\[
\max\{0, 1 - \text{Cov}[r, r_2]/\text{Var}[r_2]\} \leq \beta_{r2} \leq \min\{1, \text{Cov}[r, r_1]/\text{Var}[r_1]\}.
\]

If \(S_2\) only includes a riskless asset with a constant rate of return \(\mu_0\) and a risky asset with a random rate of return \(r_m\), then the condition above reduces to that the following “quasi-CAPM” holds:

\[
E[r] - \mu_0 = \beta'_{rm}(E[r_m] - \mu_0),
\]

\[
0 \leq \beta'_{rm} \leq \min\{1, \text{Cov}[r, r_m]/\text{Var}[r_m]\}.
\]

The last part of the theorem means that even when the market without short selling has a “two fund separation theorem”, the “CAPM” is not yet true. For an asset with an expected rate of return between the riskless rate of return and the expected rate of “market” return, its expected rate of return might be less than that in the “traditional” CAPM.

## 3 Conclusion

In the classic Sharpe’s paper, the proof of the CAPM only shows that if an asset with the random rate of return \(r_i\) does not change the “capital market line” generated by the riskless asset with a constant rate of return \(\mu_0\) and an mean-variance efficient asset with a random rate of return \(r_m\), then

\[
E[r_i] - \mu_0 = \beta_{im}(E[r_m] - \mu_0), \quad \beta_{im} = \frac{\text{Cov}[r_i, r_m]}{\text{Var}[r_m]}.
\]

This idea may be generalized to obtain a general theorem. However, in the case without short selling, the corresponding general theorem is much more complex, and the “CAPM without short selling” is also very different. A simple corresponding result would be

\[
E[r_i] - \mu_0 \leq \beta_{im}(E[r_m] - \mu_0), \quad \beta_{im} = \frac{\text{Cov}[r_i, r_m]}{\text{Var}[r_m]}.
\]

It would be a remarkable point in a stock market without short selling as in China.
Appendix 1: Sharpe’s Proof of CAPM

In [16], Sharpe said: “In order to derive conditions for equilibrium in the capital market we invoke two assumptions. First, we assume common pure rate of interest, with all investors able to borrow or funds on equal terms. Second, we assume homogeneity of investor expectation: investors are assumed to agree on the prospects of various investments—the expected values, standard deviations and correlation coefficients...” However, by a non-formal description, these two assumptions are only for showing: “Capital asset prices must, of course, continue to change until a set of prices is attained for which every asset enters at least one combination lying on the capital market line.” It is not the CAPM itself, but its precondition.

![Figure: Figure 7 of Sharpe’s paper](image)

In fact, Sharpe’s proof of the CAPM is just in a footnote of his paper. The footnote follows “Figure 7 (see our Figure)” of the paper and around Figure 7 Sharpe said: “Figure 7 illustrates the typical relationship between a single capital asset (point i) and an efficient combination of assets (point g) of which it is a part. The curve igg’ indicates all $E_M$, $\sigma_R$ values which can be obtained with feasible combination in terms of a proportion $\alpha$ of asset i and $(1 - \alpha)$ of combination g.” “In Figure 7 the curve igg’ has been drawn tangent to the capital market line (PZ) at point g. This is no accident. All such curves must be tangent to the capital market line in equilibrium, since (1) they must touch it at the point representing the efficient combinations and (2) they are continuous at the point. Under these conditions a lack of tangency would imply that the curve intersects PZ. But then some feasible combination of assets would lie to the right of the capital market line, an obvious impossibility since the capital market line represents the efficient
boundary of feasible values of $E_R$ and $\sigma_R$." "The requirement that curves such as igg' be tangent to the capital market line can be shown to lead to a relative simply formula..."

After this description, the CAPM appears in the footnote 22 of the paper: "The standard deviation of a combination of g and i will be:

$$\sigma = \sqrt{\alpha^2 \sigma_{Ri}^2 + (1 - \alpha)^2 \sigma_{Rg}^2 + 2r_{ig} \alpha (1 - \alpha) \sigma_{Ri} \sigma_{Rg}}$$

at $\alpha = 0$:

$$\frac{d\sigma}{d\alpha} = -\frac{1}{\sigma} [\sigma_{Rg}^2 - r_{ig} \sigma_{Ri}]$$

but $\sigma = \sigma_{Rg}$ at $\alpha = 0$. Thus:

$$\frac{d\sigma}{d\alpha} = -[\sigma_{Rg} - r_{ig} \sigma_{Ri}]$$

The expected return of a combination will be:

$$E = \alpha E_{Ri} + (1 - \alpha) E_{Rg}$$

Thus, at all values of $\alpha$:

$$\frac{dE}{d\alpha} = -[E_{Rg} - E_{Ri}]$$

and, at $\alpha = 0$:

$$\frac{d\sigma}{dE} = \frac{\sigma_{Rg} - r_{ig} \sigma_{Ri}}{E_{Rg} - E_{Ri}}$$

Let the equation of the capital market line be:

$$\sigma_R = s(E_R - P)$$

where $P$ is the pure interest rate. Since igg' is tangent to the line when $\alpha = 0$, and since $(E_{Rg}, \sigma_{Rg})$ lies on the line:

$$\frac{\sigma_{Rg} - r_{ig} \sigma_{Ri}}{E_{Rg} - E_{Ri}} = \frac{\sigma_{Rg}}{E_{Rg} - P}$$

or:

$$\frac{r_{ig} \sigma_{Ri}}{\sigma_{Rg}} = -\left[\frac{P}{E_{Rg} - P}\right] + \left[\frac{1}{E_{Rg} - P}\right] E_{Ri}.$$  

The last equality is just the CAPM.

**Appendix 2: Proof of Theorem 2.1**

For proving Theorem 2.1, we show the following proposition:
Proposition 1 Assume that the system of linear inequalities

$$w_1 + w_2 + \cdots + w_{k'} = 1$$
$$V_{11}w_1 + V_{12}w_2 + \cdots + V_{1k}w_{k'} - \frac{1}{2}\lambda_1\mu_1 - \frac{1}{2}\lambda_2 = 0$$
$$V_{21}w_1 + V_{22}w_2 + \cdots + V_{2k}w_{k'} - \frac{1}{2}\lambda_1\mu_2 - \frac{1}{2}\lambda_2 = 0$$
$$\vdots \quad \vdots \quad \vdots$$
$$V_{k'1}w_1 + V_{k'2}w_2 + \cdots + V_{k'k}w_{k'} - \frac{1}{2}\lambda_1\mu_1 - \frac{1}{2}\lambda_2 = 0 \quad (12)$$
$$V_{k'+1,1}w_1 + V_{k'+1,2}w_2 + \cdots + V_{k'+1,k}w_{k'} - \frac{1}{2}\lambda_1\mu_{k'+1} - \frac{1}{2}\lambda_2 \geq 0$$
$$\vdots \quad \vdots \quad \vdots$$
$$V_{k1}w_1 + V_{k2}w_2 + \cdots + V_{kk}w_{k'} - \frac{1}{2}\lambda_1\mu_k - \frac{1}{2}\lambda_2 \geq 0 \quad w_1, w_2, \ldots, w_{k'} > 0$$

admits a solution \((w_1, \ldots, w_{k'}, \lambda_1, \lambda_2) \in \mathbb{R}^{k'+2}\). Then its every solution \((w_1, \ldots, w_{k'}, \lambda_1, \lambda_2) \in \mathbb{R}^{k'+2}\) satisfies

$$V_{i1}w_1 + V_{i2}w_2 + \cdots + V_{ik}w_{k'} - \frac{1}{2}\lambda_1\mu_k - \frac{1}{2}\lambda_2 \geq 0 \quad (13)$$

if and only if there exist \(t^i_1, t^i_2, \ldots, t^i_k \in \mathbb{R}\) such that

1. for any \(l > k',\ t^i_l \geq 0\),
2. \(t^i_1 + t^i_2 + \cdots + t^i_k = 1,\ t^i_1\mu_1 + t^i_2\mu_2 + \cdots + t^i_k\mu_k = \mu_i\),
3. for any \(j = 1, \ldots, k',\ V_{ij} \geq t^i_1V_{1j} + t^i_2V_{2j} + \cdots + t^i_kV_{kj}\).

Proof: Necessity. We use the method of reduction to absurdity. If such \(t^i_1, t^i_2, \ldots, t^i_k\) do not exist, then we show that there exists a solution of (12) such that (13) is not true. Thus, for \(i > k\), define a closed convex set \(A \in \mathbb{R}^{k'+2}\) which is the collection of all \(x = (x_1, \ldots, x_{k'+1}, x_{k'+2})^T \in \mathbb{R}^{k'+2}\) such that

\[
x_j = t_1V_{1j} + t_2V_{2j} + \cdots + t_kV_{kj} + u_j - s^iV_{ij}, \quad j = 1, \ldots, k';
\]
\[
x_{k'+1} = t_1\mu_1 + t_2\mu_2 + \cdots + t_k\mu_k - s^i\mu_i;
\]
\[
x_{k'+2} = t_1 + t_2 + \cdots + t_k - s^i;
\]
\[
t_1, \ldots, t_k \in \mathbb{R},\ t_{k'+1}, \ldots, t_k, u_1, \ldots, u_{k'}, s^i \in \mathbb{R}_+,
\]
\[
u_1 + u_2 + \cdots + u_{k'} + s^i \geq 1.
\]

Then we can show \(0 \notin A\). Otherwise, if there exists

\[
\bar{t}_1, \ldots, \bar{t}_{k'} \in \mathbb{R},\ \bar{t}_{k'+1}, \ldots, \bar{t}_k, \bar{u}_1, \ldots, \bar{u}_{k'}, \bar{s}^i \in \mathbb{R}_+,
\]
\[
\bar{u}_1 + \bar{u}_2 + \cdots + \bar{u}_{k'} + \bar{s}^i \geq 1,
\]

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such that

\[ \tilde{t}_1 V_{1j} + \tilde{t}_2 V_{2j} + \cdots + \tilde{t}_k V_{kj} + \tilde{u}_j - s^i V_{ij} = 0, \quad j = 1, \ldots, k'; \]
\[ \tilde{t}_1 \mu_1 + \tilde{t}_2 \mu_2 + \cdots + \tilde{t}_k \mu_k - s^i \mu_i = 0; \quad \tilde{t}_1 + \tilde{t}_2 + \cdots + \tilde{t}_k - s^i = 0; \]

then when \( s^i > 0 \), it obviously means that there exist \( t^i_1, \ldots, t^i_k \) with the properties above; and when \( s^i = 0 \), it follows that \( \tilde{u}_1, \ldots, \tilde{u}_{k'} \) are not all equal to zero. In this case, for any \( w_1, \ldots, w'_{k'} > 0 \), we have

\[ \sum_{j=1}^{k'} \sum_{l=1}^{k} w_j V_{lj} \tilde{t}_l = \sum_{j=1}^{k'} w_j (-u_j) < 0. \]

On the other hand, however, there exists \( w_1, \ldots, w'_{k'} > 0 \) satisfying (12), and multiplying the two sides of \((l + 1)\)-th equation of (12) by \( \tilde{t}_l \) and adding all the equations, we obtain

\[ \sum_{j=1}^{k'} \sum_{l=1}^{k} w_j V_{lj} \tilde{t}_l \geq 0. \]

This contradiction shows that \( s^i = 0 \) is impossible. Hence, \( 0 \notin A \).

Thus, by the Separation Theorem of Convex Sets, there exists \( v^i = (v^i_1, \ldots, v^i_{k'+1}, v^i_{k'+2})^T \) such that

\[ \forall x \in A, \quad (v^i)^T x > 0. \quad (14) \]

Take \( s^i = 1, \ t_1 = \cdots = t_k = u_1 = \cdots = u'_{k'} = 0 \), we have

\[ a = v^i_1 V_{11} + v^i_2 V_{12} + \cdots + v^i_{k'} V_{k'1} + v^i_{k'+1} \mu_1 + v^i_{k'+2} < 0. \quad (15) \]

On the other hand, take \( t_1 \neq 0, \ t_2 = \cdots = t_{k'} = 0, \ u_1 = \cdots = u_{k'} = 0, \ s^i = 1 \). we have

\[ (v^i)^T x = t_1 (v^i_1 V_{11} + v^i_2 V_{12} + \cdots + v^i_{k'} V_{k'1} + v^i_{k'+1} \mu_1 + v^i_{k'+2}) - a > 0. \]

But \( t_1 \) may be any real number, it is possible only when

\[ v^i_1 V_{11} + v^i_2 V_{12} + \cdots + v^i_{k'} V_{k'1} + v^i_{k'+1} \mu_1 + v^i_{k'+2} = 0. \]

By the same reasoning, we have

\[ v^i_1 V_{j1} + v^i_2 V_{j2} + \cdots + v^i_{k'} V_{j_{k'}} + v^i_{k'+1} \mu_j + v^i_{k'+2} = 0, \quad j = 1, \ldots, k'. \quad (16) \]

And from \( t_{k'+1}, \ldots, t_k \geq 0 \), the same reasoning deduces

\[ v^i_1 V_{j1} + v^i_2 V_{j2} + \cdots + v^i_{k'} V_{j_{k'}} + v^i_{k'+1} \mu_1 + v^i_{k'+2} \geq 0, \quad j = k'+1, \ldots, k. \quad (17) \]

In addition, for \( x \in A \), take \( t_1 = \cdots = t_k = 0, \ u_1 \geq 1, \ u_2 = \cdots = u_{k'} = 0, \ s^i = 0 \), it follows \( v^i_1 u_1 > 0 \). It is possible only when \( v^i_1 > 0 \). By the same reasoning,

\[ v^i_j > 0, \quad j = 1, \ldots, k'. \]
Without loss of generality, suppose that \( v_1^i + \cdots + v_k^i = 1 \). Then from (16), (17), the system of linear inequations (12) admits a solution

\[
 w_1 = v_1^i, \ldots, w_{k'} = v_{k'}^i, \quad \lambda_1 = -2v_{k'+1}^i, \quad \lambda_2 = -2v_{k'+2}^i.
\]

But from (15), the inequality (13) is not true.

**Sufficiency** If there exist such \( t_1^i, t_2^i, \ldots, t_k^i \in \mathbb{R} \), then from (12) and the properties of \( t_l^i \),

\[
 \sum_{j=1}^{k'} V_{ij} w_j - \frac{1}{2} \lambda_1 \mu_i - \frac{1}{2} \lambda_2 \geq \sum_{j=1}^{k'} \left( V_{ij} w_j - \frac{1}{2} \lambda_1 \mu_i - \frac{1}{2} \lambda_2 \right) = \sum_{l=1}^k t_l^i \left( \sum_{j=1}^{k'} V_{ij} w_j - \frac{1}{2} \lambda_1 \mu_i - \frac{1}{2} \lambda_2 \right) \geq 0,
\]

i.e. (13) holds. □

In the proof of Proposition 1, we also show the following proposition:

**Proposition 2** \( S_J = \{j_1, \ldots, j_l\} \subset S_k \) is a composing set of a frontier portfolio of \( S_k \) if and only if there is no \( t_1, \ldots, t_k \in \mathbb{R} \), such that

1. for any \( l \in S_k \setminus S_J \), \( t_l \geq 0 \),
2. \( t_1^i + t_2^i + \cdots + t_k^i = 0 \), \( t_1^i \mu_1 + t_2^i \mu_2 + \cdots + t_k^i \mu_k = 0 \),
3. for some \( j \in S_J \), \( t_1^i V_{1j} + t_2^i V_{2j} + \cdots + t_k^i V_{kj} < 0 \).

**Proof of Theorem 3.1:** **Necessity.** Suppose that \( S_k \) is a frontier subset of \( S_n \) in the case without short selling. Then i) is obviously necessary. Otherwise, it is impossible to generate the portfolio frontier of \( S_n \) by \( S_k \). Meanwhile, every frontier portfolio of \( S_k \) must be that of \( S_n \). Suppose that \( S_J \subset S_k \) is a composing set of a frontier portfolio of \( S_k \). Without loss of generality, let \( S_J = S_{k'} = \{1, \ldots, k'\} \), and suppose that \( \mu_1 \neq \mu_2 \). Then when

\[
 w_1 + w_2 + \cdots + w_{k'} = 1 \\
 V_{11} w_1 + V_{12} w_2 + \cdots + V_{1k'} w_{k'} - \frac{1}{2} \lambda_1 \mu_1 - \frac{1}{2} \lambda_2 = 0 \\
 V_{21} w_1 + V_{22} w_2 + \cdots + V_{2k'} w_{k'} - \frac{1}{2} \lambda_1 \mu_2 - \frac{1}{2} \lambda_2 = 0 \\
 \vdots \\
 V_{k'1} w_1 + V_{k'2} w_2 + \cdots + V_{k'k'} w_{k'} - \frac{1}{2} \lambda_1 \mu_{k'} - \frac{1}{2} \lambda_2 = 0 \\
 V_{k'+1,1} w_1 + V_{k'+1,2} w_2 + \cdots + V_{k'+1,k'} w_{k'} - \frac{1}{2} \lambda_1 \mu_{k'+1} - \frac{1}{2} \lambda_2 \geq 0 \\
 \vdots \\
 V_{k'k} w_1 + V_{k'k+1,2} w_2 + \cdots + V_{k'k,k'} w_{k'} - \frac{1}{2} \lambda_1 \mu_k - \frac{1}{2} \lambda_2 \geq 0 \\
 w_1, w_2, \ldots, w_{k'} > 0
\]
admits a solution \((w_1, \ldots, w_{k'}, \lambda_1, \lambda_2) \in \mathbb{R}^{k'+2}\), there exist \(\lambda'_1, \lambda'_2 \in \mathbb{R}\) such that

\[
\begin{align*}
  w_1 + w_2 + \cdots + w_{k'} & = 1 \\
  V_{11}w_1 + V_{12}w_2 + \cdots + V_{1k'}w_{k'} - \frac{1}{2}\lambda'_1\mu_1 - \frac{1}{2}\lambda'_2 & = 0 \\
  V_{21}w_1 + V_{22}w_2 + \cdots + V_{2k'}w_{k'} - \frac{1}{2}\lambda'_1\mu_2 - \frac{1}{2}\lambda'_2 & = 0 \\
  \vdots & \vdots \\
  V_{k'1}w_1 + V_{k'2}w_2 + \cdots + V_{k'k'}w_{k'} - \frac{1}{2}\lambda'_1\mu_{k'} - \frac{1}{2}\lambda'_2 & = 0 \\
  V_{k'+1,1}w_1 + V_{k'+1,2}w_2 + \cdots + V_{k'+1,k'}w_{k'} - \frac{1}{2}\lambda'_1\mu_{k'+1} - \frac{1}{2}\lambda'_2 & \geq 0 \\
  \vdots & \vdots \\
  V_{k,k}w_1 + V_{k,k}w_2 + \cdots + V_{k,k}w_{k'} - \frac{1}{2}\lambda'_1\mu_k - \frac{1}{2}\lambda'_2 & \geq 0 \\
  w_1, w_2, \ldots, w_{k'} & > 0
\end{align*}
\]

From \(\mu_1 \neq \mu_2\) and second and third equations, it immediately follows \(\lambda'_{k'} = \lambda'_1\), \(\lambda'_{k'} = \lambda'_2\), which means that (12) implies (13). Therefore, from Proposition 1, ii) holds.

On the other hand, if all the rates of return on the assets in \(S'_{k'}\) are equal to \(\mu'\), then when \(\mu_i = \mu'\), any solution \((w_1, \ldots, w_{k'}, \lambda) \in \mathbb{R}^{k'+1}\) of the problem

\[
\begin{align*}
  w_1 + w_2 + \cdots + w_{k'} & = 1 \\
  V_{11}w_1 + V_{12}w_2 + \cdots + V_{1k'}w_{k'} - \frac{1}{2}\lambda & = 0 \\
  V_{21}w_1 + V_{22}w_2 + \cdots + V_{2k'}w_{k'} - \frac{1}{2}\lambda & = 0 \\
  \vdots & \vdots \\
  V_{k'1}w_1 + V_{k'2}w_2 + \cdots + V_{k'k'}w_{k'} - \frac{1}{2}\lambda & = 0 \\
  w_1, w_2, \ldots, w_{k'} & > 0
\end{align*}
\]  

as a solution of the problem

\[
\begin{align*}
  \min \sigma^2_w = \sum_{i,j=1}^{k'} V_{ij}w_iw_j \\
  \text{s.t.} \quad w_1 + w_2 + \cdots + w_{k'} & = 1 \\
  w_i & \geq 0, \quad i = 1, 2, \ldots, k'. \quad (20)
\end{align*}
\]

must be a solution of the problem

\[
\begin{align*}
  \min \sigma^2_w = \sum_{i,j=1}^{k'} (V_{ij}w_iw_j + V_{ij}w_iw_j) \\
  \text{s.t.} \quad w_1 + w_2 + \cdots + w_{k'} + w_i & = 1 \\
  w_i & \geq 0, \quad i = 1, 2, \ldots, k'. \quad (21)
\end{align*}
\]
Otherwise, it would be contradict to that $S_k$ is a frontier subset of $S_n$. Hence, we have always

$$V_{i1}w_1 + V_{i2}w_2 + \cdots + V_{ik'}w_{k'} - \frac{1}{2} \lambda \geq 0.$$  

Applying Proposition 3.1 again, ii) holds for $t_{k'+1}^i = \cdots = t_k^i = 0$ (in general, it means for any $j \in S_k \setminus S_j$, $t_j^i = 0$). If $\mu_i \neq \mu'$, then we can find $\lambda_1$, such that for any $(w_1, \ldots, w_{k'}, \lambda) \in \mathbb{R}^{k'+1}$ satisfying (18),

$$V_{i1}w_1 + V_{i2}w_2 + \cdots + V_{ik'}w_{k'} - \frac{1}{2} \lambda_1(\mu_i - \mu_0) - \frac{1}{2} \lambda \geq 0$$

holds. Hence, with a suitable transform, we can also applying Proposition 1 to obtain that ii) holds with $t_{k'+1}^i = \cdots = t_k^i = 0$.

**Sufficiency.** The proof is almost the same as that of Proposition 1, i.e., if ii) holds, then for any $(w_{ji}, \ldots, w_{ji}, \lambda_1, \lambda_2)$ and $i > k$ satisfying (9),

$$V_{ij1}w_{j1} + V_{ij2}w_{j2} + \cdots + V_{ij1}w_{ji} - \frac{1}{2} \lambda_1 \mu_i - \frac{1}{2} \lambda_2 \geq 0$$

holds. Hence, each frontier portfolio of $S_k$ must be that of $S_n$. Joining up with i), it follows that $S_k$ is a frontier subset of $S_n$. \[\square\]

**References**


